# VI The Partial-Wave Expansion

## A. The Schrodinger Eqn with a Central Potential

We wish to review methods of solution for the time-independent Schrodinger eqn, for

$$V = V(r) \quad \text{- central potential}$$

The interaction between two particles is expected to have this form by rotational invariance for spinless particles. (For particles with spin there are more exotic possibilities.)

The Hamiltonian is

$$H = -\tfrac{1}{2}\Delta + V = \tfrac{1}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{L^2}{r^2}\right) + V(r)$$

which commutes with $\vec{L}$, $[H, \vec{L}] = 0$. $H, L^2, L_z$ are a complete set of commuting observables

Separate the wave function:

$$\psi(r,\theta,\phi) = R(r)\, Y_{lm}(\theta,\phi)$$

where

$$L^2 Y_{lm} = l(l+1) Y_{lm}, \quad L_z Y_{lm} = m Y_{lm}, \quad \int d\Omega\, |Y_{lm}|^2 = 1$$

$H\psi = E\psi \Rightarrow$ $\qquad (E = \tfrac{1}{2}p^2)$

$R(r)$ obeys

$$\boxed{\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + p^2 - 2V - \frac{l(l+1)}{r^2}\right] R(r) = 0}$$

Radial equation – a one-dimensional problem with a centrifugal barrier

### Free solution of the Radial Eqn

We are interested (in scattering theory) in asymptotic behavior for $r \to \infty$, $V \to 0$. Eqn becomes "free"

i) $\ell = 0$ We have $\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + p^2\right] R(r) = 0$

or $\frac{d^2}{dr^2}(rR(r)) + p^2 = 0$

$$\Rightarrow rR(r) = e^{\pm ipr}$$

The solution regular at $r \to 0$ is

$$\boxed{rR(r) = \sin pr = \hat{j}_0(pr)}$$ "Riccati-Bessel function"

($j_0(z) = \frac{\hat{j}_0(z)}{z}$ is called "Spherical Bessel Function")

ii) $\ell \neq 0$ The equation is

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + p^2 - \frac{\ell(\ell+1)}{r^2}\right] R_\ell(r) = 0$$

If $\chi_\ell = r^{-\ell} R_\ell$, then

$$\frac{d^2}{dr^2}(r^{-\ell}R_\ell) + \frac{2(\ell+1)}{r}\frac{d}{dr}(r^{-\ell}R_\ell) + p^2(r^{-\ell}R_\ell) = 0$$

Now, differentiate $\Rightarrow$

$$\frac{d^2}{dr^2}\left(\frac{d}{dr}\chi_\ell\right) + \frac{2(\ell+1)}{r}\frac{d}{dr}\left(\frac{d}{dr}\chi_\ell\right) + \left[p^2 - \frac{2(\ell+1)}{r^2}\right]\frac{d}{dr}\chi_\ell$$

or $$\frac{d^2}{dr^2}\left(r^{-1}\frac{d}{dr}\chi_\ell\right) + \frac{2(\ell+2)}{r}\frac{d}{dr}\left(r^{-1}\frac{d}{dr}\chi_\ell\right) + p^2 r^{-1}\frac{d}{dr}\chi_\ell$$

But this is an equation of the same form, with $\ell \to \ell+1$ which means the $\chi_\ell$ satisfy a recursion relation

$$\chi_{\ell+1} = \frac{1}{r}\frac{d}{dr}\chi_\ell \Rightarrow \chi_\ell = \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \chi_0$$

or $$\boxed{rR = (-1)^\ell \frac{r^{\ell+1}}{p^\ell}\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left[\frac{1}{r}e^{\pm ipr}\right]}$$

conventional

The solution regular at $r \to 0$ is

$$\boxed{rR(r) = (-1)^\ell \frac{r^{\ell+1}}{p^\ell}\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left[\frac{\sin pr}{r}\right] = \hat{j}_\ell(pr)}$$

The functions generated by $e^{\pm iz}$, $\hat{h}^{\pm}(z)$ are called Riccati-Hankel Functions and the singular solution

$$\hat{n}(z) = \frac{1}{2}(h^{+}(z) + h^{-}(z))$$ is the Riccati-Neumann Function

$$\hat{j}(z) = \frac{1}{2i}(h^{+}(z) - h^{-}(z))$$

## Expansion of Plane Wave

$$e^{ipz} = e^{ipr\cos\theta}$$ can be expanded in terms of the $Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$

where $P_\ell(z) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dz}\right)^\ell (z^2-1)^\ell$

We have $$e^{ipr\cos\theta} = \sum_{\ell=0}^{\infty} A_\ell\, P_\ell(\cos\theta)\, \frac{\hat{j}_\ell(pr)}{pr}$$

($e^{ipr\cos\theta}$ is regular at $r=0$)

Compare terms of order $(pr\cos\theta)^n$

$$\frac{(i)^n (pr\cos\theta)^n}{n!} = A_n \frac{(2n)!(\cos\theta)^n}{2^n (n!)(n!)} \quad \frac{(pr)^n}{(2n+1)!!}$$

