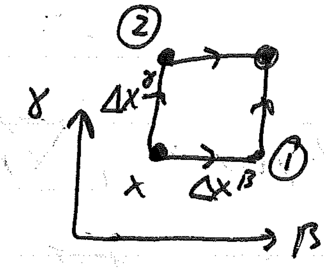
Curvature – the central concept in gravitational kinematics, because it provides a geometrical (coordinate independent) language for describing gravitational tidal forces

The simplest setting in which to discuss Riemannian curvature is: parallel transport around a small closed path

or, alternatively, consider two distinct paths from x to destination



$x \quad \Delta x^\beta$

Recall that $\Gamma_\alpha dx^\alpha$ is "connection one-form", matrix valued

Parallel transports from x to $x + \Delta x^\beta e_\beta$

$$U^\delta \Rightarrow \left[\mathbb{1} - \Gamma_\beta(x)\,\Delta x^\beta\right]^\delta{}_\eta\, U^\eta$$

(Matrix indices suppressed)

Comment: quadratic term here won't contribute

weird notation x is a coord, $\Delta x e$ is a vector

[transport to $x + \Delta x^\beta e_\beta + \Delta x^\gamma e_\gamma$ along path ①

$$①:\ \left[\mathbb{1} - \Gamma_\gamma(x + \Delta x^\beta e_\beta)\Delta x^\gamma\right]\left[\mathbb{1} - \Gamma_\beta(x)\Delta x^\beta\right]$$

$$= \mathbb{1} - \Gamma_\gamma \Delta x^\gamma - \Gamma_\beta \Delta x^\beta - \partial_\beta \Gamma_\gamma \Delta x^\beta \Delta x^\gamma + \Gamma_\gamma \Gamma_\beta \Delta x^\gamma \Delta x^\beta$$

(to quadratic order in Δx)

transport along path ②

$$②:\ \left[\mathbb{1} - \Gamma_\beta(x + \Delta x^\gamma e_\gamma)\Delta x^\beta\right]\left[\mathbb{1} - \Gamma_\gamma(x)\Delta x^\gamma\right]$$

$$= \mathbb{1} - \Gamma_\beta(x)\Delta x^\beta - \Gamma_\gamma(x)\Delta x^\gamma - \partial_\gamma \Gamma_\beta \Delta x^\gamma \Delta x^\beta + \Gamma_\beta \Gamma_\gamma \Delta x^\beta \Delta x^\gamma + \cdots$$

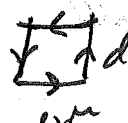
Hence ① - ②: $\quad (-\partial_\beta \Gamma_\gamma + \Gamma_\gamma \Gamma_\beta + \partial_\gamma \Gamma_\beta - \Gamma_\beta \Gamma_\gamma)\,\Delta x^\beta \Delta x^\gamma$

$$= -\left[\partial_\beta + \Gamma_\beta,\ \partial_\gamma + \Gamma_\gamma\right]\Delta x^\beta \Delta x^\gamma = -\left[\nabla_\beta, \nabla_\gamma\right]\Delta x^\beta \Delta x^\gamma$$

The effect of parallel transport around a small closed path is encoded in the curvature 2-form

$$U^\alpha \to \left[\mathbb{1} + R_{\mu\nu}\, dx^\mu dx^\nu \right]^\alpha{}_\beta\, U^\beta$$

$$R_{\mu\nu} = \left[\nabla_\mu, \nabla_\nu \right]$$



} sign depends on orientation of square

Commutator of covariant derivatives

$$\Delta U^\alpha = \left(R^\alpha{}_{\beta\mu\nu}\, dx^\mu dx^\nu \right) U^\beta$$

Here $R^\alpha{}_{\beta\mu\nu}$ really is a $\binom{1}{3}$ tensor

e.g. $$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta$$

— Both sides transform as a tensor

Important subtlety:

$\nabla_\beta \nabla_\alpha V^\gamma$ includes a term in which connection coefficient of ∇_β acts on ∇_α

$$\nabla_\beta \left(\partial_\alpha V^\gamma + \Gamma^\gamma_{\alpha\delta} V^\delta \right)$$

$$= \partial_\beta \partial_\alpha V^\gamma + \left(\partial_\beta \Gamma^\gamma_{\alpha\delta}\right) V^\delta + \Gamma^\gamma_{\alpha\delta}\, \partial_\beta V^\delta$$

$$+ \Gamma^\gamma_{\beta\eta} \left(\partial_\alpha V^\eta\right) + \Gamma^\eta_{\alpha\delta} V^\delta)$$

$$- \Gamma^\eta_{\beta\alpha} \left(\partial_\eta V^\gamma + \Gamma^\gamma_{\eta\delta} V^\delta \right)$$

Also symmetric

$$= \partial_\beta \partial_\alpha V^\gamma + \partial_\beta \Gamma^\gamma_{\alpha\delta} V^\delta + \Gamma^\gamma_{\beta\eta} \Gamma^\eta_{\alpha\delta} V^\delta +$$ (sym under $\alpha \leftrightarrow \beta$)

In components, then

WARNING
Some authors have a different convention here. (e.g. Weinberg)

$$\left[R^{\alpha}_{\ \beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\beta} \right.$$

Recall

$$\Gamma^{\alpha}_{\nu\beta} = \tfrac{1}{2} g^{\alpha\delta}\left[g_{\nu\delta,\beta} + g_{\beta\delta,\nu} - g_{\nu\beta,\delta}\right]$$

or

$$\Gamma_{\alpha\nu\beta} = \tfrac{1}{2}\left(g_{\nu\alpha,\beta} + g_{\alpha\beta,\nu} - g_{\nu\beta,\alpha}\right)$$

Warning: 1st index is transformed upper index

Now recall that we can choose locally inertial coordinates with $\Gamma = 0$. In these coordinates

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \partial_\mu \Gamma_{\alpha\nu\beta} - \partial_\nu \Gamma_{\alpha\mu\beta} \\ &= \partial_\mu \tfrac{1}{2}\left[g_{\nu\alpha,\beta} + g_{\alpha\beta,\nu} - g_{\nu\beta,\alpha}\right] \\ &\quad - \partial_\nu \tfrac{1}{2}\left[g_{\mu\alpha,\beta} + g_{\alpha\beta,\mu} - g_{\mu\beta,\alpha}\right] \\ &= \tfrac{1}{2}\left[g_{\nu\alpha,\beta\mu} - g_{\nu\beta,\alpha\mu} - g_{\mu\alpha,\nu\beta} + g_{\mu\beta,\alpha\nu}\right] \end{aligned}$$

cancel

We can "transform away" the connection coefficients by choosing the right coordinates, but we cannot "transform away" curvature — an intrinsic (tensor) quantity

$R_{\alpha\beta\mu\nu}$ is essentially 2nd derivative of g, with certain symmetry properties

And in fact it is the only meaningful tensor linear in 2nd derivs of g

Remark: E+M $D_\mu = \partial_\mu - ieA_\mu$
$[D_\mu, D_\nu] = -ieF_{\mu\nu}$ = maxwell field

what symmetries?

