PH12b 2010 Solutions HW#3

1.

The Hamiltonian of this two level system is

$$\widehat{H} = E_g \left|g\right\rangle \left\langle g\right| + E_e \left|e\right\rangle \left\langle e\right|,$$

where $E_g < E_e$. The experimentalist basis is

$$\left|+\right\rangle = \frac{1}{\sqrt{2}} \left(\left|g\right\rangle + \left|e\right\rangle\right), \qquad \left|-\right\rangle = \frac{1}{\sqrt{2}} \left(\left|g\right\rangle - \left|e\right\rangle\right).$$

a) At $t = 0$ the state is $\left|\psi(0)\right\rangle = \left|+\right\rangle$, then, the state will evolve as

$$\left|\psi(t)\right\rangle = \frac{1}{\sqrt{2}} \left(\left|g\right\rangle + e^{-i\omega t} \left|e\right\rangle\right),$$

where $\omega = (E_e - E_g)/\hbar$, and we used the fact $\left|\psi(t)\right\rangle$ is defined up to a overall phase factor, i.e. $\left|\psi(t)\right\rangle \sim e^{iE_g t/\hbar} \left|\psi(t)\right\rangle$.

Then the probability of a + outcome and a - outcome if $\left|\psi(0)\right\rangle = \left|+\right\rangle$ are

$$\begin{aligned} P_t^+(+) &= \left|\left\langle + \right|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|\left(1 + e^{-i\omega t}\right)\right|^2 = \frac{1}{2}\left(1 + \cos\omega t\right) = \cos^2\frac{\omega t}{2}, \\ P_t^+(-) &= \left|\left\langle - \right|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|\left(1 - e^{-i\omega t}\right)\right|^2 = \frac{1}{2}\left(1 - \cos\omega t\right) = \sin^2\frac{\omega t}{2}. \end{aligned}$$

$P_t^+(+) + P_t^+(-) = 1$ as expected.

b)

First, it is easy to see that if at $t = 0$ the state is $\left|\psi(0)\right\rangle = \left|-\right\rangle$, then the probability of a + outcome and a - outcome if $\left|\psi(0)\right\rangle = \left|-\right\rangle$ are

$$\begin{aligned} P_t^-(+) &= P_t^+(-), \\ P_t^-(-) &= P_t^+(+). \end{aligned}$$

Now, we want to find $P_t^+(+,2)$ and $P_t^+(-,2)$ the probability of a + outcome and a - outcome after 2 measurements each done at a interval t if $\left|\psi(0)\right\rangle = \left|+\right\rangle$. We can get a + outcome in two ways, the first measurement is + and the second + or the first measurement is - and the second is +. Remember that if the first measurement is + or - the state will collapse to + or - respectively (i.e. $\left|\psi(t)\right\rangle = \left|+\right\rangle$ or $\left|-\right\rangle$) and will start evolving again. Then,

$$\begin{aligned} P_t^+(+,2) &= P_t^+(+)\,P_t^+(+) + P_t^+(-)\,P_t^-(+) = P_t^+(+)\,P_t^+(+) + P_t^+(-)\,P_t^+(-) \\ &= \frac{1}{4}\left(1 + \cos\omega t\right)^2 + \frac{1}{4}\left(1 - \cos\omega t\right)^2 = \frac{1}{2}\left(1 + \cos^2\omega t\right) \end{aligned}$$

similarly

$$\begin{aligned} P_t^+(-,2) &= P_t^+(+)\,P_t^+(-) + P_t^+(-)\,P_t^-(-) = 2P_t^+(+)\,P_t^+(-) \\ &= \frac{1}{2}\left(1 - \cos^2\omega t\right) \end{aligned}$$

b) We are going to perform N measurements at equal t intervals. Since we want the probability that in all the measurements we get a + outcome this is equal to the probability of doing Nth times the experiment in a) and getting a + outcome in each one, because if we get a + outcome we are collapsing the state to $\left|+\right\rangle$ so this is like if we start the experiment again. Then

$$P\left(+^N\right) = P_t^+(+)^N = \cos\left(\frac{\omega t}{2}\right)^{2N}$$

d) Using $T = Nt$ we get

$$\begin{aligned} P\left(+^N\right) &= \cos\left(\frac{\omega t}{2}\right)^{2N} = \left[\cos^2\left(\frac{\omega T}{2N}\right)\right]^N \approx \left[1 - \left(\frac{\omega T}{2N}\right)^2 + O\left(\frac{1}{N^3}\right)\right]^N, \\ &\approx 1 - N\left(\frac{\omega T}{2N}\right)^2 + O\left(\frac{1}{N^2}\right) \approx 1 - \frac{\omega^2 T^2}{4}\frac{1}{N} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

Then

$$f(wT) = \frac{\omega^2 T^2}{2}.$$

We can conclude that a watched quantum state never evolve because $P\left(+^N\right) \to 1$ as $N \to \infty$ with ωT fixed.

