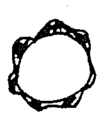
We can consider an arbitrary interval, and iterate the map many times -- then image of the interval fills I uniformly.

Ergodic behavior means that we get a uniform distribution in phase space when we consider where the particle has been. Mixing behavior means that we get a uniform distribution when we consider where it is at a given (sufficiently late) time.

The shift map is clearly mixing since interval keeps getting uniformly stretched around the torus over and over again



More generally, we can consider some distribution function $\rho(x)$ on I

Normalized $\int_0^1 \rho(x)\,dx = 1$

Under the map:

$$\rho_0 \to \rho_1(x) = \int_0^1 \rho_0(y)\,\delta(x - M(y))\,dy$$

$$\to \rho_n(x) = \int_0^1 \rho_0(y)\,\delta(x - M^n(y))\,dy$$

We have $\lim_{n\to\infty} \rho_n = 1$ - the uniform measure on I (which is invariant under the map)

Mixing implies ergodicity, but not the other way around. Indeed

$X_n = X_{n-1} + \beta \pmod 1$, β irrational is ergodic -- Every orbit fills the interval uniformly. But this map is <u>not</u> chaotic, and not mixing.

To summarize -- the simple shift map has the following properties all of which are characteristic of chaotic maps

1. It is a continuous map (on the circle) with a positive Lyapunov exponent. Nearby points are stretched apart by successive iterations of the map.

2. The eventually periodic points (those lying on orbits that contain a finite number of points) are <u>dense</u> in the phase space, but are of measure zero.

3. Each nonperiodic orbit is ergodic; it fills the phase space uniformly -- i.e. it is dense in phase space, with the probability of visiting a region tending toward an invariant measure (in this case the Lebesgue measure on the interval)

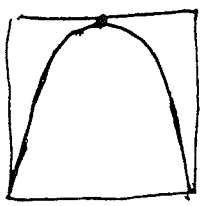
4. The map is mixing. Iterating the map causes a density on phase space to evolve toward the invariant density (at a fixed, late time)

unlike the maps associated with Hamiltonian flows, though, the shift map is not invertible (it is 2-to-1). In fact, a 1-dimensional map cannot be both invertible and chaotic.

Just to emphasize that the invariant measure does not have to be the uniform Lebesgue measure — Consider the logistic map:

$$X_n = r X_{n-1}(1 - X_{n-1})$$

(Considered by Robert May as a crude model for population biology.)



For $r=4$ -- this is a 2-to-1 map of I to I — A kind of smoothed out version of the Tent map

In fact, we can relate it to the tent map by a change of variable

$$X = \sin^2 \frac{\pi Y}{2}$$

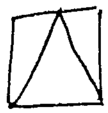
$$4X(1-X) = 4 \sin^2 \frac{\pi Y}{2} \cos^2 \frac{\pi Y}{2} = \sin^2(\pi Y)$$

so $$\sin^2 \frac{\pi Y_n}{2} = \sin^2(\pi Y_{n-1})$$

or $$\frac{\pi Y_n}{2} = \pm \pi Y_{n-1} + K\pi, \quad K \text{ an integer}$$

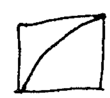
$$\Rightarrow Y_n = \pm 2 Y_{n-1} + 2K$$ — sign and integer K determined by $0 < Y_n < 1$

So
$$Y_n = \begin{cases} 2Y_{n-1} & Y_{n-1} \le \frac{1}{2} \\ -2Y_{n-1}+2 & Y_{n-1} \ge \frac{1}{2} \end{cases}$$

 This is the tent map

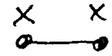
So $r=4$ logistic map is "isomorphic" to tent map — or they are "conjugate maps".
We have
$$Y_n = \mathcal{M}(Y_{n-1}) \quad \text{where } \mathcal{M} \text{ is tent map}$$
$$X = g(Y) \quad \text{where} \quad g(Y) = \sin^2\frac{\pi Y}{2} = \frac{1}{2}(1-\cos\pi Y)$$

Then
$$X_{n-1} \to g\left(\mathcal{M}\left(g^{-1}(X_{n-1})\right)\right)$$

g 
Y

Thus
$$X_n = \tilde{\mathcal{M}}(X_{n-1}) \quad \text{where}$$
$$\tilde{\mathcal{M}} = g \circ \mathcal{M} \circ g^{-1}$$
(with $\circ$ denoting composition of maps)

Corresponding to a measure on Y is a measure on X, related by
$$\tilde{\rho}(x)\,|dx| = \rho(y)\,|dy|$$



— probability of a point being in an interval does not depend on the coordinates that we use...

