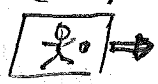
Distribute HW

Survey:	Y	N
Like Optics?	10	8
Learn something?	13	5

General Relativity

- Has its roots in Galilean relativity
  Newton's laws of motion take the same form for observers at rest and in uniform motion

- You can't tell if you are moving, if no acceleration



- But does principle apply to all laws of physics?
  What about Maxwell?

  $$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

  Presumed that ether has a preferred frame

- Einstein 1905 — universality
  There are inertial frames, in uniform motion relative to one another

All nongravitational laws of physics have a universal form independent of choice of inertial frame

c = I in all inertial frames
(You can't use light to tell if you are moving or not!)

A principle of universality — but came to be called "relativity" — because space and time are ~~perceived~~ differently by different observers

Lorentz $\begin{pmatrix} t' \\ x' \end{pmatrix} = L(v) \begin{pmatrix} t \\ x \end{pmatrix}$

But — what about gravity?

Newtons theory $\vec{F} = \frac{G m_1 m_2}{r^2}\hat{r}$

does not obey the principle — or

$\vec{F} = m\vec{\nabla}\phi \qquad \phi = \frac{M}{r}$ or $\nabla^2 \phi = 4\pi\rho$

Newtonian potential — Mass density

— ~~transforms~~ ϕ is a scalar
— But ρ is a component of a Tensor T^{00}

recall $T^{\mu\nu}$

$\partial_\mu T^{\mu\nu} = 0$ — Conservation of energy and momentum in local form

Turned out — to require a much more radical approach than most suspected

Einstein 1907

makes first serious attempt to formulate relativistic theory of gravitation

1907: "the happiest thought of my life":
someone falling freely does not feel his own weight

The powerful hint is
Principle of Equivalence of Gravitational and Inertial

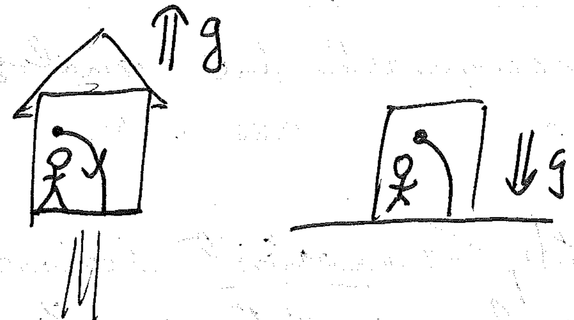
$\boxed{M_{grav} = M_{inertial}}$

Bodies of different mass fall the same way!!

verified experimentally to about 1 part in 10^{11} accuracy?!

$\Rightarrow$ A more general principle of relativity!

All freely falling observers perceive the same physical laws (in sufficiently small region of space and time)

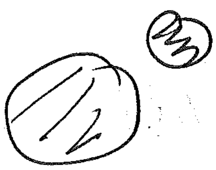


You can't tell if you are in a rocket accelerating with $\vec{a} = \vec{g}$

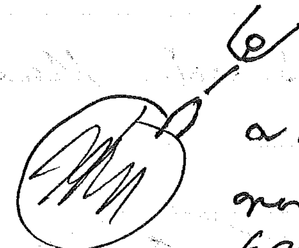
or in a grav field $\vec{g}$

So — what is gravity? An artifact of our moving the wrong way? No — there are observer independent physical effects, due to local variations in $\vec{g}$ — or tidal forces

→ e.g. the moon causes ocean tides



Powerful enough to tear apart a (freely falling) moon of Jupiter



or tear you apart as you fall into a black hole

Principle of equivalence

$\Longleftrightarrow$ Principle of "general covariance" (see below)

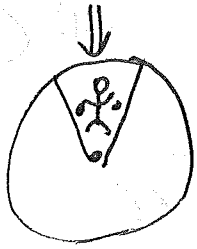
In locally inertial frame, laws of physics are same as in absence of gravitation

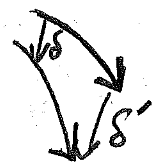
(Becomes comma → semicolon rule of tensor calculus)

Don't feel gravity if you fall freely ---

What is the principle of equivalence --
Not an invariance principle (cf Lorentz invariance)
Rather — a gauge principle — tells us how gravity couples

2 balls initially at rest, accelerate toward one another





A relative acceleration, proportional to relative separation

$$a = \ddot{\delta} \propto \delta$$

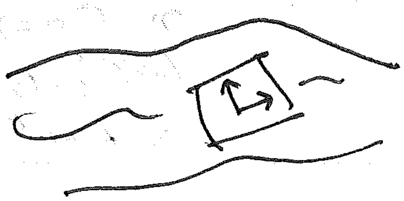
Gradually, AE recognized the analogy with Riemannian geometry.

— Relativity: Identify the <u>invariant</u> features of physics, that don't depend on state of motion of observer, or more generally, on how observer chooses coordinates

— Geometry: What are the invariant ("intrinsic") properties of a space, not depending on how space is parametrized

"Smooth Manifold"

In a local patch, looks like "flat" Euclidean space



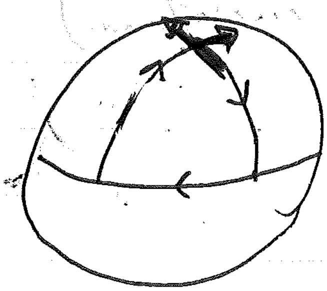
(And in Riemannian geometry, there are "measuring rods" that provide a local measure of distance between neighboring points

$$ds^2 = \delta_{ij}\, d\xi^i d\xi^j = d\vec{\xi} \cdot d\vec{\xi}\,)$$

But <u>curvature</u> is the intrinsic concept

At each point, a "tangent space"
We can carry "parallel transport" a vector around a closed path: Now points in a different direction!

Use the local metric to transport!



Carry a swinging pendulum on the earth's surface

winds up pointing a different way

- This is intrinsic curvature that could be measured by someone who never leaves the surface
- As opposed to extrinsic curvature

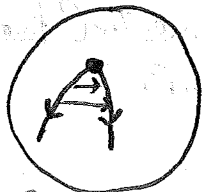
of cylinder — a property of how it is embedded

(How parallel transport on cylinder differs from parallel transport on 3 dim space)

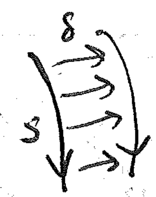
i.e. normal vs parallel transported normal

Geodesic deviation:

Example: great circle on a sphere



A geodesic is a path that is locally straight.

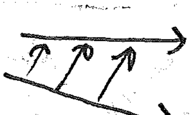


Use our measurement rod to mark off increments of constant length (parametrize by proper length)

Remark: Geodesic has extremal length.

Consider 2 nearby geodesics and $\vec{\delta}(\lambda)$ connecting them.

