Ph 12b Final Exam Solution

J. Preskill – Wednesday, 17 March 2010

1. Bound states of two narrow wells — 30 total points

A particle with mass m moves in one dimension, governed by the potential

$$V(x) = \frac{-\hbar^2\Delta}{m}\left(\delta(x+a) + \delta(x-a)\right) ,$$

where $\delta(x)$ denotes the Dirac δ-function, and $\Delta > 0$. Recall that the δ-function potential $V(x) = -(\hbar^2\Delta/m)\delta(x-a)$ causes the logarithmic derivative of the wave function $\varphi(x)$ to jump discontinuously at $x = a$:

$$\varphi'(a^+) - \varphi'(a^-) = -(2\Delta)\varphi(a) .$$

(a) *(5 points) For Δa sufficiently large, this potential has two bound states. Sketch the wave functions $\varphi_0(x)$ and $\varphi_1(x)$ of the ground state and first excited state.* **Hint***: Be sure that each bound-state wave function has the right number of nodes and the right symmetry.*

See Figure 1 and Figure 2.

(b) *(5 points) A normalizable even solution to the time-independent Schrödinger equation can be expressed as*

$$\begin{aligned}
\varphi(x) &= e^{-\kappa x} , && x > a ,\\
\varphi(x) &= A\left(e^{\kappa x} + e^{-\kappa x}\right) , && -a < x < a\\
\varphi(x) &= e^{\kappa x} , && x < -a ,
\end{aligned}$$

for some real number A, where κ is related to the energy by $E = -\hbar^2\kappa^2/2m$. Use the appropriate matching conditions at $x = a$ to derive a transcendental equation that implicitly determines κa (and hence E) in terms of Δa. Express your equation in the form

$$\Delta a = f(\kappa a)$$

for some function f.

Matching $\varphi(x)$ at $x = a$:

$$e^{-\kappa a} = A\left(e^{\kappa a} + e^{-\kappa a}\right) .$$

Matching $\varphi'(x)$ at $x = a$:

$$-\kappa e^{-\kappa a} = A\kappa\left(e^{\kappa a} - e^{-\kappa a}\right) - 2\Delta e^{-\kappa a} .$$

Eliminating A using the first equation, the second equation becomes:

$$\Delta a = \frac{1}{2}\kappa a\left(1 + \tanh\kappa a\right) .$$

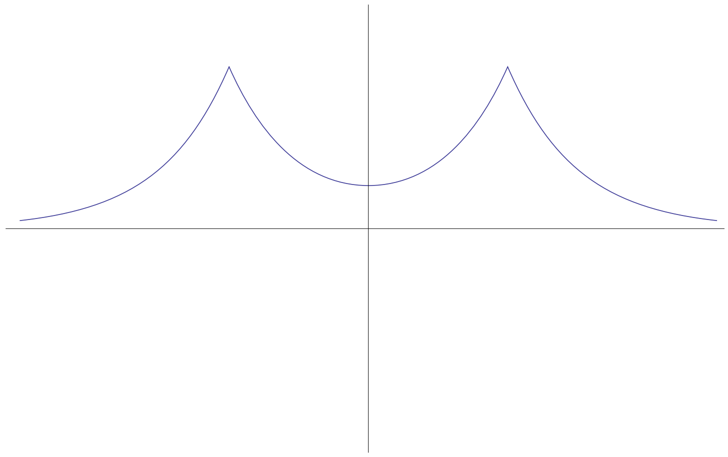

Figure 1: Sketch of the ground state $\varphi_0(x)$.

(c) *(5 points) For your solution in (b), how are κ and Δ related in the limit $a \to 0$?* **Hint***: In this limit, the two δ-functions coincide, so your result should agree with what you expect for a single δ-function with twice the strength.*

In this limit $\tanh \kappa a \to 0$, and therefore $\kappa = 2\Delta$.