So $\quad \tilde{\rho}(x) = \rho(y) \left| \frac{dy}{dx} \right|$

For the map $\quad X = \sin^2 \frac{\pi}{2} Y$

we have $\displaystyle \frac{dX}{dY} = 2 \sin\left(\frac{\pi}{2} Y\right) \frac{\pi}{2} \cos\left(\frac{\pi}{2} Y\right)$

$\displaystyle = \pi X^{\frac{1}{2}} (1 - X)^{\frac{1}{2}}$

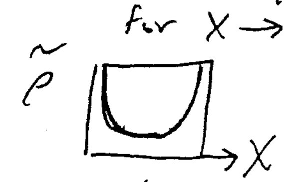
so $\quad \displaystyle \tilde{\rho}(X) = \rho(y) \frac{1}{\pi \sqrt{X(1-X)}}$

The invariant measure of the tent map, as for the shift map, is

$$\rho(y) = 1,$$

so $\quad \displaystyle \boxed{\tilde{\rho}(X) = \frac{1}{\pi \sqrt{X(1-X)}}}$

Note: singular for $X \to 0, 1$



is the invariant measure of the $r = 4$ logistic map.

Since the tent map and $r = 4$ logistic map are conjugate ($X = g(Y)$ is 1-1) there is a one-one correspondence of eventually periodic points of the two maps. So there are countably many such points for the $r = 4$ logistic map.

What happens to the Lyapunov exponent when we smooth out the map?

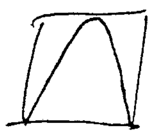
For $x_n = M(x_{n-1})$

we have $dx_n = M'(x_{n-1})\, dx_{n-1}$

Rate at which nearby points separate (or approach) is

$$\left| \frac{dx_n}{dx_{n-1}} \right| = |M'(x_{n-1})|$$

For the r=4 logistic map



$M(x) = 4x(1-x)$

$|M'|$ varies between 4 at endpoints and 0 at $x = \frac{1}{2}$

At what rate do points separate on the average?

$$\frac{dx_n}{dx_0} = \frac{dx_n}{dx_{n-1}} \frac{dx_{n-1}}{dx_{n-2}} \cdots \frac{dx_2}{dx_1} \frac{dx_1}{dx_0}$$

so $$\left| \frac{dx_n}{dx_0} \right| = |M'(x_{n-1})||M'(x_{n-2})| \cdots |M'(x_0)|$$

The Lyapunov exponent h is defined by

$$\left| \frac{dx_n}{dx_0} \right| \sim e^{nh} \quad \text{as} \quad n \to \infty$$

so $$h = \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \ln |M'(x_m)|$$

Now suppose map is ergodic -- so this time average can be expressed as integral weighted by the invariant density — we know this is so for the $r=4$ logistic map, since it is true for the tent map to which it is conjugate. Then Lyapunov exponent is

$$\boxed{h = \int_0^1 dX\, \tilde{\rho}(X)\, |\ln M'(X)|}$$

From this formula, we can show that two conjugate maps have the same h (if the isomorphism g is differentiable) — an exercise. So we know

$$h = \ln 2 \qquad \text{for } r=4 \text{ logistic map}$$

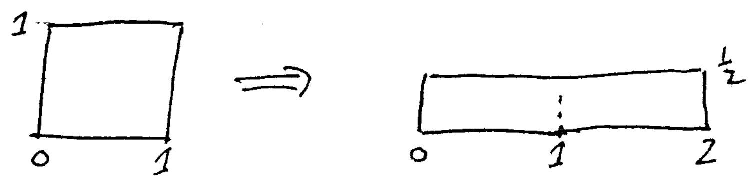
There is a lot more to say about one-dimensional maps. We'll return to the subject later.

First, though, it will be enlightening to consider some examples of 2-dimensional maps. By going to 2 dimensions, we are able to construct chaotic maps that are 1-1 -- in fact, they can also be area preserving.

