Ph 12b Midterm Exam Solutions

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1. Two-state quantum dynamics — 35 total points

Let $|e_1\rangle$ and $|e_2\rangle$ denote two normalized and mutually orthogonal states in a Hilbert space $\mathcal{H}$: $\langle e_1|e_1\rangle = \langle e_2|e_2\rangle = 1$, $\langle e_1|e_2\rangle = 0$. A certain quantum system has Hamiltonian $\hat{H}$, and the two normalized states

$$\begin{aligned} |\omega_1\rangle &= \frac{1}{2}|e_1\rangle + \frac{\sqrt{3}}{2}|e_2\rangle \ , \\ |\omega_2\rangle &= \frac{-\sqrt{3}}{2}|e_1\rangle + \frac{1}{2}|e_2\rangle \ , \end{aligned}$$

are eigenstates of $\hat{H}$ with eigenvalues $\hbar\omega_1, \hbar\omega_2$ respectively:

$$\begin{aligned} \hat{H}|\omega_1\rangle &= \hbar\omega_1|\omega_1\rangle \ , \\ \hat{H}|\omega_2\rangle &= \hbar\omega_2|\omega_2\rangle \ . \end{aligned}$$

At time $t = 0$, the system is prepared in the state $|\psi(0)\rangle = |e_1\rangle$.

(a) *(10 points) Express $|\psi(0)\rangle$ as a linear combination of energy eigenstates.*

$$|\psi(0)\rangle = |e_1\rangle = \frac{1}{2}|\omega_1\rangle - \frac{\sqrt{3}}{2}|\omega_2\rangle \ .$$

(b) *(5 points) Solve the time-dependent Schrödinger equation*

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

to find the state $|\psi(t)\rangle$ at time t. Express your answer in the form

$$|\psi(t)\rangle = f_1(t)|\omega_1\rangle + f_2(t)|\omega_2\rangle \ .$$

$$|\psi(t)\rangle = \frac{1}{2}e^{-i\omega_1 t}|\omega_1\rangle - \frac{\sqrt{3}}{2}e^{-i\omega_2 t}|\omega_2\rangle \ .$$

(c) *(10 points) Now re-express $|\psi(t)\rangle$ in the form*

$$|\psi(t)\rangle = g_1(t)|e_1\rangle + g_2(t)|e_2\rangle \ .$$

$$|\psi(t)\rangle = \left(\frac{1}{4}e^{-i\omega_1 t} + \frac{3}{4}e^{-i\omega_2 t}\right)|e_1\rangle + \left(\frac{\sqrt{3}}{4}e^{-i\omega_1 t} - \frac{\sqrt{3}}{4}e^{-i\omega_2 t}\right)|e_2\rangle \ .$$

(d) *(10 points)* *The states $|e_1\rangle$ and $|e_2\rangle$ are eigenstates of an observable $\hat{A}$:*

$$\begin{aligned}\hat{A}|e_1\rangle &= a_1|e_1\rangle \ , \\ \hat{A}|e_2\rangle &= a_2|e_2\rangle \ ,\end{aligned}$$

where $a_1 \neq a_2$, that is measured at time t. Find the probability $P(a_1)$ that the outcome of the measurement is a_1 and the probability $P(a_2)$ that the outcome is a_2.

$$\begin{aligned}P(a_1) &= \left|\frac{1}{4}e^{-i\omega_1 t} + \frac{3}{4}e^{-i\omega_2 t}\right|^2 = \frac{5}{8} + \frac{3}{8}\cos\omega t \ , \\ P(a_2) &= \left|\frac{\sqrt{3}}{4}e^{-i\omega_1 t} - \frac{\sqrt{3}}{4}e^{-i\omega_2 t}\right|^2 = \frac{3}{8} - \frac{3}{8}\cos\omega t \ ,\end{aligned}$$

where $\omega = \omega_2 - \omega_1$.

2. Particle in a box — 35 total points

A free quantum-mechanical particle with mass m moves inside a one-dimensional box with impenetrable walls located at $x = \pm a/2$.