- Antisymmetric under $\mu \leftrightarrow \nu$ Clear, since $R_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$
- Antisymmetric under $\alpha \leftrightarrow \beta$

  Also clear, since $\left(\mathbb{I} + R_{\mu\nu} dx^\mu dx^\nu\right)^\alpha{}_\beta$ is an infinitesimal rotation of a tangent vector

  $\boxed{\begin{array}{l}\text{E.g.} \\ (\mathbb{I}+\varepsilon)(\mathbb{I}+\varepsilon)\eta = \eta \\ \Rightarrow \varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} = 0\end{array}}$

- Also symmetric under $\left.\begin{array}{l}\mu \leftrightarrow \alpha \\ \nu \leftrightarrow \beta\end{array}\right]$ this is not a general property of curvature - but is true for Riemannian curvature $\Longrightarrow$ i.e. curvature derived from a metric

- one more symmetry

$$0 = R_{\alpha\beta\gamma\mu} + R_{\beta\gamma\alpha\mu} + R_{\gamma\alpha\beta\mu}$$

where does that come from?

It means that $\nabla_{[\alpha} \nabla_\beta \omega_{\gamma]} = \frac{1}{2} R_{[\alpha\beta\gamma]}{}^\delta \omega_\delta = 0$

completely antisymmetrized in $\alpha\ \beta\ \gamma$

This is analogous to $d^2\omega = 0$, where ω = 1 form
It is a covariant version of $\mathrm{div}(\mathrm{curl}\ \vec{\omega}) = 0$
But -- why is it true?
Follows from $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$

Taking into account these symmetry constraints, we can count the number of independent components of $R_{\alpha\beta\mu\nu}$ — How many numbers to characterize curvature?

In 4 dimensions

$$\left.\begin{array}{l} R_{\underbrace{\alpha\beta}_{\substack{\text{anti} \\ \Rightarrow \\ 6\text{ values}}}\underbrace{\mu\nu}_{\substack{\text{antisym} \\ \Rightarrow 6 = \frac{1}{2}4.3 \\ \text{values}}}} \\ \text{symmetry} \\ \Rightarrow \frac{1}{2}6.7 = 21 \text{ values} \end{array}\right]$$

and one more constraint

$0 = R_{1230} + R_{2310} + R_{3120}$

$\Rightarrow$ $\underset{\sim}{R}$ has 20 components

20 is the number of independent quantities we can measure by performing parallel transport (NOT for general tangent spaces transport, but for a Riemannian connection -- a connection derived from a metric - otherwise: $6 \times 6 = 36$)

Why 20? First, generalize to m dimensions

$$\frac{1}{2}\binom{m}{2}\left[\binom{m}{2}+1\right] - \binom{m}{4}$$

$$= \frac{1}{2}\left[\frac{1}{2}m(m-1)\right]\left[\frac{1}{2}m(m-1)+1\right] - \frac{1}{24}m(m-1)(m-2)(m-3)$$

$$= m(m-1)\left[\underbrace{\frac{1}{8}m(m-1) + \frac{1}{4} - \frac{1}{24}(m-2)(m-3)}_{\frac{1}{12}m^2 + \frac{1}{12}m}\right]$$

$$= \frac{1}{12}m^2(m^2-1)$$

$$= \begin{cases} 1 & m=2 \\ 6 & m=3 \\ 20 & m=4 \end{cases}$$

Just one quantity in 2 dimensions (e.g. ratio of circumference to radius is only intrinsic measure of local curvature)

Let's do the counting another way

$g_{\mu\nu,\lambda\sigma}$ has $10 \cdot 10 = 100$ components

but we can set many to zero by choosing a suitable coordinate system....

E.g. $y^a(x^\mu)$

$$\Rightarrow g_{\mu\nu} \to g_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b}$$

$$\Rightarrow g_{ab,c} = g_{\mu\nu,\lambda} \frac{\partial x^\lambda}{\partial y^c} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} + 2 g_{\mu\nu} \frac{\partial^2 x^\mu}{\partial y^a \partial y^c} \frac{\partial x^\nu}{\partial y^b}$$

$$g_{ab,cd} = g_{\mu\nu,\lambda\sigma} \frac{\partial x^\sigma}{\partial y^d} \frac{\partial x^\lambda}{\partial y^c} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} + \dots + 2 g_{\mu\nu} \frac{\partial^3 x^\mu}{\partial y^a \partial y^c \partial y^d} \frac{\partial x^\nu}{\partial y^b}$$

- $g_{\mu\nu}$ has $\frac{1}{2} n(n+1)$ components and there are n^2 $\frac{\partial x^\mu}{\partial y^a} \Rightarrow$ Enough to set $g_{ab} = \eta_{ab}$

  And we have $\frac{1}{2} n(n-1)$ "extra" functions why? – this is the number of parameters for a Lorentz transformation of the local coordinates, which leave η_{ab} invariant

- $g_{\mu\nu,\lambda}$ has $n \cdot \frac{1}{2} n(n+1)$ components, the same number as $\frac{\partial x^\mu}{\partial y^a \partial y^b} \Rightarrow$ we have enough freedom to set $g_{\mu\nu,\lambda}$

  that's why $\nabla g = 0$

- $g_{\mu\nu,\lambda\sigma}$ has $\left[\frac{1}{2}n(n+1)\right]^2$ components

But there are only $n \cdot \frac{1}{6}n(n+1)(n+2)$ $\quad \dfrac{\partial^3 x^\mu}{\partial y^a \partial y^b \partial y^c}$

$\Rightarrow$ we don't have enough freedom to transform $g_{ab,cd}$ to zero —

Remaining parameters:

$$\frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n^2(n+1)(n+2) = n^2(n+1)\left(\frac{1}{4}n + \frac{1}{4} - \frac{1}{6}n - \frac{1}{3}\right)$$
$$= n^2(n+1)\frac{1}{12}(n-1)$$
$$= \frac{1}{12}n^2(n^2-1)$$

The Riemann tensor completely characterizes the invariant information encoded in 2nd derivatives of the metric

The Bianchi Identity

In addition to the algebraic constraints just noted that reduce the number of independent components of $R_{\alpha\beta\mu\nu}$, it also obeys an important differential constraint — the Bianchi identity — it is the rough analog in GR of the source-free Maxwell eqns of E+M.

