Ph12b Solution Set 4

February 5, 2010

1 The state $|\psi\rangle_S \otimes |un\rangle_E$ is transformed to

$$\sqrt{1-p}|\psi\rangle_S \otimes |un\rangle_E + \sqrt{p}\big(a|0\rangle_S \otimes |\gamma\rangle_E + b|1\rangle_S \otimes |\eta\rangle_E\big) \tag{1}$$

Since $|\gamma\rangle_E$ and $|\eta\rangle_E$ aren't mutually orthogonal, we can't use this equation directly to compute the new density operator. Instead, we must change to an orthonormal system, as suggested in the hint. One way to do this is to keep $|\gamma\rangle_E$ and to take $|\delta\rangle_E$ as the second basis vector, where $|\delta\rangle_E$ is determined by

$$|\eta\rangle_E = (1-\epsilon)|\gamma\rangle_E + \sqrt{2\epsilon - \epsilon^2}|\delta\rangle_E \tag{2}$$

Substituting (2) into (1), we get that the new state is

$$\sqrt{1-p}|\psi\rangle_S \otimes |un\rangle_E + \sqrt{p}\Big(a|0\rangle_S + b(1-\epsilon)|1\rangle_S\Big) \otimes |\gamma\rangle_E + \sqrt{p}b\sqrt{2\epsilon - \epsilon^2}|1\rangle_S \otimes |\delta\rangle_E \tag{3}$$

Remember that if we have a state

$$\sum_i |\psi_i\rangle_S \otimes |f_i\rangle_E$$

with $|f_i\rangle_E$ normalized and mutually orthogonal, then we calculate the density operator by the formula

$$\hat{\rho} = \sum_i |\psi_i\rangle_S \langle\psi_i|_S$$

Working in the basis $|0\rangle_S, |1\rangle_S$ of S, we have

$$\sqrt{1-p}|\psi\rangle_S = \sqrt{1-p}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\sqrt{p}\big(a|0\rangle_S + b(1-\epsilon)|1\rangle_S\big) = \sqrt{p}\begin{pmatrix} a \\ b(1-\epsilon) \end{pmatrix}$$

$$\sqrt{p}\sqrt{2\epsilon - \epsilon^2}b|1\rangle_S = \sqrt{p}\sqrt{2\epsilon - \epsilon^2}\begin{pmatrix} 0 \\ b \end{pmatrix}$$

so from (3) we get that the new density operator is

$$\hat{\rho}' = (1-p)\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + p\begin{pmatrix} a^* & (1-\epsilon)b^* \end{pmatrix}\begin{pmatrix} a \\ b(1-\epsilon) \end{pmatrix} + p(2\epsilon-\epsilon^2)\begin{pmatrix} 0 & b^* \end{pmatrix}\begin{pmatrix} 0 \\ b \end{pmatrix}$$
$$= \begin{pmatrix} |a|^2 & (1-p\epsilon)b^*a \\ (1-p\epsilon)a^*b & |b|^2 \end{pmatrix}$$

Comparing this to the original density operator

$$\hat{\rho} = |\psi\rangle_S\langle\psi|_S = \begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} |a|^2 & b^*a \\ a^*b & |b|^2 \end{pmatrix}$$

lets us read off that $\lambda = 1 - p\epsilon$.

2 (a) We begin by analysing the evolution of a single term of the form $|\psi\rangle\langle\psi|$. If at time t we have $|\psi(t)\rangle\langle\psi(t)|$, then at time $t+dt$ we have

$$\begin{aligned}|\psi(t+dt)\rangle\langle\psi(t+dt)| &= \Big(|\psi(t)\rangle - i\omega dt|e\rangle\langle e||\psi(t)\rangle\Big)\Big(\langle\psi(t)| + i\omega dt\langle\psi(t)||e\rangle\langle e|\Big)\\ &= |\psi(t)\rangle\langle\psi(t)| - i\omega dt|e\rangle\langle e||\psi(t)\rangle\langle\psi(t)|\\ &\quad + i\omega dt|\psi(t)\rangle\langle\psi(t)||e\rangle\langle e| + \omega^2 dt^2|e\rangle\langle e||\psi(t)\rangle\langle\psi(t)||e\rangle\langle e|\end{aligned}$$