(a) *(10 points)* *Find the normalized wave function $\psi_0(x)$ of the energy eigenstate of lowest energy, and the normalized wave function $\psi_1(x)$ of the energy eigenstate of next-to-lowest energy. Find also the corresponding energy eigenvalues E_0 and E_1.*

The wave function is required to vanish at $x = \pm\frac{a}{2}$, so the wave number k must be an integer multiple of π/a. For wave number k, the energy is $E = \hbar^2k^2/2m$. Therefore,

$$\begin{aligned}\psi_0(x) &= \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) \ , \quad E_0 = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \ , \\ \psi_1(x) &= \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right) \ , \quad E_1 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2 \ ,\end{aligned}$$

(b) *(5 points)* *Suppose that at time $t = 0$ the particle is in the state with wave function $\psi(x,0) = \sqrt{\frac{1}{3}}\ \psi_0(x) + \sqrt{\frac{2}{3}}\ \psi_1(x)$. What is the wave function*

$\psi(x,t)$ at the subsequent time t? Applying the time evolution operator $e^{-i\hat{H}t/\hbar}$ to the initial state, we obtain:

$$\psi(x,t) = \sqrt{\frac{1}{3}}e^{-iE_0 t/\hbar}\psi_0(x) + \sqrt{\frac{2}{3}}e^{-iE_1 t/\hbar}\psi_1(x) \ .$$

(c) *(20 points) Find the expectation value*

$$\langle\psi(t)|\hat{x}|\psi(t)\rangle$$

*of the position operator $\hat{x}$ at time t. (**Hint**: An integral of the form*

$$\int dx\ x\ \sin(Ax)$$

can be done using integration by parts.)

$$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \frac{1}{3}\langle\psi_0|\hat{x}|\psi_0\rangle + \frac{2}{3}\langle\psi_1|\hat{x}|\psi_1\rangle + \frac{\sqrt{2}}{3} 2\,\mathrm{Re}\left(\langle\psi_1|\hat{x}|\psi_0\rangle e^{-i(E_1-E_0)t/\hbar}\right) \ .$$

The first two terms vanish, since each is the integral of an odd function over an even interval. We therefore have:

$$\begin{aligned}\langle\psi(t)|\hat{x}|\psi(t)\rangle \qquad &= \frac{\sqrt{2}}{3}\cdot 2\cos(\omega t)\int_{-a/2}^{a/2}(dx)\ x\cdot\frac{2}{a}\cdot\cos\left(\frac{\pi x}{a}\right)\cdot\sin\left(\frac{2\pi x}{a}\right) \ ,\\ &\left(\omega = (E_1-E_0)/\hbar = \frac{3\hbar\pi^2}{2ma^2}\right)\end{aligned}$$

We may use the identity:

$$\sin(A)\cos(B) = \frac{1}{2}\cdot(\sin(A+B)+\sin(A-B))$$

to obtain

$$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \frac{2\sqrt{2}}{3a}\cdot\cos(\omega t)\int_{-a/2}^{a/2}(dx)\ x\cdot\left(\sin\left(\frac{3\pi x}{a}\right)+\sin\left(\frac{\pi x}{a}\right)\right) \ .$$

Integrating by parts we find

$$\int(dx)\,x\cdot\sin(Ax) = -\frac{x}{A}\cdot\cos(Ax) + \frac{1}{A}\int(dx)\ \cos(Ax) = -\frac{x}{A}\cdot\cos(Ax) + \frac{1}{A^2}\sin(Ax) \ ,$$

so that

$$\begin{aligned}\langle\psi(t)|\hat{x}|\psi(t)\rangle \qquad &= \frac{2\sqrt{2}}{3a}\cdot\cos(\omega t)\cdot\left(\frac{a^2}{9\pi^2}\sin\left(\frac{3\pi x}{a}\right)+\frac{a^2}{\pi^2}\sin\left(\frac{\pi x}{a}\right)\right)\Big|_{-a/2}^{a/2}\\ &= \frac{2\sqrt{2}}{3a}\cdot\cos(\omega t)\cdot 2\cdot\left(-\frac{a^2}{9\pi^2}+\frac{a^2}{\pi^2}\right)\\ &= \frac{4\sqrt{2}}{3a}\cdot\frac{8a^2}{9\pi^2}\cdot\cos(\omega t) = \frac{32\sqrt{2}a}{27\pi^2}\cdot\cos(\omega t)\end{aligned}$$

