11. Non-integrable Systems and Chaos

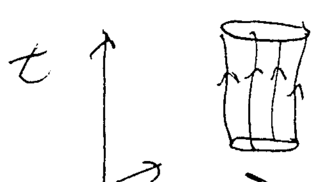
The theme of this course has been: the Main Problem of classical mechanics is to "solve" for the phase space trajectory of the system — i.e., reduce it to quadrature (the evaluation of 1 dimensional integrals that can be easily done numerically).

Now we come, finally, to a great scandal — this is almost never possible! The mechanics problems that can be solved analytically, on which we have spent so much time (Kepler, Gyroscope, Coupled oscillators) are in a sense a set of measure zero - very exceptional. If number of degrees of freedom is $N \geq 2$, a generic small perturbation

$$H = \underbrace{H_0}_{\text{solvable}} + \underbrace{\epsilon H_1}_{\text{perturbation}}$$

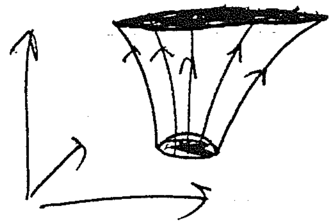
destroys the integrability of the system!

Furthermore, the non-integrable systems share a common, crucial, feature: (exponentially) sensitive dependence on initial conditions.



Consider evolving a bundle of trajectories in phase space. For an integrable system — bundle spreads linearly with time.

(Need to understand — reconciliation with Liouville's Theorem)



But for a nonintegrable system spreading is exponential $e^{t/\tau} = 10^{(\log e)\, t/\tau}$

This means, to evolve system forward in time for time $\lambda \tau$, to one place accuracy, we need $\lambda (\log_{10} e)$ accuracy in the initial data -- our inability to specify initial data to arbitrary accuracy means that long-time behavior of the system is completely unpredictable — we say it is "chaotic."

So -- what chaos challenges is not just the quaint notion that physics problems should be exactly solvable, but the whole tradition of determinism in physics — the clockwork universe. The traditional view is: give me the initial data, and the Hamiltonian, and I can tell you the state of the universe at all later (or earlier) times. Chaos means this is not so. To propagate the system forward for longer and longer times requires more and more accurate knowledge of the initial state. Since we cannot even in principle know the initial state to ∞ precision (a real no. with an ∞ number of digits — where would we write it down!), we cannot even in principle predict the behavior of the system arbitrarily far into the future.

Once we know to look, we can see chaos all around us — it is what makes the world beautiful and fascinating:

- The dancing of the flames in the fireplace —
- The waves breaking on the shore —
- The clouds drifting by ...
- "Roll of the Dice" – Deterministic yet random; Roulette wheel. Coin flip

— these are situations that develop, over and over, from similar conditions, but because of sensitive dependence on initial conditions, we never see the same pattern repeat itself. That is why these things are fun to watch!

(Note that while quantum mechanics is sometimes regarded as a retreat from determinism, since the outcome of the measurement of an observable cannot necessarily be precisely predicted, even if the state is precisely known, the Schrödinger eqn is really deterministic — given $\psi(t_0)$, we know $\psi(t) = \exp\left(-\frac{i}{\hbar}H(t-t_0)\right)\psi(t_0)$. In fact — since eqn is linear, there is no sensitive dependence on initial wave function. The role of chaos in quantum mechanics is still being debated.)

In one respect, the discovery that chaos is generic seems welcome — it provides a basis for the assumption of ergodicity in classical statistical mechanics. We have seen that integrable systems are not ergodic — For a system with N degrees of freedom there are N single-valued constants of the motion, and a generic trajectory fills only a N-dimensional torus in the 2N-dimensional phase space. But the integrable systems are highly exceptional — what is the typical behavior of a trajectory of a Hamiltonian system?

— Landau believed that N-tori are generic. In fact, he thought that all systems should be integrable and analytically solvable, a misunderstanding that mars the first section of the otherwise beautiful Mechanics Text by Landau and Lifshitz.

— At the other extreme, Fermi believed that all non integrable systems are ergodic — a generic perturbation completely destroys the "invariant tori."

