Ph 12b Quantum Physics
18 February 2010
Coherent States

Note Title 2/18/2010

Recall that Hamiltonian $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$

can we reexpressed as $\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})$, where

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^+) \qquad \hat{p} = \sqrt{\frac{m\hbar\omega}{2}}(-i)\,(\hat{a} - \hat{a}^+)$$

and $[\hat{a}, \hat{a}^+] = 1$. we'll also use dimensionless position and momentum operators

$$\hat{\xi} = \frac{1}{\sqrt{2}}\,(\hat{a} + \hat{a}^+) \qquad \hat{p}_\xi = \frac{-i}{\sqrt{2}}\,(\hat{a} - \hat{a}^+)$$

such that $[\hat{\xi}, \hat{p}_\xi] = -i$, in terms of which

$$\hat{H} = \hbar\omega\left(\frac{1}{2}\hat{p}_\xi^{\,2} + \frac{1}{2}\xi^2\right).$$

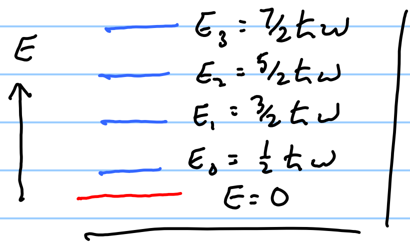
Using commutation relations $\begin{array}{l}[\hat{N},\hat{a}] = -\hat{a} \\ [\hat{N},\hat{a}^+] = \hat{a}^+\end{array}$, where $\hat{N} = \hat{a}^+\hat{a}$,

we found:

Ground state $|0\rangle$ annihilated by $\hat{a}$: $\hat{a}|0\rangle = 0$

Excited state $|n\rangle$ is $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$.

where $\hat{N}|n\rangle = n|n\rangle$ and hence $\hat{H}|n\rangle = E_n|n\rangle$, $E_n = \hbar\omega(n+\frac{1}{2})$



Energy eigenvalues are evenly spaced

"Creation operator" $\hat{a}^+$ raises n

"Annihilation operator" $\hat{a}$ lowers n

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle.$$

In position representation, ground state wave function is a Gaussian, centered at origin:

$\varphi_0(\xi) = \langle\xi|0\rangle$ satisfies $0 = \hat{a}\,\varphi_0(\xi) = \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial\xi}\right)\varphi_0(\xi)$

$$\Longrightarrow \boxed{\varphi_0(\xi) = \frac{1}{\pi^{1/4}}e^{-\xi^2/2}}$$

Normalized so $\int_{-\infty}^{\infty} d\xi\,|\varphi_0(\xi)|^2 = 1$

The ground state is also a Gaussian in the momentum representation, as seen by Fourier transforming, or by noting that we can represent $\xi = i\frac{d}{dp_\xi} \Rightarrow$

$$0 = \hat{a}\,\tilde{\varphi}_0(p_\xi) = \frac{1}{\sqrt{2}}\left(i\frac{d}{dp_\xi} + i p_\xi\right)\tilde{\varphi}_0(p_\xi) \Rightarrow$$

$$\boxed{\tilde{\varphi}_0(p_\xi) = \frac{1}{\pi^{1/4}}\, e^{-p_\xi^2/2}}$$

Ground state is minimum uncertainty wave packet, with

$$\langle \hat{\xi} \rangle_0 = \langle \hat{p}_\xi \rangle_0 = 0\,, \qquad \langle \hat{\xi}^2 \rangle_0 = \langle \hat{p}_\xi^2 \rangle_0 \quad \frac{1}{2}$$

Excited state wave functions are obtained by applying powers of $\hat{a}^+$ to the ground state.