It can be expressed as

$$0 = \nabla_{[\lambda} R_{\mu\nu]\alpha\beta}$$ where [ ] denotes antisymmetrization,

[or $\nabla_{[\lambda} \nabla_\mu \nabla_{\nu]} = 0$ (?)]

To derive it, note that

$$[\nabla_{[\lambda}, \nabla_\mu] \nabla_{\nu]} \omega_\alpha = \nabla_{[\lambda} [\nabla_\mu, \nabla_{\nu]}] \omega_\alpha$$

Now write out both sides —

we note that e.g.

$$[\nabla_\alpha, \nabla_\beta] T^{\gamma_1 \gamma_2 \cdots} = R_{\alpha\beta}{}^{\gamma_1}{}_{\delta_1} T^{\delta_1 \gamma_2 \cdots} + R_{\alpha\beta}{}^{\gamma_2}{}_{\delta_2} T^{\gamma_1 \delta_2 \cdots}$$

(derivatives acting on T cancel — see discussion on p. 19)

$$LHS = \underset{\lambda\mu\nu}{\text{Antisym}} \left(R_{\lambda\mu\nu}{}^{\gamma} \nabla_\gamma \omega_\alpha + R_{\lambda\mu\alpha}{}^{\beta} \nabla_\nu \omega_\beta \right)$$

cancel after antisymmetrization

$$= RHS = \underset{\lambda\mu\nu}{\text{Antisym}} \left(\nabla_\lambda (R_{\mu\nu\alpha}{}^{\beta} \omega_\beta) = (\nabla_\lambda R_{\mu\nu\alpha}{}^{\beta}) \omega_\beta + R_{\mu\nu\alpha}{}^{\beta} \nabla_\lambda \omega_\beta \right)$$

$0 = R_{[\lambda\mu\nu]}{}^{\gamma}$

$$\Downarrow$$

$$\nabla_{[\lambda} R_{\mu\nu]\alpha}{}^{\beta} = 0$$

Since $R_{\mu\nu\alpha\beta}$ is already antisym in $\mu \leftrightarrow \nu$, we may write this as:

More general than Riemannian geometry — i.e. depends on how R is constructed from Γ, not on how Γ is constructed from g

$$\boxed{0 = \nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla_\mu R_{\nu\lambda\alpha\beta} + \nabla_\nu R_{\lambda\mu\alpha\beta}}$$

Remark: One easy way to derive the identity is to use the locally inertial coords, where $\nabla = \partial$ and

$$R_{\mu\nu\alpha\beta} = \tfrac{1}{2}[g_{\beta\mu,\nu\alpha} - g_{\beta\nu,\mu\alpha} - g_{\alpha\mu,\beta\nu} + g_{\alpha\nu,\mu\beta}]$$

We can check that the 12 terms cancel pairwise:

$$g_{\beta\mu,\nu\alpha\lambda} - g_{\beta\nu,\mu\alpha\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\alpha\nu,\mu\beta\lambda}$$
$$+ g_{\beta\nu,\lambda\alpha\mu} - g_{\beta\lambda,\nu\alpha\mu} - g_{\alpha\nu,\beta\lambda\mu} + g_{\alpha\lambda,\nu\beta\mu}$$
$$+ g_{\beta\lambda,\mu\alpha\nu} - g_{\beta\mu,\lambda\alpha\nu} - g_{\alpha\lambda,\beta\mu\nu} + g_{\alpha\mu,\lambda\beta\nu}$$
$$= 0$$

Contracted Bianchi Identities

We can obtain other tensor from Riemann by contractions — filling some of its slots with the metric

$$R_{\mu\alpha} = g^{\nu\beta} R_{\mu\nu\alpha\beta}$$ Ricci Tensor

can't contract here because antisymmetric

Has 10 components — NOT enough to fully characterize 4-dim curvature (but its 6 components are enough in 3-dim)

Then $R_{\mu\alpha} = R_{\alpha\mu}$
— symmetric tensor

Hence: $R = g^{\mu\nu} R_{\mu\nu}$ Curvature Scalar

All there is in 2 dimensions! (the "trace" of the Ricci tensor)

What does the Bianchi identity for Riemann imply for Ricci?

$$0 = g^{\nu\beta} \,(\text{Bianchi})$$

or:

$$0 = g^{\nu\beta}\left[\nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla_\mu R_{\nu\lambda\alpha\beta} + \nabla_\nu R_{\lambda\mu\alpha\beta}\right]$$
$$= \nabla_\lambda R_{\mu\alpha} - \nabla_\mu R_{\lambda\alpha} + \nabla^\beta R_{\lambda\mu\alpha\beta}$$

(remember: $\nabla g = 0$)

$$0 = \nabla_\lambda R_{\mu\alpha} - \nabla_\mu R_{\lambda\alpha} + \nabla^\beta R_{\lambda\mu\alpha\beta}$$

and hence $\quad 0 = g^{\mu\alpha}\,[\text{above}]$

$$= \nabla_\lambda R - \nabla^\alpha R_{\lambda\alpha} - \nabla^\beta R_{\lambda\beta}$$

i.e. $$0 = \nabla^\alpha R_{\beta\alpha} - \tfrac{1}{2}\nabla_\beta R$$

$$= \nabla^\alpha \left[R_{\beta\alpha} - \tfrac{1}{2} g_{\beta\alpha} R \right]$$

we may define:

$$G^{\alpha\beta} = R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R$$

= Einstein Tensor = "trace-reversed Ricci Tensor"

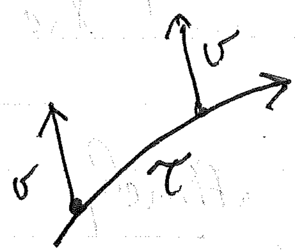
— so called because

$$G^\alpha{}_\alpha = -R^\alpha{}_\alpha = -R$$

Then $\boxed{\nabla_\beta G^{\alpha\beta} = 0}$ is the contracted Bianchi identity

Curvature and Geodesic Deviation

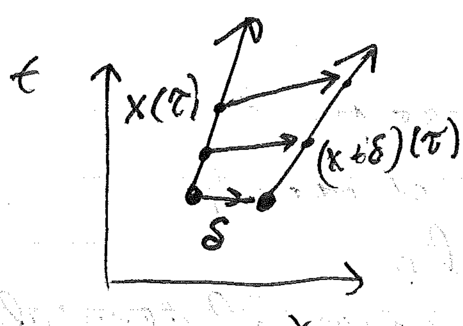
The connection between gravitation physics and Riemannian curvature is: curvature describes the tidal forces that cause neighboring geodesics (freely falling bodies) to converge or diverge. Let's find the precise relation



Let τ be the affine parameter of a geodesic with tangent vector $\vec{u} = \frac{d}{d\tau}$