(d) *(5 points) A normalizable odd solution to the time-independent Schrödinger equation can be expressed as*

$$\begin{aligned} \varphi(x) &= e^{-\kappa x} \ , & x > a \ , \\ \varphi(x) &= A\left(e^{\kappa x} - e^{-\kappa x}\right) \ , & -a < x < a \\ \varphi(x) &= -e^{\kappa x} \ , & x < -a \ . \end{aligned}$$

Use the appropriate matching conditions at $x = a$ to derive a transcendental equation that implicitly determines κa (and hence E) in terms of Δa. Express your equation in the form

$$\Delta a = g(\kappa a)$$

for some function g.

Matching $\varphi(x)$ at $x = a$:

$$e^{-\kappa a} = A\left(e^{\kappa a} - e^{-\kappa a}\right) \ .$$

Matching $\varphi'(x)$ at $x = a$:

$$-\kappa e^{-\kappa a} = A\kappa\left(e^{\kappa a} + e^{-\kappa a}\right) - 2\Delta e^{-\kappa a} \ .$$

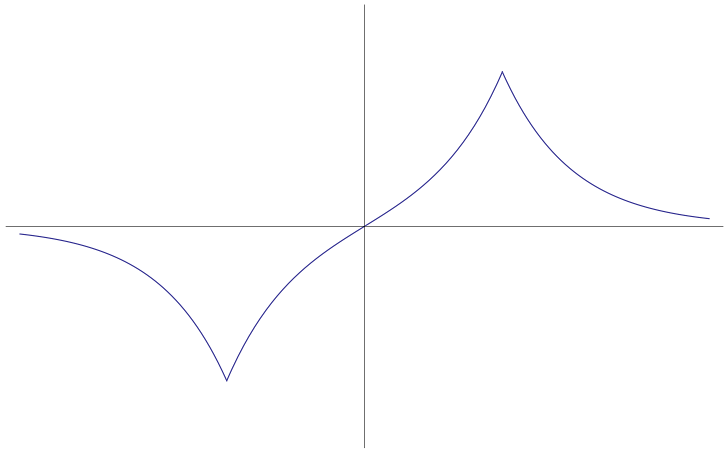

Figure 2: Sketch of the first excited state $\varphi_1(x)$.

Eliminating A using the first equation, the second equation becomes:

$$\Delta a = \frac{1}{2}\kappa a\,(1 + \coth \kappa a) \ .$$

(e) *(5 points)* *In the limit $a \to 0$ considered in (c), the two delta functions merge to one, and there is only one bound state. Thus there is a second bound state only for sufficiently large Δa. Find the value $(\Delta a)_1$ such that a second bound state exists if and only if $\Delta a > (\Delta a)_1$.* **Hint***: Under what condition does the equation found in (d) have a solution with $\kappa > 0$? It may be useful to note that the function $x(1 + \coth(x))$ is monotonically increasing for $x \in [0, \infty]$.*

Since $x(1 + \coth x)|_{x=0} = 1$ and $x(1 + \coth x) > 1$ for $x > 0$, there is a normalizable first excited state for $\Delta a > 1/2$.

(f) *(5 points)* *Now consider the limit $\Delta a \gg 1$; show that the splitting between the energy E_0 of the ground state and the energy E_1 of the first excited state has the form*

$$E_1 - E_0 = C\frac{\hbar^2\Delta^2}{2m}\left(e^{-2\Delta a} + O(e^{-4\Delta a})\right) \ ,$$

and find the constant C. **Hint***: Use the approximations*

$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - 2e^{-2x} + O(e^{-4x}) \ ,$$
$$\coth x \equiv \frac{e^x + e^{-x}}{e^x - e^{-x}} = 1 + 2e^{-2x} + O(e^{-4x}) \ ,$$

and note that the equation

$$B = x\left(1 \mp e^{-2x}\right)$$

has the approximate solution

$$x = B\left(1 \pm e^{-2B} + O(e^{-4B})\right)$$

for $B \gg 1$*.*

With these approximations, the condition for the ground state becomes

$$2\Delta a \approx \kappa a\left(2 - 2e^{-2\kappa a}\right) \ ,$$

or

$$\kappa a \approx \Delta a\left(1 + e^{-2\Delta a}\right) \ ,$$

and the condition for the first excited state becomes

$$2\Delta a \approx \kappa a\left(2 + 2e^{-2\kappa a}\right) \ ,$$

or

$$\kappa a \approx \Delta a\left(1 - e^{-2\Delta a}\right) \ .$$

Therefore, the energies of ground state and first excited state are:

$$E_0 \approx \frac{-\hbar^2\Delta^2}{2m}\left(1 + 2e^{-2\Delta a}\right) \ , \quad E_1 \approx \frac{-\hbar^2\Delta^2}{2m}\left(1 - 2e^{-2\Delta a}\right) \ ;$$