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\xi - d/d\xi\right) \Rightarrow \varphi_n(\xi) = \frac{1}{\sqrt{n!}}\,\frac{1}{2^{n/2}}\left(\xi - d/d\xi\right)\frac{1}{\pi^{1/4}}\, e^{-\xi^2/2}$$

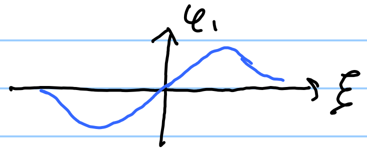
or $$\boxed{\varphi_n(\xi) = \frac{1}{\pi^{1/4}\, 2^{n/2}\sqrt{n!}}\, H_n(\xi)\, e^{-\xi^2/2}}$$

where $H_n(\xi)$ is the nth Hermite polynomial:

$$H_n(\xi) = e^{\xi^2/2}\left(\xi - d/d\xi\right)^n e^{-\xi^2/2},$$

a degree-n polynomial with n real zeros.
For example, the first excited state is

$$\varphi_1(\xi) = \frac{1}{\pi^{1/4}\sqrt{2}}\left(\xi - d/d\xi\right)e^{-\frac{1}{2}\xi^2} = \frac{\sqrt{2}}{\pi^{1/4}}\,\xi\, e^{-\xi^2/2},$$



an odd function with one node (at the origin).

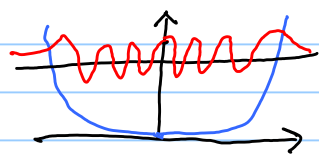
$\varphi_n(\xi)$ has n nodes; it is even for n even and odd for n odd (as is true in general for 1D potential problems)

Furthermore, these functions are orthogonal:

$$\int_{-\infty}^{\infty} d\xi\, \varphi_n(\xi)\varphi_m(\xi) = \frac{1}{\sqrt{\pi}\,2^n n!}\int_{-\infty}^{\infty} d\xi\, H_n(\xi)H_m(\xi)e^{-\xi^2} = \delta_{n,m}$$

The harmonic oscillator energy eigenstates are a <u>countable</u> basis for $L^2(\mathbb{R})$ (square integrable on $[-\infty,\infty]$).

If $V \to$ constant as $|x| \to \infty$, then bound state wave functions fall off exponentially as $e^{-\kappa|x|}$. Here $V \to \infty$, and bound states fall off faster, like a Gaussian.

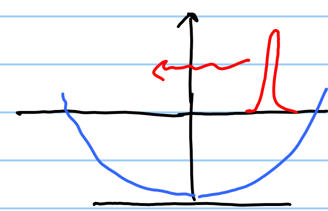


As E_n increases, turning points spread and function wiggles faster.

$$\langle \hat{\xi}^2 \rangle_n = \langle n|\tfrac{1}{2}(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger)|n\rangle = \langle n|\tfrac{1}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})|n\rangle$$

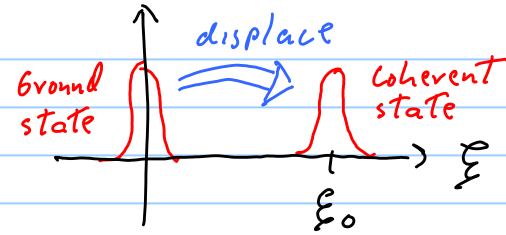
$$\left(\text{since } \langle n|(\hat{a}^\dagger)^2|n\rangle = \langle n|\hat{a}^2|n\rangle = 0\right) = \langle n|a^\dagger a + \tfrac{1}{2}|n\rangle = n + \tfrac{1}{2}$$

and also $\langle \hat{p}_\xi^2 \rangle_n = n + \frac{1}{2}$.



Correspondence Principle

How does the behavior of the quantum oscillator match up with the known behavior of the classical oscillator? As for the free particle, we should consider the motion of narrow wavepacket states. The "most classical" motion occurs for a minimum uncertainty wavepacket, for which x and p are as sharply defined as the uncertainty relations allow. Unlike the case of the free particle, we'll find that for the harmonic oscillator, these minimum uncertainty wavepackets <u>do not spread</u>. They are called "coherent states;" as expected, they oscillate with period $T = 2\pi/\omega$.