Flat space



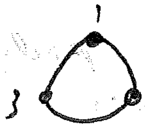
$\delta(\lambda) = a\lambda + b$ or $\dddot{\delta} = 0$

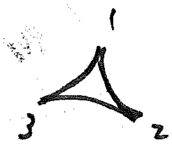
Curved space $\ddot{\delta} \propto \delta$

or $\ddot{\delta} = (\ )\,\delta$

where curvature determines how the geodesics attract (positive curvature)

or repel (negative curvature)

E.g.  Positive curvature $\Rightarrow \theta_1 + \theta_2 + \theta_3 > \pi$

 Negative curvature $\theta_1 + \theta_2 + \theta_3 < \pi$

<u>Summary of Riemannian Geometry:</u>

- Space is approximately flat locally looks Euclidean
- Can parallel transport a vector, but it can rotate when we transport around a closed path

 or -- transporting along different paths gives different results

"Path-dependence of parallel transport"

- Geodesic deviation $\ddot{\delta} = -(\text{curvature})\cdot\delta$

Gravitational Analogy:

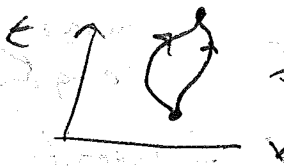
- Space is locally <u>Minkowski</u>

  Locally Euclidean chart $\leftrightarrow$ local inertial frame (free fall)

This only characterizes spatial curvature (and wouldn't hold in 1+1 dimensions!)

- Gyroscope $\Rightarrow$ parallel transport in spacetime

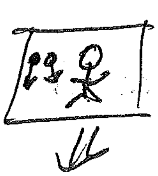
  Path dependent

 $\Rightarrow$ Measure curvature

- Free fall = <u>geodesic</u>

  Geodesic deviation (in spacetime)

  $\ddot{\delta} = -(\text{curvature})\,\delta \rightsquigarrow$ Tidal forces (compress you or pull you apart)



Tensors

In both RG and GR — the crux is to separate intrinsic (coordinate-independent) properties from special properties that hold only in particular coordinate systems —

In math, coordinate-independent properties are expressed in terms of Tensors

In principle — laws of physics should be expressed in coordinate-independent way

In practice — we need to use coordinates to solve the equations!

We can express a Tensor in particular coordinates. Then, what makes it a Tensor is how it transforms when we change the coordinate system

"A Tensor is something that transforms like a Tensor" (!)

With tensor technology, the principle of equivalence becomes:
"The principle of general covariance"

PoE: "the laws of physics look the same to all freely falling observers"

PoGC:

An equation ① Holds in "flat space" ② Is a tensor equation $\Longrightarrow$ It holds in curved space (including gravity)

E.g. $\underset{\sim}{T}$ = Tensor

If $\underset{\sim}{T}\Big|_{\text{local flat coords}} = 0$ then $\underset{\sim}{T} = 0$

(If a tensor vanishes in one coord system, it vanishes in any coord system.)

And that is why we care about tensors in physics -- they enable us to promote laws that form in particular coord systems to laws that hold in general (any coordinate system)

But what is a tensor (i.e. how does it transform)

x^μ = old set of coordinates

$x'^\mu(x)$ = new set

Δx consider infinitesimal displacement

$$\Delta x^\mu \to \Delta x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\Delta x^\nu$$ "Contravariant" index

Consider gradient

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$$

"Covariant" index

A general tensor has up (contravariant) and down (covariant) indices

$$T'^{\mu_1\mu_2\cdots}_{(x')\,\alpha_1\alpha_2\cdots} = \frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}}\frac{\partial x'^{\mu_2}}{\partial x^{\nu_2}}\frac{\partial x^{\beta_1}}{\partial x'^{\alpha_1}}\frac{\partial x^{\beta_2}}{\partial x'^{\alpha_2}}\cdots T^{\nu_1\nu_2\cdots}_{(x)\,\beta_1\beta_2\cdots}$$

Evidently

$T(x)=0 \Rightarrow T'(x')=0$ Key property of a tensor

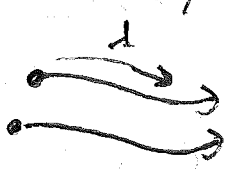
But what is the coordinate-independent meaning of a tensor

Two basic types of tensors

@ A vector (field) $\vec{V}$

Assigns a tangent vector to each point on the manifold

One way to think about a vector field



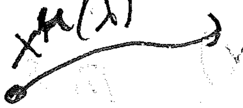
Consider parametrized flow in the space (like the flow parametrized by t in Hamiltonian mechanics)

then $\dfrac{d}{d\lambda}$ is a vector field

It assigns to each point a tangent vector
- direction = direction of flow
- length = rate of flow

This has a meaning independent of choice of coordinates (however, length of vector depends on how the flow is parametrized)

What if we do choose particular coords?

$x^\mu(\lambda)$  Describes flow in vicinity of a point

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} = \frac{dx^\mu}{d\lambda} e_\mu$$

This is expansion of the vector field in terms of a particular basis, the basis e_μ in the tangent space defined by the coordinates x^μ

If we used different coordinates, then the same vector field could be expressed as

$$\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} = \frac{d}{d\lambda} = \frac{\partial x'^\nu}{\partial \lambda}\frac{\partial}{\partial x'^\nu}$$

$$\vec{V} = V^\mu e_\mu \qquad = \qquad V'^\nu e'_\nu$$

components, basis

$$V'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} V^\mu$$ — components of vector transform contravariantly

$\vec{V}$ is coord independent

Its components in a basis "transform like a vector"

The other basic type of tensor is a <u>one-form</u>

It is a linear map that takes a vector to a real number

$\langle \omega, \vec{V} \rangle$ = number

i.e. a linear map of tangent space to $\mathbb{R}$ at each point of the manifold

<u>Example</u>. ϕ = scalar field (a number at each point)

$d\phi$ is a 1-form

$$\langle d\phi, \frac{d}{d\lambda} \rangle = \frac{d\phi}{d\lambda}$$

— Rate at which ϕ changes along the flow defined by vector field $\frac{d}{d\lambda}$

Note that:

vectors and 1-forms are dual one to the other

1-form: vector $\rightarrow$ number

vector: 1-form $\rightarrow$ number

We can expand 1-form in a basis determined by choice of coordinates

$$d\phi = \frac{\partial \phi}{\partial x^\mu} \, dx^\mu$$

basis 1-form

$$\boxed{\left\langle dx^\mu , \frac{\partial}{\partial x^\nu} \right\rangle = \delta^\mu{}_\nu}$$