(See below)

$$A_n = (i)^n \frac{2^n (2n+1)!!\, n!}{(2n)!} = (i)^n (2n+1)$$

Hence $$\boxed{e^{ipr\cos\theta} = \sum_{\ell=0}^{\infty} (i)^\ell (2\ell+1) P_\ell(\cos\theta) \frac{\hat{j}_\ell(pr)}{pr}}$$

## Asymptotic Behavior

For large $r$ dominant term is that in which each derivative hits exponential

$$h_\ell^{\pm}(pr) \xrightarrow[r\to\infty]{} \left(-\frac{d}{dr}\right)^\ell e^{\pm ipr} = e^{\pm i(pr - \frac{1}{2}\ell\pi)}$$

$$j_\ell(pr) \xrightarrow[r\to\infty]{} \sin(pr - \tfrac{1}{2}\ell\pi)$$

The asymptotic $r\to 0$ behavior of regular solution

$$\hat{j}_\ell(pr) = \frac{r^{\ell+1}}{p^\ell}\left(-\frac{1}{r}\frac{d}{dr}\right)^\ell \frac{\sin pr}{r}$$

$$= \frac{r^{\ell+1}}{p^\ell}\left(-\frac{1}{r}\frac{d}{dr}\right)^\ell \sum_{n=0}^{\infty} \frac{p^{2n+1} r^{2n}}{(2n+1)!}(-1)^n$$

$$\sim \frac{r^{\ell+1}}{p^\ell} p^{2\ell+1} \frac{2n(2n-2)\cdots}{(2n+1)!} = \frac{(pr)^{\ell+1}}{(2n+1)!!} = (pr)^{\ell+1}\frac{2^n n!}{(2n+1)!}$$

## B. Partial-Wave Scattering States

Our "improper" eigenstates can be used to expand wave packets, like plane waves. Properly normalized eigenstates are

$$\boxed{\langle \vec{x} | E,\ell,m\rangle = (i)^\ell \left(\frac{2}{\pi p}\right)^{1/2} \frac{\hat{j}_\ell(pr)}{r} Y_{\ell m}(\Omega)}$$

convention

This is continuum normalization, because

$$\int dr\, \hat{j}_\ell(pr)\hat{j}_\ell(p'r) = \frac{\pi}{2}\delta(p-p') = \frac{\pi p}{2}\delta(E-E')$$

Now we consider

$$|E,\ell,m+\rangle = \Omega_+ |E,\ell,m\rangle$$

Stationary Scattering State
This satisfies exact Schrodinger Eqn,

$$H|E,\ell,m+\rangle = E|E,\ell,m+\rangle \quad \text{(Intertwining relation)}$$

and also eigenstate of $L^2$, $L_z$, because

$$[\vec{L}, \Omega_+] = 0.$$

The angular wave function is unchanged by potential

$|E,\ell,m+\rangle$ is a linear combination of collapsing and expanding spherical waves, but only the outgoing wave is affected by the potential.

$$\boxed{\langle \vec{x} | E,\ell,m+\rangle = (i)^\ell \left(\frac{2}{\pi p}\right)^{1/2} \frac{\psi_{\ell,p}(r)}{r} Y_{\ell m}(\Omega)}$$

A (normalized) plane wave is

$$\langle \vec{x} | p\hat{z} \rangle = \frac{1}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} (i)^{\ell} (2\ell+1) P_{\ell}(\cos\theta) \frac{\hat{j}_{\ell}(pr)}{pr}$$

The state $|p\hat{z}+\rangle$ can also be expanded

$$\langle \vec{x} | p\hat{z}+ \rangle = \langle \vec{x} | \Omega_+ | p\hat{z} \rangle = \frac{1}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} (i)^{\ell} (2\ell+1) P_{\ell}(\cos\theta) \frac{\psi_{\ell,p}(r)}{pr}$$

But, as we derived on p. (4.9)

$$\langle \vec{x} | p\hat{z}+ \rangle \xrightarrow[r\to\infty]{} \frac{1}{(2\pi)^{3/2}} \left[ e^{ipz} + f(p,\theta) \frac{e^{+ipr}}{r} \right] + \cdots$$

(if $V$ falls rapidly enough as $r \to \infty$)

If we write $\quad f(p,\theta) = \sum_{\ell} (2\ell+1) f_{\ell}(p) P_{\ell}(\cos\theta)$

(defining $f_{\ell}$)

Then, from asymptotic behavior of solutions to free radial eqn, we know that

$$\boxed{\psi_{\ell,p}(r) \xrightarrow[r\to\infty]{} \hat{j}_{\ell}(pr) + p f_{\ell}(p) \hat{h}_{\ell}^{+}(pr)}$$

(because $\hat{h}_{\ell}^{+} \xrightarrow[r\to\infty]{} e^{i(pr - \ell(\pi)} = (-i)^{\ell} e^{ipr}$

~~But if $\langle E, \ell', m' | S | E, \ell, m \rangle = \delta(E'-E)\delta_{\ell'\ell}\delta_{m'm} S_{\ell}(p)$ unitarity of $S$ requires $S_{\ell} = e^{2i\delta_{\ell}}$, $\delta_{\ell}$ real~~

It is customary to write

$$\boxed{p f_{\ell} = \frac{e^{2i\delta_{\ell}} - 1}{2i} = e^{i\delta_{\ell}} \sin\delta_{\ell}}$$