Denote by $\frac{D}{D\tau}\vec{v} = \nabla_{\vec{u}}\vec{v}$

the covariant derivative along the geodesic of the vector $\vec{v}$ (e.g τ is the proper time of a freely falling observer, and $\nabla_{\vec{u}}\vec{v}$ describes how $\vec{v}$ changes relative to locally inertial coordinate system)

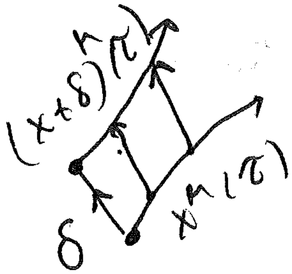


In flat space, consider 2 inertial bodies, each moving on a trajectory parametrized by its own proper time – the deviation is $\delta^\mu(\tau) = \delta^\mu_0 + (\tau - \tau_0)u^\mu_0$ – linear in τ

In Minkowski coordinates

In a spacetime diagram in flat space, the trajectories are straight lines, with equally spaced increments of τ on each line

Tidal forces are characterized by the leading departure from linearity in τ, to leading order in the small separation δ.



Both trajectories are affinely parametrized geodesics

$$0 = \frac{d^2}{d\tau^2} x^\mu + \Gamma^\mu_{\nu\lambda}(x) \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$0 = \frac{d^2}{d\tau^2}(x+\delta)^\mu + \Gamma^\mu_{\nu\lambda}(x+\delta)\frac{d}{d\tau}(x+\delta)^\nu \frac{d}{d\tau}(x+\delta)^\lambda$$

And so — to first order in δ

$$0 = \frac{d^2\delta^\mu}{d\tau^2} + \partial_\sigma \Gamma^\mu_{\nu\lambda}\,\delta^\sigma \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{d\delta^\lambda}{d\tau}$$

But — this is the coordinate change in δ which could arise from a screwy choice of coordinates — We are interested in

$$\nabla_u \nabla_u \delta = \frac{D^2}{D\tau^2}\delta$$ — covariant deriv along geodesic, which has geometric meaning

i.e. u^λ (25)

$$\frac{D}{D\tau}\delta^\mu = \nabla_u \delta^\mu = \frac{d}{d\tau}\delta^\mu + \Gamma^\mu_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\delta^\sigma$$

$$\frac{D^2}{D\tau^2}\delta^\mu = \frac{d^2}{d\tau^2}\delta^\mu + \partial_\alpha \Gamma^\mu_{\lambda\sigma}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau}\delta^\sigma + \Gamma^\mu_{\lambda\sigma}\frac{d^2x^\lambda}{d\tau^2}\delta^\sigma + 2\Gamma^\mu_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\frac{d\delta^\sigma}{d\tau} + \Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\lambda\sigma}\frac{dx^\beta}{d\tau}\frac{dx^\lambda}{d\tau}\delta^\sigma$$

Now -- we substitute in the above expressions for: $\frac{d^2}{d\tau^2}\delta^\mu$ and $\frac{d^2}{d\tau^2}x^\lambda$

$$\frac{D^2}{D\tau^2}\delta^\mu = \left[-\partial_\sigma \Gamma^\mu_{\nu\lambda}\delta^\sigma\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} - 2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{d\delta^\lambda}{d\tau}\right] + \Gamma^\mu_{\lambda\sigma}\left[-\Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right]\delta^\sigma + \partial_\alpha\Gamma^\mu_{\lambda\sigma}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau}\delta^\sigma + 2\Gamma^\mu_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\frac{d\delta^\sigma}{d\tau} + \Gamma^\mu_{\alpha\beta}\Gamma^\alpha_{\lambda\sigma}\frac{dx^\beta}{d\tau}\frac{dx^\lambda}{d\tau}\delta^\sigma$$

cancel

$$= \underbrace{\left[\partial_\nu \Gamma^\mu_{\lambda\sigma} - \partial_\sigma\Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\gamma\nu}\Gamma^\gamma_{\lambda\sigma} - \Gamma^\mu_{\gamma\sigma}\Gamma^\gamma_{\nu\lambda}\right]}_{R^\mu{}_{\lambda\nu\sigma}}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}\delta^\sigma$$

Our conclusion is

$$\boxed{\frac{D^2}{D\tau^2}\delta^\mu = R^\mu{}_{\lambda\nu\sigma}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}\delta^\sigma}$$

or $$\boxed{\nabla_u\nabla_u\delta^\mu = R^\mu{}_{\lambda\nu\sigma}u^\lambda u^\nu\delta^\sigma}$$

— the equation of geodesic deviation

Why is it Riemann — which we defined in terms of parallel transport around a closed path — that characterizes geodesic deviation?

It is because $R^\mu{}_{\lambda\nu\sigma}$ is the tensor that completely characterizes the coordinate-independent content of the 2nd derivatives of the metric ... and these 2nd derivatives control both the path dependence of parallel transport and geodesic deviation in an infinitesimal region!

simplify the derivation by choosing locally inertial coordinates ($\Gamma = 0$)

then $$0 = \frac{d^2\delta^\mu}{d\tau^2} + \partial_\sigma \Gamma^\mu_{\nu\lambda}\, \delta^\sigma u^\nu u^\lambda$$

and $$\frac{D^2}{D\tau^2}\delta^\mu = \frac{d^2}{d\tau^2}\delta + \partial_\alpha \Gamma^\mu_{\lambda\sigma}\, u^\alpha u^\lambda \delta^\sigma$$

$$= \partial_\alpha \Gamma^\mu_{\lambda\sigma}\, u^\alpha u^\lambda \delta^\sigma - \partial_\sigma \Gamma^\mu_{\nu\lambda}\, \delta^\sigma u^\nu u^\lambda$$

$$= R^\mu{}_{\lambda\alpha\sigma}\, u^\lambda u^\alpha \delta^\sigma$$

The Einstein Field Eqn

Recall — there are two sides to the theory of gravitation — Both guided by the principle of equivalence $\Longrightarrow$

 You don't feel gravity when freely falling

↳ General covariance: laws with no preferred coordinate system

① How does matter move? — On geodesics

Tidal forces: $\nabla_u \nabla_u \delta^\mu = R^\mu{}_{\lambda\nu\sigma} u^\lambda u^\nu \delta^\sigma$ — geodesic deviation

Spacetime curvature tells matter how to move

② But how does spacetime know how to curve?

cf ① Lorentz force law: How charges move
② Maxwell eqns: How charges determine E & B

A Field eqn for gravitation
em tensor $T \Rightarrow$ metric g (up to a change of coordinates)

Again — general covariance can guide us — an eqn with a meaning independent of coordinates...

