Ph 12b

Homework Assignment No. 3
Due: 5pm, Thursday, 28 January 2010

1. "A watched quantum state never moves." Consider a simple model of an atom with two energy levels — the ground state $|g\rangle$ has energy E_g and the excited state $|e\rangle$ has energy $E_e > E_g$, where $\omega = (E_e - E_g)/\hbar$; the Hamiltonian of this system is

$$\hat{H} = E_g|g\rangle\langle g| + E_e|e\rangle\langle e|.$$

An experimentalist is equipped to perform a measurement that projects the state of the atom onto the orthonormal basis

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + |e\rangle\right), \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle - |e\rangle\right),$$

and to prepare the atom in the state $|+\rangle$.

a) Suppose that the state $|+\rangle$ is prepared at time 0 and that the measurement projecting onto $\{|+\rangle, |-\rangle\}$ is performed at time t. Find the probability $P_t(+)$ of the + measurement outcome and the probability $P_t(-)$ of the − measurement outcome.

b) Suppose that the measurement projecting onto $\{|+\rangle, |-\rangle\}$ is performed twice in succession. The state $|+\rangle$ is prepared at time 0, the first measurement is performed at time t, and the second measurement is performed at time $2t$. Find the probability of a + outcome and the probability of a − outcome in the *second* measurement.

c) Now suppose that N measurements, equally spaced in time, are performed in succession. The state $|+\rangle$ is prepared at time 0, the first measurement is performed at time t, the second measurement at time $2t$, and so on, with the Nth measurement performed at time Nt. Find the probability $P(+^N)$ that the + outcome occurs in *every one* of the N measurements.

d) For the same situation as in part (c), denote the total elapsed time by $T = Nt$, so that the time interval between the measurements is $t = T/N$. Show that $P(+^N)$ can be expressed as

$$P(+^N) = 1 - f(\omega T)/N + O(1/N^2),$$

and find the function $f(\omega T)$. Thus, taking the limit $N \to \infty$ with ωT fixed, we conclude that if the atom is observed continuously its state never evolves.

2. A better bomb test. The bomb-testing protocol explained in class uses a beam splitter described by the unitary transformation

$$\hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

A single photon can enter the beam splitter through either port *in1* or port *in2* and can exit through either port *out1* or port *out2*. If the input state of the photon is $v_{in} = a|in1\rangle + b|in2\rangle$ and the output state is $v_{out} = c|out1\rangle + d|out2\rangle$, then

$$v_{out} = \begin{pmatrix} c \\ d \end{pmatrix} = \hat{U} v_{in} = \frac{1}{\sqrt{2}} \begin{pmatrix} a-b \\ a+b \end{pmatrix},$$

An interferometer can be fashioned from two beam splitters and two mirrors, where photon detectors are placed at the exit ports of the second beam splitter, labeled *exit1* and *exit2*. The state of a photon that exits the interferometer, $v_{exit} = e|exit1\rangle + f|exit2\rangle$, is given by

$$v_{exit} = \begin{pmatrix} e \\ f \end{pmatrix} = \hat{U} v_{out} = \hat{U}^2 v_{in},$$

where

$$\hat{U}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus if the input state is $|in1\rangle$, then the output state is $|exit2\rangle$, and the photon is detected at port *exit2* with probability one.

In the bomb test, the mirror at port *out2* is replaced by the bomb to be tested, and a photon enters the interferometer at port *in1*. If the bomb is a dud, which acts like a perfect mirror, the interferometer functions normally, and the photon exits at port *exit2*. But if the bomb is good, a photon in the state $|out2\rangle$ is absorbed, exploding the bomb. Thus the test has three possible outcomes — either the photon is absorbed and bomb explodes, or the bomb does not explode and the photon is detected at port *exit1* or port *exit 2*. The probabilities for these three outcomes are

$$P(explode) = 1/2, \quad P(exit1) = 1/4, \quad P(exit2) = 1/4.$$

If the photon is detected at port *exit1*, then we know that the bomb must be good, even though no photon encountered the bomb. If the photon is detected at port *exit2*, then the test is inconclusive – we do not find out whether the bomb is good or a dud. However, if we repeat the test many times and obtain the inconclusive result every time, we conclude that the bomb is very likely to be a dud.

a) If the bomb is good, what is the probability of obtaining the inconclusive result N times in succession?

b) Suppose that the bomb is good, and that we repeat the test as many times as necessary so that eventually either the bomb explodes or the test yields a conclusive result. What is the probability that the bomb eventually explodes?