— the truth, remarkably subtle and complex, is somewhere in between. There are trajectories of a perturbed integrable system that fill a region of dimension 2N, rather than an N torus, while there are also trajectories that remain confined on N-tori. When the perturbation is very weak ($\epsilon \ll 1$), N-tori are "Typical," but more and more "chaotic" trajectories appear as ϵ increases.

Some history

1680's : Newton, trying to understand the motion of the earth-moon-sun system, considers the problem of three bodies. It is "the only problem that ever made my head hurt."

18th and 19th century: Celestial mechanics, the motion of the solar system, is a central problem of mathematical physics, and the question of the stability of the solar system receives much attention. Perturbation series are formulated by Lagrange, Laplace, and many others — the issue of stability becomes: do these series converge?

1890's : Henri Poincaré understands why the problem of 3 bodies is so hard — He considers behavior of a bundle of orbits (rather than individual orbits) — discovers chaos — very complicated orbits are possible — and perturbation theory diverges ---

1917 : Albert Einstein understands why the old quantum theory won't work for the Helium atom (one nucleus + two electrons a 3 body problem) — It is non integrable so there are not invariant tori that fill phase space — can't be expressed in action-angle variables. But no one pays attention!

1954: Kolmogorov (and Arnold (1962), Moser (1962) — known as KAM) show that weak perturbations leave "most" invariant tori distorted but intact.

1963: Meteorologist Edward Lorenz, studying simple systems of non-linear differential equations that crudely model the atmosphere, recognizes sensitive dependence on initial conditions. He argues that long-term weather forcasting (say $\gtrsim$ 30 days) is impossible because of the "Butterfly Effect": whether a butterfly in Australia flaps its wings can determine whether it rains in Chicago 30 days later.

Late 70's: Chaos finally starts to catch on, explosion of interest signals a "paradigm shift." This is due to a convergence of a variety of factors:

- Numerical experiments, made possible by increasing computational power.
- Recognition that chaotic systems can have "universal" features that don't depend on the particular system that is being studied — so some things can be predicted. To some extent, chaos can be tamed: the dice are loaded.

- Recognition of relevance of chaos to a central problem: the onset of turbulent flow in (dissipative) fluids — another notoriously hard problem.
- Many laboratory experiments exhibiting chaotic behavior, and confirming some of the predicted "universal" features.

Chaos and Randomness

Chaotic dynamics makes it possible for a deterministic dynamical system (like dice) to exhibit apparently "random" behavior. We would like to understand better how such "deterministic randomness" is possible.

To gain insight, it helps to consider the very simplest dynamical systems. We usually think of a dynamical system in terms of continuous flow in phase space, governed by 1st-order differential equations of the form

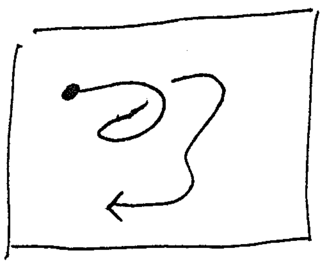
$$\frac{d}{dt}\vec{x}(t) = \vec{F}[\vec{x}(t)]$$

— Hamilton's equations are a special case.

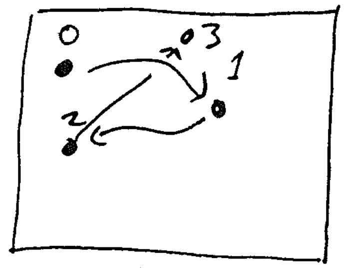
But sometimes it is useful to consider, instead of a flow, a map. In effect, a map is a dynamical system with discrete time, labeled by an integer n. Stepping from time n to $n+1$, dynamical variables behave as

$$\vec{x}_{n+1} = \vec{M}(\vec{x}_n)$$

or $x_n = M^n(x_0)$
– nth iteration of the map



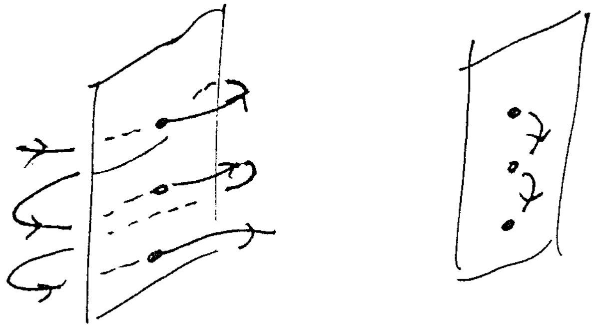
Flow



Map

For a map, an initial condition determines an orbit that is a sequence of discrete points (rather than a continuous flow). The map can be continuous, though, in the sense that nearby points are mapped to nearby points.