$\delta_{\ell}$ is the "phase shift", and we will see in a moment that it is required to be real (in elastic scattering)

$\rho = \psi^* \psi \qquad \dot{\rho} + \vec{\nabla}\cdot j = 0$

$\vec{j} = \frac{1}{2i}(\psi^* \vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi) \Rightarrow$ In a stationary state

$\vec{\nabla}\cdot\vec{j} = \frac{1}{2i}[\psi^* \Delta \psi - \psi \Delta \psi^*] = 0$

But this does not constrain modulus of outgoing wave

6.6

From the asymptotic behavior

$$\hat{j}_\ell(pr) \to \frac{1}{2i}\left[e^{i(pr - \frac{1}{2}\ell\pi)} - e^{-i(pr - \frac{1}{2}\ell\pi)}\right]$$

$$\hat{h}^+_\ell(pr) \to e^{i(pr - \frac{1}{2}\ell\pi)},$$

we have

$$\psi_{\ell,p}(r) \xrightarrow[r\to\infty]{} \frac{1}{2i}\left[(1 + 2ipf_\ell)\, e^{i(pr - \frac{1}{2}\ell\pi)} - e^{-i(pr - \frac{1}{2}\ell\pi)}\right].$$

This has the same incoming waveform as $e^{ikz}$, but outgoing wave has been perturbed by potential. However, "conservation of probability" requires coef. of ingoing, outgoing waves to have same modulus

$$1 + 2ipf_\ell = e^{2i\delta_\ell} \Rightarrow f_\ell = \frac{e^{2i\delta_\ell} - 1}{2ip} = e^{i\delta_\ell}\sin\delta_\ell$$

(conservation of probability is equivalent to unitarity, and in fact, it is shown in Taylor, Chapter 6, that $e^{2i\delta_\ell}$ is just eigenvalue of S-Matrix in $\ell$th partial wave

$$\langle E'\ell' m' | S | E \ell m\rangle = \delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{m'm}\, e^{2i\delta_\ell}$$

The phase shift can be interpreted as a time delay. An attractive potential speeds up particle $\Rightarrow \delta_\ell > 0$. Repulsive potential slows down particle $\Rightarrow \delta_\ell < 0$. Phase delay $\delta_\ell \to 0$ for <u>fast particle</u>, $E \to \infty$

## <u>Total Cross Section</u>

$$f(p,\theta) = \sum_\ell \frac{(2\ell+1)}{p} e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)$$

$$\sigma_T = \int d\Omega |f|^2 = \frac{2\pi}{p^2}\int_{-1}^{1} d\cos\theta \sum_{\ell\ell'} \sin\delta_\ell \sin\delta_{\ell'} (2\ell+1)(2\ell'+1) P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$\boxed{\sigma_T(p) = \frac{4\pi}{p^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell(p)} \qquad \text{since } \int d\cos\theta\, P_\ell P_{\ell'} = \frac{2}{2\ell+1}$$

Thus we have a <u>unitarity</u> <u>bound</u> on total cross section from each partial wave. We can understand the bound semiclassically - Total cross section for particles of impact parameter $b$ or less is

$$\sigma_T(b) \leq \pi b^2 = \pi\left(\frac{l}{p}\right)^2$$

$$\text{and } \sigma_T(b_{l+1}) - \sigma_T(b_l) \leq \frac{\pi}{p^2}(2l+1)$$

This differs from the quantum mechanical result by a factor of four, because of wave effects (Diffraction). Classically $\sigma_T$ is maximized if whole wave is absorbed, but in QM, it is 4 times better if phase is changed by $\pi$ ($\delta_l = \pi/2$)

By similar semiclassical reasoning, we expect that for a potential with range $R$, there will be little scattering if $pR \ll l$
(particle doesn't penetrate centrifugal barrier and feel potential). Thus, we have, complementing the Born approximation, an approximation scheme which works at low energy (truncated partial wave expansion)

For $pR \ll 1$, only first partial wave is important.
$l = 0 \Rightarrow$ isotropic scattering

## C. The Partial-Wave Amplitude

We can solve the radial equation iteratively if we convert it to an integral equation, i.e., resolve the Lippmann-Schwinger equation into its partial waves:

$$|E,l,m+\rangle = |E,l,m\rangle + G^0(E+i\epsilon)V|E,l,m+\rangle$$

The easiest thing to do is work directly with the one dimensional equation

$$\left[\frac{d^2}{dr^2} + p^2 - \frac{l(l+1)}{r^2}\right](rR) = 2V(rR),$$

and solve it by conventional Green's Function methods.

The usual approach is, given the eqn

$$(H-E)\psi = \varrho$$

and appropriate B.C., choose a complete set of Eigenfunctions satisfying the B.C. and solve by eigenfunction expansion

$$\text{"}(H-E)^{-1}\text{"} = \sum \frac{|n\rangle\langle n|}{E_n - E} \Rightarrow \psi = \psi_0 + \sum \frac{|n\rangle\langle n|\varrho\rangle}{E_n - E}$$

(where $\psi_0$ satisfies the homogeneous eqn and the B.C.)