To prepare a coherent state, imagine displacing the ground state wave function away from the origin, obtaining a Gaussian centered at $\xi = \xi_0$. This displaced Gaussian is no longer a stationary state — the oscillator's restoring force pushes it back toward the origin. To study this motion, we expand the displaced Gaussian in terms of energy eigenstates.

How do we displace a wave function? Since $\hat{P}_\xi = -i \frac{d}{d\xi}$, note that

$$\hat{U}(\xi_0) \equiv \exp(-i\xi_0 \hat{P}_\xi) = \exp\left(-\xi_0 \frac{d}{d\xi}\right) = \sum_{m=0}^{\infty} \frac{1}{m!}(-\xi_0)^m \left(\frac{d}{d\xi}\right)^m,$$

which acting on a function $\varphi(\xi)$ yields

$$\hat{U}(\xi_0)\,\varphi(\xi) = \sum_{m=0}^{\infty} \frac{1}{m!}(-\xi_0)^m \frac{d^m}{d\xi^m}\varphi(\xi) = \varphi(\xi - \xi_0)$$

(Taylor series).

Thus $\hat{U}(\xi_0)$ is a <u>displacement</u> <u>operator</u>. It rigidly shift a function centered at origin to function centered at ξ_0. We say that the momentum $\hat{P}_\xi$ <u>generates</u> displacements:

$$\hat{U}(\xi_0) = \lim_{N\to\infty}\left(\hat{I} - i\frac{\xi_0}{N}\hat{P}_\xi\right)^N, \text{ where } (\hat{I} - i\delta\hat{P}_\xi) \text{ displaces by } \delta.$$

Since $\hat{P}_\xi = \frac{-i}{\sqrt{2}}(\hat{a} - \hat{a}^+)$, displacement operator is

$$\hat{U}(\xi_0) = \exp\left(-\frac{\xi_0}{\sqrt{2}}(\hat{a} - \hat{a}^+)\right)$$

Can we express this in a more useful way?

What is the exponential of a sum of operators?

If $\hat{A}$ is Hermitian, with eigenbasis $\{|\psi_n\rangle\}$, $\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$,
then $e^{\hat{A}}$ is diagonal in the same basis:

$$\hat{A} = \sum_n |\psi_n\rangle a_n \langle\psi_n| \Rightarrow e^{\hat{A}} = \sum_n |\psi_n\rangle e^{a_n}\langle\psi_n|$$

If $\hat{A}$ and $\hat{B}$ commute, they can be <u>simultaneously</u> diagonalized: $\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$, $\hat{B}|\psi_n\rangle = b_n|\psi_n\rangle$

$$\Rightarrow e^{\hat{A}+\hat{B}} = \sum_n |\psi_n\rangle e^{a_n+b_n}\langle\psi_n| = \sum_n |\psi_n\rangle e^{a_n}e^{b_n}\langle\psi_n|$$

$$= \sum_{n,m} (|\psi_m\rangle e^{a_m}\langle\psi_m|)\left(\sum_n |\psi_n\rangle e^{b_n}\langle\psi_n|\right) = e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}$$

But -- if $\hat{A}$ and $\hat{B}$ do not commute, then eigenbases of $\hat{A}$, $\hat{B}$, $\hat{A}+\hat{B}$ are all distinct. What then?

<u>Claim</u>: <u>If</u> $\hat{A}$ and $\hat{B}$ both commute with their commutator $[\hat{A},\hat{B}]$, <u>then</u> $\boxed{e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}}$

To prove this, consider the operator

$$\hat{\mathcal{O}}(\lambda) = e^{-\lambda\hat{B}}e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})} \quad \text{where } \lambda \in [0,1]$$

We'll derive a differential equation satisfied by $\hat{\mathcal{O}}(\lambda)$, then integrate from $\lambda=0$ to $\lambda=1$ to prove the claim.