Recall Newton $\nabla^2\phi = 4\pi G\rho$] we should obtain this in a suitable limit:
- weak field $g \sim \eta + h$, $|h| \ll 1$
- slow sources, $v \ll c$

we already made the identification

$h_{00} = -2\phi$, from geodesic eqn

and $\Gamma^\alpha_{00} \simeq -\frac{1}{2} h_{00,}{}^\alpha \simeq \nabla^\alpha \phi$

So $\Gamma^{\alpha}_{00} \simeq \nabla^{\alpha}\phi$

And $= \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \cdots$

$$\Rightarrow R^{\alpha}{}_{00\nu} \simeq \partial_0 \Gamma^{\alpha}_{\nu 0} - \partial_\nu \Gamma^{\alpha}_{00}$$

$$\Rightarrow R_{00} \simeq -\partial_0 \Gamma^{\alpha}_{\alpha 0} + \partial_\alpha \Gamma^{\alpha}_{00}$$

$$\simeq -\partial_0 \Gamma^{0}_{00} + \partial_\alpha \Gamma^{\alpha}_{00} = \partial_i \Gamma^{i}_{00} = \partial_i \partial^i \phi$$

So in the Newtonian limit

$$R_{00} = \nabla^2 \phi = 4\pi G \rho \simeq 4\pi G T_{00}$$

So we can guess: $R_{\mu\nu} = 4\pi G T_{\mu\nu}$ (??)

Einstein made such a guess but there is a problem with

$R \propto T$

Ricci

Bianchi:
$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\partial^\nu R = 0$
Hence $\partial^\nu T^{\lambda}{}_{\lambda} = 0$ — NOT true in general an tensor

We have $\nabla \cdot T = 0$, but not $\nabla \cdot R = 0$

At least, it is not an identity — but so what?

$T^{ij} \sim 0$
$\Rightarrow G^{ij} \sim 0$
$\Rightarrow R^{ij} \sim \frac{1}{2} g^{ij} R$
$\Rightarrow R = \frac{3}{2} R - R^{00}$
$\Rightarrow R = 2R^{00}$

The key to constructing a consistent field eqn:

Recall that $\nabla \cdot G = 0$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \quad \text{(contracted Bianchi identity)}$$

Now $G^{00} = R^{00} + R^{00} = 2\nabla^2\phi$

Hence $G^{00} = 8\pi G T^{00}$ — and we are led to propose

or — it is easier to use $R^{\mu\nu} = K(T^{\mu\nu} - \frac{1}{2} g T)$

$$\boxed{G = 8\pi G T}$$ The Einstein field eqn —

We may as well choose our units so that $G = c = 1$, and then $\boxed{\underset{\sim}{G} = 8\pi \underline{T}}$

ordinarily $\frac{GM}{r} \sim v^2$ so $\frac{GM}{c^2} \sim$ distance

In these units -- mass and distance have the same dimensions (because a mass can be interpreted as a source of curvature: $\frac{1}{\text{distance}^2} \sim \frac{\text{mass}}{\text{distance}^3}$).

We can write the Einstein eqn differently:

$$\left(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R\right) = 8\pi T^{\mu\nu}$$

Take trace $\quad -R = 8\pi T^{\mu}{}_{\mu}$

$$\Rightarrow R^{\mu\nu} = 8\pi T^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu} R$$
$$= 8\pi\left(T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} T^{\lambda}{}_{\lambda}\right)$$

(which is actually how Einstein wrote it in 1915)

If there is no matter (source-free gravitation), then $T^{\mu\nu} = 0$

and $\boxed{\boxed{R^{\mu\nu} = 0}}$ the empty-space einstein eqn

of the 20 components of the Riemann curvature, 10 vanish in "the vacuum," but 10 do not.... this equation describes: pure spacetime geometry

[Mach was wrong: spacetime not determined by the distant fixed stars.]

it can wiggle, shake and "warp" w/o any sources

$\boxed{R^{\mu\nu} = 0}$ — It sure looks simple

But recall $\Gamma = g^{-1}(\partial g + \partial g - \partial g)$
$R = (\partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma)$] these eqns are actually highly nonlinear in the metric

And indeed, are capable of exhibiting much weird and wonderful behavior!

Far more difficult than source-free Maxwell eqns

Comment: the cosmological term

Some history of the field eqn (Pais, "Subtle is the Lord")

1907 (Bern) P. of equivalence for uniform acceleration,
and - the red shift
- Bending of light (½ of GR)

1907-1911, he was busy with q. theory (in Zurich)

1911 (Prague) → he understands the significance of the P. of Equiv more broadly — there is no absolute acceleration

1912 — First attempts at dynamics of grav. field
[c = speed of light = a scalar field $\Delta c = K c \rho$] Still only static fields

He realizes that energy momentum, the source for gravitation, should have a contribution from gravity itself ⇒ A nonlinear theory!

Any other symmetric tensor aside from $R_{\mu\nu}$?
How about the metric

$$\nabla \cdot g = 0 \text{ , since } \nabla g = 0$$

$$\boxed{G + \Lambda g = 8\pi T}$$

"cosmological constant"
not introduced by Einstein until later...

We could also write this as

$$G = 8\pi \left(T - \frac{\Lambda}{8\pi} g \right)$$

and interpret as a constant (vacuum) contribution to energy momentum

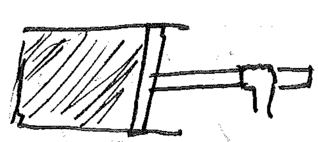
$$T^{\mu\nu} = \mathrm{diag}(\rho, P, P, P)$$

For $\Lambda > 0$, the vacuum has

$$\rho_{vac} = \frac{\Lambda}{8\pi} > 0$$

$$P_{vac} = -\rho_{vac} < 0$$

Negative pressure?
Really the same as tension
We work to pull out the piston if we are creating pos. energy density



Cosmological term modifies Newtonian limit

$$G^{00} = \frac{1}{2} \nabla^2 \phi = 8\pi (\rho + \rho_{vac})$$

Best limit – from limit on how fast the expansion of the universe is decellerating (accelerating)

$$|\Lambda| < 10^{-29}\, g/cm^3 \qquad (!)$$

1917 – Einstein pioneers relativistic cosmology
Λ -- for a static universe

A great mistake because:

① His universe was unstable

② He failed to predict universe is dynamic

But not such a great mistake – it made sense to include Λ.

cf Newton's similar mistake: universe must be ∞ so won't collapse? (Homogeneous collapse can occur)

What about other conserved tensors?

There are more – involving more powers of Riemann ($\Rightarrow$ more derivs of metric)

$$\text{E.g.}\quad \mathcal{L} = \sqrt{g}\left(c_1 + c_2 R + c_3 R_{\mu\nu} R^{\mu\nu} + c_4 R^2 + \cdots \right)$$

$c_3/c_2 \sim (\text{Length})^2$ What length?