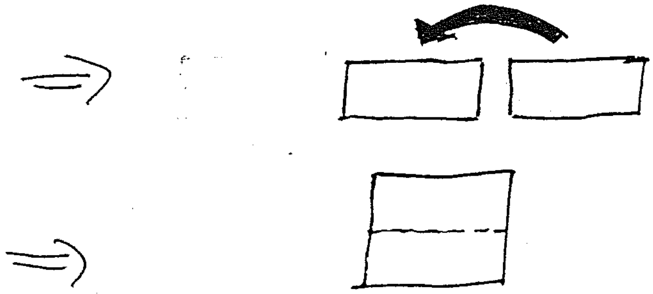
The simplest example of an area preserving chaotic map is the "Baker's map" — similar to the shift map (and shares with it the property of being discontinuous on a set of measure zero).

The bakers map takes unit square to self.

$$M: \quad I \times I \longrightarrow I \times I$$



First, stretch horizontally and contract vertically



Then cut in half and stack the right half over the left half

— clearly an invertible map

Can be expressed as —

$x < \frac{1}{2}$:

$$X_n = 2\,X_{n-1}$$
$$Y_n = \tfrac{1}{2}\,Y_{n-1}$$

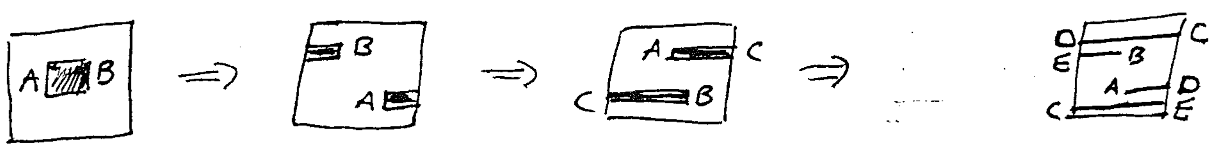
$x > \frac{1}{2}$

$$X_n = 2X_{n-1} - 1$$
$$Y_n = \tfrac{1}{2}Y_{n-1} + \tfrac{1}{2}$$

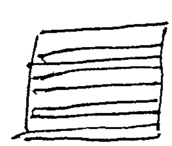
This map obviously preserves the area element

Because of the cutting, it is discontinuous at $x = \frac{1}{2}$

Suppose we iterate the map many times -- stretch-cut-stack, stretch-cut-stack, over and over



Many iterations



A long thin filament crosses the square horizontally many times.

For this map there are two distinct Lyapunov exponents — a positive one corresponding to the horizontal stretching, and a negative one corresponding to vertical contraction. It is obvious that they are

$$h_1 = \ln 2$$
$$h_2 = -\ln 2$$

Because of the one positive Lyapunov exponent — the map is chaotic.

In general, how are Lyapunov exponents in higher dimension maps to be <u>defined</u>?

We can <u>linearize</u> the map near a particular orbit to find how nearby points deviate from that orbit $\Longrightarrow$

$$x_0 \rightarrow x_1 \rightarrow x_2$$
$$x_0+\epsilon_0 \rightarrow x_1+\epsilon_1 \rightarrow x_2+\epsilon_2$$

$$x_1+\epsilon_1 = M(x_0+\epsilon_0) = x_1 + DM(x_0)\cdot\epsilon_0$$

where $DM(x_0)$ is the linearized map

$$DM(x_0) = \left.\begin{bmatrix} \partial M^{(1)}/\partial x^{(1)} & \cdots & \partial M^{(1)}/\partial x^{(N)} \\ \vdots & & \vdots \\ \partial M^{(N)}/\partial x^{(1)} & \cdots & \partial M^{(N)}/\partial x^{(N)} \end{bmatrix}\right|_{x_0}$$

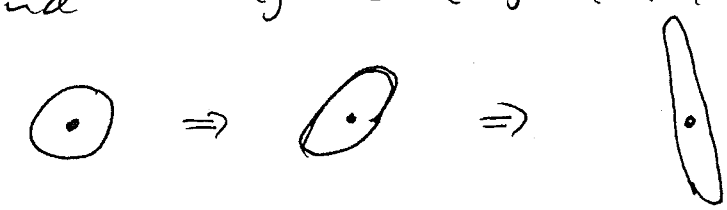
$$DM^n(x_0) = DM(x_{n-1})\cdot DM(x_{n-2}) \cdots DM(x_0)$$

— matrix product

The idea now is to diagonalize the matrix $DM^n(X_0)$ as $n \to \infty$. The eigenvalues behave like λ_j^n $j=1,\dots,N$

The λ_j's are the "Lyapunov numbers"

and $h_j = \ln \lambda_j$ are the Lyapunov exponents.