Notice that the right hand side is linear in $|\psi(t)\rangle\langle\psi(t)|$. Since a density operator is a linear combination of $|\psi_i(t)\rangle\langle\psi_i(t)|$ terms, by summing the previous equation over such terms, for an arbitrary density operator $\hat{\rho}(t)$ we have

$$\hat{\rho}(t+dt) = \hat{\rho}(t) - i\omega dt|e\rangle\langle e|\hat{\rho}(t) + i\omega dt\hat{\rho}(t)|e\rangle\langle e| + \omega^2 dt^2|e\rangle\langle e|\hat{\rho}(t)|e\rangle\langle e|$$

Now taking the limit of $\frac{\hat{\rho}(t+dt)-\hat{\rho}(t)}{dt}$ as $dt \to 0$ gives

$$\frac{d\hat{\rho}}{dt} = -i\omega|e\rangle\langle e|\hat{\rho} + i\omega\hat{\rho}|e\rangle\langle e| \tag{4}$$

as required.

(b) In the basis $\{|g\rangle, |e\rangle\}$, we have $|e\rangle = \binom{0}{1}$ so $|e\rangle\langle e| = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and Eq. (4) becomes

$$\frac{d}{dt}\begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix} = -i\omega\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix} + i\omega\begin{pmatrix} \rho_{gg} & \rho_{ge} \\ \rho_{eg} & \rho_{ee} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & i\omega\rho_{ge} \\ -i\omega\rho_{eg} & 0 \end{pmatrix}$$

Componentwise, we get the 4 differential equations

$$\begin{aligned}\frac{d}{dt}\rho_{gg} &= 0\\ \frac{d}{dt}\rho_{ge} &= i\omega\rho_{ge}\\ \frac{d}{dt}\rho_{eg} &= -i\omega\rho_{eg}\\ \frac{d}{dt}\rho_{ee} &= 0\end{aligned}$$

The solutions of these equations, in terms of initial conditions at $t = 0$, are

$$\begin{aligned} \rho_{gg}(t) &= \rho_{gg}(0) \\ \rho_{ge}(t) &= e^{i\omega t}\rho_{ge}(0) \\ \rho_{eg}(t) &= e^{-i\omega t}\rho_{eg}(0) \\ \rho_{ee}(t) &= \rho_{ee}(0) \end{aligned}$$

so in terms of components of $\hat{\rho}(0)$, $\hat{\rho}(t)$ is given by

$$\hat{\rho}(t) = \begin{pmatrix} \rho_{gg}(0) & e^{i\omega t}\rho_{ge}(0) \\ e^{-i\omega t}\rho_{eg}(0) & \rho_{ee}(0) \end{pmatrix}$$

(c) Using the same approach as in part (a), if at time t we had a pure state $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$, then at time $t + dt$ we have the density operator

$$\begin{aligned} \hat{\rho}(t+dt) = &\left(|g\rangle\langle g||\psi(t)\rangle + \sqrt{1-\Gamma dt}|e\rangle\langle e||\psi(t)\rangle\right)\left(\langle\psi(t)||g\rangle\langle g| + \sqrt{1-\Gamma dt}\langle\psi(t)||e\rangle\langle e|\right) \\ &+ \left(\sqrt{\Gamma dt}|g\rangle\langle e||\psi(t)\rangle\right)\left(\sqrt{\Gamma dt}\langle\psi(t)||e\rangle\langle g|\right) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \hat{\rho}(t+dt) = &|g\rangle\langle g|\hat{\rho}(t)|g\rangle\langle g| + \sqrt{1-\Gamma dt}|e\rangle\langle e|\hat{\rho}(t)|g\rangle\langle g| + \sqrt{1-\Gamma dt}|g\rangle\langle g|\hat{\rho}(t)|e\rangle\langle e| \\ &+ (1-\Gamma dt)|e\rangle\langle e|\hat{\rho}(t)|e\rangle\langle e| + \Gamma dt|g\rangle\langle e|\hat{\rho}(t)|e\rangle\langle g| \end{aligned}$$

