Feynman Formulation of Quantum Mechanics

In $\hbar \to 0$ limit, we have seen that phase of the wavefunction ψ is the classical action S. This suggests a deep connection between QM and the least action principle. There is such a connection, embodied in the Feynman formulation, which we will now derive.

We will consider, for notational simplicity, QM with one degree of freedom; the generalization is obvious. Let's begin with an explanation of our notation, and the facts which will be needed in the derivation:

- We will use Heisenberg picture: $\underset{\sim}{q}(t)$ $\underset{\sim}{p}(t)$ evolve in time.

- Dirac Notation: e.g. $|q,t\rangle$ is eigenstate of $\underset{\sim}{q}(t)$ with eigenvalue q

$$\underset{\sim}{q}(t)|q,t\rangle = q|q,t\rangle$$

$$\underset{\sim}{p}(t)|p,t\rangle = p|p,t\rangle$$

- How do these eigenstates evolve? Recall that we found

$$e^{-i\underset{\sim}{H}t/\hbar}\,\underset{\sim}{q}(t)\,e^{i\underset{\sim}{H}t/\hbar} = \underset{\sim}{q}(0) \quad \text{is time independent.}$$

$$\text{so} \quad \underset{\sim}{q}(t) = e^{i\underset{\sim}{H}t/\hbar}\,\underset{\sim}{q}(0)\,e^{-i\underset{\sim}{H}t/\hbar}$$

Thus, if $\underset{\sim}{q}(0)|q,0\rangle = q|q,0\rangle$, then

$$\underset{\sim}{q}(t)\,e^{i\underset{\sim}{H}t/\hbar}|q,0\rangle = q\,e^{i\underset{\sim}{H}t/\hbar}|q,0\rangle$$

$$\Rightarrow \quad \boxed{\begin{array}{l} |q,t\rangle = e^{i\underset{\sim}{H}t/\hbar}|q,0\rangle \\ |p,t\rangle = e^{i\underset{\sim}{H}t/\hbar}|p,0\rangle \end{array}}$$

and similarly (second line)

- Inner products are denoted e.g. $\langle q; t' | q, t \rangle$: this is probability <u>amplitude</u> for particle in eigenstate of $\underset{\sim}{q}$ with eigenvalue q to be measured at q' at time t' (Specifying this for all $q' t'$ is a complete specification of dynamics)

  This is just a representation of the schrodinger wave function since $i\hbar \frac{d}{dt'} \langle q' t' | = \langle q' t' | \underset{\sim}{H}$

- Completeness: $$\int_{-\infty}^{\infty} dq \; | q, t \rangle \langle q, t | = \underset{\sim}{\mathbb{1}}$$
  $$= \int_{-\infty}^{\infty} dp \; | p, t \rangle \langle p, t |$$
  At each fixed t, the eigenstates of $\underset{\sim}{q}$ or $\underset{\sim}{p}$ are a complete basis

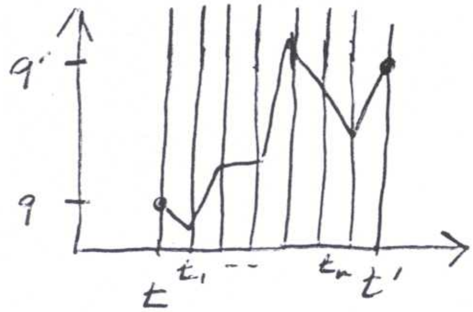
- $$\langle q t | p t \rangle = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} e^{ipq/\hbar}$$ Plane Waves

  This is the position rep of a momentum eigenstate, at fixed time

  Note δ-function normalization:
  $$\langle p' t | p t \rangle = \frac{1}{2\pi\hbar} \int dq \; e^{i(p-p')q/\hbar} = \delta(p-p')$$

  $$\langle p t | q t \rangle = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} e^{-ipq/\hbar}$$

Now we are ready to proceed with the derivation. To save writing, I will set $\hbar = 1$ for now. It will be obvious where the $\hbar$'s belong, and we can put them back in the end.



We begin by partitioning the time interval $[t, t']$ into $n+1$ equal intervals, each of width
$$\Delta t = \frac{t'-t}{n+1}.$$
That is, we define intermediate times
$$t_1 = t + \Delta t$$
$$t_2 = t + 2\Delta t, \text{ etc}$$

(We will eventually allow n to get very large, and consider the limit $\Delta t \to 0$.)

