Ph 106b Midterm Exam
Due: Thursday 14 February 2008, 4pm

- This exam is to be taken in one continuous time interval not to exceed **3 hours**, beginning when you first open the exam. (You may take one 15 minute break during the exam, which does not count as part of the 3 hours.)

- You may consult the textbooks *Chaos in Dynamical Systems* by Ott and Classical Mechanics by Goldstein, the distributed lecture notes, your own lecture notes, and the problem sets and solutions. If you wish, you may use a calculator, computer, or integral table for doing calculations. However, this probably won't be necessary. **No other materials or persons are to be consulted.**

- There are three problems, each with multiple parts, and 100 possible points; the value of each problem is indicated. You are to work all of the problems.

- The completed exam is to be deposited in the box outside 448 Lauritsen, no later than 4:00 pm on Thursday 14 February 2008. **No late exams will be accepted.**

- Good luck!

1. A one-dimensional map (30 total points)

Consider a one-dimensional map on the circle (*i.e.*, the periodically identified unit interval) defined by

$$M(x) = ax \pmod 1 ,$$

where a is a positive integer.

a) (10 points) What is the Lyapunov exponent of the map M?

b) (10 points) For this map M, how many orbits are there of length p, where p is a prime number?

c) (10 points) Are the periodic points of M dense in the unit circle? Explain your answer.

2. A two-dimensional map (30 total points)

Consider the two-dimensional map

$$M : \begin{cases} w \to w' = w + J \pmod 1 , \\ J \to J' = J , \end{cases}$$

where w is a periodic variable with period 1 and J is a real number. We may interpret this map as describing a rotor with constant angular velocity; we take a "snapshot" of the position of the rotor in phase space at regular time intervals. The angular position of the rotor (measured counterclockwise relative to the vertical) is $2\pi w$, and J is proportional to $L\delta t$, where L is the angular momentum of the rotor, and δt is the interval between successive snapshots.

a) (10 points) Find *all* of the periodic orbits of the map M. Are the periodic points dense in the (w, J) phase space?

b) (10 points) Suppose that (w_0, J_0) is *not* a periodic point of M. Then the orbit containing this point densely fills an invariant curve in the (w, J) phase space. Describe this curve.

c) (10 points) Suppose that (w_0, J_0) is not a periodic point of M. Evaluate the time average of $J^2 \sin^2(\pi w)$

$$\langle J^2 \sin^2(\pi w)\rangle_{\text{time}} \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} J_n^2 \sin^2(\pi w_n) ,$$

where $(w_n, J_n) = M^n(w_0, J_0)$.

3. Another two-dimensional map (40 total points)

Now consider a new map

$$M : \begin{cases} w \to w' = w + J \pmod 1 , \\ J \to J' = J + \frac{F}{2\pi} \sin(2\pi(w + J)) , \end{cases}$$

which reduces for $F = 0$ to the map considered in Problem 2. This map also describes a rotor, but now the rotor receives a vertical impulse proportional to F an instant before each snapshot is taken.

a) (5 points) Find the fixed points of the new map, for $F \neq 0$.

b) (10 points) The effect of the map on the separation between two nearby points in phase space is described by a *linearized map*. That is, if the points (w, J) and $(w+\delta w, J+\delta J)$ are mapped to (w', J') and $(w'+\delta w', J'+\delta J')$, then, to linear order,

$$\begin{pmatrix} \delta w' \\ \delta J' \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} \delta w \\ \delta J \end{pmatrix} ,$$

where $\boldsymbol{A}$ is a 2×2 matrix. Find this matrix $\boldsymbol{A}$, expressed as a function of w' and J'.

c) (5 points) Show that the linearized map preserves the phase space area element $dw \wedge dJ$.

d) (10 points) Consider the fixed points found in (a), with $-\frac{1}{2} < J < \frac{1}{2}$. For each such fixed point, find the range of values of F for which the fixed point is elliptic, and the range of values for which the fixed point is hyperbolic. Assume $F > 0$.

e) (10 points) Consider the period-two orbits of the $F = 0$ map that you found in part (a) of Problem 2, with $0 < J < 1$. Describe *qualitatively* what happens to these orbits for $0 < F << 1$. What do the new invariant curves look like? Are there chaotic regions in phase space? Do not calculate anything, but **draw a picture** of the orbits in the (w, J) plane.