Moreover, since this equation is linear in $\hat{\rho}(t)$, it will still hold for an arbitrary mixture of states. Since $|g\rangle$, $|e\rangle$ form an orthonormal basis of the atom's Hilbert space, we have $|g\rangle\langle g| + |e\rangle\langle e| = 1$ which we use with the previous equation to get

$$\begin{aligned} \frac{\hat{\rho}(t+dt) - \hat{\rho}(t)}{dt} = &\frac{\sqrt{1-\Gamma dt} - 1}{dt}|e\rangle\langle e|\hat{\rho}(t)|g\rangle\langle g| + \frac{\sqrt{1-\Gamma dt} - 1}{dt}|g\rangle\langle g|\hat{\rho}(t)|e\rangle\langle e| \\ &- \Gamma|e\rangle\langle e|\hat{\rho}(t)|e\rangle\langle e| + \Gamma|g\rangle\langle e|\hat{\rho}(t)|e\rangle\langle g| \end{aligned}$$

and taking the limit $dt \to 0$, this becomes

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}|e\rangle\langle e|\hat{\rho}|g\rangle\langle g| - \frac{\Gamma}{2}|g\rangle\langle g|\hat{\rho}|e\rangle\langle e| - \Gamma|e\rangle\langle e|\hat{\rho}|e\rangle\langle e| + \Gamma|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|$$

Again using $|g\rangle\langle g| + |e\rangle\langle e| = 1$, this can be simplified to

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}|e\rangle\langle e|\hat{\rho} - \frac{\Gamma}{2}\hat{\rho}|e\rangle\langle e| + \Gamma|g\rangle\langle e|\hat{\rho}|e\rangle\langle g| \tag{5}$$

proving the atom's master equation.

(d) Since $|g\rangle = \binom{1}{0}$ and $|e\rangle = \binom{0}{1}$, we can rewrite Eq. (5) in the matrix form as

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\rho_{gg} & \rho_{ge}\\ \rho_{eg} & \rho_{ee}\end{pmatrix} &= -\frac{\Gamma}{2}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho_{gg} & \rho_{ge}\\ \rho_{eg} & \rho_{ee}\end{pmatrix} - \frac{\Gamma}{2}\begin{pmatrix}\rho_{gg} & \rho_{ge}\\ \rho_{eg} & \rho_{ee}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\\ &\quad + \Gamma\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\rho_{gg} & \rho_{ge}\\ \rho_{eg} & \rho_{ee}\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix}\Gamma\rho_{ee} & -\frac{\Gamma}{2}\rho_{ge}\\ -\frac{\Gamma}{2}\rho_{eg} & -\Gamma\rho_{ee}\end{pmatrix}\end{aligned}$$

Componentwise, we get the 4 differential equations

$$\begin{aligned}\frac{d}{dt}\rho_{gg} &= \Gamma\rho_{ee}\\ \frac{d}{dt}\rho_{ge} &= -\frac{\Gamma}{2}\rho_{ge}\\ \frac{d}{dt}\rho_{eg} &= -\frac{\Gamma}{2}\rho_{eg}\\ \frac{d}{dt}\rho_{ee} &= -\Gamma\rho_{ee}\end{aligned}$$

The one for ρ_{gg} is coupled with ρ_{ee}, so we start by solving the other three,

$$\begin{aligned}\rho_{ge}(t) &= e^{-\Gamma t/2}\rho_{ge}(0)\\ \rho_{eg}(t) &= e^{-\Gamma t/2}\rho_{eg}(0)\\ \rho_{ee}(t) &= e^{-\Gamma t}\rho_{ee}(0)\end{aligned}$$

Now integrating the equation for $d\rho_{gg}/dt$ from 0 to t we get

$$\rho_{gg}(t) = \rho_{gg}(0) + (1 - e^{-\Gamma t})\rho_{ee}(0)$$

so in terms of components of $\hat{\rho}(0)$, $\hat{\rho}(t)$ is given by

$$\hat{\rho}(t) = \begin{pmatrix}\rho_{gg}(0) + (1 - e^{-\Gamma t})\rho_{ee}(0) & e^{-\Gamma t/2}\rho_{ge}(0)\\ e^{-\Gamma t/2}\rho_{eg}(0) & e^{-\Gamma t}\rho_{ee}(0)\end{pmatrix}$$