Thus, the energy splitting is:

$$E_1 - E_0 \approx \frac{\hbar^2\Delta^2}{2m}\left(4e^{-2\Delta a}\right)$$

i.e., $C = 4$.

2. Variational method — 30 total points

The variational method is a technique for estimating the ground state energy E_0 *of a Hamiltonian* $\hat{H}$*. We choose a family of states, and then find the one in the family that makes* $\langle \hat{H} \rangle$ *as small as possible. The minimal value of* $\langle \hat{H} \rangle$ *is our estimate of* E_0*. We can also use a variant of this method to estimate the energy* E_1 *of the first excited state.*

(a) *(5 points)* *Show that for any normalized state* $|\psi\rangle$*,*

$$\langle \psi|\hat{H}|\psi\rangle \geq E_0 \ .$$

Hint: *Expand the state* $|\psi\rangle$ *as a sum over* $\hat{H}$ *eigenstates.*

If $|\psi\rangle = \sum_i a_i|\psi_i\rangle$, with $\hat{H}|\psi_i\rangle = E_i|\psi_i\rangle$, then $\langle\psi|\hat{H}|\psi\rangle = \sum_i |a_i|^2 E_i$. And since $E_i \geq E_0$ if E_0 is the ground state energy, we have $\langle\psi|\hat{H}|\psi\rangle \geq \sum_i |a_i|^2 E_0 = E_0$ (with the last equality holding because the state $|\psi\rangle$ is normalized).

(b) *(5 points)* *Consider a Hamiltonian* $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ *for a particle in one dimension, where* V *is an even function. If this potential has at least two bound states, then the ground state wave function* $\psi_0(x)$ *is an even function of* x*, and the first-excited-state wave function* $\psi_1(x)$ *is an odd function of* x*. Noting that* $\langle\varphi|\psi\rangle = 0$ *where* $\psi(x)$ *is any even function of* x *and* $\varphi(x)$ *is any odd function of* x*, show that for any normalized odd wave function* $\varphi(x)$*,*

$$\langle\varphi|\hat{H}|\varphi\rangle \geq E_1 \ .$$

Again we expand $|\varphi\rangle = \sum_i |\psi_i\rangle\langle\psi_i|\varphi\rangle$, but now note that $\langle\psi_i|\varphi\rangle = 0$ whenever ψ_i is an even function; therefore only the odd eigenstates have nonzero coefficients in the sum over i. For each odd eigenstate, the energy satisfies $E_i \geq E_1$ where E_1 is the energy of the first excited state; therefore we have $\langle\psi|\hat{H}|\psi\rangle \geq \sum_i |a_i|^2 E_1 = E_1$ (with the last equality holding because the state $|\varphi\rangle$ is normalized).

Now consider the one-dimensional Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + F|\hat{x}| \ ,$$

where $|\hat{x}|$ *denotes the absolute value of* $\hat{x}$*. As you know from a homework problem, the ground-state energy* E_0 *and first-excited-state energy* E_1 *for this Hamiltonian are*

$$E_0 \approx \ 1.0188\left(\frac{\hbar^2 F^2}{2m}\right)^{1/3} , \quad E_1 \approx 2.3381\left(\frac{\hbar^2 F^2}{2m}\right)^{1/3} .$$