But it is often simpler to find directly the solution to $(H-E)\,G(r,r') = \delta(r-r')$ which satisfies the B.C. In the present case our B.C. are

$$\psi_{l,p} \to 0 \quad \text{as } r \to 0$$

$$\psi_{l,p} \to \hat{j}_l(pr) + c\,\hat{h}^+(pr) \quad \text{as } r \to \infty$$

Hence

$$\boxed{\psi_{l,p} = \hat{j}_l(pr) + \int_0^\infty G^0_{lp}(r,r')\big(2V(r')\big)\,\psi_{l,p}(r')}$$

where

$$\left[\frac{d^2}{dr^2} + p^2 - \frac{l(l+1)}{r^2}\right] G_{el}(r,r') = \delta(r-r')$$

and

$$G = \begin{cases} \hat{j}_l(pr)\,A(r') & r<r' \\ \hat{h}^+_l(pr)\,B(r') & r>r' \end{cases}$$

solves homog. eqn and gives right B.C. for $r \neq r'$

We must also satisfy jump condition

$$\left.\frac{dG}{dr}\right|_{r=r'+\epsilon} - \left.\frac{dG}{dr}\right|_{r=r'-\epsilon} = 1$$

or $\quad -p\,\hat{j}_\ell'(pr')A(r') + p\,\hat{h}_\ell^{+\prime}(pr')B(r') = 1$

If $G$ has an eigenfunction expansion as described above we should have $G(r,r') = G(r';r)^*$, and this becomes

$$-pA\left(\hat{j}_\ell'(pr')\hat{h}_\ell^+(pr') - \hat{h}_\ell^{+\prime}(pr')\hat{j}_\ell(pr')\right) = 1$$

To establish this recall the Wronskian

If $\quad (d^2/dr^2 + V(r))\,u_1 = 0 \quad$ or $\quad Du_1 = 0$

$\quad\;\; (d^2/dr^2 + V(r))\,u_2 = 0 \qquad\quad Du_2 = 0$

then $\quad u_2 Du_1 - u_1 Du_2 = 0 = \frac{d}{dr}\left(u_2\frac{d}{dr}u_1 - u_1\frac{d}{dr}u_2\right)$

i.e. $\quad W(u_2,u_1) = u_2u_1' - u_1u_2' = \underline{\text{constant}}$ the Wronskian

(If $u_1, u_2$ are bound, then constant $= 0$ at $r=\infty$, and

$u_1'/u_1 = u_2'/u_2 \Rightarrow \quad u_1 = Cu_2$ - linear dependence

there is no bound state degeneracy in one dimension)

But there are usual two independent unbound solutions - left moving and right moving.

To evaluate $W(\hat{j}_\ell, \hat{h}_\ell^+)$ consider $r \to \infty$

$$\sin(z-\tfrac{1}{2}\ell\pi)\, i e^{i(z-\frac{1}{2}\ell\pi)} - \cos(z-\tfrac{1}{2}\ell\pi)\, e^{i(z-\frac{1}{2}\ell\pi)}$$

$$= -\sin^2 - \cos^2 = -1$$

Hence $A = \dfrac{-1}{p}$

$$\boxed{\overset{\circ}{G}_{\ell p}(r,r') = -\frac{1}{p}\,\hat{j}_\ell(pr_<)\,\hat{h}_\ell^+(pr_>)}$$

By iteration of the equation $\psi_{\ell,p} = \hat{j}_\ell + G^0_{\ell,p}(2V)\psi_{\ell,p}$ we obtain the partial-wave born expansion

$$\psi_{\ell,p} = \hat{j}_\ell + G^0_{\ell,p}(2V)\hat{j}_\ell + G^0_{\ell,p}(2V)\,G^0_{\ell,p}(2V)\,\hat{j}_\ell + \cdots$$

and this can be be transcribed into an expansion for the partial-wave amplitude $f_\ell$:

We recall

$$-\frac{1}{(2\pi)^2} f(\vec{p}\,';\vec{p}) = \langle \vec{p}\,'|T|\vec{p}\rangle_{\text{on shell}} = \langle \vec{p}\,'|V+VGV|\vec{p}\rangle = \langle \vec{p}\,'|V|\vec{p}+\rangle$$

and that

$$\langle \vec{x}|\vec{p}+\rangle = \frac{1}{(2\pi)^{3/2}} \sum_\ell (i)^\ell (2\ell+1) P_\ell(\cos\theta)\,\frac{\psi_{\ell,p}(r)}{pr} \qquad \text{if } \vec{p} = p\hat{z}$$

i.e. the argument of $P_\ell$ is $\hat{x}\cdot\hat{p}$

$$\langle \vec{x}|\vec{p}\,'\rangle = \frac{1}{(2\pi)^{3/2}} \sum_\ell (i)^\ell (2\ell+1) P_\ell(\hat{x}\cdot\hat{p}\,')\,\frac{\hat{j}_\ell(p'r)}{pr}$$

To evaluate $\langle \vec{p}\,'|V|\vec{p}+\rangle$, we must calculate

$$\int d\Omega_x\; P_\ell(\hat{x}\cdot\hat{p})\,P_{\ell'}(\hat{x}\cdot\hat{p}\,')$$

to do this we may apply the formula

$$P_\ell(\hat{x}\cdot\hat{p}) = \frac{4\pi}{2\ell+1}\sum_m Y_{\ell m}(\hat{x})\,Y^*_{\ell m}(\hat{p}) \qquad \text{Addition theorem}$$