We wish to improve the test by reducing the probability of exploding a good bomb. For this purpose, we use an unbalanced beam splitter described by the unitary transformation

$$\hat{V} = \begin{pmatrix} \cos(\pi/2N) & -\sin(\pi/2N) \\ \sin(\pi/2N) & \cos(\pi/2N) \end{pmatrix}.$$

where $N > 2$ is a positive integer. The interferometer is configured so that the photon passes through such a beam splitter N times before finally exiting; thus the state of the exiting photon is related to the state of the input photon by

$$v_{exit} = \hat{V}^N v_{in}.$$

c) Verify that

$$\hat{V}^N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus a photon that enters at port *in1* exits at port *exit2*.

d) To test a bomb, we configure the interferometer so that for each of the first $N-1$ beam splitters, a photon emerging from the *out2* port of the beam splitter encounters the bomb, exploding it if the bomb is good. (A photon emerging from either port of the final beam splitter is detected by one of the photon detectors.) If the bomb is good, what is the probability that it does not explode? Show that for large N your answer has the form

$$P(not\ explode) = 1 - c/N + O(1/N^2),$$

and find the number c.

e) If the bomb does not explode, the photon is detected at either port *exit2*, in which case the test is inconclusive, or at port *exit1*, in which case the unexploded bomb is known to be good. Assuming the bomb is good and does not explode, what is the probability $P(exit1|not\ explode)$ of a conclusive result? Show that for large N your answer has the form

$$P(exit1|not\ explode) = 1 - d/N^2 + O(1/N^3),$$

and find the number d.

Thus by choosing N sufficiently large, we ensure that the probability of exploding a good bomb is low and that the probability of a conclusive test result is high.

3. Distinguishing nonorthogonal states. Alice tells Bob that she will prepare a qubit in one of the two quantum states

$$|\psi_1\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix},$$

where $0 \le \alpha \le \pi/4$, and hence

$$0 \le \langle\psi_2|\psi_1\rangle = \sin(2\alpha) \le 1.$$

To decide which state to prepare, Alice flips a fair coin; then she prepares $|\psi_1\rangle$ if result of the coin flip is "heads" and she prepares $|\psi_2\rangle$ if the result of the coin flip is "tails." After preparing the qubit, Alice sends it to Bob, who performs a measurement that projects onto the orthonormal basis

$$|e_1\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

where $0 \le \theta \le \pi/2$. If Bob's measurement outcome is $|e_1\rangle$, he guesses that Alice prepared $|\psi_1\rangle$, and if Bob's outcome is $|e_2\rangle$, he guesses that Alice prepared $|\psi_2\rangle$.

a) Show that the probability that Bob's guess is wrong can be expressed as

$$P_{\text{error}} = \frac{1}{2}\left(|\langle e_2|\psi_1\rangle|^2 + |\langle e_1|\psi_2\rangle|^2\right).$$

b) Compute P_{error}, expressing your answer as a function of $\sin^2\theta$ and α.

c) How should Bob choose θ to minimize P_{error}, and what is the minimal error probability $P^{\min}_{\text{error}}$ he achieves by this choice?

d) What is $P^{\min}_{\text{error}}$ for $\alpha = 0$ and for $\alpha = \pi/4$? Explain why your answers make sense.

e) What is $P^{\min}_{\text{error}}$ for $\alpha = \pi/8$? For this case, sketch the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ and the measurement axes $|e_1\rangle$ and $|e_2\rangle$ that minimize Bob's error probability.

4. A second look. A particle is confined inside a one-dimensional box with walls at $x = -L/2$ and $x = L/2$. Its normalized wavefunction is

$$\psi(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right), \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right].$$

a) A bright beam of light is focused on the interval $[a, b]$, where $-L/2 \leq a \leq b \leq L/2$. If the particle lies in this interval, it will scatter the light and be detected. Conversely, if no scattered light is detected, then the particle is known to lie in the complement of this interval, $[-L/2, a] \cup [b, L/2]$. What is the probability that scattered light is detected?

b) What is the numerical value of your answer from part (*a*) in the case where the light beam illuminates the interval $[L/4, 3L/8]$?

c) Now suppose that the interval $[-L/4, L/4]$ is illuminated and no scattered light is detected. What is the updated wavefunction of the particle right after this measurement?

d) Right after the measurement described in part (*c*), the interval $[L/4, 3L/8]$ is illuminated. What is the probability that scattered light is detected?

e) Charlie thinks that the answers to parts (*b*) and (*d*) should agree. He notes that in the measurement described in part (*c*) there was no interaction between the light and the particle, and he argues that therefore the quantum state of the particle outside the illuminated interval $[-L/4, L/4]$ should not have been altered. Is there something wrong with Charlie's argument?