We will use the variational method to estimate E_0 *and* E_1 *for this Hamiltonian. For estimating* E_0*, we use the family of normalized even wave functions* $\{|\psi_a\rangle\}$*, where*

$$\psi_a(x) = \frac{1}{\pi^{1/4}a^{1/2}} \ e^{-x^2/2a^2} \ ,$$

and for estimating E_1*, we use the family of normalized odd wave functions* $\{|\varphi_a\rangle\}$*, where*

$$\varphi_a(x) = \frac{\sqrt{2}}{\pi^{1/4}a^{3/2}} \ x \ e^{-x^2/2a^2} \ .$$

The following integrals may be useful for working this problem:

$$\int_{-\infty}^{\infty} dx \ e^{-x^2} = \sqrt{\pi} \ , \quad \int_{-\infty}^{\infty} dx \ x^2 e^{-x^2} = \frac{\sqrt{\pi}}{2} \ , \quad \int_{-\infty}^{\infty} dx \ x^4 e^{-x^2} = \frac{3\sqrt{\pi}}{4} \ ,$$

$$\int_{-\infty}^{\infty} dx \ |x| e^{-x^2} = \int_{-\infty}^{\infty} dx \ |x|^3 e^{-x^2} = 1 \ .$$

(c) *(5 points)* *Compute $\langle\psi_a|\hat{H}|\psi_a\rangle$ as a function of a.*

Since $\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + F|\hat{x}|$, we have (after integrating by parts)

$$\begin{aligned}
\langle\psi_a|\hat{H}|\psi_a\rangle &= \int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m}\left|\frac{d}{dx}\psi_a(x)\right|^2 + F\,|\psi_a(x)|^2\right) \\
&= \frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m}\left(\frac{x}{a^2}\right)^2 + F|x|\right) e^{-x^2/a^2} \ .
\end{aligned}$$

Using the integrals listed above, we find:

$$\langle\psi_a|\hat{H}|\psi_a\rangle = \frac{\hbar^2}{2ma^2}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} + Fa\cdot\frac{1}{\sqrt{\pi}} = \frac{\hbar^2}{4ma^2} + \frac{Fa}{\sqrt{\pi}} \ .$$

(d) *(5 points)* *Estimate E_0 by minimizing the function found in (c). (If done correctly, this calculation overestimates the value of E_0 by less than 1%.)*

The function to be minimized is

$$E_0(a) \equiv \frac{1}{2}Aa^{-2} + Ba \ , \quad A = \frac{\hbar^2}{2m} \ , \quad B = \frac{F}{\sqrt{\pi}} \ .$$

It is stationary for

$$0 = \frac{d}{da}E_0(a) = -Aa^{-3} + B \ , \quad \text{or} \quad a = (A/B)^{1/3} \ ;$$

thus,

$$E_0(a) = \left(\frac{1}{2} + 1\right)\left(AB^2\right)^{1/3} \ .$$

Finally, we obtain

$$E_{0,\min} = \frac{3}{2}\left(\frac{\hbar^2}{2m}\cdot\frac{F^2}{\pi}\right)^{1/3} = 1.024\ \left(\frac{\hbar^2F^2}{2m}\right)^{1/3} \ .$$

Our estimate is too high by about 0.6%.

(e) *(5 points)* *Compute $\langle\varphi_a|\hat{H}|\varphi_a\rangle$ as a function of a.*

$$\begin{aligned}
\langle\varphi_a|\hat{H}|\varphi_a\rangle &= \int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m}\left|\frac{d}{dx}\varphi_a(x)\right|^2 + F\,|\varphi_a(x)|^2\right) \\
&= \frac{2}{a^3\sqrt{\pi}}\int_{-\infty}^{\infty} dx \left(\frac{\hbar^2}{2m}\left(1 - \frac{x^2}{a^2}\right)^2 + F|x|^3\right) e^{-x^2/a^2} \ .
\end{aligned}$$

Using the integrals listed above, we find:

$$\langle\varphi_a|\hat{H}|\varphi_a\rangle = \frac{\hbar^2}{2ma^2}\cdot\frac{2}{\sqrt{\pi}}\cdot\left(\sqrt{\pi} - 2\frac{\sqrt{\pi}}{2} + \frac{3\sqrt{\pi}}{4}\right) + Fa\cdot\frac{2}{\sqrt{\pi}} = \frac{3\hbar^2}{4ma^2} + \frac{2Fa}{\sqrt{\pi}} \ .$$

(*f*) *(5 points) Estimate E_1 by minimizing the function found in (e). (If done correctly, this calculation overestimates the value of E_1 by less than 1%.)*

Compared to part (*c*), A is now 3 times larger and B is twice as large; hence AB^2 is larger by the factor $3 \cdot 2^2 = 12$, and so we find:

$$E_{1,\min} = \frac{3}{2}\left(\frac{12}{\pi}\right)^{1/3}\left(\frac{\hbar^2 F^2}{2m}\right)^{1/3} = 2.3448\ \left(\frac{\hbar^2 F^2}{2m}\right)^{1/3} .$$

Our estimate is too high by about 0.3%.