Hence $\left\langle d\phi , \frac{\partial}{\partial x^\nu} \right\rangle = \left\langle \frac{\partial \phi}{\partial x^\mu} dx^\mu , \frac{\partial}{\partial x^\nu} \right\rangle = \frac{\partial \phi}{\partial x^\nu}$

change of ϕ along flow determined by increasing x^ν coord

coordinate-free view of a tensor:

$$T(\underbrace{\cdot , \cdot , \cdot \ldots}_{\text{vector slots}} ; \underbrace{\cdot , \cdot , \cdot , \ldots}_{\text{one-form slots}}) = \text{tensor}$$

(Multi-)
Linear map: vectors + one forms to numbers

E.g. $T(\cdot) =$ 1-form

vector

$T(\cdot) =$ vector

one-form

In components

$$\underset{\sim}{T} = T^{\overbrace{\mu\nu}^{\text{contravariant (vector) components}}}{}_{\underbrace{\alpha\beta}_{\text{covariant (1-form) components}}} \, e_\mu e_\nu \, dx^\alpha dx^\beta$$

$\underset{\sim}{T}$ devours the vectors + 1-forms in its slots by contraction of indices

$$\underset{\sim}{T}(\vec{V}, \omega)$$

$$= T(V^\alpha e_\alpha, \omega_\mu dx^\mu)$$

$$= T^\mu{}_\alpha V^\alpha \omega_\mu = \text{number}$$

the same number in any coord system (as scalar)

A tensor can also devour a tensor (of the same or lower rank).

e.g. $T^{\mu\nu}{}_{\alpha\beta} S_\mu{}^\alpha$ has 2 unfilled slots — Also a tensor

An $\binom{n}{m}$ tensor has n up indices (slots for 1-forms)
m down indices (slots for vectors)

Metric Tensor

Is the fundamental tensor of Riemannian geometry — it is a symmetric $\binom{0}{2}$ tensor

$$(\vec{u} \cdot \vec{v}) = g(\vec{u}, \vec{v}) = \text{number}$$

Interpretation $\angle$ the inner product of two tangent vectors

In components $\underset{\sim}{g} = g_{\mu\nu} dx^\mu dx^\nu$

Note: It's a symmetric tensor $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ or $g_{\mu\nu} = g_{\nu\mu}$

$$g(\vec{u}, \vec{v}) = g_{\mu\nu} U^\mu V^\nu$$

There are locally Euclidean (or Minkowski) coordinates such that

$$\underset{\sim}{g} = \eta_{\alpha\beta}\, dy^\alpha dy^\beta$$

$y(x)$ = locally inertial coords

Hence $\underset{\sim}{g} = \underbrace{\eta_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}} \; \overline{dx^\mu dx^\nu}$ Not a 2-form Denotes map of 2 vectors to numbers

$g_{\mu\nu}$ — in terms of locally flat coordinates

The metric tensor (or any $\binom{0}{2}$ tensor) defines an isomorphism relating 1-forms and vectors

vector slot

$$\vec{V} \Longleftrightarrow g(\vec{V}, \bullet) = \omega_{\vec{V}}$$

i.e. $\omega_{\vec{V}}(\vec{u}) = g(\vec{V}, \vec{u})$

In component language, we can "raise" and "lower" indices with g

$\omega_\alpha = g_{\alpha\beta} V^\beta$ — lowers an index, vector $\rightarrow$ 1-form

We can also define a $\binom{2}{0}$ tensor such that

$$V^{\beta} = g^{\beta\alpha}\omega_{\alpha}$$
$$= g^{\beta\alpha} g_{\alpha\gamma} V^{\gamma}$$

$$\Rightarrow \quad g^{\beta\alpha} g_{\alpha\gamma} = \delta^{\beta}_{\gamma}$$

assumes nonsingular metric [$g^{\beta\alpha}$ is matrix inverse of $g_{\alpha\gamma}$

then $f = g^{\alpha\beta} e_{\alpha} e_{\beta} = f(\cdot,\cdot)$ is a $\binom{2}{0}$ tensor

such that

$$\vec{V} = f(\tilde{w}_{\vec{V}}, \cdot)$$

inverts the map vector → 1-form

So – we can raise and lower indices of a tensor with $g_{\alpha\beta}$

By convention, if e.g. $T^{\alpha\beta}$ is a $\binom{2}{0}$ tensor then $T^{\alpha}{}_{\beta}$ denotes the $\binom{1}{1}$ tensor

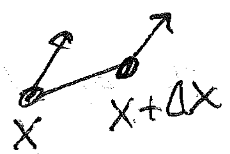
$$T^{\alpha}{}_{\beta} = g_{\beta\gamma} T^{\alpha\gamma}$$

Covariant Derivative

We have already discussed differentiation of a scalar

$$\langle d\phi, \frac{d}{d\lambda} \rangle = \frac{d\phi}{d\lambda}$$

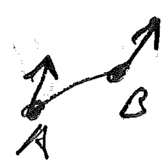
How do we differentiate vectors?



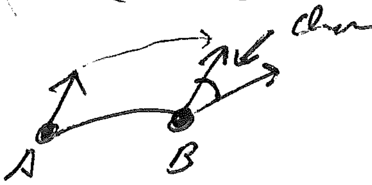
This is tricky — because vectors at neighboring points lie in *different* tangent spaces

To compare, we need a notion of *parallel* *transport* (a "connection") that relates these two vector spaces

- A connection is a coordinate independent concept — it tells us that a vector at A becomes a certain vector when carried to B



- *Given* a connection, there is a coord-independent notion of a derivative of a vector field



Compare vector at B to result of transporting vector at A to B — this is how the vector *changes* along the flow from A to B.

The covariant derivative of a vector field is a $\binom{1}{1}$ tensor — it has a slot for a vector — tells us how the vector field changes along flow determined by vector in the slot

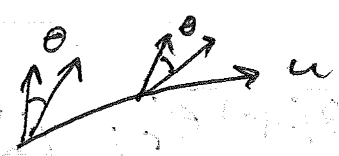
$$\nabla_{\vec{u}} \vec{V}$$

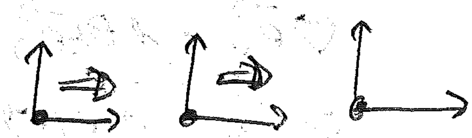
Is a vector field that tells us how $\vec{V}$ changes (relative to parallel transport) along flow $\vec{u}$

— In Riemannian geometry, the notion of parallel transport is provided by the locally flat structure (by the metric tensor)

If $\vec{V}, \vec{W}$ are parallel transported along $\vec{u}$

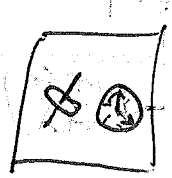
then $g(\vec{V}, \vec{W})$ = constant along $\vec{u}$

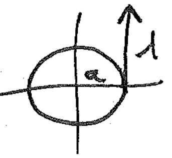




[In Euclidean space, we know what it means to transport a frame w/o rotating it

Or — a free falling observer with gyroscope and clock can operationally perform parallel transport!