Now we reexpress the amplitude $\langle q', t' | q, t\rangle$ by inserting the operator $\underset{\sim}{\mathbb{I}}$ many times:

$$\boxed{\begin{aligned}\langle q' t' | q t\rangle = &\int dq_1\, dp_1 \cdots dq_n\, dp_n\, dp_0 \\ &\langle q', t' | p_n, t_n\rangle \langle p_n, t_n | q_n, t_n\rangle \langle q_n, t_n | p_{n-1}, t_{n-1}\rangle \langle p_{n-1}, t_{n-1} | q_{n-1}, t_{n-1}\rangle \\ &\quad \cdots \langle p_1, t_1 | q_1, t_1\rangle \langle q_1, t_1 | p_0, t\rangle \langle p_0, t | q, t\rangle\end{aligned}}$$

and we recall that

$$\boxed{\langle p_j t_j | q_j t_j\rangle = (2\pi)^{-\frac{1}{2}} e^{-i q_j p_j}} \qquad \text{(no sum on } j\text{)}$$

Also
$$| q_j, t_j\rangle = e^{i\underset{\sim}{H}\Delta t} | q_j, t_{j-1}\rangle$$

so
$$\langle q_j t_j | p_{j-1} t_{j-1}\rangle = \langle q_j, t_{j-1} | e^{-i\underset{\sim}{H}\Delta t} | p_{j-1} t_{j-1}\rangle$$

Now let $\underset{\sim}{p}$ in $\underset{\sim}{H}(\underset{\sim}{q}, \underset{\sim}{p})$ act to the right, while $\underset{\sim}{q}$ acts to the left (assuming no ordering ambiguity)

so
$$= e^{-iH(q_j, p_{j-1})\Delta t} \langle q_j, t_{j-1} | p_{j-1} t_{j-1}\rangle$$

or

$$\boxed{\langle q_j t_j | p_{j-1} t_{j-1}\rangle = (2\pi)^{-\frac{1}{2}} e^{i q_j p_{j-1}} e^{-iH(q_j, p_{j-1})\Delta t}}$$

Putting together the boxed equations gives ....

$$\langle q',t'|q,t\rangle = \int \prod_{j=1}^{n} \frac{dq_j}{(2\pi)^{\frac{1}{2}}} \prod_{j=0}^{n} \frac{dp_j}{(2\pi)^{\frac{1}{2}}} \frac{dp_0}{2\pi}$$
$$\exp\left[i \sum_j \left(p_j(q_{j+1}-q_j) - (\Delta t)H(q_{j+1},p_j)\right)\right]$$

We write $(q_{j+1}-q_j) = \dot{q}_j \Delta t$, and take the $\Delta t \to 0$ limit, obtaining

$$\boxed{\langle q',t'|q,t\rangle = \lim_{\Delta t \to 0} \int \prod_j \left(\frac{dq_j\, dp_j}{2\pi}\right) \exp\left[i \sum_j \left(p_j \dot{q}_j - H(q_j,p_j)\right)\Delta t\right]}$$

$dp_0/2\pi$

This limit defines a <u>functional integral</u> (or path integral). i.e., we define

$$\int (dq)\, e^{\int f(q,\dot{q})dt}$$

$$= \lim_{\Delta t \to 0} \int \left(\prod_j \frac{dq_j}{(2\pi)^{\frac{1}{2}}}\right) \exp\left(\sum_j f(q_j,\dot{q}_j)\Delta t\right)$$

and we have

$$\boxed{\langle q',t'|q,t\rangle = \int (dq)(dp)\, e^{iS[q,p]/\hbar}}$$

($\hbar$ has been restored here.)

where $S[q,p] = \int dt \left[p\dot{q} - H(q,p)\right]$

is the classical action of a path going from q at time t to q' at time t'

The path integral is a "sum over histories;" i.e. a coherent sum of all trajectories with the given endpoints, weighted by the factor $e^{iS/\hbar}$. This weighted sum gives the quantum mechanical transition amplitude $\langle q';t'|q,t\rangle$

Since the momentum appears only quadratically in S and its time derivative does not appear at all, we can do the (dp) integral. We need to evaluate

$$\int \frac{dp_j}{(2\pi\hbar)^{1/2}}\, e^{\frac{i}{\hbar}[p_j\dot{q}_j - H(q_j, p_j)]\Delta t}$$