3. Two correlated oscillators — 40 total points

Consider a system of two harmonic oscillators, labeled 1 and 2. The Hilbert space of the system is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ is the Hilbert space of oscillator 1 and $\mathcal{H}_2$ is the Hilbert space of oscillator 2. For notational convenience, an operator $\hat{\mathcal{O}}_1 \otimes \hat{I}_2$ that acts nontrivially only on oscillator 1 will be denoted as simply $\hat{\mathcal{O}}_1$, with the identity operator $\hat{I}_2$ acting on oscillator 2 left implicit. Similarly, an operator $\hat{I}_1 \otimes \hat{\mathcal{O}}_2$ that acts nontrivially only on oscillator 2 will be denoted as simply $\hat{\mathcal{O}}_2$, with the identity operator $\hat{I}_1$ acting on oscillator 1 left implicit. Obviously, any operator acting only on oscillator 1 commutes with any operator acting only on oscillator 2.

The annihilation and creation operators for oscillator 1 are $\hat{a}_1$ and $\hat{a}_1^\dagger$, satisfying $[\hat{a}_1, a_1^\dagger] = 1$, and the annihilation and creation operators for oscillator 2 are $\hat{a}_2$ and $\hat{a}_2^\dagger$, satisfying $[\hat{a}_2, a_2^\dagger] = 1$. As usual, the corresponding dimensionless position and momentum operator acting on oscillators 1 and 2 are

$$\hat{\xi}_1 = \tfrac{1}{\sqrt{2}}\left(\hat{a}_1 + \hat{a}_1^\dagger\right) , \qquad \hat{p}_{\xi_1} = \frac{-i}{\sqrt{2}}\left(\hat{a}_1 - \hat{a}_1^\dagger\right) ,$$
$$\hat{\xi}_2 = \tfrac{1}{\sqrt{2}}\left(\hat{a}_2 + \hat{a}_2^\dagger\right) , \qquad \hat{p}_{\xi_2} = \frac{-i}{\sqrt{2}}\left(\hat{a}_2 - \hat{a}_2^\dagger\right) .$$

A two-mode squeezed state *of the two oscillators can be expressed as*

$$|\psi_\gamma\rangle = N_\gamma \exp\left(\gamma\ \hat{a}_1^\dagger \otimes \hat{a}_2^\dagger\right)|0\rangle_1 \otimes |0\rangle_2 ,$$

where γ is a real number such that $\gamma^2 < 1$, $|0\rangle_1$ is the ground state of oscillator 1, $|0\rangle_2$ is the ground state of the oscillator 2, and N_γ is a normalization factor chosen to ensure that $\langle\psi_\gamma|\psi_\gamma\rangle = 1$.

(*a*) *5 points) Show that $|\psi_\gamma\rangle$ can be expressed in the form*

$$|\psi_\gamma\rangle = N_\gamma \sum_{n=0}^{\infty} C_n\ |n\rangle_1 \otimes |n\rangle_2,$$

where $|n\rangle_1$ is the eigenstate with eigenvalue n of $\hat{a}_1^\dagger \hat{a}_1$, and $|n\rangle_2$ is the eigenstate with eigenvalue n of $\hat{a}_2^\dagger \hat{a}_2$. Find C_n, and determine the value of N_γ.