Finally, we have obtained:

$$\langle\psi(t)|\hat{x}|\psi(t)\rangle = \frac{32\sqrt{2}a}{27\pi^2}\cdot\cos\left(\frac{3\hbar\pi^2}{2ma^2}\cdot t\right) .$$

3. Two qubits — 30 total points

A qubit is a quantum system whose Hilbert space is two dimensional. Consider two qubits labeled A and B, where $\{|e_0\rangle, |e_1\rangle\}$ is a basis for qubit A, and $\{|f_0\rangle, |f_1\rangle\}$ is a basis for qubit B. Suppose the state vector for the composite system AB is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\cos\theta|e_0\rangle\otimes|f_0\rangle+\sin\theta|e_0\rangle\otimes|f_1\rangle+\sin\theta|e_1\rangle\otimes|f_0\rangle+\cos\theta|e_1\rangle\otimes|f_1\rangle\big).$$

(a) *(15 points) Alice has no access to qubit B; she can perform measurements only on qubit A. For any measurement that Alice might perform on qubit A, the probability distribution for the measurement outcomes is determined by the density operator $\hat{\rho}$ for qubit A. Express this density operator as a 2×2 matrix in the basis $\{|e_0\rangle, |e_1\rangle\}$.*

We may write

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\varphi_0\rangle\otimes|f_0\rangle + \frac{1}{\sqrt{2}}|\varphi_1\rangle\otimes|f_1\rangle,$$

where

$$|\varphi_0\rangle = \cos\theta|e_0\rangle + \sin\theta|e_1\rangle, \quad |\varphi_1\rangle = \sin\theta|e_0\rangle + \cos\theta|e_1\rangle;$$

Thus the density operator is

$$\hat{\rho} = \frac{1}{2}|\varphi_0\rangle\langle\varphi_0| + \frac{1}{2}|\varphi_1\rangle\langle\varphi_1|.$$

We may express kets as column vectors and bras as row vectors, so that

$$|\varphi_0\rangle\langle\varphi_0| = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} = \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix},$$

$$|\varphi_1\rangle\langle\varphi_1| = \begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}\begin{pmatrix}\sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\sin^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \cos^2\theta\end{pmatrix},$$

and therefore

$$\hat{\rho} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\sin 2\theta\\ \frac{1}{2}\sin 2\theta & \frac{1}{2}\end{pmatrix}.$$

(*b*) *(5 points) Suppose Alice performs an orthogonal measurement in the basis* $\{|e_0\rangle, |e_1\rangle\}$*. What is the probability that Alice's outcome is* $|e_0\rangle$*?*

$$P(e_0) = \langle e_0|\hat{\rho}|e_0\rangle = 1/2.$$

(*c*) *(10 points) Now suppose that Bob, who* does *have access to qubit* B*, measures his qubit in the basis* $\{|f_0\rangle, |f_1\rangle\}$*, and reports to Alice that he obtained the outcome* $|f_0\rangle$*. After learning of Bob's outcome, Alice performs her measurement in the basis* $\{|e_0\rangle, |e_1\rangle\}$*. In this case, what is the probability that Alice's outcome is* $|e_0\rangle$*?*

When Bob gets outcome $|f_0\rangle$, he prepares Alice's qubit in the state $|\varphi_0\rangle$. Therefore, when Alice measures, she gets outcome $|e_0\rangle$ with probability

$$P(e_0) = |\langle e_0|\varphi_0\rangle|^2 = \cos^2\theta.$$

[Note that this is different than the answer to (*b*). To recover the answer to (*b*), we should average over the two possible measurement outcomes that Bob could have obtained. If Bob had found $|f_1\rangle$, then Alice would have found $|e_0\rangle$ with probability $|\langle e_0|\varphi_1\rangle|^2 = \sin^2\theta$. Averaging over Bob's two possible outcomes, each occurring with probability 1/2, yields $P(e_0) = \frac{1}{2}\left(\cos^2\theta + \sin^2\theta\right) = 1/2$, in agreement with (*b*).]