In fact, for a flow in a compact phase space, it is often possible to associate a map with the flow, that captures some of its dynamical content. In an N-dimensional phase space, embed an $N-1$ dimensional surface. A one-dimensional flow generically intersects the surface at isolated points

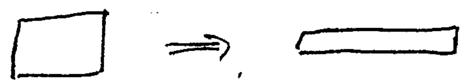


Consider the successive points of intersection of flow and surface, in which the flow pierces the surface in the same "sense" (with the same "sign" tangent vector to the flow). Thus, a smooth flow in N dimensions determines a continuous invertible map in $N-1$ dimensions. (Note that actual time of flow elapsed between crossings of the surface will vary, so $x_n \longrightarrow x_{n+1}$ does not correspond, for each n, to the same increment of time.)

The map is called a "Poincare section" of the flow. This technique is especially useful for Hamiltonian flow with 2 degrees of freedom. Then phase space is 4-dimensional, but conserved energy restricts orbit to lie in 3-d $H = E$ surface. The Poincare section reduces 3-d flow to 2-d map, which is easy to visualize.

The simplest maps are one-dimensional, and nonlinear 1-d maps already provide examples of chaotic behavior (although not when the 1-d map is the Poincare section of a Hamiltonian flow — Hamiltonian systems with one degree of freedom are integrable).

Chaos means exponentially sensitive dependence on initial conditions. In Hamiltonian flow,



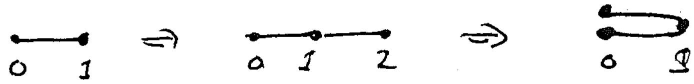
phase space volumes are preserved, so stretching in one direction means contraction in transverse directions.

(As we'll see, Poincaré section of flow is area preserving.)

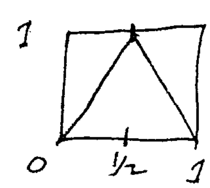
If $H=E$ surface is compact, flow must fold back as it stretches, so that nearby points continue to separate while flow remains confined.

For a 1-d map, say from unit interval to itself $I = [0,1] \rightarrow I$ we can't have stretching for a continuous map if map is to be 1-1.
(Another reason why we can't have chaos in Hamiltonian system with one degree of freedom — associated Poincaré map is 1-1)
But we'll consider some noninvertible chaotic maps.

<u>Example</u>: The "tent map"



It stretches I by factor of 2, and then folds it over. Called the tent map because of the shape of its graph:



Suppose we iterate the tent map many times. In each iteration, sufficiently nearby points separate by a factor of two

$$(\Delta x)_n = 2(\Delta x)_{n-1}\ , \quad \text{for } (\Delta x)_{n-1} \ll 1$$

(just need to avoid the kink of the map at $x = \frac{1}{2}$)

This is an example of a Lyapunov exponent. Exponent h is defined by

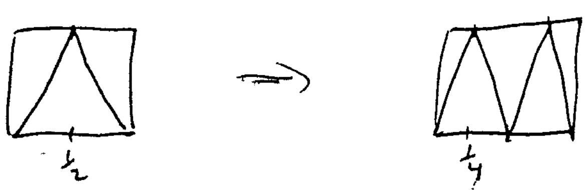
$$(\Delta x)_n \sim e^{hn}(\Delta x)_0\ , \quad \text{as } n \to \infty$$

(for "typical" infinitesimal initial interval). Here we have

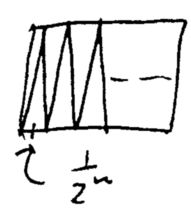
$$h = \ln 2$$

Chaotic maps have positive Lyapunov exponents.