Expanding the exponential:

$$|\psi_\gamma\rangle = N_\gamma \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \left(\hat{a}_1^\dagger\right)^n |0\rangle_1 \otimes \left(\hat{a}_2^\dagger\right)^n |0\rangle_2 = N_\gamma \sum_{n=0}^{\infty} \gamma^n |n\rangle_1 \otimes |n\rangle_2.$$

Therefore $C_n = \gamma^n$, and

$$1 = N_\gamma^2 \sum_{n=0}^{\infty} \gamma^{2n} = \frac{N_\gamma^2}{1-\gamma^2} \to N_\gamma = \sqrt{1-\gamma^2}.$$

(b) *(5 points) The expectation value of an operator $\hat{\mathcal{O}}_1$ that acts nontrivially only on oscillator 1 can be expressed as*

$$\langle\psi_\gamma|\hat{\mathcal{O}}_1 \otimes \hat{I}_2|\psi_\gamma\rangle = \mathrm{tr}\left(\hat{\mathcal{O}}_1 \hat{\rho}^{(1)}\right),$$

where $\hat{\rho}^{(1)}$ is the density operator for oscillator 1. Similarly, the expectation value of an operator $\hat{\mathcal{O}}_2$ that acts nontrivially only on oscillator 2 can be expressed as

$$\langle\psi_\gamma|\hat{I}_1 \otimes \hat{\mathcal{O}}_2|\psi_\gamma\rangle = \mathrm{tr}\left(\hat{\mathcal{O}}_2 \hat{\rho}^{(2)}\right).$$

In fact, in this case both oscillators have the same density operator, which can be expressed as

$$\hat{\rho}^{(1)} = \hat{\rho}^{(2)} = \sum_n \rho_n |n\rangle\langle n| \ .$$

Find $\{\rho_n\}$.

We note that

$$\langle\psi_\gamma|\hat{\mathcal{O}}_1 \otimes \hat{I}_2|\psi_\gamma\rangle = \sum_{n=0}^{\infty} N_\gamma^2 \gamma^{2n} \langle n|\hat{\mathcal{O}}_1|n\rangle.$$

Therefore,

$$\rho_n = N_\gamma^2 \gamma^{2n} = \left(1-\gamma^2\right)\gamma^{2n}.$$

(c) *(10 points) Using $\hat{\rho}^{(1)}$ from part (b), check that*

$$\langle\psi_\gamma|\hat{\xi}_1|\psi_\gamma\rangle = 0 = \langle\psi_\gamma|\hat{p}_{\xi_1}|\psi_\gamma\rangle \ ,$$

and compute

$$\langle\psi_\gamma|\hat{\xi}_1^2|\psi_\gamma\rangle, \quad \langle\psi_\gamma|\hat{p}_{\xi_1}^2|\psi_\gamma\rangle.$$

Of course, since $\hat{\rho}^{(2)} = \hat{\rho}^{(1)}$, we get the same values for the expectation values of position and momentum, and their squares, for oscillator 2. **Hint***: It may be useful to notice that*

$$\sum_{n=0}^{\infty} n\gamma^{2n} = \gamma^2 \frac{d}{d\gamma^2} \sum_{n=0}^{\infty} \gamma^{2n} = \gamma^2 \frac{d}{d\gamma^2}\left(1-\gamma^2\right)^{-1} = \frac{\gamma^2}{\left(1-\gamma^2\right)^2}.$$

We have $\langle n|\hat{a}|n\rangle = 0 = \langle n|\hat{a}^\dagger|n\rangle$, and therefore $\langle n|\hat{\xi}|n\rangle = 0 = \langle n|\hat{p}_\xi|n\rangle$. Thus

$$\langle\psi_\gamma|\hat{\xi}_1|\psi_\gamma\rangle = \sum_n \rho_n \langle n|\hat{\xi}_1|n\rangle = 0,$$

and

$$\langle\psi_\gamma|\hat{p}_{\xi_1}|\psi_\gamma\rangle = \sum_n \rho_n \langle n|\hat{p}_{\xi_1}|n\rangle = 0.$$

Furthermore,

$$\begin{aligned}\langle n|\hat{\xi}^2|n\rangle &= \frac{1}{2}\langle n|\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger|n\rangle \\ &= \frac{1}{2}\langle n|[\hat{a},\hat{a}^\dagger] + 2\hat{a}^\dagger\hat{a}|n\rangle = n + \frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\langle n|\hat{p}_\xi^2|n\rangle &= -\frac{1}{2}\langle n|\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger|n\rangle \\ &= \frac{1}{2}\langle n|[\hat{a},\hat{a}^\dagger] + 2\hat{a}^\dagger\hat{a}|n\rangle = n + \frac{1}{2},\end{aligned}$$