(e) Combining the two differential equations gives a new differential equation, and we can combine them at the componentwise level, getting

$$\begin{aligned}\frac{d}{dt}\rho_{gg} &= \Gamma\rho_{ee}\\ \frac{d}{dt}\rho_{ge} &= \left(-\frac{\Gamma}{2} + i\omega\right)\rho_{ge}\\ \frac{d}{dt}\rho_{eg} &= \left(-\frac{\Gamma}{2} - i\omega\right)\rho_{eg}\\ \frac{d}{dt}\rho_{ee} &= -\Gamma\rho_{ee}\end{aligned}$$

As in part (d), we start by solving the uncoupled differential equations, getting

$$\rho_{ge}(t) = e^{\left(-\frac{\Gamma}{2}+i\omega\right)t}\rho_{ge}(0)$$
$$\rho_{eg}(t) = e^{\left(-\frac{\Gamma}{2}-i\omega\right)t}\rho_{eg}(0)$$
$$\rho_{ee}(t) = e^{-\Gamma t}\rho_{ee}(0)$$

after which we integrate the equation for ρ_{gg}, getting

$$\rho_{gg}(t) = \rho_{gg}(0) + (1 - e^{-\Gamma t})\rho_{ee}(0)$$

To summarize, in terms of components of $\hat{\rho}(0)$, $\hat{\rho}(t)$ is given by

$$\hat{\rho}(t) = \begin{pmatrix} \rho_{gg}(0) + (1 - e^{-\Gamma t})\rho_{ee}(0) & e^{\left(-\frac{\Gamma}{2}+i\omega\right)t}\rho_{ge}(0) \\ e^{\left(-\frac{\Gamma}{2}-i\omega\right)t}\rho_{eg}(0) & e^{-\Gamma t}\rho_{ee}(0) \end{pmatrix}$$

3 (a) Since we have 1/2 probability for each of the states $|\psi_H\rangle$, $|\psi_T\rangle$, the density operator is

$$\begin{aligned} \hat{\rho} &= \frac{1}{2}|\psi_H\rangle\langle\psi_H| + \frac{1}{2}|\psi_T\rangle\langle\psi_T| \\ &= \frac{1}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} \\ &= \begin{pmatrix}3/4 & 1/4\\ 1/4 & 1/4\end{pmatrix} \end{aligned}$$

(b) Eigenvalues λ of $\hat{\rho}$ satisfy the characteristic equation

$$(\frac{3}{4} - \lambda)(\frac{1}{4} - \lambda) - \frac{1}{4}\frac{1}{4} = 0$$

which has the solutions

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{8}} \tag{6}$$

(c) From the equation

$$\begin{pmatrix}3/4 - \lambda_\pm & 1/4 \\ 1/4 & 1/4 - \lambda_\pm\end{pmatrix}\begin{pmatrix}u_\pm\\v_\pm\end{pmatrix} = 0$$

we get $(1 \mp \sqrt{2})u_\pm + v_\pm = 0$. If we focus on normalized eigenvectors, $u_\pm^2 + v_\pm^2 = 1$, and combining these two equations gives, after some algebra,

$$\begin{pmatrix}u_\pm\\v_\pm\end{pmatrix} = \begin{pmatrix}\sqrt{2 \pm \sqrt{2}}/2 \\ \pm\sqrt{2 \mp \sqrt{2}}/2\end{pmatrix}$$

By comparing these with the values of $\cos(\pi/8)$, $\sin(\pi/8)$ (which you can calculate using the half-angle formulas), you may notice that

$$\begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix}$$
$$\begin{pmatrix} u_- \\ v_- \end{pmatrix} = \begin{pmatrix} \sin(\pi/8) \\ -\cos(\pi/8) \end{pmatrix}$$

Another reason to suspect that the angle $\pi/8$ would have something to do with the solution is just by noticing that $|\psi_H\rangle$, $|\psi_T\rangle$ are vectors with phases 0 and $\pi/4$, and since they contribute equally to the density operator, it is to be expected that the density operator has some sort of symmetry around the axis with angle $\pi/8$.