Iterating tent map twice gives the map:

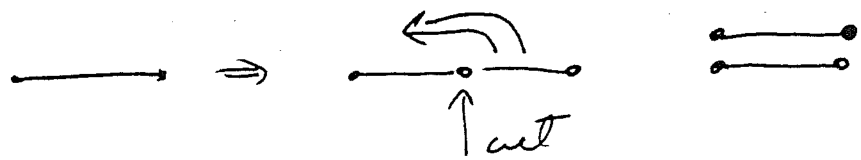


Iterating n times gives:

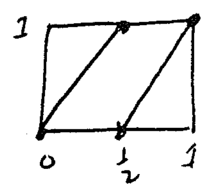


Any initial interval eventually gets stretched over all of I, folding back and forth many times, filling I uniformly

Closely related to the tent map is the "shift map":

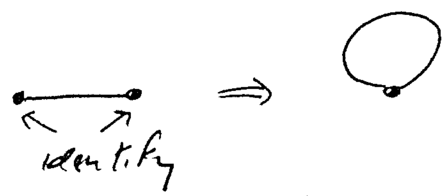


We cut in half after stretching, instead of folding back.

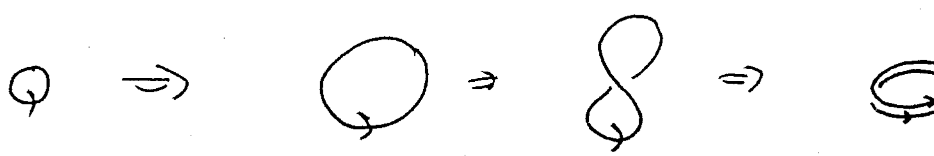


This is continuous *except* at the point $x = \frac{1}{2}$ (where we place the *cut*).

Alternatively,

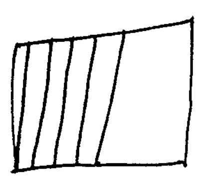


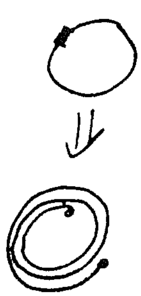
we can identify the endpoints of the interval, and think of the shift map as a map from the circle to the circle:



we stretch the circle, twist it, and fold it over (a 2 to 1 map).

Iterate many times:





Takes an interval on circle and stretches it around the circle many times -- Again, Lyapunov exponent $h = \ln 2$

We can write the shift map as

$$X_n = 2X_{n-1} \pmod 1$$

$$= \begin{cases} 2X_{n-1}, & X_{n-1} \le \frac{1}{2} \\ 2X_{n-1} - 1 & X_{n-1} > \frac{1}{2}. \end{cases}$$

An instructive way to think about this map is to express X in base 2, as a sequence of binary digits

$$X = .b_1 b_2 b_3 b_4 \cdots$$

$$= b_1 \frac{1}{2} + b_2 \frac{1}{2^2} + b_3 \frac{1}{2^3} + \cdots, \qquad b_i = 0 \text{ or } 1$$

Then, under the map

$$X \longrightarrow .b_2 b_3 b_4 \cdots ;$$

the map "throws away" the first binary digit, and shifts all of the others one unit to the left. This is the origin of the name "shift map." (The map discards one bit of information because it is a 2 to 1 map.)

This representation of the shift map makes its properties clear. We see that the map has many orbits that are periodic or "eventually periodic" — meaning that the orbit has a finite number of points. After awhile, the map keeps repeating itself. The initial points that lie on such periodic orbits have binary expansions that terminate or repeat. They are the rational numbers. These are a countable set, densely embedded in I.