Best guess

$$L_{P\ell} = \sqrt{\hbar} \left(= \left(\frac{G\hbar}{c^3}\right)^{\frac{1}{2}} \simeq 1.6\times 10^{-33}\,cm \right)$$

(Planck in 1899!)

Whew!!

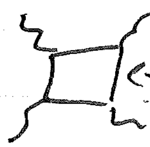
Neglect at long length scales

But – what about Λ?
Too small by $\sim 10^{122}$

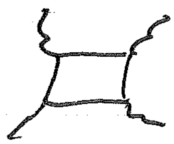
(Most serious disagreement between theory & exp in physics

Similar issue in electromagnetism:

$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ — what about $(F_{\mu\nu}F^{\mu\nu})^2$, $(\epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta})^2$ etc?

 ← electron $\Rightarrow$ $\frac{\alpha^2}{m_e^4}$ suppression

Negligible at ordinary distances

cf  Find term with 4 derivatives $\sim G$

Natural expansion parameter $\left({}^{L_{pl}}/_{L}\right)^2$

Another comment on electromagnetism — it has a Bianchi identity too!

Analog of $\epsilon^{\sigma\lambda\mu\nu} \nabla_\lambda R_{\mu\nu\alpha\beta} = 0$

is $\epsilon^{\sigma\lambda\mu\nu} \partial_\lambda F_{\mu\nu} = 0$ the source free Maxwell eqns!

$F_{\mu\nu}$ is a source $\Rightarrow$

$\partial_\mu F^{\mu\nu} = J^\nu$ consistent with $\partial_\nu J^\nu = 0$ if $F^{\mu\nu}$ antisymmetric

(or... antisymmetry of $F^{\mu\nu}$ means that source must be conserved)

1912 (Zurich)

Recalls Gaussian theory of surfaces

Grossman tells him of Ricci, Levi-Civita, Riemann

- Riemannian geom as the crucial tool
- Eqns for $g_{\mu\nu}$
- Guess $T_{\mu\nu} = S_{\mu\nu} \Leftarrow$ some Tensor constructed from $g_{\mu\nu}$

Great progress — but some unfortunate misconceptions

Erroneous claim

Cannot obtain $\Delta\phi = 4\pi G\rho$ as a weak field limit!

He still doesn't understand general covariance: Coordinates must be restricted

- $\nabla g = 0 \Rightarrow$ Cannot construct a Tensor eqn from g
- Field eqns cannot determine $g_{\mu\nu}$ completely! (he does not understand that it is not suppose to!

Meanwhile — Nordström 1912 — a special relativistic theory

$\Phi = \ln \psi \qquad -\psi\Box\psi = K T^\alpha{}_\alpha$

1913 Einstein (+ Fokker) point out —

This is $R = \text{const}\, T$ where $g_{\mu\nu} = \psi^2 \eta_{\mu\nu}$

Ricci scalar

But E. thinks this is not enough coord freedom

1914 (Berlin)

- Another wrong paper on why field eqns cannot be generally covariant

- He develops a theory with only linear covariance

- until July, 1915, he still believed old theory

Nov 1915 - In October, he is dissatisfied

A new paper every week communicated to Prussian Academy

Nov 4: $\boxed{\text{o } T_{\mu\nu} \text{ ought not to determine } g_{\mu\nu} \text{ completely}}$

- more general covariance, but unimodular trans only

Nov. 11: A forgettable paper? — demanding that matter satisfy $T^{\mu}{}_{\mu} = 0$ (!)] Requires $\sqrt{g} = 1$

He writes $\boxed{R_{\mu\nu} = -K T_{\mu\nu}}$

Nov. 18: still $\sqrt{g} = 1$ — but

- Perihelion of mercury
- Bending of light: Twice his 1907 prediction (Good thing war prevented earlier eclipse expedition)

Nov. 25 $\boxed{R^{\mu\nu} = K\left(T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} T\right)}$ TADA!

Still does not know about Bianchi

He thinks $0 = T^{\mu\nu}{}_{;\nu}$ constrains metric

(Realizes $\sqrt{g} = 1$ on Nov. 18 was just a gauge choice)

Note: Hilbert Nov. 20 had it first

$0 = \delta\left[\int \sqrt{g}\, d^4x \left[\mathcal{L}_{matter} - \frac{1}{2K} R\right]\right]$

He doesn't know Bianchi either

H and E exchanged 6 letters Nov. 7-19

- In Oct 1916, E. finally realizes that
  $T^{\mu\nu}{}_{;\nu} = 0$ follows from $G^{\mu\nu} \propto T^{\mu\nu}$

- In Aug 1917, Weyl shows that $G^{\mu\nu}{}_{;\nu}$
  follows from Hilbert's variational principle

- Only in 1927 does a paper appear that
  points out that Bianchi identity had long
  been known (since 1880) in diff geom literature

Remark: $0 = T^{\mu\nu}{}_{;\nu}$, $0 = G^{\mu\nu}{}_{;\nu}$] special case of Noether's thm?

Gravitational Waves

Even w/o any matter, gravitation is interesting

$\boxed{R^{\mu\nu} = 0}$ still allows
Riemann $R_{\mu\nu\alpha\beta} \neq 0$
So space can shake, rattle, and roll

$\Uparrow$

Nonlinear and complicated
(cf "the Grand Challenge" of coalescing black holes)

Much easier are the linearized ("weak field") eqns

Propagating ripples of spacetime curvature

The traditional approach — use coordinates that are "near Minkowski"

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad |h| \ll 1$

(But not longer assume $v \ll c$, as in our discussion of Newtonian limit)

The Einstein eqn $R_{\mu\nu} = 0$ becomes a linear eqn for h, which we can interpret as eqn for $h_{\alpha\beta}$ in flat space (with linear Lorentz invariance)

$$\boxed{h_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\, \Lambda^{\nu}{}_{\beta}\, h_{\mu\nu}}$$

— (Since the "background" η by itself is Lorentz invariant)

How h transforms under linear coord trans.

Also — consider small shifts in the coords

$$x'^{\alpha} = x^{\alpha} + \varepsilon^{\alpha} \qquad \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta^{\alpha}{}_{\beta} + \varepsilon^{\alpha}{}_{,\beta}$$

$$\Rightarrow \quad g'_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \varepsilon_{\alpha,\beta} - \varepsilon_{\beta,\alpha} + \cdots$$

"Gauge symmetry"

$$\boxed{h_{\alpha\beta} \to h_{\alpha\beta} - \partial_{\beta}\varepsilon_{\alpha} - \partial_{\alpha}\varepsilon_{\beta}}$$

descended from freedom to choose coords as we please

$R_{\mu\nu}$ & G is gauge-invariant to first order

$$R_{\alpha\beta\mu\nu} \stackrel{\sim}{=} \tfrac{1}{2}\left(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}\right)$$

(This is because $R \to (1+\epsilon) R$ ... it is a tensor there is no $R \to R + O(\epsilon)$ ...)