We note that:

$$\frac{d}{d\lambda}e^{\lambda\hat{A}} = \frac{d}{d\lambda}\sum_n |\psi_n\rangle e^{\lambda a_n}\langle\psi_n| = \sum_n |\psi_n\rangle a_n e^{\lambda a_n}\langle\psi_n|$$

$$= \sum_{n,m} |\psi_m\rangle a_m \langle\psi_m|\psi_n\rangle e^{\lambda a_n}\langle\psi_n| = \hat{A}e^{\lambda\hat{A}} = e^{\lambda\hat{A}}\hat{A}.$$

We can differentiate using the product rule.

$$\frac{d}{d\lambda}\left[e^{-\lambda\hat{B}}e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})}\right]$$

$$= e^{-\lambda\hat{B}}(-\hat{B})e^{-\lambda\hat{A}}e^{\lambda(\hat{A}+\hat{B})} + e^{-\lambda\hat{B}}e^{-\lambda\hat{A}}(-\hat{A})e^{\lambda(\hat{A}+\hat{B})} + e^{-\lambda\hat{B}}e^{-\lambda\hat{A}}(\hat{A}+\hat{B})e^{\lambda(\hat{A}+\hat{B})}$$

$$= e^{-\lambda\hat{B}}\left(-[\hat{B}, e^{-\lambda\hat{A}}]\right)e^{\lambda(\hat{A}+\hat{B})}$$

what is this commutator?

In general,

$$[\hat{B}, \hat{A}^n] = [\hat{B},\hat{A}]\hat{A}^{n-1} + \hat{A}[\hat{B},\hat{A}]\hat{A}^{n-2} + \hat{A}^2[\hat{B},\hat{A}]\hat{A}^{n-3} + \cdots + \hat{A}^{n-1}[B,\hat{A}]$$

But if $\hat{A}$ commutes with $[\hat{B},\hat{A}]$, then

$$[\hat{B},\hat{A}^n] = n[\hat{B},\hat{A}]\hat{A}^{n-1}$$ and hence

$$[\hat{B}, e^{\lambda\hat{A}}] = [\hat{B}, \sum_n \frac{\lambda^n}{n!}\hat{A}^n] = \lambda[\hat{B},\hat{A}]\sum_n \frac{\lambda^{n-1}}{(n-1)!}\hat{A}^{n-1} = \lambda[\hat{B},\hat{A}]e^{\lambda\hat{A}}$$

$$\Rightarrow \frac{d}{d\lambda}\hat{\Theta}(\lambda) = -\lambda[\hat{A},\hat{B}]\hat{\Theta}(\lambda)$$ (since $[\hat{A},\hat{B}]$ commutes with all the exponentials)

Integrating this equation from $\lambda=0$ with the boundary condition $\hat{\Theta}(0)=\hat{I}$, we obtain

$$\hat{\Theta}(\lambda) = \exp\left(-\tfrac{1}{2}\lambda^2[\hat{A},\hat{B}]\right)$$

Thus $\Theta(1) = e^{-\hat{B}}e^{-\hat{A}}e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, or

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]},$$ proving the claim.

Also, notice that $e^{\hat{A}+\hat{B}} = e^{\hat{B}+\hat{A}} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$

(interchanging $\hat{A}$ and $\hat{B}$ in the formula), so that

$$e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$$

$$\Rightarrow \boxed{e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A},\hat{B}]}}$$ <u>if</u> $\hat{A}$ and $\hat{B}$ both commute with $[\hat{A},\hat{B}]$

Okay, now back to the displacement operator ---

$\hat{U}(\xi_0) = \exp\left(-\frac{\xi_0}{\sqrt{2}}(\hat{a}-\hat{a}^+)\right)$ consider displacing by $\xi_0 = \sqrt{2}\,\alpha \Rightarrow$

$$\hat{D}(\alpha) = \hat{U}(\sqrt{2}\,\alpha) = e^{\alpha(\hat{a}^+ - \hat{a})} = e^{\alpha \hat{a}^+} e^{-\alpha \hat{a}} e^{-\frac{1}{2}\alpha^2[\hat{a}^+, -\hat{a}]}$$

$$= e^{-\frac{1}{2}\alpha^2} e^{\alpha \hat{a}^+} e^{-\alpha \hat{a}}$$

If we displace the ground state, use

$$\hat{a}|0\rangle = 0 \Rightarrow e^{-\alpha \hat{a}}|0\rangle = |0\rangle \Rightarrow$$

$$\hat{D}(\alpha)|0\rangle = e^{-\frac{1}{2}\alpha^2} e^{\alpha \hat{a}^+}|0\rangle \equiv |\alpha\rangle$$