2.

a) The probability of an inconclusive result is $P(exit2) = 1/4$. Then, the probability of obtaining the inconclusive result N times in succession is just

$$P = (1/4)^N.$$

b) If the bomb is good and we repite the test as many times as necessary until that eventually either the bomb explodes or the test yields a conclusive result, then the probability $P_{Explode}$ that the bomb eventually explodes is

$$\begin{aligned} P_{Explode} &= P(explode) + P(inconclusive)P_{Explode}, \\ &= \frac{1}{2} + \frac{1}{4}P_{Explode}, \end{aligned}$$

where $P(explode) = 1/2$ and $P(inconclusive) = P(exit2) = 1/4$. Then we get,

$$P_{Explode} = \frac{2}{3}.$$

Similarly, the probability $P_{Conclusive}$ that the test eventually yields to a conclusive result is

$$\begin{aligned} P_{Conclusive} &= P(conclusive) + P(inconclusive)P_{Conclusive}, \\ &= \frac{1}{4} + \frac{1}{4}P_{Conclusive}, \end{aligned}$$

$$\Longrightarrow P_{Conclusive} = \frac{1}{3}.$$

Note that $P_{Conclusive} + P_{Explode} = 1$ as expected.

c) First notice that

$$\begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}\begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} = \begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ \sin(a+b) & \cos(a+b) \end{pmatrix},$$

then it is easy to see that

$$\widehat{V}^N = \begin{pmatrix} \cos\left(\frac{\pi}{2N}\right) & -\sin\left(\frac{\pi}{2N}\right) \\ \sin\left(\frac{\pi}{2N}\right) & \cos\left(\frac{\pi}{2N}\right) \end{pmatrix}^N = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

d) If the input state is $V_{in} = (1,0)^T$ then after the **first** beam splitter $V_{out} = \left(\cos\left(\frac{\pi}{2N}\right), \sin\left(\frac{\pi}{2N}\right)\right)^T$. This means that if the bomb is good then it will explode with the probability of finding the state $|out2\rangle$ that is $P_1\left(explode\right) = \sin\left(\frac{\pi}{2N}\right)^2$. If it does not explode then, $P_1(\text{not explode}) = \cos\left(\frac{\pi}{2N}\right)^2$ and the post-measurement state will collapse to $|out1\rangle$. This means that the input state for the next beam splitter is again $V_{in} = (1,0)^T$, so we are just repeating the experiment again. Therefore, the probability that it does not explode after $N-1$ beam splitters is the same than the probability that it does not explode after making the single beam splitter experiment $N-1$ times. Then

$$P\left(\text{not explode}\right)=\left[\cos\left(\frac{\pi}{2N}\right)^{2}\right]^{N-1}\approx\left[1-\frac{\pi^{2}}{4N^{2}}+O\left(\frac{1}{N^{4}}\right)\right]^{N-1}\approx 1-\frac{\pi^{2}}{4N}+O\left(\frac{1}{N^{2}}\right),$$

then

$$c=\frac{\pi^{2}}{4}.$$

e) If the bomb does not explode the state that enters the last beam splitter will be $|out1\rangle$ and it will be transform by the last beam splitter to $|out1\rangle\rightarrow\cos\left(\frac{\pi}{2N}\right)|exit1\rangle+\sin\left(\frac{\pi}{2N}\right)|exit2\rangle$ then $P\left(exit1\right)=\cos\left(\frac{\pi}{2N}\right)^{2}$. And

$$P\left(exit1|\text{not explode}\right)=P\left(exit1\right)=\cos\left(\frac{\pi}{2N}\right)^{2}\approx 1-\frac{\pi^{2}}{4N^{2}}+O\left(\frac{1}{N^{4}}\right).$$

Then,

$$d=\frac{\pi^{2}}{4}.$$

3.

Alice basis is:

$$|\psi_{1}\rangle=\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},\qquad|\psi_{2}\rangle=\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix},$$

where $0\leq\alpha\leq\pi/4$.

Bob basis is:

$$|e_{1}\rangle=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},\qquad|e_{2}\rangle=\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix},$$

where $0\leq\theta\leq\pi/2$.

a) The experiment has four possible outcomes with probabilities

prepare/measure	$\langle e_1\|$	$\langle e_2\|$
$\|\psi_1\rangle$	$P_{\psi_1}\|\langle e_1\|\psi_1\rangle\|^2$	$P_{\psi_1}\|\langle e_2\|\psi_1\rangle\|^2$
$\|\psi_2\rangle$	$P_{\psi_2}\|\langle e_1\|\psi_2\rangle\|^2$	$P_{\psi_2}\|\langle e_2\|\psi_2\rangle\|^2$

where $P_{\psi_1}=1/2=P_{\psi 2}$ is the probability that the state was prepare by Alice in $|\psi_1\rangle$ or $|\psi_2\rangle$.