$r(\lambda)=\sqrt{a^2+\lambda^2}$ $\quad \frac{dr}{d\lambda}=\frac{\lambda}{\sqrt{a^2+\lambda^2}}$ $\quad \frac{d^2r}{d\lambda^2}=\frac{1}{\sqrt{a^2+\lambda^2}}-\frac{\lambda^2}{(a^2+\lambda^2)^{3/2}}=\frac{a^2}{(a^2+\lambda^2)^{3/2}}$

$\tan\theta(\lambda)=\lambda/a$

$\theta(\lambda)=\arctan(\lambda/a) \Rightarrow \frac{d\theta}{d\lambda}=\frac{(1/a)}{1+\lambda^2/a^2}$

We will need to be able to express a covariant derivative in a particular coordinate system ("curvilinear coords")

$X^\alpha \Rightarrow$ Basis vectors $\quad e_\alpha=\frac{\partial}{\partial X^\alpha}$

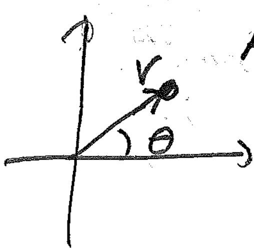
In Euclidean (Minkowski) coordinates

$$\nabla\left(\frac{\partial}{\partial y^\alpha}\right)=0$$

But not so for general coords

Example:

polar coordinates in the plane



$(ds^2=dr^2+r^2d\theta^2)$

$\nabla e_x=\nabla e_y=0$

$$\nabla\vec{e}_\alpha=dx^\beta\,\Gamma^\gamma_{\alpha\beta}\,\vec{e}_\gamma$$

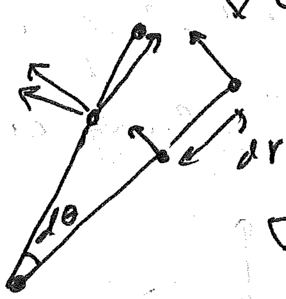
$\vec{e}_r\cdot\vec{e}_r=1$

$\vec{e}_\theta\cdot\vec{e}_\theta=r^2$

i.e.

$$\nabla\vec{e}_r=dr\,\Gamma^r_{rr}\vec{e}_r+dr\,\Gamma^\theta_{rr}\vec{e}_\theta+d\theta\,\Gamma^r_{r\theta}\vec{e}_r+d\theta\,\Gamma^\theta_{r\theta}\vec{e}_\theta$$

$$\nabla\vec{e}_\theta=dr\,\Gamma^r_{\theta r}\vec{e}_r+dr\,\Gamma^\theta_{\theta r}\vec{e}_\theta+d\theta\,\Gamma^r_{\theta\theta}\vec{e}_r+d\theta\,\Gamma^\theta_{\theta\theta}\vec{e}_\theta$$



$\nabla\vec{e}_r=d\theta\frac{1}{r}\vec{e}_\theta$ $\qquad \Gamma^\theta_{r\theta}=1/r$

$\nabla\vec{e}_\theta=dr\,(r^{-1})\,\vec{e}_\theta+d\theta(-r)\,\vec{e}_r;$ $\quad \Gamma^\theta_{\theta r}=r^{-1}$

$\Gamma^r_{\theta\theta}=-r$

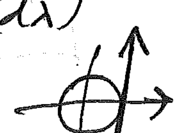
Christoffel $\Rightarrow$

$\Gamma^r_{\theta\theta}=-\frac{1}{2}g_{\theta\theta,r}=-r$

$\Gamma^\theta_{\theta r}=\frac{1}{2}g^{\theta\theta}[g_{\theta\theta,r}]=\frac{1}{r}$

$\frac{d^2r}{d\lambda^2}=\Gamma^r_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2=-r\left(\frac{d\theta}{d\lambda}\right)^2$

$\frac{d^2\theta}{d\lambda^2}=2\Gamma^\theta_{r\theta}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda}$



using the locally inertial notion of parallel transport:

$$\nabla_\beta \vec{e}_\gamma = \Gamma^\alpha_{\beta\gamma}\vec{e}_\alpha \quad \text{or} \quad \boxed{\nabla\vec{e}_\gamma = dx^\beta \Gamma^\alpha_{\beta\gamma}\vec{e}_\alpha}$$

The Affine connection

or $\frac{d}{d\lambda}\vec{e}_\gamma = \frac{dx^\beta}{d\lambda}\Gamma^\alpha_{\beta\gamma}\vec{e}_\alpha$

$\boxed{\text{Warning}}$: $\Gamma^\alpha_{\beta\gamma}$ is <u>not</u> a tensor (does <u>not</u> transform like one)

In particular $\left[\begin{array}{l}\Gamma^\alpha_{\beta\gamma} = 0 \text{ in local flat coordinates} \\ \text{but } \neq 0 \text{ in general coordinates}\end{array}\right.$

It is telling us something about the coordinates, not about any geometrical property

In components: $\vec{V} = V^\alpha \vec{e}_\alpha$

Two contributions to $\nabla_{\vec{u}}\vec{V}$

- How <u>components</u> change along $\vec{u}$
- How <u>basis vectors</u> change along $\vec{u}$ (encoded in the connection)

$$\frac{d}{d\lambda}(V^\alpha\vec{e}_\alpha) = \left(\frac{d}{d\lambda}V^\alpha\right)\vec{e}_\alpha + V^\alpha\frac{d}{d\lambda}\vec{e}_\alpha$$

$$= \left(\frac{dx^\beta}{d\lambda}\frac{\partial}{\partial x^\beta}V^\alpha\right)\vec{e}_\alpha + V^\alpha\frac{dx^\beta}{d\lambda}\Gamma^\delta_{\beta\alpha}\vec{e}_\delta$$

$$= \frac{dx^\beta}{d\lambda}\left(\frac{\partial}{\partial x^\beta}V^\alpha + V^\gamma\Gamma^\alpha_{\beta\gamma}\right)\vec{e}_\alpha$$

$$= u^\beta(\nabla\vec{V})^\alpha{}_\beta\,\vec{e}_\alpha \quad \text{where } (\nabla\vec{V})^\alpha_\beta = \frac{\partial}{\partial x^\beta}V^\alpha + \Gamma^\alpha_{\beta\gamma}V^\gamma$$

i.e. $\nabla \vec{V} = dx^\beta \left(V^\alpha{}_{;\beta}\right) \vec{e}_\alpha$

In components we denote the covariant derivative

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha_{\beta\gamma} V^\gamma$$

ordinary derivative

connection (parallel transport)

Another view:

If the y^a's are locally flat coords that satisfy $\nabla \vec{e}_a = 0$

where $\vec{e}_a = \frac{\partial}{\partial y^a}$

Then $\nabla(V^a \vec{e}_a) = dy^b \left(\frac{\partial}{\partial y^b} V^a\right) \vec{e}_a$

