Ph 12b

Homework Assignment No. 5
Due: 5pm, Thursday, 18 February 2010

1. Minimal uncertainty I: particle in one dimension (10 points).

If we measure the Hermitian operator $\hat{A}$ in the state vector $|\psi\rangle$, the variance of the measurement outcomes is

$$(\Delta A)^2 = \left| \langle\psi| \left(\hat{A} - \langle\hat{A}\rangle\right)^2 |\psi\rangle \right|^2 ,$$

where $\langle\hat{A}\rangle$ denotes $\langle\psi|\hat{A}|\psi\rangle$; the standard deviation ΔA is also called the "uncertainty" of the observable $\hat{A}$ in the state $|\psi\rangle$. The *uncertainty principle*, derived in class, is an inequality relating the product of the uncertainties for two Hermitian operators $\hat{A}$ and $\hat{B}$ to the expectation value of their commutator:

$$\Delta A\ \Delta B \geq \frac{1}{2} \left| \langle\psi| \left[\hat{A}, \hat{B}\right] |\psi\rangle \right| . \tag{1}$$

The online lecture notes for the Feb. 4 lecture include a discussion of when eq.(1) is satisfied as an equality — we have

$$\Delta A\ \Delta B = \frac{1}{2} \left| \langle\psi| \left[\hat{A}, \hat{B}\right] |\psi\rangle \right| . \tag{2}$$

if and only if there exists a real number γ and a complex number λ such that

$$\left(\hat{A} - i\gamma\hat{B} - \lambda\right)|\psi\rangle = 0.$$

For a particle moving in one dimension, with position operator $\hat{x}$ and wave-number operator $\hat{k} = -i\frac{d}{dx}$, eq.(2) becomes

$$\Delta x\ \Delta k = 1/2;$$

we say that a wavefunction satisfying this condition has *minimal uncertainty.*

a) Find the most general complex-valued function $\psi(x)$ on the real line satisfying

$$\left(-i\frac{d}{dx} - i\gamma x - \lambda\right)\psi(x) = 0,$$

where γ is a real number and λ is a complex number.

b) We say that a wavefunction $\psi(x)$ is *normalizable* if

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 < \infty.$$

What conditions must γ and λ satisfy to ensure that $\psi(x)$ is normalizable?

c) Find a physical interpretation for γ, and for the real and imaginary parts of λ.

2. Minimal uncertainty II: the qubit (15 points).

Recall that the 2×2 *Pauli spin matrices* are the Hermitian operators

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

acting on a *qubit* (a quantum system with a two-dimensional Hilbert space). The operators σ_1 and σ_3 have minimal uncertainty in the state $|\psi\rangle$ if

$$\Delta\sigma_1\ \Delta\sigma_3 = \frac{1}{2}\left|\langle\psi|\,[\hat{\sigma}_1, \hat{\sigma}_3]\,|\psi\rangle\right|, \tag{3}$$

which is satisfied if and only if there exist real γ and complex λ such that

$$(\hat{\sigma}_1 - i\gamma\hat{\sigma}_3 - \lambda)\,|\psi\rangle = 0. \tag{4}$$

a) For $0 \le \gamma \le 1$, it is convenient to write γ as $\gamma = \sin\theta$, where $0 \le \theta \le \pi/2$. Express in terms of θ the nonnegative value of λ for which eq.(4) has a solution.

b) For the value of λ found in (*a*), express $|\psi\rangle$ in terms of θ.

c) Evaluate $\langle\psi|\hat{\sigma}_1|\psi\rangle$, $\langle\psi|\hat{\sigma}_3|\psi\rangle$, and $\langle\psi|\frac{i}{2}[\hat{\sigma}_1, \hat{\sigma}_3]|\psi\rangle$. Check that eq.(3) is satisfied.

d) For $\lambda \ge 1$, it is convenient to write γ as $\gamma = 1/\sin\theta$, where $0 < \theta < \pi/2$. Thus eq.(3) is satisfied if and only if there exists λ such that

$$(\hat{\sigma}_3 + i\sin\theta\hat{\sigma}_1 - \lambda)\,|\psi\rangle = 0. \tag{5}$$

Again, find the nonnegative value of λ for which eq.(5) has a solution, find the corresponding value of $|\psi\rangle$, and check eq.(3).

3. Measuring position with limited resolution (15 points).

No real apparatus can measure the position of a particle with perfect accuracy. To model a position measurement with imperfect resolution, consider two systems — one is the particle whose position is to be measured, and the other is used as a "meter" to facility the measurement. The Hilbert space of the meter, as for the particle, is the space of square integrable functions on the real line.

To perform the measurement, we begin by preparing the meter in the state $|\varphi\rangle = \int_{-\infty}^{\infty} dy\ \varphi(y)|y\rangle$. Then the meter interacts with the particle, with the interation described by the unitary transformation $\hat{U}$, where

$$\hat{U}(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |x+y\rangle;$$

that is, the meter is shifted by an amount equal to the position of the particle. Thus, if the initial state of the particle is $|\psi\rangle = \int_{-\infty}^{\infty} dx\ \psi(x)|x\rangle$, then after the interaction the joint state of the particle and meter becomes

$$\begin{aligned} \hat{U}(|\psi\rangle \otimes |\varphi\rangle) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \psi(x)\varphi(y)|x\rangle \otimes |y+x\rangle \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \psi(x)\varphi(y-x)|x\rangle \otimes |y\rangle. \end{aligned} \tag{6}$$

Finally, we imagine that the measurement of the particle is completed by performing a measurement on the meter that projects it onto the position eigenstate $|y\rangle$.

The joint state right before the measurement of the meter can be expressed as

$$\int_{-\infty}^{\infty} dy\ \sqrt{p(y)}\ |\psi_y\rangle \otimes |y\rangle$$

where

$$\sqrt{p(y)}\ |\psi_y\rangle = \int_{-\infty}^{\infty} dx\ \psi(x)\varphi(y-x)|x\rangle.$$

Here $p(y)$ is the probability that the measurement of the meter yields the outcome $|y\rangle$, and $|\psi_y\rangle$ is the normalized state of the particle right after the meter is measured; thus

$$p(y) = \int_{-\infty}^{\infty} dx\ |\psi(x)|^2|\varphi(y-x)|^2,$$

and the probability distribution for the particle's position x right after a measurement that yields outcome y is

$$p(x|y) = |\langle x|\psi_y\rangle|^2 = \frac{1}{p(y)}|\psi(x)|^2|\varphi(y-x)|^2.$$

Now suppose that the initial states of the particle and the meter are Gaussian wave packets:

$$\psi(x) = \frac{1}{(2\pi a^2)^{1/4}}e^{-(x-x_0)^2/4a^2}, \quad \varphi(y) = \frac{1}{(2\pi b^2)^{1/4}}e^{-y^2/4b^2}.$$

a) Compute $p(y)$, $\langle y\rangle$, and $(\Delta y)^2 = \langle (y - \langle y\rangle)^2\rangle$, where

$$\langle A\rangle \equiv \int_{-\infty}^{\infty} dy\ p(y)A(y).$$

b) Compute $p(x|y)$, $\langle x\rangle_y$, and $[(\Delta x)^2]_y = \langle (x - \langle x\rangle_y)^2\rangle_y$, where

$$\langle A\rangle_y \equiv \int_{-\infty}^{\infty} dx\ p(x|y)A(x).$$

Do your results seem reasonable for the case of a "narrow" meter ($b^2 \ll a^2$) and a "broad" meter ($b^2 \gg a^2$)?