Now it is easy to expand the shifted Gaussian in terms of energy eigenstates

$$|\alpha\rangle = e^{-\frac{1}{2}\alpha^2} e^{\alpha \hat{a}^+}|0\rangle = e^{-\frac{1}{2}\alpha^2} \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n (\hat{a}^+)^n |0\rangle$$

$$= e^{-\frac{1}{2}\alpha^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

If we measure the energy, probability of outcome E_n is

$$P(n) = e^{-\alpha^2} \frac{\alpha^{2n}}{n!}, \text{ a Poisson distribution}$$

with $\langle n \rangle = e^{-\alpha^2} \sum_{n=0}^{\infty} n \frac{\alpha^{2n}}{n!} = \alpha^2 e^{-\alpha^2} \sum_{n=1}^{\infty} \frac{(\alpha^2)^{n-1}}{(n-1)!} = \alpha^2$

$\Rightarrow \langle E \rangle = \hbar\omega(\alpha^2 + \frac{1}{2})$

shifting the ground state by

$\xi_0 = \sqrt{2}\,\alpha$ increases energy by $\hbar\omega(\frac{1}{2}\xi_0^2) = \hbar\omega\alpha^2$.

The coherent state $|\alpha\rangle$ is an eigenstate of $\hat{a}$:

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \Rightarrow \hat{a}|\alpha\rangle = e^{-\alpha^2/2} \sum_{n=1}^{\infty} \frac{\alpha\,\alpha^{n-1}}{\sqrt{(n-1)!}} = \alpha|\alpha\rangle$

It is normalized because we obtained it by applying unitary operator $\hat{D}(\alpha) = \exp(\alpha(\hat{a}^+ - \hat{a}))$ to ground state $|0\rangle$

we could instead consider complex number α, and displace with unitary

$\hat{D}(\alpha) = \exp(\alpha\hat{a}^+ - \alpha^* a)$ (unitary because $\alpha\hat{a}^+ - \alpha^* a$ is anti-Hermitian)

$= e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+} e^{-\alpha^* a}$

$\Rightarrow |\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^+}|0\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

where now α is complex. For α imaginary,

$\alpha = i|\alpha| \Rightarrow \hat{D}(\alpha) = \exp(i|\alpha|(\hat{a}^+ + \hat{a}))$

$= \exp(i\sqrt{2}\,|\alpha|\,\hat{\xi})$, we are displacing the momentum

$= \exp(-\sqrt{2}\,|\alpha|\,d/dp_\xi)$ by $\sqrt{2}\,|\alpha|$

(since $\hat{\xi} = i\,d/dp_\xi$ satisfies $[\hat{\xi}, \hat{p}_\xi] = i$).

In general, for complex α, coherent state α is obtained from $|0\rangle$ by displacing a linear combination of position and momentum.

How does a coherent state evolve?

$$e^{-i\hat{H}t/\hbar}|n\rangle = e^{-i\omega t/2} e^{-in\omega t}|n\rangle.$$

And then $e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-i\omega t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha e^{-in\omega t}}{\sqrt{n!}}|n\rangle$

The phase of α rotates uniformly as a function of time. $\quad = e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle$