So $\nabla \vec{e}_\alpha = \nabla\left(\frac{\partial}{\partial x^\alpha}\right) = \nabla\left(\frac{\partial y^a}{\partial x^\alpha} \vec{e}_a\right)$

$$= dy^b \left(\frac{\partial}{\partial y^b} \frac{\partial y^a}{\partial x^\alpha}\right) \vec{e}_a \quad \text{(we used } \nabla \vec{e}_a = 0\text{)}$$

$$= dx^\beta \left(\frac{\partial}{\partial x^\beta} \frac{\partial y^a}{\partial x^\alpha}\right)\left(\frac{\partial x^\gamma}{\partial y^a}\right) \frac{\partial}{\partial x^\gamma} = dx^\beta \Gamma^\gamma_{\beta\alpha} \vec{e}_\gamma$$

where $\boxed{\Gamma^\gamma_{\beta\alpha} = \frac{\partial x^\gamma}{\partial y^a} \frac{\partial^2 y^a}{\partial x^\beta \partial x^\alpha}}$

We see that
- $\Gamma^{\gamma}_{\beta\alpha}$ has $\alpha \leftrightarrow \beta$ symmetry
- $\Gamma = 0$ in coordinates that differ from y's by a linear transformation
- Also.. easy to see how $\Gamma^{\gamma}_{\beta\alpha}$ transforms

Covariant derivative of a 1-form

We will define it by demanding that ∇ acting on a tensor obey Leibniz product rule

$$\nabla\, T \otimes S = \nabla T \otimes S + T \otimes \nabla S$$

We have $\langle dx^{\alpha}, \vec{e}_{\beta} \rangle = \delta^{\alpha}{}_{\beta}$

basis one-form — basis vector

$$\nabla \langle dx^{\alpha}, \vec{e}_{\beta} \rangle = 0 = \langle \nabla dx^{\alpha}, \vec{e}_{\beta} \rangle + \langle dx^{\alpha}, \nabla \vec{e}_{\beta} \rangle$$

But $\nabla \vec{e}_{\beta} = dx^{\gamma}\, \Gamma^{\alpha}_{\gamma\beta}\, \vec{e}_{\alpha}$

$$\Rightarrow \langle \nabla dx^{\alpha}, \vec{e}_{\beta} \rangle = -dx^{\gamma}\, \Gamma^{\alpha}_{\gamma\beta}$$

or
$$\boxed{\nabla dx^{\alpha} = -dx^{\gamma}\, \Gamma^{\alpha}_{\gamma\beta}\, dx^{\beta}}$$

And so

$$\nabla(\omega_{\alpha} dx^{\alpha}) = dx^{\beta} \left(\omega_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\, \omega_{\gamma} \right) dx^{\alpha}$$

or
$$\boxed{\omega_{\alpha;\beta} = \omega_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta}\, \omega_{\gamma}}$$

WARNING: minus sign

Now that we know ∇dx^α and ∇e_β, we can covariantly differentiate any tensor

$$\nabla \left(T^\beta{}_\gamma \, e_\beta \, dx^\gamma\right)$$

$$= dx^\alpha \Big(T^\beta{}_{\gamma,\alpha} \, e_\beta dx^\gamma$$
$$+ T^\beta{}_\gamma \Gamma_{\alpha\beta}{}^\delta e_\delta dx^\gamma$$
$$- T^\beta{}_\gamma e_\beta \Gamma^\gamma{}_{\alpha\delta} dx^\delta \Big)$$

or $$T^\beta{}_{\gamma;\alpha} = T^\beta{}_{\gamma,\alpha} + \Gamma_\alpha{}^\beta{}_\delta T^\delta{}_\gamma - \Gamma^\delta{}_{\alpha\gamma} T^\beta{}_\delta$$

$+\Gamma$ for up index $-\Gamma$ for down index.

What is the covariant derivative of $\underset{\sim}{g}$?

Claim $\nabla g = 0$ .. why?

Because g is used to define parallel transport

If $\vec{v}$ and $\vec{w}$ are parallel transported along $\vec{u}$

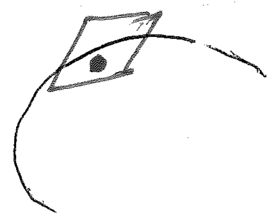
Then $\nabla_{\vec{u}} \vec{v} = \nabla_{\vec{u}} \vec{w} = 0$

And $g(\vec{u}, \vec{w}) =$ constant along $\vec{u}$

$$\Rightarrow \nabla g(\vec{u}, \vec{w}) = 0 = \nabla g(\vec{u}, \vec{w}) + \underbrace{g(\nabla \vec{u}, w)}_{0} + \underbrace{g(\vec{u}, \nabla \vec{w})}_{0}$$

For any $\vec{u}$

More on $\nabla g = 0$

- 

It is a statement about how accurately the Euclidean metric on a tangent plane matches the actual metric.

Because in the locally flat coordinates, we have ($\Gamma = 0$)

$$g_{ab;c} = 0$$

The metric on the tangent plane agrees with the actual metric to linear order in ΔX

(Since ∇g vanishes in the locally inertial coordinates, and is a tensor $\Rightarrow$ vanishes in any coords
"metric is covariantly constant"

- We can use the property $\nabla g = 0$ to express the connection in terms of the metric. (makes sense, since as we've seen, metric provides the local notion of parallel)

We have

$$0 = \nabla_\alpha g_{\beta\gamma} = g_{\beta\gamma,\alpha} - \Gamma^\delta_{\alpha\beta} g_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} g_{\beta\delta}$$

$$0 = \nabla_\beta g_{\gamma\alpha} = g_{\gamma\alpha,\beta} - \Gamma^\delta_{\beta\gamma} g_{\delta\alpha} - \Gamma^\delta_{\beta\alpha} g_{\gamma\delta}$$

$$0 = \nabla_\gamma g_{\alpha\beta} = g_{\alpha\beta,\gamma} - \Gamma^\delta_{\gamma\alpha} g_{\delta\beta} - \Gamma^\delta_{\gamma\beta} g_{\alpha\delta}$$

$$g_{\beta\gamma,\alpha} + g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} = 2\Gamma^\delta_{\alpha\beta} g_{\delta\gamma}$$

or

$$\boxed{\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left[g_{\alpha\delta,\beta} + g_{\delta\beta,\alpha} - g_{\alpha\beta,\delta}\right]}$$

Remark — an alternative derivation of $\nabla g = 0$ uses expressions for g and Γ in terms of locally flat coordinates