The offdiagonal outcomes in the table represent the wrong guesses of Bob, therefore the probability that Bob guess is wrong is

$$P_{error}=\frac{1}{2}\left(|\langle e_1|\psi_2\rangle|^2+|\langle e_2|\psi_1\rangle|^2\right).$$

b) The overlaps between the basis are

$$\langle e_1|\psi_2\rangle=\sin(\alpha+\theta),\qquad\langle e_2|\psi_1\rangle=\sin(\alpha-\theta),$$

then

$$P_{error}=\frac{1}{2}\left(\sin(\alpha+\theta)^2+\sin(\alpha-\theta)^2\right)=\sin^2\alpha+\sin^2\theta\cos 2\alpha.$$

c) Because $0\leq 2\alpha\leq\pi/2$ then $\cos 2\alpha\geq 0$. This implies that P_{error} is minimized when $\sin^2\theta=0$, then

$$\theta_{\min}=0,\qquad P_{error}^{\min}\left(\alpha\right)=\sin^2\alpha.$$

d)

For $\alpha=0$, $P_{error}^{\min}\left(0\right)=0$, this make sense since for $\alpha=0$, $|\psi_i\rangle=|e_i\rangle$ and $\langle\psi_2|\psi_1\rangle=0$ so the basis are orthogonal.

For $\alpha=\pi/4$, $P_{error}^{\min}\left(\pi/4\right)=1/2$, this make sense since for $\alpha=\pi/4$, the two states are the same $|\psi_1\rangle=|\psi_2\rangle=\binom{1}{1}/\sqrt{2}$ and there is a equal chance of Alice having prepared $|\psi_1\rangle$ or $|\psi_2\rangle$.

e) Evaluating we get

$$P_{error}^{\min}(\pi/8) = \frac{2-\sqrt{2}}{4} \approx 0.146$$

The sketch is just $|e_1\rangle$ in the x-axis, $|e_2\rangle$ in the y-axis, $|\psi_1\rangle$ making an angle of $\pi/8$ radians with the x-axis, and $|\psi_2\rangle$ making an angle of $\pi/8$ with the y-axis. All four vectors have unit length.

4.

The walls of the one-dimensional box are at $x = -L/2$ and $L/2$. The wavefunction is

$$\psi(x) = \langle x|\, \psi\rangle = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right), \qquad x \in \left[-\frac{L}{2}, \frac{L}{2}\right].$$

a) The probability density of the particle to be at some x is

$$P(x) = |\psi(x)|^2 = \psi^*(x)\psi(x) = \frac{2}{L}\cos^2\left(\frac{\pi x}{L}\right).$$

The probability that scattered light is detected is

$$P_{scattered} = \int_a^b P(x)dx = \frac{(b-a)}{L} + \frac{1}{2\pi}\left(\sin\left(\frac{2\pi}{L}b\right) - \sin\left(\frac{2\pi}{L}a\right)\right)$$

b) $[a,b] = [L/4, 3L/8]$, substituting in a) we get

$$P_{scattered} = \frac{1}{8} - \frac{2-\sqrt{2}}{4\pi} \approx 0.0784.$$

c) (Elaborate Answer) First we have to define the operator that make the measurement. The operator that detects a particle at some point x' is given by $|x'\rangle\langle x'|\, dx'$. Then the Hermitian operator that detects a particle in the interval $[a,b]$ is

$$O = \int_a^b |x'\rangle\langle x'|\, dx'.$$

The wave function is

$$|\psi\rangle = \int_{-L/2}^{L/2} \psi(y)\, |y\rangle\, dy.$$

The eigenvalue equation for this operator is $O\,|\psi\rangle = \lambda\,|\psi\rangle$ this is

$$\begin{aligned} O\,|\psi\rangle &= \int_a^b \int_{-L/2}^{L/2} \psi(y)\,|x'\rangle\langle x'\,|y\rangle\, dy dx' = \int_a^b \int_{-L/2}^{L/2} \psi(y)\,|x'\rangle\,\delta\left(x'-y\right) dy dx' \\ &= \int_a^b \psi(y)\,|y\rangle\, dy = \lambda \int_{-L/2}^{L/2} \psi(y)\,|y\rangle\, dy \end{aligned}$$

the last line can just be satisfy if $\lambda = 0$ and $\psi(y)$ is zero in the interval $[a,b]$ *i.e.* just have support in the complement of the interval $[a,b]$; or if $\lambda = 1$ and $\psi(y)$ just have support in the interval $[a,b]$.

