Ph 106b

Homework Assignment No. 2
Due: Thursday, 24 January 2008

1. Finite-length orbits of the shift map. The "shift map" is a 2-to-1 map M on the unit interval $I = [0, 1]$ defined by

$$M(x) = \begin{cases} 2x \ , & \text{if } x < 1/2 \ , \\ 2x - 1 \ , & \text{if } x \geq 1/2 \ . \end{cases}$$

If x is expressed as a binary expansion $x = .b_1 b_2 b_3 b_4 b_5 \ldots$, *i.e.*,

$$x = \sum_{n=1}^{\infty} b_n 2^{-n} \ ,$$

then the action of the shift map is

$$M(.b_1 b_2 b_3 b_4 b_5 \ldots) = .b_2 b_3 b_4 b_5 b_6 \ldots \ ;$$

the bit furthest to the left is erased, and all other bits are shifted one place to the left. (That's why we call it the "shift map.")

a) If an orbit of a map returns to its starting point after n iterations of the map, *and not before*, we say that the orbit is of length n. For the shift map, count the periodic orbits of length 4. (Don't include the orbits of length 1 or 2, and don't overcount—two orbits that sweep through the same points in the same cyclic order are considered to be the same orbit.)

b) For the shift map, how many orbits are there of length p, where p is a prime number? (Why is it more difficult to answer this question in general, if p is not prime?)

2. An ergodic map that is not chaotic. Consider the invertible map on the unit interval

$$M(x) = x + \alpha \ (\text{mod } 1) \equiv \begin{cases} x + \alpha \ , & \text{if } x + \alpha < 1 \ , \\ x + \alpha - 1 \ , & \text{if } x + \alpha \geq 1 \ . \end{cases}$$

where $\alpha \in (0, 1)$ is a real number.

a) Suppose α is a *rational* number, which can be expressed as $\alpha = p/q$, where p and q are positive integers with no common factor. For $x \in I = [0, 1]$, what is the length of the orbit of x under the map M?

b) Now suppose that α is *irrational*. Show that for any $x \in I$ and for any two distinct nonnegative integers n and m, $M^n(x) \neq M^m(x)$, where M^n denotes the nth iteration of the map m (and M^0 is the identity map).

c) For irrational α, show that for any $x \in [0, 1]$, the orbit $\{M^n(x), n = 0, 1, 2, 3 \ldots\}$ is dense in I. That is, show that for any positive ε and for any $y \in I$, there exists an n such that $|M^n(x) - y| < \varepsilon$. (Use the *Bolzano-Weierstrass Theorem*, which says that any infinite set in I has a point of accumulation.)

3. Periodic points of the baker's map. The "baker's map" is an invertible map M on $I \times I$ defined by

$$M(x, y) = \begin{cases} (2x, y/2) \ , & \text{if } x < 1/2 \ , \\ (2x - 1, y/2 + 1/2) \ , & \text{if } x \geq 1/2 \ . \end{cases}$$

If the binary expansions of x and y are $x = .b_1b_2b_3b_4b_5\ldots$, $y = .c_1c_2c_3c_4c_5\ldots$, we can represent the point (x, y) as two "back-to-back" sequences $\ldots c_5c_4c_3c_2c_1.b_1b_2b_3b_4b_5\ldots$. Then the action of M can be expressed as

$$M : \ldots c_5c_4c_3c_2c_1.b_1b_2b_3b_4b_5\ldots \to \ldots c_4c_4c_3c_2c_1b_1.b_2b_3b_4b_5b_6\ldots \ ;$$

each bit shifts one place to the left. We say that the point $P = (x, y)$ is a *periodic point* of the map M if $M^n(P) = P$ for some n, where M^n denotes the nth iteration of the map. For the baker's map, show that the periodic points are dense. That is, show that there is a periodic point of the map that is arbitrarily close to any point in the unit square $I \times I$.

4. Numerical evaluation of a Lyapunov exponent.. The *Lyapunov exponent* h of a one-dimensional map M is defined by

$$M^n(x_0 + \epsilon) - M^n(x_0) \simeq Ce^{hn}\epsilon \ ,$$

for ϵ small and n large (where C is a constant). Consider the $r = 4$ logistic map

$$M(x) = 4x(1 - x)$$

defined on the unit interval I.

a) Numerically calculate

$$M^n(.1 + 10^{-17}) - M^n(.1)$$

for $n = 1, 2, 3, \cdots, 50$. How far apart are the images of .1 and $.1 + 10^{-17}$ after 50 iterations of M? (Be sure to specify enough precision in your computations!)

b) Estimate the Lyapunov exponent by fitting the data from (*a*) to the form $Ce^{hn}\epsilon$. Compare to the exact value of $h = \ln 2$ derived in class.