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial y^{a}} \frac{\partial^2 y^{a}}{\partial x^{\alpha}\partial x^{\beta}}$$

$$g_{\alpha\beta} = \eta_{ab} \frac{\partial y^{a}}{\partial x^{\alpha}} \frac{\partial y^{b}}{\partial x^{\beta}}$$

$$\Rightarrow g_{\alpha\beta,\gamma} = \eta_{ab}\left(\frac{\partial^2 y^{a}}{\partial x^{\alpha}\partial x^{\gamma}}\frac{\partial y^{b}}{\partial x^{\beta}} + \frac{\partial y^{a}}{\partial x^{\alpha}}\frac{\partial^2 y^{b}}{\partial x^{\beta}\partial x^{\gamma}}\right)$$

$$\frac{\partial^2 y^{a}}{\partial x^{\beta}\partial x^{\gamma}} = \Gamma^{\delta}_{\beta\gamma}\frac{\partial y^{a}}{\partial x^{\delta}} \Rightarrow$$

$$g_{\alpha\beta,\gamma} = \Gamma^{\delta}_{\alpha\gamma} g_{\delta\beta} + \Gamma^{\delta}_{\beta\gamma} g_{\alpha\delta} \Rightarrow \nabla g = 0$$

Remark: $\nabla g = 0$ means we can raise and lower indices either before or after ∇ acts -- or contract indices

Parallel Transport; Connection 1-form

$\vec{V}$ is parallel transported along $\vec{u}$ if $\nabla_{\vec{u}} \vec{V} = 0$,

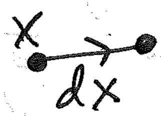
or in components

$$0 = u^{\alpha}\nabla_{\alpha}V^{\beta} = u^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}}V^{\beta} + \Gamma^{\beta}_{\alpha\gamma}V^{\gamma}\right)$$

writing $u^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$, we have $\frac{dV^{\beta}}{d\lambda} = -\Gamma^{\beta}_{\alpha\gamma}\frac{dx^{\alpha}}{d\lambda}V^{\gamma}$

could discuss here:

- general covariance as comma → semicolon
- Enough freedom in $\frac{\partial^2 y^{a}}{\partial x^{\alpha}\partial x^{\beta}}$ to set $g_{\alpha\beta,\gamma} = 0$



this means that as we move from x to $x+dx$, the components of the parallel transported vector $\vec{V}$ change according to

$$V^\beta(x+dx) = V^\beta(x) - \Gamma^\beta_{\alpha\gamma}(x)\,dx^\alpha\, V^\gamma(x)$$

$$= \left[\delta^\beta{}_\gamma - (\Gamma_\alpha dx^\alpha)^\beta{}_\gamma\right] V^\gamma(x)$$

In other words, how the components change is described by a matrix

$$\left[\mathbb{1} - \Gamma_\alpha(x)\,dx^\alpha\right]$$

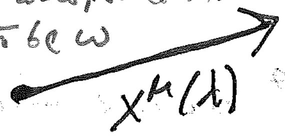
Here $\Gamma_\alpha dx^\alpha$ is the matrix-valued "connection one-form"

It takes tangent vectors to matrices (but remember it is not a tensor)

Geodesic what is it?

A geodesic is a curve that is locally straight — e.g. the trajectory of a freely falling ("inertial") body. One way to describe it: use the local Minkowski coordinates e.g. $d\lambda = \frac{dt}{\omega}$

dt = time as measured by observer who perceives freq of photon to be ω



Parametrize the geodesic by λ. Convenient to chose λ so geodesic is <u>linear</u> in λ in the locally flat coordinates

Spacelike $\lambda \propto$ proper distance

timelike $\lambda \propto$ proper time

null: ? [coordinate length in local Minkowski coordinate system]

λ is then an "affine parameter" of the geodesic

For an affine parametrization, a straight path in space time (in locally flat coords) obeys

$$0 = \frac{d^2}{d\lambda^2} y^a(\lambda)$$

and
$$\frac{dy^a}{d\lambda} = \frac{\partial y^a}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}$$

$$\Rightarrow 0 = \frac{d}{d\lambda}\left(\frac{\partial y^a}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}\right)$$

$$= \frac{\partial y^a}{\partial x^\alpha}\frac{d^2x^\alpha}{d\lambda^2} + \frac{dx^\beta}{d\lambda}\frac{\partial^2 y^a}{\partial x^\alpha \partial x^\beta}\frac{dx^\alpha}{d\lambda}$$

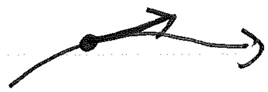
$$\Rightarrow \frac{d^2x^\alpha}{d\lambda^2} + \underbrace{\frac{\partial x^\alpha}{\partial y^a}\frac{\partial^2 y^a}{\partial x^\beta \partial x^\gamma}}_{\Gamma^\alpha_{\beta\gamma}}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

$$\Rightarrow \boxed{0 = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}}$$

Eqn of a geodesic in arbitrary coords

WARNING: Only if path is affinely parametrized!

An alternative (more geometrical) viewpoint



The path is locally straight if the tangent to the path at B agrees with tangent at A, parallel transported to B

Hence if $\vec{V} = \frac{d}{d\lambda}$ is the tangent to the curve

$$\nabla_{\vec{V}} \vec{V} = c(\lambda) \vec{V}$$

Depending on the parametrization, the length of tangent may vary — though its direction is covariantly constant

The affine parametrization is the choice of λ along the curve so that "length" of the tangent remains constant. For this parametrization

$$\boxed{\nabla_{\vec{V}} \vec{V} = 0}$$

WARNING: ditto

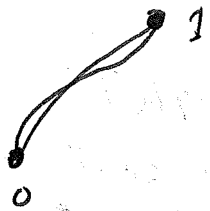
In components

$$\vec{V} = \frac{dx^\alpha}{d\lambda}\frac{\partial}{\partial x^\alpha}, \quad 0 = \left(\nabla_{\vec{V}} \vec{V}\right)^\beta = V^\alpha \nabla_\alpha V^\beta$$

$$= \frac{dx^\alpha}{d\lambda}\left(\frac{\partial}{\partial x^\alpha} V^\beta + \frac{dx^\alpha}{d\lambda}\Gamma^\beta_{\alpha\gamma} V^\gamma\right)$$

$$= \frac{d^2}{d\lambda^2} x^\beta + \Gamma^\beta_{\gamma\alpha}\frac{dx^\gamma}{d\lambda}\frac{dx^\alpha}{d\lambda}$$

Remark: A geodesic is a curve of extremal length (which we would expect for the "straightest" path)



Proper distance $L = \int ds = \int_0^1 d\lambda \left(g_{\alpha\beta}(x) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{\frac{1}{2}}$