The vast majority of real numbers in I (all but a set of measure zero) are irrational, and so do not lie on periodic orbits. Thus, the shift map provides a nice illustration of how a deterministic system can behave randomly. Suppose we can know the value of X to only some fixed accuracy (say 9 bits). Each time we iterate the map, we lose one bit of information, and we gain one new bit. But there is no way to anticipate what the new bit will be — it can be either 0 or 1, and each has probability $\frac{1}{2}$ of occuring. After 9 iterations, the initial sequence of 9 bits is completely lost. Nine new bits have replaced them — and what these nine new bits will be is completely unpredictable. Since we can never know the initial data precisely — eventually the result of iterating the map will be a random sequence of bits that cannot be predicted or calculated ahead of time.

"Algorithmic complexity theory" attaches a precise meaning to the notion of a random number. Imagine writing a computer program for a computer that is to generate a sequence of binary digits. For a rational number, there are of course programs that can generate the whole infinite binary expansion — e.g., if there are n digits that repeat, the program prints the same n digits over and over again. There are also irrational, even transcendental numbers ($\sqrt{2}, \pi, --$) for which a program of finite length generates all the digits.

But a binary sequence that cannot be the output of any program of finite length is a random sequence. New "surprises" keep happening as more and more bits appear.

Notice though, that the number of computer programs of finite length is countable. So the sequences that can be computed are a countable set — a subset of measure zero of I. All real numbers, except for a set of measure zero, are random.

The shift map, though deterministic, passes the test of randomness. If we know the first n bits of a sequence, there is no way to predict what comes next. So the orbit under the shift map is not computable. If we specify initial data to nine bits, and sample the position every nine iterations, the orbit is completely random. There is no more concise "model" of the orbit than just a table of the data, and the future cannot be computed from the past. So — the map

$$X_n = 2 X_{n-1} \pmod{1},$$

surely deterministic, produces a random output. In a sense, the randomness doesn't come from the dynamical law; the randomness of the output comes from the randomness of the input — from the inherent randomness of the real numbers. For the real numbers are a well-defined set of mostly undefinable objects.

Does that mean that all dynamical systems produce random output — even the integrable ones? No — it is the exponential dependence on initial conditions that makes output at later times depend on more and more bits of the input. As an example of a nonchaotic map — consider

$$X_n = X_{n-1} + \beta \pmod{1}$$

where β is a computable irrational number. Here all of the orbits are nonperiodic, but there is no chaos — nearby points stay close together. If we know X_0 to m digits, we can always find X_n to m digits, no matter how large m and n are. As n gets longer, we need to know β to higher accuracy, since

$$X_n = n\beta + X_0 \pmod 1$$

That just means we compute longer, but there is no need to specify initial data to higher accuracy — that is the typical behavior of linear dynamical systems (or of integrable ones, which become linear after a canonical transformation — e.g. to action-angle variables).

To know $X_n = n\beta + X_0$ to accuracy 2^{-m} (to m bits), need to know

$$n\beta = 2^{\log_2 n}\beta$$

to this accuracy. To keep accuracy fixed, we need to know additional digits of β at a rate growing like $\log n$ — i.e. logarithmically with the time. It is characteristic of integrable systems that computing final condition to fixed accuracy requires computational time going like $\log t$. — time needed to compute frequencies ν to higher accuracy.

For chaotic map

$$X_n = 2^n X_0 \pmod 1$$

we need all of the bits of 2^n (as well as increasing no of bits of X_0) to compute to fixed accuracy -- so no. of operations (powers of 2) grows linearly with the "time" - this is characteristic of chaotic systems.

The chaotic orbit is its own briefest description (since it is random) and also its own fastest computer. "Calculating" the next time step is equivalent to reading in another bit of the initial data. The dynamical system cannot really calculate the output, it can only copy the input.

The shift map is clearly ergodic. For almost any initial point, the orbit fills the interval densely and uniformly. (Almost every point is random.) Exceptions are the periodic points, but they are a countable set. So, for almost any initial data, a time average over the orbit is equivalent to an average over I with respect to the uniform measure. For some function $A(x)$ on the interval

$$\langle A \rangle_{time} = \frac{1}{n} \sum_{m=0}^{n} A(M^m(x_0))$$

$$= \langle A \rangle_{space} = \int_0^1 dx\, A(x)$$

(where $M: I \to I$ is the shift map). But the shift map actually has a stronger property than ergodicity, called mixing.