Hence $\int d\Omega_x\, P_\ell(\hat{x}\cdot\hat{p})\,P_{\ell'}(\hat{x}\cdot\hat{p}\,') = \left[\frac{(4\pi)}{2\ell+1}\right]^2 Y^*_{\ell m}(\hat{p})\,Y_{\ell m}(\hat{p}\,')\,\delta_{\ell\ell'}$

$$= \left(\frac{4\pi}{2\ell+1}\right) P_\ell(\hat{p}\cdot\hat{p}\,')$$

$$\langle \vec{p}\,'|V|\vec{p}+\rangle = \frac{1}{(2\pi)^3}\frac{(4\pi)}{(2\ell+1)} P_\ell(\hat{p}\cdot\hat{p}\,')\,\frac{1}{p^2}\int dr\;\hat{j}_\ell(pr)\,\psi_{\ell,p}(r)\,2V(r)$$

$$= -\frac{1}{(2\pi)^2}\sum_\ell (2\ell+1) P_\ell(\hat{p}\cdot\hat{p}\,')\,f_\ell(p)$$

Thus

$$\boxed{f_\ell(p) = -\frac{1}{p^2}\int_0^\infty dr\, \hat{j}_\ell(pr)(2V)\,\psi_{\ell p}(r)}$$

which has the expansion

$$\boxed{f_\ell(p) = -\frac{1}{p^2}\left[\int \hat{j}_\ell\, 2V \hat{j}_\ell + \int \hat{j}_\ell (2V) \hat{j}_\ell\, G_{\ell,p}(2V)\hat{j}_\ell + \cdots\right]}$$

Partial Wave Born Series

The 1st Born approximation can be good only if $\delta_\ell$ is small, for the Born approx is real, and

$f_\ell = \frac{1}{p} e^{i\delta_\ell}\sin\delta_\ell$ is nearly real only for $\delta_\ell$ small

We expect the approximation to be good if

multi amb. can be removed $\left\{\begin{array}{l} \lambda\to 0 \\ p\to\infty \\ \ell\to\infty \end{array}\right.$ in all these cases, $V$ in radial equation becomes negligible

Threshold behavior

How do partial wave amplitudes behave as $p\to 0$?

We have

$$\psi_{\ell,p}(r) = \hat{j}_\ell(pr) + \int_0^\infty dr'\, G^0_{\ell,p}(r,r')(2V(r'))\,\psi_{\ell,p}(r')$$

Since $G^0_{\ell p} = -\frac{1}{p}\hat{j}_\ell(pr_<)\hat{h}^+_\ell(pr_>) \underset{pr_>\to 0}{\sim} -\frac{1}{p}(pr_<)^{\ell+1}(pr_>)^{-\ell}$

$= r_<^{\ell+1} r_>^{-\ell}$ independent of $p$

As long as $V(r)\to 0$ as $r\to\infty$ faster than a power of $\frac{1}{r}$, we expect $\psi_{\ell,p}\sim p^{\ell+1}$; same as $\hat{j}_\ell$

$$f_\ell(p) = -\frac{1}{p^2}\int_0^\infty dr\,\hat{j}_\ell\, 2V\psi_{\ell,p}$$

$$\boxed{f_\ell(p) \xrightarrow[p\to 0]{} -a_\ell p^{2\ell}}$$

(We will be able to do a more precise analysis of the threshold behavior when we know more about analytic structure of $f_\ell$.)

As $p \to 0$, $f_0$ dominates and $\sigma_T \to 4\pi a_0^2$

scattering length

For a finite range potential, at least, we have

$$f_\ell = \frac{1}{p} e^{i\delta_\ell} \sin\delta_\ell \to -a_\ell p^{2\ell} \quad \text{as } p \to 0$$

Hence $\delta_\ell \to n\pi - a_\ell p^{2\ell+1}$

We also expect $f_\ell \to 0$ as $p \to \infty$
(because potential becomes unimportant in radial eqn)

We may define $\delta_\ell(p=\infty) = 0$, so that the "n" in

$$\boxed{\delta_\ell \xrightarrow[p\to 0]{} n\pi}$$

has significance. Eventually, we will learn that $n$ is simply the no. of bound states of angular momentum $\ell$

## D. The Regular Solution

The B.C. we imposed on the solution $\psi_{\ell,p}$ to the radial eqn was

$$\psi_{\ell,p}(r=0) = 0$$
$$\psi_{\ell,p} \xrightarrow[r\to\infty]{} \hat{j}_\ell(pr) + p f_\ell(p) \hat{h}_\ell^+(pr)$$

Such a B.C. is rather awkward, involving behavior at both $r=0$ and $r=\infty$

It is more convenient to specify

$$\phi_{\ell,p} \xrightarrow[r\to 0]{} 0$$
$$\phi_{\ell p}' \xrightarrow[r\to 0]{} p$$

for $\ell=0$

in general:

$$\phi_{\ell,p} \xrightarrow[r\to 0]{} \hat{j}_\ell(pr) \sim \frac{2^\ell \ell!}{(2\ell+1)!}(pr)^{\ell+1}$$

(same leading behavior)

(why is it necessary to have $\phi_{\ell,p}(0)=0$ ?
For $\ell\neq 0$, it is required for square integrability, because $\phi \sim \begin{cases} r^{\ell+1} \\ r^{-\ell} \end{cases}$ and $R \sim \frac{\phi}{r}$

But for $\ell=0$ it is because the irregular solution solves an inhomogeneous Schrodinger equation e.g. $-(\Delta+p^2)\psi = 4\pi\delta(\vec{r})$ )

What is the advantage of this B.C. We will find that it guarantees that $\phi_{\ell,p}(r)$ is an entire function of $\lambda$ — In fact it is the Fredholm solution (i.e. the numerator of the Fredholm soln) in disguise.