Vary with endpoints fixed $\Rightarrow$
Euler Lagrange

$$\frac{d}{d\lambda} \frac{g_{\alpha\beta} \frac{dx^\beta}{d\lambda}}{\left(g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}\right)^{\frac{1}{2}}} = \frac{\partial}{\partial x^\alpha} (\ ) = \frac{\frac{1}{2} g_{\beta\gamma,\alpha} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}}{\left(g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}\right)^{\frac{1}{2}}}$$

Now choose affine parametrization $\int d\lambda = \int (g_{\alpha\beta} dx^\alpha dx^\beta)^{\frac{1}{2}}$

$$\Rightarrow \quad g_{\alpha\beta,\gamma} \frac{dx^\gamma}{d\lambda} \frac{dx^\beta}{d\lambda} + g_{\alpha\beta} \frac{d^2 x^\beta}{d\lambda^2} = \frac{1}{2} g_{\beta\gamma,\alpha} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}$$

$$\text{or} \quad g_{\alpha\beta} \frac{d^2 x^\beta}{d\lambda^2} + \left(g_{\alpha\beta,\gamma} - \frac{1}{2} g_{\beta\gamma,\alpha} \right) \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}$$

This is the geodesic eqn:

$$\left[\frac{d^2 x_\alpha}{d\lambda^2} + \underbrace{\frac{1}{2} \left(2 g_{\alpha\beta,\gamma} - g_{\beta\gamma,\alpha} \right)}_{g_{\alpha\delta} \Gamma^\delta_{\beta\gamma}} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} \right.$$

Be careful:

$$g_{\alpha\beta} \frac{d^2 x^\beta}{d\lambda^2} \overset{?}{=} \frac{d^2 x_\alpha}{d\lambda^2}$$

There are 3 different types of geodesic

- Timelike
- Spacelike
- null

Parallel Transport $\Rightarrow$

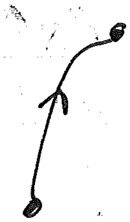
$$g\left(\frac{d}{d\lambda}, \frac{d}{d\lambda}\right) = \text{constant} = \begin{matrix} - \\ + \\ 0 \end{matrix}$$

(Affine param $\Rightarrow$ tangent is covariantly constant)

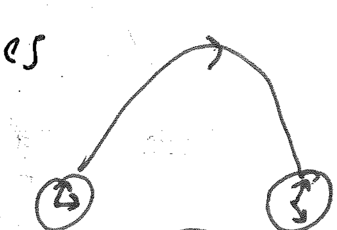
Since it is constant, timelike remains timelike, etc. (Freely falling observers always remain on timelike trajectory, and photon, once null, is always null.)

Timelike case: $\int d\tau$ is a local max

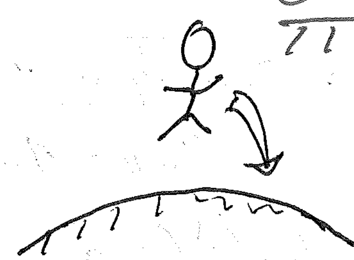
"Principle of cosmic laziness" – we want the trip to take as long as possible as measured on local clock where initial and final points are fixed in spacetime



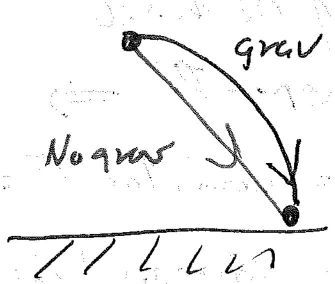
In special relativity, we should stay at rest – Moving causes time dilation



In GR: clocks run faster higher up, slower lower down (gravitational red shift)



Maximize elapsed time on the clock by shooting it up and letting it fall (free fall all the time)



Projectile falls in a parabola --

It is better to stay up high longer -- where my clock would run faster!

Newtonian Limit of Geodesic Eqn

Let's look more closely at how to reconcile the geodesic eqn with motion in a Newtonian gravitational field

Newton applies: - when fields are weak
- when motion is slow

$$0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^{\alpha}_{\beta\gamma} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

τ = proper time is affine parameter

$$\frac{dt}{d\tau} \sim 1 >> \frac{dx^i}{d\tau} \sim v$$

$$\Rightarrow \quad \frac{d^2 x^i}{dt^2} \simeq -\Gamma^i_{00}$$

We use "nearly Minkowski coordinates"

with $\quad g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad h_{\alpha\beta} << 1$ weak field

$$\Gamma^i_{00} = \frac{1}{2} g^{i\lambda}(2 g_{0\lambda,0} - g_{00,\lambda}) = -\frac{1}{2} h_{00,}{}^{i}$$ (To linear order in h)

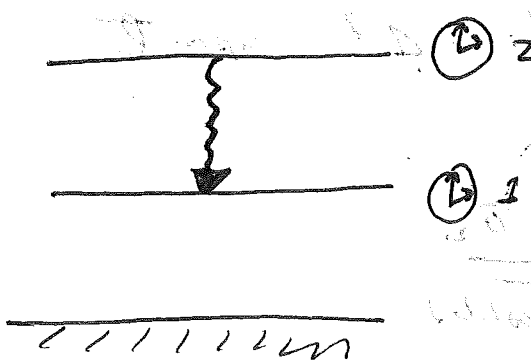
We have $\frac{d^2 x^i}{dt^2} = \frac{1}{2} h_{00,}{}^{i}$

cf Newton $\frac{d^2 \vec{x}}{dt^2} = -\vec{\nabla}\phi$ Hence $h_{00} = -2\phi$ + constant

Hence $g_{00} = -(1+2\phi)$

[0 if $\phi \to 0$ and $g \to \eta$ far from all sources

From this, we may infer the gravitational red shift (Einstein, 1907)



Consider 2 clocks in the static field of the earth

Proper time is related to coordinate time by

$$\Delta\tau^2 = -g_{00}\Delta t^2$$

But who cares about coordinate time?

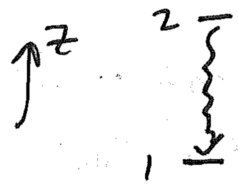
That coordinate time runs at different rates at two different heights has observable consequences.

Crudely speaking: ticks of clock in any position are spaced by a universal $\Delta\tau$

But $\Delta t = \frac{\Delta\tau}{\sqrt{-g_{00}}}$

$\simeq (1-\phi)\Delta\tau$

means that clock runs "faster" (smaller Δt between ticks) higher up where ϕ is larger that is $\quad \dfrac{\Delta t_1}{\Delta t_2} \simeq \dfrac{1-\phi_1}{1-\phi_2} \simeq 1+\phi_2-\phi_1$



E.g. send a sequence of photons, spaced Δt apart

Since photon is null

$$0 = g_{00}\,dt^2 + g_{33}\,dz^2 \Rightarrow dt = \sqrt{\frac{g_{33}}{-g_{00}}}\,dz$$

$$\text{or} \quad \int dt = \int \sqrt{\frac{g_{33}}{-g_{00}}}\,dz$$

is <u>coordinate</u> time for photon to travel from sender to receiver.

