Problem 1 - Differential Cross Section

a) First, we know that for $r > r_0$, $V = 0$. Thus, $\frac{m}{2}v^2 = E$, so that $v = \sqrt{2mE}$. Similarly, when $r < r_0$, $\frac{m}{2}v^2 - C = E$, so that $v' = \sqrt{2m(E+C)}$. Since in both of these regions, the potential is constant, the particles travel in straight lines – they only change directions at the boundary.

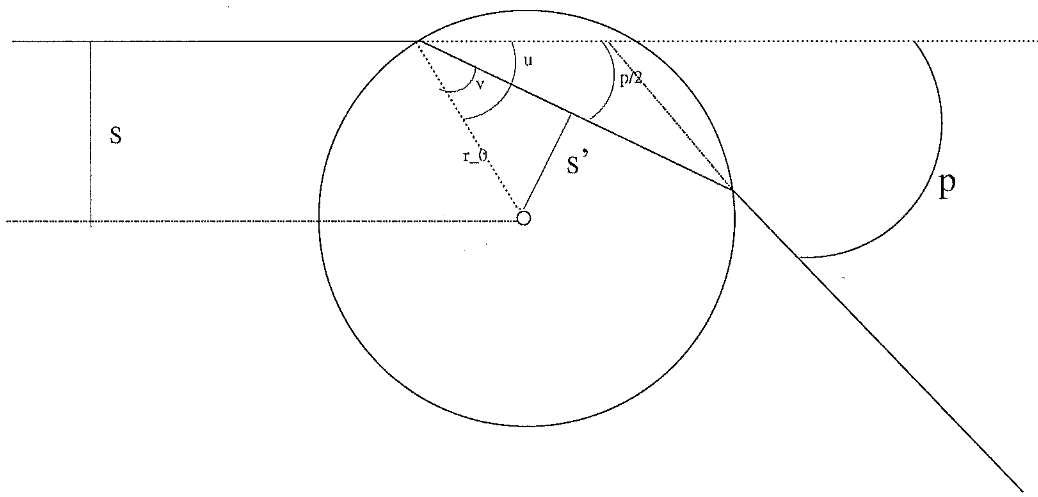

Figure 1:

Now, we would like to calculate p as a function of s. To do so, notice that by conservation of angular momentum about $r = 0$, we have

$$mvs = mv's'$$

$$\Rightarrow s' = s\sqrt{\frac{2mE}{2m(E+C)}} = s\sqrt{\frac{E}{E+C}}.$$

Now, from the figure, we see that $sin(v) = \frac{s'}{r_0}$ and $sin(u) = \frac{s}{r_0}$, so

$$\frac{p}{2} = u - v = sin^{-1}(\frac{s}{r_0}) - sin^{-1}(\frac{s'}{r_0})$$

$$= sin^{-1}(\frac{s}{r_0}) - sin^{-1}(\frac{s\sqrt{\frac{E}{E+C}}}{r_0})$$

$$\Rightarrow p = 2(sin^{-1}(\frac{s}{r_0}) - sin^{-1}(\frac{s\sqrt{\frac{E}{E+C}}}{r_0})).$$

b) Now, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{s}{sin(p)}|\frac{ds}{dp}|.$$

So, lets find an expression for $s(p)$:

$$cos(\frac{p}{2}) = cos(sin^{-1}(\frac{s}{r_0}) - sin^{-1}(\frac{s\sqrt{\frac{E}{E+C}}}{r_0})$$

$$= cos(sin^{-1}(\frac{s}{r_0}))cos(sin^{-1}(\frac{s\sqrt{\frac{E}{E+C}}}{r_0})) + sin(sin^{-1}(\frac{s}{r_0}))sin(sin^{-1}(\frac{s\sqrt{\frac{E}{E+C}}}{r_0}))$$

$$= \sqrt{(1-\frac{s^2}{r_0^2})(1-(\frac{E}{E+C})(\frac{s^2}{r_0^2}))} + \sqrt{\frac{E}{E+C}}\frac{s^2}{r_0^2}.$$

A little more work shows that

$$s = \frac{nr_0 sin(\frac{p}{2})}{\sqrt{n^2+1-2ncos(\frac{p}{2})}},$$

where we introduced the notation $n = \sqrt{\frac{E+C}{E}}$.
Now, we can take the derivative to get

$$\frac{d\sigma}{d\Omega} = \frac{s}{sin(p)}|\frac{ds}{dp}| = \frac{n^2r_0^2}{4cos(\frac{p}{2})}\frac{(n^2cos(p/2) + cos(p/2) - ncos^2(p/2) - n)}{(n^2+1-2ncos(p/2))^2}.$$

Foucault Pendulum

Let ω be the natural frequency of the pendulum, λ_0 the latitude of the pendulum, and Ω the angular velocity of the earth.