The sphere near X_0 gets distorted into an ellipsoid

After many interations, the largest eigenvalue dominates — if it is positive there will be exponential stretching in the direction associated with the largest eigenvalue

In general, exponents can be different for different trajectories.

Like the shift map, the Baker's map has a nice symbolic representation. This time -- write X and Y as binary expansion

$$X = .b_1 b_2 b_3 b_4 \dots$$
$$Y = .c_1 c_2 c_3 c_4 \dots$$

Now put the two sequences "back to back" by writing Y backwards to the left of the decimal point:

$$\dots c_4 c_3 c_2 c_1 . b_1 b_2 b_3 b_4 \dots$$

Now $X < \frac{1}{2}$ $(b_1 = 0)$ $\Rightarrow$ $X \to 2X = .b_2 b_3 b_4 b_5 \dots$

$Y \to \frac{1}{2} Y = .0 c_1 c_2 c_3 \dots$

$$X > \tfrac{1}{2} \quad (b_1 = 1) \qquad X \to 2X - 1 = .b_2 b_3 b_4 b_5 \cdots$$
$$Y \to \tfrac{1}{2} Y + \tfrac{1}{2} = .1\, c_1 c_2 c_3 \cdots$$

In both cases -- the map just moves the decimal of the double sequence one unit to the right:

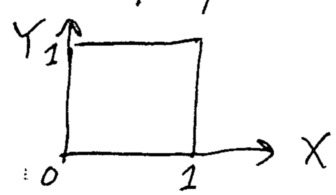
$$\cdots c_4 c_3 c_2 c_1 . b_1 b_2 b_3 b_4 \cdots \Rightarrow \cdots c_3 c_2 c_1 b_1 . b_2 b_3 b_4 b_5 \cdots$$

— We see that the baker's map is quite similar to the shift map, and it has similar properties. But note that we never throw away a bit — this is an <u>invertible</u> map.

- The periodic points of the map are dense in the unit square (an exercise).
- But for a typical point, iterating the map forward or backward eventually generates random output that is completely unpredictable — orbit fills the square.
- The map is <u>mixing</u>, where the invariant density is the uniform density on the unit square (as well as ergodic, with all but a set of measure zero of points lying on orbits that uniformly fill the square).

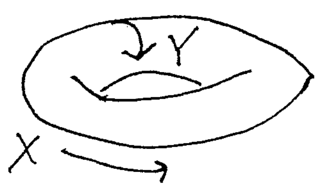
The baker's map improves on the shift map in that it demonstrates that the above properties can hold for a map that is one to one. Next we want to see that the discontinuity in the Baker's map is not necessary.

We recall that the shift map could be construed as continuous if we regarded it as a map from the circle to the circle, i.e. periodically identified X with $X+1$. Similarly, we can construct a continuous 2-dimensional chaotic map on the torus — i.e. by periodically identifying X with $X+1$ and Y with $Y+1$.



The periodically identified square is topologically a product of circles $S^1 \times S^1$

— or a torus



The linear transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \pmod 1$$

is a continuous transformation from torus to torus if the matrix elements are integers. Then

$$\begin{pmatrix} X+1 \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \pmod 1$$

$$\begin{pmatrix} X \\ Y+1 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \pmod 1$$

which ensures that, e.g. (ϵ_x, ϵ_y) and $(1-\epsilon_x', \epsilon_y')$ – nearby points on the torus – are mapped to nearby points.

This map is also area preserving if it has determinant one. A simple example is

$$\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (\text{mod } 1)$$

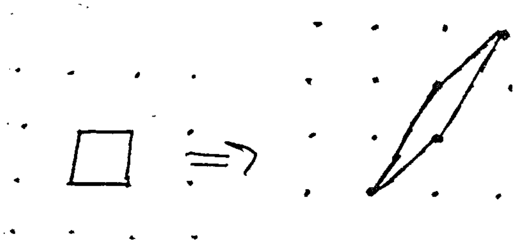
– the "cat map"

