Ph 12b

Homework Assignment No. 6
Due: 5pm, Thursday, 25 February 2010

1. Geometric phase (15 points).

Recall that the coherent state $|\alpha\rangle$ of a harmonic oscillator, where α is an arbitrary complex number, can be obtained by applying a unitary *displacement operator*

$$\hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$$

to the ground state:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle.$$

If α is real, then

$$\hat{D}(\alpha) = \exp\left(-i\left(\alpha\sqrt{2}\right)\hat{p}_\xi\right),$$

where $\hat{p}_\xi$ is the dimensionless momentum operator, and in that case $\hat{D}(\alpha)$ displaces the oscillator's dimensionless position $\hat{\xi}$ by $\alpha\sqrt{2}$. If α is imaginary, then

$$\hat{D}(\alpha) = \exp\left(i\left(-i\alpha\sqrt{2}\right)\hat{\xi}\right),$$

and in that case $\hat{D}(\alpha)$ displaces $\hat{p}_\xi$ by $-i\alpha\sqrt{2}$. If both the real and the imaginary part of α are nonzero, then $\hat{D}(\alpha)$ displaces both the position and the momentum. Because $\hat{\xi}$ and $\hat{p}_\xi$ do not commute ($[\hat{\xi}, \hat{p}_\xi] = i$, and $[\hat{a}, \hat{a}^\dagger] = 1$), displacements in different directions in the complex plane do not commute.

a) In class we derived the identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]} e^{\hat{A}+\hat{B}}$$

which holds if $\hat{A}$ and $\hat{B}$ both commute with the commutator $[\hat{A}, \hat{B}]$. Use this identity to show

$$\hat{D}(\beta)\hat{D}(\alpha) = e^{i\phi(\beta,\alpha)}\hat{D}(\beta + \alpha),$$

and find $\phi(\beta, \alpha)$.

b) Suppose that the oscillator is displaced by $\alpha(t)$, where $\alpha(0) = 0$ and $\alpha(t)$ varies continuously with the time t. To understand how this continuously varying displacement affects the oscillator's state, regard the cumulative displacement as a product of many infinitesimal displacements, in each of which α advances from $\alpha(t)$ to $\alpha(t) + dt\frac{d\alpha}{dt}$. Suppose that as t varies from 0 to t_{fin}, $\alpha(t)$ follows a path P from 0 to the final value α_{fin}. Then if the initial state vector of the oscillator at $t = 0$ is $|\psi(0)\rangle$, the final state vector $|\psi(t_{\text{fin}})\rangle$ at $t = t_{\text{fin}}$ is

$$|\psi(t_{\text{fin}})\rangle = e^{i\phi(P)}\hat{D}(\alpha_{\text{fin}})|\psi(0)\rangle,$$

where $\phi(P)$ can be written as

$$\phi(P) = \int_P (A_1 d\alpha_1 + A_2 d\alpha_2)\,.$$

Here we have expressed $\alpha = \alpha_1 + i\alpha_2$ in terms of its real and imaginary parts, and $(A_1(\alpha), A_2(\alpha))$ can be regarded as a vector field in the (α_1, α_2) plane. Find this vector field (A_1, A_2). You will find it convenient to use the result of part (*a*) in the case where β is the infinitesimal increment $d\alpha$.

c) Now suppose that the path followed by $\alpha(t)$ is a closed path C that returns to the origin at the final time $t = t_{\text{fin}}$. In this case, the displacement operator $\hat{D}(\alpha_{\text{fin}})$ is the identity operator, but the phase factor $e^{i\phi(C)}$ may be nontrivial. It is called a "geometric phase" because it depends only on the path followed in the α plane, not on the initial state of the oscillator. Show, by converting a line integral around the closed path C to a surface integral over the enclosed region S, that the geometric phase can be written as

$$\exp\left(ic(\text{Area})\right),$$

and find the constant c. Here

$$\text{Area} = \pm \int_S d\alpha_1 d\alpha_2,$$

with a $+$ sign if C encloses S in a counter-clockwise sense, and a $-$ sign if C encloses S in a clockwise sense.