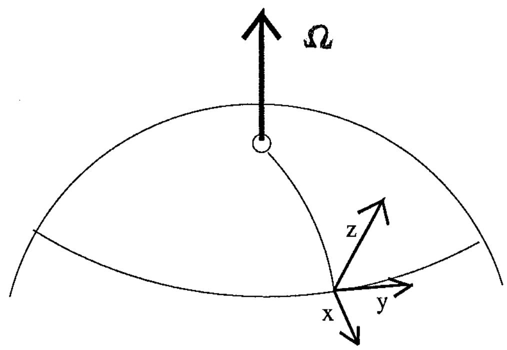

Figure 2: look.

In the approximation of small oscillations, we have $\dot{z}$ small compared to $\dot{x}$ and $\dot{y}$. Thus, the horizontal component of the coriolis force will be $2m\Omega_z(\dot{y}\hat{x} - \dot{x}\hat{y})$. This gives the equations of motion as

$$\ddot{x} = -\omega^2 x + 2\dot{y}\Omega_z$$

$$\ddot{y} = -\omega^2 y - 2\dot{x}\Omega_z,$$

where $\Omega_z = |\Omega| sin(\lambda_0)$, and λ_0 is the latitude.

Now, set $\eta = x + iy$, and the two equations reduce to one complex equation,

$$\ddot{\eta} + i2\Omega_z\dot{\eta} + \omega^2\eta = 0,$$

which we can solve using the ansatz $\eta = e^{\lambda t}$. The solution is $\lambda = -i\Omega_z \pm i\sqrt{\Omega_z^2 + \omega^2} = -i\Omega_z \pm i\omega$, where the second equality is true to first order in $\frac{\Omega_z}{\omega}$, and we know this quantity is small. Equivalently, the solution is

$$\eta = e^{-i\Omega_z t}(c_1 e^{i\omega t} + c_2 e^{-i\omega t}),$$

which makes it clear that the effect of the coriolis force is to cause the $x - y$ plane to rotate at angular velocity $-\Omega_z = |\Omega| sin(\lambda_0)$.

Torque Free Precession

Choose a basis such that $\hat{e}_3$ points along the direction of the line through both the mountains, and $\hat{e}_1$ and $\hat{e}_2$ lie in the plane perpendicular to that line. Then,

$$I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}MR^2 + 2mR^2 & 0 & 0 \\ 0 & \frac{2}{5}MR^2 + 2mR^2 & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{pmatrix}.$$

The Euler equations are then

$$I_{11}\dot{\omega_1} = (I_{11} - I_{33})\omega_3\omega_2$$
$$I_{22}\dot{\omega_2} = -(I_{11} - I_{33})\omega_3\omega_1$$
$$I_{33}\dot{\omega_3} = 0.$$

These have the solution

$$\omega_1 = Acos(\Omega t)$$
$$\omega_2 = Asin(\Omega t)$$
$$\omega_3 = constant,$$

where $\Omega = \frac{I_{33}-I_{11}}{I_{11}}\omega_3 = \frac{-2mR^2}{\frac{2}{5}MR^2+2mR^2}\omega_3$.
Now, $\omega_3 = \frac{\omega}{\sqrt{2}}$, so we can numbers given to find

$$|\Omega| = 3.7 \times 10^{-12} radday^{-1}.$$

Thus, the north pole will wander by $3.7 \times 10^{-12} \times 365 \frac{days}{year} \times 100 years = 1.35 \times 10^{-7} rad$, which translates to
a distance of $\frac{6\times10^6 m}{\sqrt{2}} \times 1.35 \times 10^{-7} rad = 0.6m$.

Spinning Cube

We define our coordinates as follows: the center of mass of the cube is at the origin, and the x_i axes each point along directions perpendicular to the faces of the cube.

The off-diagonal elements are zero.

$$I_{jk} = -\frac{m}{a^3}\left(\int_{\frac{-a}{2}}^{\frac{a}{2}} dx_i\right)\left(\int_{\frac{-a}{2}}^{\frac{a}{2}} x_j dx_j\right)\left(\int_{\frac{-a}{2}}^{\frac{a}{2}} x_k dx_k\right) = 0,$$

where $j \neq k$ and $i \neq j, k$.

The diagonal elements are all the same, and equal to

$$I_1 = \frac{m}{a^3}\int_{\frac{-a}{2}}^{\frac{a}{2}}\int_{\frac{-a}{2}}^{\frac{a}{2}}\int_{\frac{-a}{2}}^{\frac{a}{2}} (x_2^2 + x_3^2) dx_1 dx_2 dx_3$$

$$= \frac{1}{6}ma^2.$$

So,

$$I = \begin{pmatrix} \frac{1}{6}ma^2 & 0 & 0 \\ 0 & \frac{1}{6}ma^2 & 0 \\ 0 & 0 & \frac{1}{6}ma^2 \end{pmatrix},$$

and the moment of inertia about the body diagonal is

$$\left(\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\right)\begin{pmatrix} \frac{1}{6}ma^2 & 0 & 0 \\ 0 & \frac{1}{6}ma^2 & 0 \\ 0 & 0 & \frac{1}{6}ma^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \frac{1}{6}ma^2.$$