Physics 106a
Final Exam Solutions

J. Preskill

1 a)

Rolls without slipping ⇒

$$r(\theta+\psi) = R(\theta-\phi)\ ; \quad \dot\phi = \alpha t$$

$$\Rightarrow r\dot\psi = (R-r)\dot\theta - R\alpha t$$

$$K.E. = \tfrac{1}{2}m(R-r)^2\dot\theta^2 + \tfrac{1}{4}mr^2\dot\psi^2$$

$$= \tfrac{3}{4}m(R-r)^2\dot\theta^2 - \tfrac{1}{2}m\alpha t R(R-r)\dot\theta + \tfrac{1}{4}mR^2\alpha^2t^2$$

b) $L = T + mg(R-r)\cos\theta$

$$\Rightarrow \frac{\partial L}{\partial\theta} = -mg(R-r)\sin\theta = \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{d}{dt}\left[\tfrac{3}{2}m(R-r)^2\dot\theta - \tfrac{1}{2}mR(R-r)\alpha t\right]$$

$$\longrightarrow \tfrac{3}{2}(R-r)\ddot\theta = \tfrac{1}{2}R\alpha - g\sin\theta$$

c) $\ddot\theta = 0 \Rightarrow \sin\theta = R\alpha/2g\ ; \quad \alpha < 2g/R$

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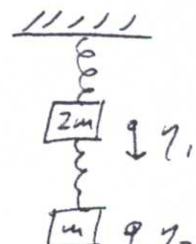
2 a) $T = \frac{1}{2}(2m)\dot\eta_1^2 + \frac{1}{2}m\dot\eta_2^2$



Since gravitational potential energy depends linearly on height of the blocks, gravity does not contribute to quadratic part of V.

$$V = \tfrac{1}{2}K\left[\eta_1^2 + (\eta_2-\eta_1)^2\right], \quad L = T - V$$

b) we have $\underset{\sim}{T} = \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}$ and $\underset{\sim}{V} = \begin{pmatrix} 2K & -K \\ -K & K \end{pmatrix}$.

So

$$\det(\omega^2\underline{T} - \underline{V}) = \det\begin{pmatrix} 2m\omega^2 - 2K & K \\ K & m\omega^2 - K \end{pmatrix}$$

$$= 2(m\omega^2 - K)^2 - K^2 = 2m^2\omega^4 - 4Km\omega^2 + K^2 = 0.$$

Thus, $\omega^2 = \frac{1}{4m^2}\left[4Km \pm 16K^2m^2 - 8K^2m^2\right] = \frac{K}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)$

c) Let $\boxed{\omega_1^2 = K/m\left(1+\frac{1}{\sqrt{2}}\right)}$. Then $\omega_1^2 \underset{\sim}{T} - \underset{\sim}{V} = K\begin{pmatrix} \sqrt{2} & 1 \\ 1 & 1/\sqrt{2} \end{pmatrix}$,

and $(\omega_1^2 \underset{\sim}{T} - \underset{\sim}{V})\vec{X}^{(1)} = 0 \Rightarrow \vec{X}^{(1)} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$.

Normalizing: $\boxed{\vec{X}^{(1)} = \frac{1}{\sqrt{4m}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}}$

Similarly $\boxed{\omega_2^2 = K/m\ 1 - \frac{1}{\sqrt{2}}} \Rightarrow \omega_2^2 \underset{\sim}{T} - \underset{\sim}{V} = K\begin{pmatrix} -\sqrt{2} & 1 \\ 1 & -1/\sqrt{2} \end{pmatrix}$,

so $(\omega_2^2 \underset{\sim}{T} - \underset{\sim}{V})\vec{X}^{(2)} = 0 \Rightarrow \vec{X}^{(2)} \propto \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

Normalizing: $\boxed{\vec{X}^{(2)} = \frac{1}{\sqrt{4m}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}}$

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3 a) $[Q,Q] = [P,P] = 0;\quad [Q,P] = \cos^2\alpha\,[q,p] - \sin^2\alpha\,[p,q]$
$= [q,p] = 1$

b) $p = \frac{1}{\cos\alpha}[P - \sin\alpha\, q] = \partial G_2/\partial q$

$Q = \frac{1}{\cos\alpha}[q - \sin\alpha\, P] = \partial G_2/\partial P$

$\Rightarrow \quad G_2(q,P) = \frac{1}{\cos\alpha}\left[qP - \frac{1}{2}\sin\alpha\,(P^2+q^2)\right]$

c) $G_2 \sim qP - \frac{1}{2}\alpha(p^2+q^2) + O(\alpha^2)$

$\Rightarrow p^2+q^2$ conserved

4a) T is a generating function of the third type $\Rightarrow$

$q = -\frac{\partial T}{\partial p}$ and $\bar{H}(Q,P,t) = H(q,p,t) + \frac{\partial T}{\partial t}$

we require $\bar{H} = 0$ or $\boxed{\frac{\partial T}{\partial t} + H\left(-\frac{\partial T}{\partial p}, p, t\right) = 0}$

b) $H = \frac{1}{2m}p^2 + mgq \Rightarrow \boxed{\frac{\partial T}{\partial t} + \frac{1}{2m}p^2 - mg\frac{\partial T}{\partial p} = 0}$

c) Let $T(p,t) = W(p) - Qt \Rightarrow Q = \frac{1}{2m}p^2 - mg\frac{dW}{dp}$

or $T = \frac{1}{mg}\int dp\left(\frac{1}{2m}p^2 - Q\right) - Qt \Rightarrow \boxed{T = \frac{1}{6m^2 g}p^3 - \frac{Q}{mg}p - Qt}$

d) $\boxed{P = -\frac{\partial T}{\partial Q} = \frac{p}{mg} + t}$ $\qquad q = -\frac{\partial T}{\partial p} = \frac{-p^2}{2m^2 g} + \frac{Q}{mg} \Rightarrow \boxed{Q = \frac{1}{2m}p^2 + mgq}$

Q is the energy, and P is the time when $p=0$ (i.e., the particle is at rest and its height is a maximum.)

$\boxed{p = mg(P-t)}$

$q = -\frac{1}{2m^2 g}p^2 + \frac{Q}{mg} \Rightarrow \boxed{q = -\frac{1}{2}g(t-P)^2 + \frac{Q}{mg}}$