Therefore,

$$\begin{aligned}\langle\psi_\gamma|\hat{\xi}_1^2|\psi_\gamma\rangle &= \sum_n \rho_n \langle n|\xi_1^2|n\rangle = \left(1-\gamma^2\right)\sum_n \gamma^{2n}\left(n + \frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{\gamma^2}{1-\gamma^2} = \frac{1}{2}\left(\frac{1+\gamma^2}{1-\gamma^2}\right).\end{aligned}$$

The computation of $\langle\psi_\gamma|\hat{p}_{\xi_1}^2|\psi_\gamma\rangle$ is identical, and yields

$$\langle\psi_\gamma|\hat{p}_{\xi_1}^2|\psi_\gamma\rangle = \frac{1}{2}\left(\frac{1+\gamma^2}{1-\gamma^2}\right).$$

(d) *(10 points) Compute the expectation value*

$$\langle\psi_\gamma|\hat{a}_1 \otimes \hat{a}_2 + \hat{a}_1^\dagger \otimes \hat{a}_2^\dagger|\psi_\gamma\rangle\ .$$

Since you are now evaluating the expectation value of an operator that acts nontrivially on both oscillator 1 and oscillator 2, you'll need to use the expression for the joint state $|\psi_\gamma\rangle$ from part (a), rather than the density operators for the individual oscillators.

Since $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, we have

$$\left(\hat{a}_1 \otimes \hat{a}_2\right)|n\rangle_1 \otimes |n\rangle_2 = n\left(|n-1\rangle_1 \otimes |n-1\rangle\right)\ .$$

Therefore,

$$\begin{aligned}
\langle\psi_\gamma|\hat{a}_1\otimes\hat{a}_2|\psi_\gamma\rangle &= \left(1-\gamma^2\right)\sum_n \left(\langle n-1|\otimes\langle n-1|\right)\gamma^{n-1}n\gamma^n\left(|n-1\rangle\otimes|n-1\rangle\right) \\
&= \left(1-\gamma^2\right)\sum_n n\gamma^{2n-1} = \left(1-\gamma^2\right)\gamma^{-1}\frac{\gamma^2}{\left(1-\gamma^2\right)^2} = \frac{\gamma}{1-\gamma^2}\ .
\end{aligned}$$

And $\langle\psi_\gamma|\hat{a}_1^\dagger\otimes\hat{a}_2^\dagger|\psi_\gamma\rangle$ is just the complex conjugate of the above, so

$$\langle\psi_\gamma|\hat{a}_1\otimes\hat{a}_2+\hat{a}_1^\dagger\otimes\hat{a}_2^\dagger|\psi_\gamma\rangle = \frac{2\gamma}{1-\gamma^2}\ .$$

(*e*) *(5 points) Using the notation*

$$\hat{\xi}_\pm = \hat{\xi}_1 \pm \hat{\xi}_2\ ,\quad \hat{p}_{\xi_\pm} = \hat{p}_{\xi_1} \pm \hat{p}_{\xi_2}\ ,$$

combine your results from (*c*) *and* (*d*) *to evaluate the four expectation values*

$$\Delta\xi_\pm^2 = \langle\psi_\gamma|\hat{\xi}_\pm^2|\psi_\gamma\rangle\ ,\quad \Delta p_{\xi_\pm}^2 = \langle\psi_\gamma|\hat{p}_{\xi_\pm}^2|\psi_\gamma\rangle\ .$$

We have

$$\hat{\xi}_\pm^2 = \hat{\xi}_1^2 \pm 2\hat{\xi}_1\hat{\xi}_2 + \hat{\xi}_2^2\ ,\quad \hat{p}_{\xi_\pm}^2 = p_{\xi_1}^2 \pm 2\hat{p}_{\xi_1}\hat{p}_{\xi_2} + p_{\xi_2}^2\ ,$$

and we already know that

$$\langle\hat{\xi}_1^2\rangle = \langle\hat{\xi}_2^2\rangle = \langle\hat{p}_{\xi_1}^2\rangle = \langle\hat{p}_{\xi_2}^2\rangle = \frac{1}{2}\left(\frac{1+\gamma^2}{1-\gamma^2}\right);$$