Then the eigenvalues and the degenerate eigenvector of the operator are

$$\begin{aligned} &\lambda = 0, \quad |x'\rangle\langle x'|\, dx', \qquad x' \in [-L/2, a) \cup (b, L/2], \\ &\lambda = 1, \qquad\quad |x'\rangle\langle x'|\, dx', \qquad x' \in [a,b]. \end{aligned}$$

From the lecture notes "Quantum postulates III" we know that if the outcome $\lambda = 0$ is obtained, the normalized state right after the measurement is

$$\left|\psi_{after}\right\rangle = \frac{\widehat{Q}_0 \left|\psi\right\rangle}{\sqrt{\left\langle\psi\right| \widehat{Q}_0 \left|\psi\right\rangle}},$$

where $\widehat{Q}_0$ is the orthogonal projector to the space of eigenvectors with $\lambda = 0$, i.e.

$$\widehat{Q}_0 = \int_{-L/2}^{a} \left|x'\right\rangle \left\langle x'\right| dx' + \int_{b}^{L/2} \left|x'\right\rangle \left\langle x'\right| dx'.$$

Then we have

$$\begin{aligned}
\widehat{Q}_0 \left|\psi\right\rangle &= \int_{-L/2}^{L/2} \int_{-L/2}^{a} \psi(y) \left\langle x'\right| y\rangle \left|x'\right\rangle dx' dy + \int_{-L/2}^{L/2} \int_{b}^{L/2} \psi(y) \left\langle x'\right| y\rangle \left|x'\right\rangle dx' dy, \\
&= \int_{-L/2}^{L/2} \int_{-L/2}^{a} \psi(y) \delta(x'-y) \left|x'\right\rangle dx' dy + \int_{-L/2}^{L/2} \int_{b}^{L/2} \psi(y) \left|x'\right\rangle \delta(x'-y) dx' dy, \\
&= \int_{-L/2}^{a} \psi(x') \left|x'\right\rangle dx' + \int_{b}^{L/2} \psi(x') \left|x'\right\rangle dx',
\end{aligned}$$

where we used $\left\langle x'\right| y\rangle = \delta(x'-y)$, and $\delta(x'-y)$ is a Dirac Delta function.This just means that the wave function after the measurement is proportional to

$$\left\langle x\right| \phi\rangle \equiv \left\langle x\right| \widehat{Q}_0 \left|\psi\right\rangle = \begin{cases} \psi(x) & x \in [-L/2, a) \cup (b, L/2] \\ 0, & x \in [a, b]. \end{cases} .$$

Finally we have to normalized the wave function after the measurement, this is done by the factor $\sqrt{\left\langle\psi\right| \widehat{Q}_0 \left|\psi\right\rangle} = \sqrt{\left\langle\psi\right| \widehat{Q}_0 \widehat{Q}_0 \left|\psi\right\rangle} = \sqrt{\left\langle\phi\right| \phi\rangle}$ where we used $\widehat{Q}_0 \widehat{Q}_0 = \widehat{Q}_0$. Now, it is easy to see that

$$\left\langle\phi\right| \phi\rangle = 1 - P_{scattered}\left(a, b\right).$$

Therefore,

$$\psi_{after}\left(x\right) = \left\langle x\right| \psi_{after}\rangle = \frac{1}{\sqrt{1 - P_{scattered}\left(a, b\right)}} \begin{cases} \psi(x) & x \in [-L/2, a) \cup (b, L/2] \\ 0, & x \in [a, b]. \end{cases} .$$

For $[a, b] = [-L/4, L/4]$, $P_{scattered}\left(-L/4, L/4\right) = 1/2 - 1/\pi \approx 0.1817$ we have

$$\psi_{after}\left(x\right) = \begin{cases} \sqrt{\frac{1}{(1/2-1/\pi)}} \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) & x \in [-L/2, a - L/4) \cup (L/4, L/2] \\ 0, & x \in [-L/4, L/4]. \end{cases}$$

d) Because the wave function after the measurement in the interval $[a, b] = [L/4, 3L/8]$ is proportional to the original wave function we have that

$$P^{after}_{scattered}(L/4, 3L/8) = \frac{P_{scattered}(L/4, 3L/8)}{1/2 - 1/\pi} = \frac{0.0784}{0.1817} \approx 0.431,$$

where $P_{scattered}(L/4, 3L/8)$ comes from b).

e) Charlie's argument is wrong. Although the light and particle do not interact this is a measurement because we are getting the information that the particle is not in a certain interval. From the postulates of Quantum mechanics we know that any measurement will collapse the wavefunction to a new wavefunction as explain in c). In this process we need to normalize again the wavefunction and this will change the probability as we calculate in d). If Charlie were right then the probability of finding the particle in the entire box after the measurement will not be one which is obviously wrong.