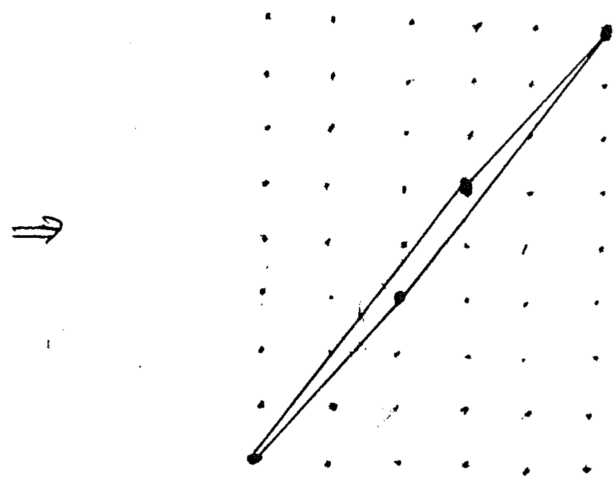
– inverted by... $\begin{pmatrix} X \\ Y \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (\text{mod } 1)$

This matrix has eigenvalues λ and λ^{-1}, where

$$\lambda = \tfrac{1}{2}(3+\sqrt{5}) > 1$$

– so it stretches the torus along one axis and contracts it along the orthogonal axis





$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

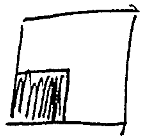
$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$T^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$

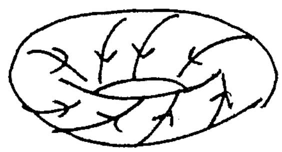
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

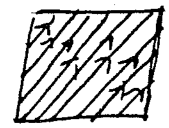
$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 8 \end{pmatrix}$



After map acts many times square in the lower left wraps around square many times





— or around the torus many times

This map has the mixing property, where the invariant density is the uniform measure on the torus. Arnold called it the "cat map" because it destroys a cat who lives on the torus.

The cat map has periodic orbits; these are the fixed points of the nth iteration for some n:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \pmod 1$$

Since the matrix has integer matrix elements, this can have solutions only when both X and Y are rational. Also true, though less obvious, is that there is a solution, for some n, whenever X and Y are rational. Thus, the periodic points of the cat map are

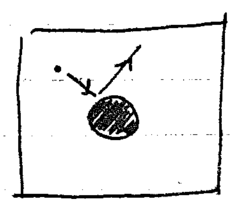
(X, Y) where X and Y are both rational

— Periodic points are dense on the torus.

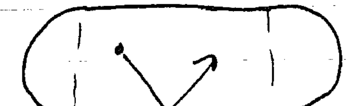
Thus the cat map provides an example of a continuous, invertible, area preserving chaotic map on the torus. It is mixing, and its periodic points are dense.

There are also "real" dynamical systems that exhibit these properties, though this is much harder to prove.

Simplest example — a torus (periodically identified square) with a circular hole removed. And a pointlike or circular billiard ball that bounces elastically off of the hole. For all but a set of measure zero of initial conditions, the orbit of the center of the ball fills the region uniformly — Further, neighboring orbits diverge — the mapping is mixing as well as ergodic. The ball "forgets" the initial conditions, and the late time behavior is random.



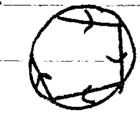
Another example: point mass in a "stadium" formed by parallel walls capped at each end by semicircles.

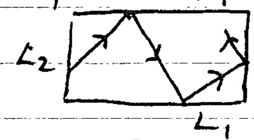


(- these results shown by Sinai around 1970)

- There are a set of measure zero (of initial data) for which orbit is periodic and has Lyapunov exponent = 0

In contrast, billiards in a rectangle or circle is completely integrable





- Circle: E and angular momentum are conserved

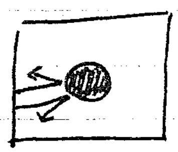
  periods of the motion $\quad \nu^{-1}$ = time between bounces

  ν^{-1} = time to advance around circle

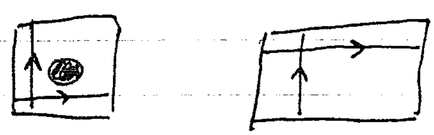
- Rectangle $\quad E_1 = p_1^2/2m$ and $E_2 = p_2^2/2m$ are conserved

  periods of the motion $\quad \nu_1^{-1} = 2L_1/v_1$

  $\nu_2^{-1} = 2L_2/v_2$ $\qquad v_{1,2}$ = horizontal, vertical components of velocity



In the case of the missing disk, it is obvious that bouncing off the disk widens the angular separation between the nearby orbits

Periodic orbits:  $v_1 = 0$ or $v_2 = 0$