It remains to evaluate the expectation values of the cross terms. But

$$\begin{aligned}
2\hat{\xi}_1\hat{\xi}_2 &= \hat{a}_1\otimes\hat{a}_2 + \hat{a}_1\otimes\hat{a}_2^\dagger + \hat{a}_1^\dagger\otimes\hat{a}_2 + \hat{a}_1^\dagger\otimes\hat{a}_2^\dagger\ , \\
2\hat{p}_{\xi_1}\hat{p}_{\xi_2} &= -\hat{a}_1\otimes\hat{a}_2 + \hat{a}_1\otimes\hat{a}_2^\dagger + \hat{a}_1^\dagger\otimes\hat{a}_2 - \hat{a}_1^\dagger\otimes\hat{a}_2^\dagger\ ,
\end{aligned}$$

while

$$\langle\hat{a}_1\otimes\hat{a}_2^\dagger\rangle = \langle\hat{a}_1^\dagger\otimes\hat{a}_2\rangle = 0\ .$$

Therefore, by combining (*c*) and (*d*) we have

$$\begin{aligned}
\langle\hat{\xi}_+^2\rangle &= \langle\hat{p}_{\xi_-}^2\rangle = \frac{1+\gamma^2}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2} = \frac{(1+\gamma)^2}{1-\gamma^2} = \frac{1+\gamma}{1-\gamma}\ , \\
\langle\hat{\xi}_-^2\rangle &= \langle\hat{p}_{\xi_+}^2\rangle = \frac{1+\gamma^2}{1-\gamma^2} - \frac{2\gamma}{1-\gamma^2} = \frac{(1-\gamma)^2}{1-\gamma^2} = \frac{1-\gamma}{1+\gamma}\ .
\end{aligned}$$

If your calculations are correct up to this point, you should have found that, in the limit $\gamma \to 1$, $\Delta\xi_+$ and Δp_{ξ_-} diverge while $\Delta\xi_-$ and Δp_{ξ_+} approach zero. (If that is not the case, check your work.) Therefore, in this limit, the two oscillators have the same position and opposite momentum. It is possible for both the relative position and the total momentum to have arbitrarily small uncertainty, because $\hat{\xi}_-$ and $\hat{p}_{\xi_+}$ are commuting observables. Conversely, in the limit $\gamma \to -1$, $\Delta\xi_-$ and Δp_{ξ_+} diverge while $\Delta\xi_+$ and Δp_{ξ_-} approach zero. Again, this is possible because $\hat{\xi}_+$ and $\hat{p}_{\xi_-}$ commute.

(*f*) *(5 points) What are the minimum values of the products $\Delta\xi_+ \cdot \Delta p_{\xi_+}$ and $\Delta\xi_- \cdot \Delta p_{\xi_-}$ allowed by the uncertainty principle? Verify that these minimum values are attained by $|\psi_\gamma\rangle$ for any $\gamma \in (-1, 1)$.*

From (*e*) we have

$$\Delta\xi_+ \cdot \Delta p_{\xi_+} = \Delta\xi_- \cdot \Delta p_{\xi_-} = 1 \ ,$$

while the commutators are

$$\left[\hat{\xi}_+, \hat{p}_{\xi_+}\right] = \left[\hat{\xi}_1, \hat{p}_{\xi_1}\right] + \left[\hat{\xi}_2, \hat{p}_{\xi_2}\right] = 2i = \left[\hat{\xi}_-, \hat{p}_{\xi_-}\right] \ .$$

According to the uncertainty principle

$$\Delta\xi_+ \cdot \Delta p_{\xi_+} \leq \frac{1}{2} \left| \left\langle \left[\hat{\xi}_+, \hat{p}_{\xi_+}\right] \right\rangle \right| = 1 \ ;$$

same thing for $\Delta\xi_- \cdot \Delta p_{\xi_-}$.