Because B.C. and radial eqn are real, solution will be real $\Rightarrow$ Green's function is real

Take
$$\boxed{g_{\ell p}(r,r') = \begin{cases} \frac{1}{p}\left(\hat{j}_\ell(pr)\,\hat{n}_\ell(pr') - \hat{n}_\ell(pr)\,\hat{j}_\ell(pr')\right) & r>r' \\ 0 & r<r' \end{cases}}$$

then $\phi_{\ell,p}$ satisfying

$$\boxed{\phi_{\ell,p}(r) = \hat{j}_\ell(pr) + \lambda\int_0^r dr'\, g_{\ell p}(r,r')\, 2V(r')\, \phi_{\ell p}(r')}$$

solves the radial eqn and satisfies our B.C.

(see below)
Actually, easy to verify that $\phi \sim r^{\ell+1} \Rightarrow \int g V \phi \sim r^{\ell+3-p}$ if $V \sim \frac{1}{r^p}$

An integral eqn of the form
$$\phi(r) = \phi^0(r) + \lambda\int_0^r dr'\, K(r,r')\,\phi(r')$$
is called a Volterra Eqn.
The iterative solution converges for any value of $\lambda$
To see this, note that this is a Fredholm eqn with
$$K(r,r') = 0 \quad \text{for } r \leq r',$$
and this property guarantees that (It is a "triangular matrix.")
$$\det(\mathbb{1} - \lambda K) = 1$$

e.g. $\quad \mathrm{tr}\, K = \int dx\, K(x,x) = 0$

$\quad\quad \mathrm{tr}\, K^2 = \int dx\, dx'\, K(x,x') K(x',x) = 0$

$\quad\quad$ etc.

In fact, we will explicitly show that the series converges, and in the process deduce bounds on $\phi_{\ell p}$ which we will use later

In deriving these bounds, we will allow $p$ in the radial eqn

$$\boxed{\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \lambda (2V) + p^2\right] \phi_{\ell p}(r) = 0}$$

to be a complex number, and consider the analytic behavior of $\phi_{\ell p}(r)$ as a function of $\lambda$ and $p$. (Eventually, we will allow $\ell$ to wander into the complex plane also)

- Consider first the case $\ell = 0$.

We have $\quad g_{0p}(r,r') = \frac{1}{p} \sin pr \cos pr' - \cos pr \sin pr' = \frac{1}{p} \sin p(r-r')$

we first note that $\sin z = \frac{1}{2i}\left(e^{-b} e^{ia} - e^{b} e^{-ia}\right) \quad z = a+ib,\ b>0$

clearly

$$|\sin z| \le e^{|\mathrm{Im}\, z|}$$

$$|\sin z| \le |z|\, e^{|\mathrm{Im}\, z|}$$

By continuity, there must be a bound of the form

$$|\sin z| \le \beta \frac{|z|}{1+|z|} e^{|\mathrm{Im}\, z|}$$

$= \cancel{\sin a \cosh b + i \cos a \sinh b}$

$\cancel{|\sin z|^2 = \sin^2 a \cosh^2 b + \cos^2 a \sinh^2 b}$

$\le \cancel{\cosh)}$

Now consider the series $\phi = \sum_n \lambda^n \phi^{(n)}$

We know that

$$\phi_p^{(n)}(r) = \int_0^r dr_n \frac{1}{p}\sin p(r-r_n)\, 2V(r')\, \phi_{0p}^{(n-1)}(r')$$

$$|\phi_{op}^{(n)}(r)| \le \beta\int_0^r dr_n \frac{|r-r_n'|}{1+|p(r-r)|} e^{|\mathrm{Im}\, p|(r-r_n')} |2V(r_n')|\, |\phi_{op}^{n-1}(r_n)|$$

$$\le \beta \frac{r}{1+|pr|} e^{|\mathrm{Im}\, p| r} \int dr_n |2V(r_n)|\, e^{-|\mathrm{Im}\, p| r_n} |\phi_{op}^{n-1}(r_n)|$$

Now, apply this bound interatively:

$$\le \beta e^{|\mathrm{Im}\, p| r} \frac{r}{1+|pr|}\int_0^r dr_n |2V(r_n)| \frac{\beta r_n}{1+|pr_n|}\int_0^{r_n} dr_{n-1} |2V(r_{n-1})| \frac{\beta r_{n-1}}{1+|pr_{n-1}|}\cdots$$