Using

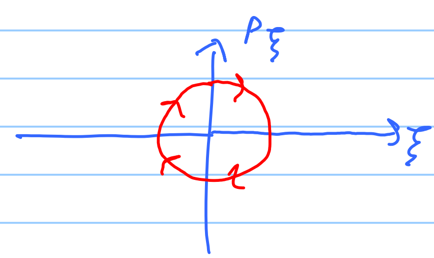
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \langle\alpha|\hat{a}^+ = \langle\alpha|\alpha^*,$$

we see that

$$\langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}(\alpha + \alpha^*) = \sqrt{2}\,\mathrm{Re}\,\alpha = \sqrt{2}|\alpha|\cos\omega t$$

$$\langle\alpha|\hat{p}_\xi|\alpha\rangle = \frac{-i}{\sqrt{2}}(\alpha - \alpha^*) = \sqrt{2}\,\mathrm{Im}\,\alpha = -\sqrt{2}|\alpha|\sin\omega t$$

If $\alpha = |\alpha|$ at $t = 0$



The time dependence of expectation value matches classical trajectory of the oscillator. Furthermore, the shape of the wave packet does not change — it remains a minimum uncertainty Gaussian.

Compute uncertainty:

$$\hat{\xi} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^+) \Rightarrow \hat{\xi}^2 = \frac{1}{2}(\hat{a} + \hat{a}^+)^2 = \frac{1}{2}(\hat{a}^2 + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^{+2})$$

$$= \frac{1}{2}(\hat{a}^2 + 2\hat{a}^+\hat{a} + [\hat{a},\hat{a}^+] + (\hat{a}^+)^2)$$

$$\Rightarrow \langle\alpha|\hat{\xi}^2|\alpha\rangle = \frac{1}{2}(\alpha^2 + 2\alpha^*\alpha + \alpha^{*2} + 1)$$

$$= \left[\frac{1}{\sqrt{2}}(\alpha + \alpha^*)\right]^2 + \frac{1}{2} = \langle\alpha|\hat{\xi}|\alpha\rangle^2 + \frac{1}{2}$$

$$\Rightarrow \quad \Delta\xi^2 = \langle \hat{\xi}^2 \rangle - \langle \hat{\xi} \rangle^2 = \tfrac{1}{2}$$

Similarly, $\hat{p}_\xi^2 = \frac{(-i)^2}{2}(\hat{a}-\hat{a})^2 = -\frac{1}{2}(\hat{a}^2 - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}^{+2})$

$$= -\tfrac{1}{2}\left(\hat{a}^2 - 2\hat{a}^+\hat{a} + \hat{a}^{+2} - [\hat{a},\hat{a}]\right)$$

$$\Rightarrow \langle\alpha|\hat{p}_\xi^2|\alpha\rangle = -\tfrac{1}{2}(\alpha^2 - 2\alpha^*\alpha + \alpha^{*2} - 1)$$

$$= \left[\frac{-i}{\sqrt{2}}(\alpha - \alpha^*)\right]^2 + \tfrac{1}{2} = \langle\alpha|\hat{p}_\xi|\alpha\rangle^2 + \tfrac{1}{2}$$

$$\Rightarrow \quad \Delta p_\xi^2 = \langle \hat{p}_\xi^2 \rangle - \langle \hat{p}_\xi \rangle^2 = \tfrac{1}{2}$$

and $\quad \Delta\xi\,\Delta p_\xi = \frac{1}{2}$ at all times. The minimum uncertainty is maintained, as wavepacket executes classical motion.

Debye-Waller Factor

Suppose a harmonic oscillator is in its ground state, until I suddenly hit it with a hammer, an impulse that transfers momentum δp_ξ to the oscillator.

A classical oscillator would surely get excited --- but what happens to a quantum oscillator? It <u>might</u> stay in its ground state.

As we've already discussed, displacing momentum by δp_ξ transforms ground state $|0\rangle$ to

$$\hat{D}\left(i\frac{\delta p_\xi}{\sqrt{2}}\right)|0\rangle = \left|\alpha = i\frac{\delta p_\xi}{\sqrt{2}}\right\rangle.$$

If we measure in the energy eigenstate basis, after the impulse, the probability of being in the ground state is:

$$P(0) = |\langle 0|\alpha\rangle|^2 = e^{-|\alpha|^2} = e^{-(\delta p_\xi)^2/2} = \exp\left[-\frac{1}{2}\frac{(\delta p)^2}{m\hbar\omega}\right]$$

which is called the "Debye-Waller factor."