4 (a) Let M be the matrix $(\psi_{ab})_{a,b=1}^N$. By the singular value decomposition, there exist unitary matrices $U = (u_{ai})_{a,i=1}^N$ and $V = (v_{jb})_{j,b=1}^N$ such that $M = UDV$ with $D = (d_{ij})_{i,j=1}^N$ a diagonal real $N \times N$ matrix with nonnegative eigenvalues. In componentwise notation, $M = UDV$ becomes

$$\psi_{ab} = \sum_{i,j=1}^N u_{ai} d_{ij} v_{jb}$$

which we can use to express $|\psi\rangle$ as

$$\begin{aligned} |\psi\rangle &= \sum_{a,b=1}^N \psi_{ab} |e_a\rangle \otimes |f_b\rangle \\ &= \sum_{a,i,j,b=1}^N u_{ai} d_{ij} v_{jb} |e_a\rangle \otimes |f_b\rangle \\ &= \sum_{i,j=1}^N d_{ij} \left(\sum_{a=1}^N u_{ai} |e_a\rangle \right) \otimes \left(\sum_{b=1}^N v_{jb} |f_b\rangle \right) \end{aligned}$$

Now we use the condition that U and V are unitary. Remember that unitary matrices correspond to transformations from one orthonormal basis to another. Treating U as a unitary matrix acting on $\mathcal{H}_A$, we see that

$$|e_i'\rangle = \sum_{a=1}^N u_{ai} |e_a\rangle$$

defines a new orthonormal basis of $\mathcal{H}_A$, and treating V as a unitary matrix acting on $\mathcal{H}_B$, we see that

$$|f_j'\rangle = \sum_{b=1}^N v_{jb} |f_b\rangle$$

defines a new orthonormal basis of $\mathcal{H}_B$ (note that here the summation is in the second index of v_{jb}, so this transform actually corresponds to V^T, but that is irrelevant because the transpose of a unitary matrix is again unitary).

With these new bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, the state $|\psi\rangle$ is represented as

$$|\psi\rangle = \sum_{i,j=1}^{N} d_{ij}|e'_i\rangle \otimes |f'_j\rangle$$

But remember that D is a diagonal real matrix with nonnegative eigenvalues! Thus, $d_{ij} = 0$ for $i \neq j$. Denote also $p_i = d_{ii}^2$. Then the previous equation becomes

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{p_i}|e'_i\rangle \otimes |f'_i\rangle \tag{7}$$

Since we know that the state is normalized, $\langle\psi||\psi\rangle = 1$, which can, by the formula given in the problem, be expressed componentwise as

$$\sum_{i=1}^{N} |\sqrt{p_i}|^2 = 1$$

Thus, $\sum_{i=1}^{N} p_i = 1$.

(b) From Eq. (7), since $|f'_i\rangle$ are orthonormal vectors, we get

$$\hat{\rho}_A = \sum_{i=1}^{N} p_i|e'_i\rangle\langle e'_i| \tag{8}$$

Since $\{|e'_i\rangle\}$ form an orthonormal basis, in that basis $\hat{\rho}_a$ is a diagonal matrix,

$$\hat{\rho}_A = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_N \end{pmatrix} \tag{9}$$

Analogously, the density operator for system B can be written in basis $\{|f'_i\rangle\}$ as

$$\hat{\rho}_B = \sum_{i=1}^{N} p_i|f'_i\rangle\langle f'_i| = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_N \end{pmatrix} \tag{10}$$

(c) Since from part (b) we have representations of $\hat{\rho}_A$ and $\hat{\rho}_B$ as diagonal matrices, their eigenvalues are just the elements on the diagonal. Thus, reading off from Eqs. (9) and (10) we see that $\hat{\rho}_A$ and $\hat{\rho}_B$ have the same eigenvalues $p_1, p_2, \ldots, p_N$.