This integral over the triangle is $\frac{1}{n!}$ times integral over a rectangle:

$$|\phi_{op}^{(n)}(r)| \le \frac{1}{n!}\beta^{n+1} e^{|\mathrm{Im}\, p| r} \frac{r}{1+|pr|}\left[\int_0^r dr' |2V| \frac{r'}{1+|pr'|} dr'\right]^n$$

$$\Rightarrow \boxed{|\phi_{op}(r)| \le \sum |\lambda|^n |\phi^{(n)}| \le \beta e^{|\mathrm{Im}\, p| r}\frac{r}{1+|pr|} \exp\left[|\lambda|\beta\int_0^r dr' \frac{|2V| r'}{1+|pr'|}\right]}$$

Regular at origin

This series converges absolutely for all $r, \lambda, p$ if

$$\left(\int_0^\infty dr' r' |V(r')| < \infty\right)$$

Condition at $r \to 0$ is same as for regular B.C. to be satisfied

In fact, because $\phi_{op}(r)$ is the uniform limit of analytic functions, we know that

i) $\phi_{op}$ is entire function of $\lambda$, for $r, p$ fixed

ii) $\phi_{op}$ is entire function of $p$ for $r, \lambda$ fixed

This is require for $\phi$ to be well behaved as $r \to \infty$

But in fact, if we consider only $|p| \neq 0$,

$$\int_a^\infty dr' \, |V(r')| < \infty \text{ suffices}$$

(<u>Almost</u> includes coulomb potential)

Thus, we have found the unique analytic continuation of $\varphi_{l,p}(r)$ into the complex $p$ plane. Of course the *physical region* of interest in scattering theory is $p$ real and positive.

- What about $l \neq 0$

There are similar bounds for all the Riccati-Bessel functions

$$\hat{h}_l^{\pm}(z) \le \alpha_l \left(\frac{|z|}{1+|z|}\right)^{-l} e^{\mp \operatorname{Im} z} \leftarrow \text{no absolute value}$$

$$\hat{j}_l(z) \le \beta_l \left(\frac{|z|}{1+|z|}\right)^{l+1} e^{|\operatorname{Im} z|}$$

$$\hat{n}_l(z) \le \gamma_l \left(\frac{|z|}{1+|z|}\right)^{-l} e^{|\operatorname{Im} z|}$$

Hence $$|g_{l,p}(r,r')| \le \Delta_l \left(\frac{|pr|}{1+|pr|}\right)^{l+1} \left(\frac{|pr'|}{1+|pr'|}\right)^{-l} e^{|\operatorname{Im} p|(r-r')}$$

Thus, we can obtain a bound

$$\boxed{|\varphi_{lp}(r)| \le \Delta_l \left(\frac{|pr|}{1+|pr|}\right)^{l+1} e^{|\operatorname{Im} p| r} \exp\left[|\lambda| \Delta_l \int_0^r dr' \frac{|2V| r'}{1+|pr'|}\right]}$$

for the general $l$ case

Note that the solution satisfies our B.C. ?

E. <u>The Jost Function</u>

Because $\varphi_{l,p}(r)$ is real, its asymptotic behavior must be

$$\varphi_{l,p}(r) \xrightarrow[r\to\infty]{} \frac{1}{2i}\left(f_l^*(p) e^{i(pr - \frac{1}{2}l\pi)} - f_l(p) e^{-i(pr - \frac{1}{2}l\pi)}\right)$$

If $f_l(p)$ is nonzero, we have

$$\boxed{\frac{\varphi_{l,p}(r)}{f_l(p)} \longrightarrow \frac{1}{2i}\left[\frac{f_l^*(p)}{f_l(p)} e^{i(pr - \frac{1}{2}l\pi)} - e^{-i(pr - \frac{1}{2}l\pi)}\right]}$$

But this obeys the $r \to \infty$ B.C. satisfied by $\psi_{\ell,p}$.
Therefore, we identify

$$\boxed{\psi_{\ell,p}(r) = \frac{\phi_{\ell,p}(r)}{f_\ell(p)}} \qquad \boxed{S_\ell(p) = e^{2i\delta_\ell(p)} = \frac{f_\ell(p)^*}{f_\ell(p)}}$$

$f_\ell(p)$ is the Jost Function
(Let's try not to confuse it with the partial wave scattering amplitude $f_\ell(p)$)

(It is actually the Fredholm determinant of the radial Eqn, and the expression $\psi = \frac{\phi}{f}$ is a ratio of two entire functions of $\lambda$.)

We can express $f_\ell$ in terms of $\phi_{\ell,p}$, because we know

$$\phi_{\ell p} = \hat{j}_\ell(pr) + \lambda \int_0^r dr' \, g_{\ell p}(r,r') \, 2V(r') \, \phi_{\ell p}(r')$$

and

$$g_{\ell p}(r,r') = \frac{-1}{2ip}\left[\hat{h}_\ell^-(pr)\,\hat{h}_\ell^+(pr') - \hat{h}_\ell^+(pr)\,\hat{h}_\ell^-(pr')\right]$$

Since $f_\ell(p)$ is coefficient of $-\frac{1}{2i} e^{-i(pr - \frac{1}{2}\ell\pi)}$ as $r \to \infty$ in $\phi$, we have

$$\boxed{f_\ell(p) = 1 + \frac{\lambda}{p}\int_0^\infty dr' \, \hat{h}_\ell^+(pr') \, 2V(r') \, \phi_{\ell p}(r')}$$

$f = 1 \Rightarrow$ No scattering

This expression will enable us to continue $f$ into the complex $p$ plane.