E.g. suppose $H = \frac{p^2}{2m} + V(q)$

Then $p_j\dot{q}_j - H = -\frac{1}{2m}(p_j^2 - 2m\,p_j\dot{q}_j) - V(q_j)$

$$= -\frac{1}{2m}(p_j - m\dot{q}_j)^2 + \tfrac{1}{2}m\dot{q}_j^2 - V(q_j)$$

$$= -\frac{1}{2m}p_j'^2 + L(q_j, \dot{q}_j)$$

where $p_j' = p_j - m\dot{q}_j$

Thus, the integral becomes:

$$e^{\frac{i}{\hbar}L(q_j,\dot{q}_j)\Delta t} \int \frac{dp_j'}{(2\pi\hbar)^{\frac{1}{2}}} \exp\left[-i\frac{p_j'^2\Delta t}{2m\hbar}\right]$$

This is a Gaussian integral, but a peculiar one, because the argument of the exponential is imaginary. However, the integral

$$\lim_{\epsilon\to 0^+} \int \frac{dp_j}{(2\pi\hbar)^{\frac{1}{2}}} e^{-\epsilon p_j^2} e^{-ip_j^2\Delta t/2m\hbar} = \left(\frac{m}{i\Delta t}\right)^{\frac{1}{2}}$$

is well-defined. In fact, some such $\epsilon \to 0$ prescription is needed for the Feynman path integral to make mathematical sense.

We are left with

$$\langle q', t' | q, t\rangle = \lim_{\Delta t\to 0} \int \prod_j \left[\frac{dq_j}{(2\pi\hbar)^{\frac{1}{2}}}\left(\frac{m}{i\Delta t}\right)^{\frac{1}{2}}\right] e^{\frac{i}{\hbar}\sum_j L(q_j,\dot{q}_j)\Delta t}$$

$$\times \left(\frac{m}{2\pi\hbar i\Delta t}\right)^{\frac{1}{2}} \text{ from } dp_0 \text{ integral}$$

$$\langle q't'|qt\rangle = \lim_{\Delta t\to 0} \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\, \prod_j \int \frac{dq_j}{(2\pi i\hbar\Delta t/m)^{\frac{1}{2}}} \exp\left[\frac{i}{\hbar}\sum_j L(q_j,\dot q_j)\Delta t\right]$$

$$\equiv \int (dq)'\, e^{i/\hbar \int dt\, L(q,\dot q)}$$

[The $(dq)'$ indicates that the integral is normalized somewhat differently than in the definition on p (8.12)]

This is the way Feynman's formula is usually written. It is important because....

1) It provides a deeper understanding of the Least Action Principle in classical mechanics. Classical physics reemerges as $\hbar \to 0$. The phase factor $e^{iS/\hbar}$ changes very rapidly when a small change is made in the path, and averages to zero, except where $\delta S = 0$. At the classical trajectory which makes the action stationary, this "destructive interference" does not occur, and we have

$$\langle q't'|qt\rangle \sim e^{iS_0/\hbar} \quad (\text{as } \hbar \to 0)$$

where S_0 is the classical action. (This agrees with our earlier result on the classical limit of the Schrödinger eqn.)

Now it is easier to understand how a classical particle can "know" which trajectory makes S an extremum. It really tries all trajectories, weighted by $e^{iS/\hbar}$. As $\hbar\to 0$, constructive interference occurs only along the classical trajectory

For systems with $S \simeq \hbar$, fluctuations about the classical trajectory ("quantum fluctuations") are not suppressed, and classical mechanics cannot provide an adequate description.

2) The Feynman path integral is easily generalized to more complicated systems — e.g., relativistic field theory. For such systems it is sometimes far easier to carry out Feynman's procedure than to perform canonical quantization, to which, we have seen, it is mathematically equivalent.

3) Feynman's formula reduces quantum mechanics to quadrature! That is, quantum mechanical calculations can be reduced to evaluation of a path integral. This reduction is of practical importance. Nowadays the most intractable QM problems (in particle physics and condensed matter physics) are solved numerically by high speed computers which estimate the path integral.

4) Even if we can't do the integral exactly, we can expand it systematically in powers of $\hbar$. This is a generalization of the WKB expansion that can be carried out for complicated quantum systems — the Feynman diagram expansion. The path integral provides an especially efficient method for formulating this perturbation expansion.