The energy transfered to the oscillator is

$\delta E = \frac{(\delta p)^2}{2m}$ and the expectation value of the energy increases by this amount:

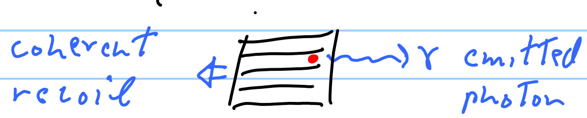
$$\langle \alpha | \hat{H} | \alpha \rangle - \langle 0 | \hat{H} | 0 \rangle = \hbar\omega \langle \alpha | a^{+} a | \alpha \rangle = \hbar\omega |\alpha|^2.$$

The argument of the exponential in the deBye-Waller is the energy transfer in units of the oscillator level spacing:

$$P(0) = \exp\left(-\delta E / \hbar\omega\right).$$

Consider, for example, an atomic nucleus in a crystal. The nucleus is in the quantum ground state of a harmonic potential well, arising from interactions with other atoms. But now the nucleus suddenly emits (or absorbs) photon with momentum $p_\gamma = \hbar\omega_\gamma$, transferring momentum p_γ and energy $\delta E = p_\gamma^2 / 2m$ to the nucleus.

With probability $P(0) = \exp(-\delta E / \hbar\omega)$, there is no nuclear recoil — the nucleus stays in its vibrational ground state. But what about conservation of momentum?



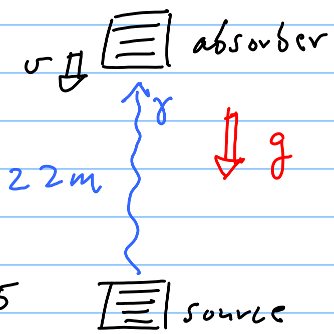
If the oscillator remains unexcited, then the entire crystal carries the compensating momentum.

We say that the recoil is "coherent." This is called the "Mössbauer effect."

Because the crystal has a very large mass M, the recoil energy $p_\gamma^2/2M$ is negligible, and as a result the energy of the emitted photon is very sharply defined.

Correspondingly, if the crystal is moving at a small but nonzero velocity, the nucleus is unable to absorb the photon.

In a famous experiment, Pound and RebKa used the Mössbauer effect to measure the gravitational red shift.



A photon climbing $L = 22$ m has frequency shifted by

$$\hbar \Delta\omega = \frac{\hbar\omega}{c^2} gL \Rightarrow \frac{\Delta\omega}{\omega} = \frac{gL}{c^2} \simeq 2.5 \times 10^{-15}$$

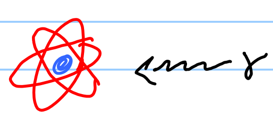
They detected this red shift by noting they could compensate for it with a Doppler shift of the frequency as seen by a moving crystal, with velocity v:

$$v/c \sim 2.5 \times 10^{-15} \Rightarrow v \sim 7 \times 10^{-5} \text{ cm/sec}.$$

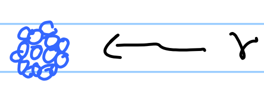
For the experiment to work, need spread in frequency of emitted and absorbed photon to have $\Delta\omega/\omega \sim 10^{-15}$.

(Actually -- they used ^{57}Fe, for which $\Delta\omega/\omega \sim 10^{-11}$, but they we still able to observe an enhancement of the absorption rate for $v \neq 0$.)

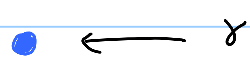
In general, if we wish to probe the internal structure of a physical system, we need to slam it with energy $E \gtrsim \hbar\omega$. For $E \ll \hbar\omega$, it will recoil coherently, without any internal change in state.



Probing an atom: For $E \ll 1$eV, atom recoils coherently, remaining in ground state.



Probing a nucleus: For $E \ll 1$ MeV, coherent recoil.



Probing the proton: For $E \ll 1$GeV, coherent recoil.