If metric is static and sender and receiver are at fixed coordinate positions, this is the same time of travel for <u>each</u> photon

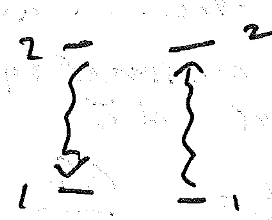
So photons are received Δt apart

Therefore

$$\frac{\Delta\tau_1}{\sqrt{-g_{00}(1)}} = \Delta t = \frac{\Delta\tau_2}{\sqrt{-g_{00}(2)}}$$

$$\text{or } \frac{\nu_2}{\nu_1} = \frac{\Delta\tau_1}{\Delta\tau_2} = \sqrt{\frac{g_{00}(1)}{g_{00}(2)}} = 1+\phi_1-\phi_2$$

$$\text{or } \boxed{\frac{\nu_2-\nu_1}{\nu_1} = \phi_1-\phi_2}$$



- Falling photon detected shifted blue
- Rising photon detected shifted red

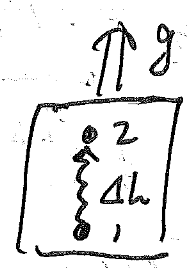
<u>Crudely</u>: $\Delta E = -E\Delta\phi$: photon loses "kinetic" energy as it gains potential energy

(where $E=h\nu$)

Or (Einstein 1907):

E. didn't use this argument until 1911

In accelerating frame:



- 1 and 2 are at rest when photon is emitted

- when received, 2 is pulling away at $v \sim g\Delta t \sim \frac{g\Delta h}{c}$

$\Rightarrow$ Red shift $\quad \frac{\nu_2}{\nu_1} = \gamma\left(1-\frac{v}{c}\right) \sim \left(1-\frac{v}{c}\right) \sim 1 - \frac{g\Delta h}{c^2}$

$$= 1 - \frac{\Delta\phi}{c^2}$$

or $\boxed{\frac{\Delta\nu}{\nu} = \frac{-\Delta\phi}{c^2}}$

Pound - Rebka

$\Delta h \sim 23m \qquad \frac{\Delta\nu}{\nu} = \frac{-\Delta\phi}{c^2} = \frac{g\Delta h}{c^2} \sim \frac{980 \times 2260}{(3\times 10^{10})^2} \simeq 2.5\times 10^{-15}$

$\Uparrow$

small

Use Mossbauer and compensate with Doppler shift

More on geodesics: Symmetries and conservation laws

If the metric has symmetries $\Rightarrow$ constant of the motion along a geodesic
("1st integral" of the geodesic eqn)

Four velocity $\frac{d}{d\tau}$ (where τ = "proper time" is the affine parameter of timelike geodesic)

i.e. $u^\alpha = \frac{dx^\alpha}{d\tau}$

and $0 = \nabla_{\vec{u}} \vec{u} = u^\alpha \nabla_\alpha u^\beta \Rightarrow \frac{du^\beta}{d\tau} + \Gamma^\beta_{\alpha\gamma} u^\alpha u^\gamma = 0$

$$\frac{du^\beta}{d\tau} = \underbrace{-\Gamma^\beta_{\alpha\gamma}}\, u^\alpha u^\gamma$$

$$\frac{1}{2} g^{\beta\delta}\left(g_{\alpha\delta,\gamma} + g_{\delta\gamma,\alpha} - g_{\alpha\gamma,\delta}\right)$$

$$\Rightarrow \left[\frac{du_\delta}{d\tau} = -\frac{1}{2}\underbrace{\left(g_{\alpha\delta,\gamma} + g_{\delta\gamma,\alpha}\right.}_{2g_{\delta\gamma,\alpha}\ \text{(because of symmetry)}}\left. - g_{\alpha\gamma,\delta}\right)u^\alpha u^\gamma\right.$$

Is it legit to lower this index?

$$= -\left(g_{\delta\gamma,\alpha} - \frac{1}{2} g_{\alpha\gamma,\delta}\right)u^\alpha u^\gamma$$

This is what we found from E-L eqn for geodesic

Okay -- but suppose we write this differently

$$\Rightarrow 0 = u^\alpha \nabla_\alpha u_\beta = u^\alpha\left(u_{\beta,\alpha} - \Gamma^\gamma_{\beta\alpha} u_\gamma\right)$$

$$\Rightarrow \frac{du_\beta}{d\tau} = \Gamma^\gamma_{\beta\alpha} u_\gamma u^\alpha$$

$$= \frac{1}{2} g^{\gamma\delta}\left(g_{\beta\delta,\alpha} + g_{\delta\alpha,\beta} - g_{\beta\alpha,\delta}\right)u_\gamma u^\alpha$$

$$= \frac{1}{2}\left(g_{\beta\delta,\alpha} + g_{\delta\alpha,\beta} - g_{\beta\alpha,\delta}\right)u^\delta u^\alpha$$

cancel

$$\boxed{\frac{du_\beta}{d\tau} = \frac{1}{2} g_{\delta\alpha,\beta}\, u^\delta u^\alpha}$$

Another useful general form of geodesic eqn

But -- how do we reconcile this with eqn above

Conservation Law:

If $g_{\alpha\beta}(x)$ is independent of x^γ,

then u_γ = constant along geodesic

WARNING: u_γ is constant but u^γ is <u>not</u>!

i.e. x^γ = "cyclic" $\Rightarrow$

$$u_\gamma = \vec{u}\cdot\vec{e}_\gamma = \vec{u}\cdot\frac{\partial}{\partial x^\gamma} = \text{const}$$

Another, more general "conservation law"

$$\vec{u}\cdot\vec{u} = -1 = \text{constant} \quad (\text{timelike})$$
$$= 0 = \text{constant} \quad (\text{null})$$

i.e. $\nabla_{\vec{u}}(\vec{u}\cdot\vec{u}) = 2\vec{u}\cdot(\nabla_{\vec{u}}\vec{u}) = 0$

(Same for 4-momentum $\vec{p}$) vanishes along geodesic

Example: Static (t-independent) metric
$\Rightarrow u_0$ (or p_0) = constant

constant = $-1 = g_{00}(u^0)^2 + g_{ij}u^i u^j$ (stronger assumption the metric is "stationary" — $g_{i0}=0 \Rightarrow$ static)

$$-1 = g^{00}(u_0)^2 + g_{ij}u^i u^j$$

E.g. $g_{33}\left(\frac{dz}{d\tau}\right)^2 = -1 + \underbrace{\left(\frac{1}{-g_{00}}\right)}u_0^2$

Here g_{00} can depend on z

should correspond to conservation of energy in Newtonian field (cf Example 24.2 in B+T)

what is the more general form of this connection (Killing vector field)