d) Suppose that the time-dependent Hamiltonian

$$\hat{H}(t) = \hbar\Omega(-i)\left(\hat{a}e^{-i\nu t} - \hat{a}^\dagger e^{i\nu t}\right)$$

is applied to the oscillator during the time interval $t \in [0, 2\pi/\nu]$. Then the infinitesimal time-evolution operator

$$\hat{U}(t + dt, dt) = \exp\left(-\frac{i}{\hbar} dt \; \hat{H}(t)\right)$$

is an infinitesimal displacement, in a direction that rotates uniformly as a function of t. Thus, the evolution can be described as a continuously varying displacement in the α plane. Show that the cumulative displacement vanishes at $t = 2\pi/\nu$, compute the area enclosed by the path, and find the geometric phase.

e) Now suppose that the oscillator is coupled to two qubits, labeled A and B. The joint Hilbert space of the qubits and oscillator is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{\text{oscillator}}$, the tensor product of the two-qubit Hilbert space and the oscillator's Hilbert space, and the Hamiltonian acting on qubit and oscillator is

$$\hat{H}(t) = \hbar\Omega \left[(\sigma_3 \otimes I + I \otimes \sigma_3) \otimes (-i)\left(\hat{a}e^{-i\nu t} - \hat{a}^\dagger e^{i\nu t}\right)\right].$$

If this Hamiltonian acts for a time $2\pi/\nu$, the geometric phase acquired by the oscillator depends on the state of the qubits; hence in effect the Hamiltonian applies a two-qubit unitary $\hat{V}(\Omega, \nu)$ transformation, which depends on Ω and ν. Denoting by $\{|0\rangle, |1\rangle\}$ the basis in which σ_3 is diagonal, so that

$$\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|,$$

the transformation $\hat{V}(\Omega, \nu)$ is diagonal in the orthonormal basis

$$|0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle.$$

What are the eigenvalues of $\hat{V}(\Omega, \nu)$? For fixed Ω, find the largest positive value of ν (and correspondingly the shortest possible duration $2\pi/\nu$) such that the eigenvalues are $(i, 1, 1, i)$.

[This transformation is called a *geometric phase gate*, and it can be applied to a pair of two-level atomic ions in an electromagnetic trap. The qubits are encoded in the internal states of the two ions. The oscillator arises from the motion of the ions in the harmonic trapping potential — specifically, it is a normal mode of vibration in which both ions participate. The motion of the two ions is coupled because of their mutual Coulomb repulsion.]

2. Squeezing an Oscillator (15 points).

In class we constructed the coherent states of a harmonic oscillator, and showed that these are minimum-uncertainty wave packets. In this problem, you will construct another type of minimum-uncertainty wave packet for the oscillator, called a "squeezed state."

a) Suppose that $\hat{B}$ is an anti-Hermitian operator; that is, $\hat{B}^\dagger = -\hat{B}$. Show, to linear order in a power series expansion in ϵ, that $e^{\epsilon\hat{B}}$ is unitary, if ϵ is real.

b) For $\hat{B}$ anti-Hermitian, compute $\left(e^{\epsilon\hat{B}}\right)^\dagger A \left(e^{\epsilon\hat{B}}\right)$ to linear order in ϵ, assuming again that ϵ is real. (Express your answer in terms of $\hat{A}$ and the commutator $[\hat{A},\hat{B}]$.)