Quite analogous to freedom $A_\mu \to A_\mu + \partial_\mu \Lambda$ (we can fix the gauge as we please)

Analog of Lorentz gauge is Hilbert gauge

$$\boxed{\partial_\mu A^\mu = 0 \Rightarrow \Box A^\mu = 0}$$

$$\partial_\mu (h^{\mu\nu} - \tfrac{1}{2} \eta^{\mu\nu} h)$$

trace reversed "$\bar{h}$"

And then

$$\boxed{\begin{array}{c} \Box \bar{h}^{\mu\nu} = 0 \\ \partial_\mu \bar{h}^{\mu\nu} = 0 \end{array}}$$

source free

Does not fix gauge completely

A wave eqn: $\bar{h}^{\mu\nu} = e^{\mu\nu} e^{iK\cdot x}$

$$\Rightarrow \quad K^2 = 0$$

$$K_\mu e^{\mu\nu} = 0$$

Remaining freedom: $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2} \eta_{\mu\nu} h$$
$$\to h_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu + \eta_{\mu\nu} \partial^\lambda \epsilon_\lambda$$

$$\epsilon_\mu = e_\mu e^{ik\cdot x}$$

or $e_{\mu\nu} \to e_{\mu\nu} - iK_\mu e_\nu - iK_\nu e_\mu + i\eta_{\mu\nu} K\cdot e$

Enough freedom to impose:

$K_\alpha = K(1\,0\,0\,1)$

$\hat{z}$ direction

$$\left.\begin{array}{l} e^\mu{}_\mu = 0 \\ e_{0\mu} = e_{3\mu} = 0 \end{array}\right]$$

"Transverse Traceless gauge"

Now we need to figure out the gauge-invariant meaning of the soln:

E.g. construct Riemann, and consider geodesic deviation of (Tidal forces on) test particles

---

But Roger + Kip use an elegant approach that short-circuits much of this analysis, by working directly with invariant (Tensor) quantities ...

Its like writing wave eqn for $F_{\mu\nu}$ instead of A_μ — in some ways it is easier to work with vector than tensor

Vacuum propagation $R_{\mu\nu} = 0$

what do we know about Riemann?

Recall one-contracted Bianchi identity

$$0 = \nabla_\lambda R_{\mu\alpha} - \nabla_\mu R_{\lambda\alpha} + \nabla^\beta R_{\lambda\mu\alpha\beta}$$

$$R_{\mu\nu} = 0 \Rightarrow \boxed{\nabla^\beta R_{\lambda\mu\alpha\beta}}$$

So Riemann is divergence free in each index

Going back to uncontracted Bianchi

$$0 = \nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla_\mu R_{\nu\lambda\alpha\beta} + \nabla_\nu R_{\lambda\mu\alpha\beta}$$

Divergence free $\Rightarrow$

$$0 = \nabla^\lambda (\text{above}) = \nabla^\lambda \nabla_\lambda R_{\mu\nu\alpha\beta} + \nabla^\lambda \nabla_\mu R_{\nu\lambda\alpha\beta} + \nabla^\lambda \nabla_\nu R_{\lambda\mu\alpha\beta}$$

Now -- *if* we could commute ∇^λ through ∇_μ or ∇_ν -- we could use divergence free condition

But -- they don't commute $[\nabla_\lambda, \nabla_\mu]^\alpha{}_\beta = R^\alpha{}_{\beta\lambda\mu}$

On the other hand, if curvature is small -- neglect terms quadratic in Riemann and obtain

$$\boxed{(\nabla \cdot \nabla)\underset{\sim}{R} = 0}$$

i.e. $\nabla^\lambda \nabla_\lambda R_{\mu\nu\alpha\beta} = 0$

WARNING: weak field approximation

In "nearly Minkowski" coordinates, again neglecting effects higher order in curvature

$$\nabla \cdot \nabla = \Box \Rightarrow \boxed{\Box \underset{\sim}{R} = 0}$$

A wave equation

A soln -- wave propagating in z direction

$$\boxed{R_{\alpha\beta\gamma\delta}(t-z)}$$

Each component is a function of $t-z$

But -- how many independent components -- How many polarizations?

Plug $R_{\mu\nu\alpha\beta}(t-z)$ into Bianchi identity

Linearized $\quad \partial_\mu R_{\nu\lambda\alpha\beta} + \partial_\nu R_{\lambda\mu\alpha\beta} + \partial_\lambda R_{\mu\nu\alpha\beta} = 0$

$$\partial_0 R_{xy\alpha\beta} = 0 = \partial_z R_{xy\alpha\beta}$$

$$\partial_z R_{x0\alpha\beta} + \partial_0 R_{zx\alpha\beta} = 0 \Rightarrow 0 = \partial_0 (R_{zx\alpha\beta} - R_{x0\alpha\beta})$$

$$\partial_z R_{y0\alpha\beta} + \partial_0 R_{zy\alpha\beta} = 0 \Rightarrow 0 = \partial_0 (R_{zy\alpha\beta} - R_{y0\alpha\beta})$$

$R_{xy\alpha\beta} = \text{constant} \Rightarrow = 0$ (localized wavepacket)

$$\boxed{R_{xy\alpha\beta} = 0 \qquad R_{x0\alpha\beta} = R_{zx\alpha\beta} \qquad R_{y0\alpha\beta} = R_{zy\alpha\beta}}$$

$\mu\nu$ or $\alpha\beta$ can have the values

$$\boxed{\begin{array}{l} xy \to 0 \\ zx \to x0 \\ zy \to y0 \end{array}} \qquad \begin{array}{ccc} 0 \leftarrow xy & yz & zx \\ 0x & 0y & 0z \end{array}$$

We can eliminate $yz \quad zx \quad xy$

Now only components are R_{i0j0}

What further restrictions from the vacuum Einstein Eqns?

$$0 = R^\mu{}_{\alpha\mu\beta}$$

$$R^i{}_{0ix} = R_{z0zx} = R_{z0x0} = 0$$

$$R^i{}_{0iy} = R_{z0zy} = R_{z0y0} = 0$$

$$R^i{}_{0iz} = R_{x0xz} + R_{y0yz} = -R_{x0x0} - R_{y0y0} = 0$$

$$R^i{}_{0i0} = R_{x0x0} + R_{y0y0} + R_{z0z0} = 0$$

The remaining 6 eqns are trivial identities -- so we learn

$R_{z0x0} = 0$

$R_{z0y0} = 0$

$R_{z0z0} = 0$

$\boxed{\text{The waves are "Transverse"}}$

$R_{z0\alpha\beta} = 0$

"no z-component"