Now consider a harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}\left(\hat{p}_\xi\right)^2 + \frac{1}{2}\hat{\xi}^2 ,$$

where $\left[\hat{\xi},\hat{p}_\xi\right] = i$, and define the operator $\hat{a}$ (and its adjoint $\hat{a}^\dagger$) by

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}+\hat{a}^\dagger\right) \ , \quad \hat{p}_\xi = \frac{-i}{\sqrt{2}}\left(\hat{a}-\hat{a}^\dagger\right) \ .$$

Also define a unitary operator

$$\hat{S}(r) = \exp\left[\frac{1}{2}r\left(\hat{a}^2 - (\hat{a}^\dagger)^2\right)\right] \ ,$$

where r is real. ($\hat{S}(r)$ is called the "squeeze operator.")

c) Compute, *to linear order in* ϵ,

$$\hat{S}(\epsilon)^\dagger\hat{a}\hat{S}(\epsilon) \quad \text{and} \quad \hat{S}(\epsilon)^\dagger\hat{a}^\dagger\hat{S}(\epsilon) \ .$$

d) Using the result from (*c*), compute, again to linear order in ϵ,

$$\hat{S}(\epsilon)^\dagger\hat{\xi}\hat{S}(\epsilon) \quad \text{and} \quad \hat{S}(\epsilon)^\dagger\hat{p}_\xi\hat{S}(\epsilon) \ .$$

e) A squeeze operator with finite r can be composed from many infinitesimal squeeze operators:

$$\hat{S}(r) = \lim_{N\to\infty}\left(\hat{S}(r/N)\right)^N \ .$$

Using the result from (*d*), compute

$$\hat{S}(r)^\dagger\hat{\xi}\hat{S}(r) \quad \text{and} \quad \hat{S}(r)^\dagger\hat{p}_\xi\hat{S}(r) \ .$$

Let $|0\rangle$ denote the ground state of the harmonic oscillator, satisfying $\hat{a}|0\rangle = 0$. Now consider a state $|r\rangle$ defined by

$$|r\rangle \equiv \hat{S}(r)|0\rangle \ .$$

(This state is called a "squeezed state," and r is called the "squeeze factor.")

f) Compute $\langle r|\hat{\xi}|r\rangle$, and $\langle r|\hat{p}_\xi|r\rangle$, and compute the uncertainties $(\Delta\xi)_r$ and $(\Delta p_\xi)_r$ in the state $|r\rangle$. Why is $|r\rangle$ called a "squeezed state?"

3. Anharmonic oscillator (10 points).

The one-dimensional harmonic oscillator, with Hamiltonian

$$\hat{H} = \hbar\omega\left(\frac{1}{2}\hat{p}_\xi^2 + \frac{1}{2}\hat{\xi}^2\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

has an unusual property — all spacings between consecutive energy levels are equal:

$$E_{n+1} - E_n = \hbar\omega.$$

This property no longer holds if there is a small quartic term in the potential energy, so that the Hamiltonian becomes

$$\hat{H} = \hbar\omega\left(\frac{1}{2}\hat{p}_\xi^2 + \frac{1}{2}\hat{\xi}^2 + k\hat{\xi}^4\right),$$

where k is a small positive dimensionless number. The quartic term produces a correction to the energy of the nth excited state, which becomes

$$E'_n = E_n + k\hbar\omega\langle n|\hat{\xi}^4|n\rangle + \cdots.$$

Here we have written down only the constant and linear terms in a power series expansion in k, but we will assume that k is sufficiently small that the higher order corrections can be safely neglected.

a) Compute $\langle n|\hat{\xi}^4|n\rangle$, and hence find the leading k-dependent correction to the energy E'_n. You will find it convenient to express $\hat{\xi}$ in terms of the annihilation operator $\hat{a}$ and the creation operator $\hat{a}^\dagger$.

b) Because of the anharmonic term in the potential, the energy splitting $E'_{21} \equiv E'_2 - E'_1$ between the second and first excited states differs from the energy splitting $E'_{10} = E'_1 - E'_0$ between the first excited state and the ground state. A useful measure of anharmonicity is the ratio

$$\Delta(k) = \frac{E'_{21} - E'_{10}}{E'_{10}}.$$

Compute $\Delta(k)$, to linear order in k.