3 components remain $\begin{bmatrix} R_{x0y0} \\ R_{x0x0} \\ R_{y0y0} \end{bmatrix}$

And also $R_{x0x0} + R_{y0y0} = 0$

$\boxed{\text{The waves are "traceless"}}$

Transverse-traceless: "TT"

In the "gauge fixed" language, we have

$$\bar{h}_{\mu\nu} = e^{(+)}_{\mu\nu}\, h_+(t-z) + e^{(\times)}_{\mu\nu}\, h_\times(t-z)$$

"h-plus" "h-cross"

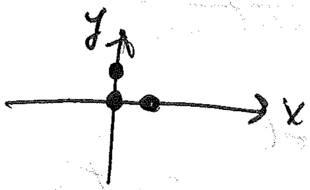
$$e^{(+)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e^{(\times)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And $R^x{}_{0x0} = -\frac{1}{2}\ddot{h}_+(t-z) = -R^y{}_{0y0}$

$R^y{}_{0x0} = -\frac{1}{2}\ddot{h}_\times(t-z) = R^x{}_{0y0}$

R_{i0j0} is <u>transverse</u> (No $\hat{z}$ component)
and <u>traceless</u> $[\, R_{z0z0} = R_{z0x0} = R_{z0y0} = 0 \,]$

A gravitational wave has 2 independent polarizations: plus and cross



What tidal acceleration does the wave produce?

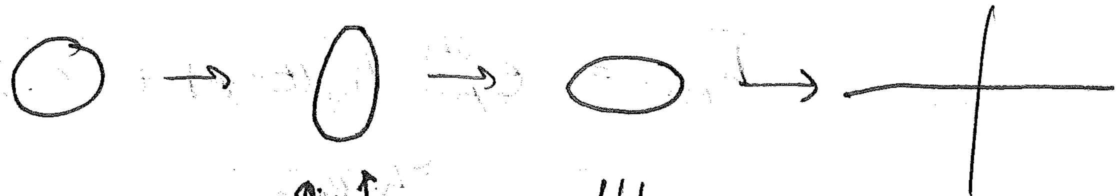
Geodesic deviation $\frac{D^2\delta^\mu}{D\tau^2} = R^\mu{}_{\nu\lambda\sigma} u^\nu u^\lambda \delta^\sigma$

For particles initially at rest
$\frac{d^2}{dt^2}\delta^i = -R^i{}_{0j0}\,\delta^j \Rightarrow$

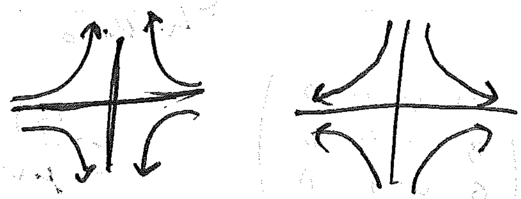
$$\boxed{\begin{aligned} \ddot{\delta}^x &= \tfrac{1}{2}\ddot{h}_+\,\delta^x + \tfrac{1}{2}\ddot{h}_\times\,\delta^y \\ \ddot{\delta}^y &= \tfrac{1}{2}\ddot{h}_\times\,\delta^x - \tfrac{1}{2}\ddot{h}_+\,\delta^y \end{aligned}}$$

Acting on a circle of free particles

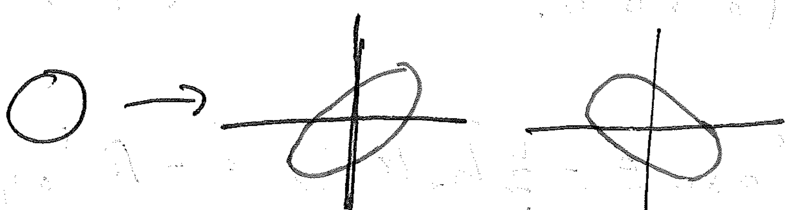
h_+



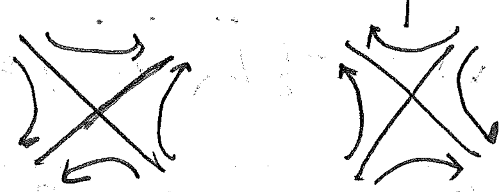
Quadrupole tidal field



$h_\times$



The same quadrupole field -- only rotated 45°



(Origin of the names $+$ and $\times$)

Are they really independent?

$$e_{\mu\nu} \to R e R^T \quad \text{under rotations}$$

and $e^{(+)}$ and $e^{(\times)}$ are orthogonal basis states of a 2 dim representation of the 2D rotation group

$$\begin{pmatrix}1 & \varepsilon\\ -\varepsilon & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -\varepsilon\\ \varepsilon & 1\end{pmatrix} = \begin{pmatrix}+\\ \times\end{pmatrix} \to \begin{pmatrix}1 & 2\varepsilon\\ -2\varepsilon & 1\end{pmatrix}\begin{pmatrix}+\\ \times\end{pmatrix}$$

$$(1+i\varepsilon\sigma_y)(\sigma_z)(1-i\varepsilon\sigma_y)$$

$$= \sigma_z + i\varepsilon[\sigma_y,\sigma_z] = \sigma_z - 2\varepsilon\sigma_x$$

Helicity states

$$\frac{1}{\sqrt{2}}\left(e^{(+)} \pm i e^{(\times)}\right); \ \lambda = \pm 2$$

Are the waves "real"...?
They have a subtle property — unlike the case of an em wave, we can't express its energy-momentum as the integral of a local density!

Why not? Equivalence principle — the wave is "invisible" in the local Lorentz frame

It is a general feature that in GR energy is not the sum of its parts

Chapel Hill 1957
— a spirited discussion

Feynman's beads on a stick

(the stick remains stiff — but the beads move)

Friction ⇒ the stick heats up.

Must really be

<u>Generation of the Waves</u>

Solve, in the gauge-fixed formalism

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

Do a multipole expansion
No "gravitational dipole," so leading term is quadrupole

$$L = \frac{1}{5}\langle \dddot{I}_{ij}\, \dddot{I}^{ij} \rangle$$

cf Binary pulsar $\quad L \sim M^2 \ell^4 \omega^6$

Measure
- Period
- Doppler shifts
- Precession

$\Downarrow$

Predict L

period 7h 45min

$M_1 \sim M_2 \sim 1.4 M_\odot$

Precess at $4°$/year

$5 \times 10^{-9}\ \mathrm{yr}^{-1}$ rate of orbit decay

measured to better than 1%

$$\frac{dP}{dt} \simeq 7.2 \times 10^{-5}\ \mathrm{sec/yr}$$