First let's consider the behavior of $f_\ell$ in the limits $\lambda \to 0$, $p \to \infty$

Recalling the bounds $\left|\hat{h}_\ell^+(pr)\right| \le \alpha_\ell \left(\frac{|pr|}{1+|pr|}\right)^{-\ell} e^{-(\operatorname{Im} p) r}$

$$|\phi_{\ell p}| \le C_\ell e^{|\lambda| c} \left(\frac{|pr|}{1+|pr|}\right)^{\ell+1} e^{|\operatorname{Im} p| r}$$

$(C \to \frac{1}{|p|}$ as $p \to \infty)$

we find $|f_\ell(p) - 1| \leq \frac{\text{const}}{|p|} \, \int dr' \, |2V| \frac{|pr'|}{1+|pr'|} e^{(|\text{Im}\,p| - \text{Im}\,p)r}$

$|\lambda| e^{|\lambda| c}$

we see first of all, that for $\int dr' |2V| \frac{|pr'|}{1+|pr'|} < \infty$ ] Bounded uniformly in $|p|$

$f_\ell(p)$ can be continued into the half plane $\boxed{\text{Im}\, p \geq 0}$, i.e., the integral converges. (the physical region is at the boundary of the domain of analyticity)

$f_\ell(p)$ is also an entire function of $\lambda$ (It can be integrated term by term, giving a uniformly convergent series of analytic functions)

Now

i) It is obvious that $|f_\ell(p)| \sim |\lambda|$ for $\lambda$ small

ii) If $\int dr |V| < \infty$, we have $|f_\ell| \sim \frac{1}{|p|}$ for $p \to \infty$

In either limit, there is no scattering in each partial wave channel, but approach to the limit may occur at different rates for different values of $\ell$.

## Partial Wave Born Series

The zeros of $f_\ell(p) = \det(\mathbb{1} - \lambda K(p))$ determine radius of convergence of partial wave Born series. Since $f_\ell \to 1$ as $|p| \to \infty$, convergence always occurs at sufficiently high energy

This is the partial wave analog of our earlier finding, that the Born series converges at high energy.

## F. Analytic Continuation of Jost Function and S-Matrix

We have already found that $f_\ell(p)$ can be analytically continued into $\text{Im}\, p > 0$. What are its properties as an analytic function of $p$?

First, note that, because the radial Schrodinger eqn is invariant under $p \to -p$, $\varphi_{\ell,-p}$ is a solution if $\varphi_{\ell,p}$ is. In fact

$$\left.\begin{aligned} \hat{j}_\ell(-pr) &= (-1)^{\ell+1}\hat{j}_\ell(pr) \\ \hat{n}_\ell(-pr) &= (-1)^{\ell}\hat{n}_\ell(pr) \end{aligned}\right\} \quad \begin{aligned} \hat{h}^+_\ell(-pr) &= (-1)^\ell \hat{h}^-_\ell(pr) \\ &= (-1)^\ell \hat{h}^{+*}_\ell(pr) \end{aligned}$$

$$g_{\ell,-p}(r,r') = g_{\ell p}(r,r')$$

and therefore, from the integral equation

$$\varphi_{\ell p}(r) = \hat{j}_\ell(pr) + \int^r dr'\, g_{\ell p}(r,r')\, 2V(r')\, \varphi_{\ell,p}(r'),$$

we have

$$\boxed{\varphi_{\ell,-p}(r) = (-1)^{\ell+1}\varphi_{\ell p}(r)}$$

and the integral representation

$$f_\ell(p) = 1 + \frac{1}{p}\int_0^\infty dr'\, \hat{h}^+_\ell(pr')\, 2V(r')\, \varphi_{\ell p}(r')$$

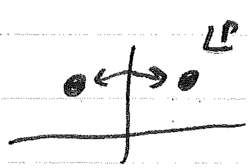
implies that, <u>if $p$ is real</u>

$$\boxed{f_\ell(-p) = f_\ell(p)^*}$$

Thus $f_\ell$ obeys a sort of reflection principle (but about the imaginary axis), and the analytic continuation obeys

$$\boxed{f_\ell(p) = f_\ell(-p^*)^*}$$

In particular $f_\ell$ is real when $p$ is pure imaginary



What is the analytic continuation of $S$

For $p$ real we have $S_\ell = \dfrac{f_\ell(p)^*}{f_\ell(p)}$

and its analytic continuation is $\boxed{S_\ell = \dfrac{f_\ell(-p)}{f_\ell(p)}}$

But we have not yet shown that $f_\ell$ can be continued into the lower half plane, and therefore, we have not yet shown that $S_\ell$ can be continued off the real axis. We will return to this problem shortly.

But let us first assume that $f_\ell(p)$ can be continued for $\operatorname{Im} p < 0$, and ask what the resulting analytic structure of $S$ would be.

Does $S_\ell$ have poles in $\operatorname{Im} p > 0$? $S_\ell$ has a pole if $f_\ell(p) = 0$, $f_\ell(-p) \neq 0$

Thus

$$\varphi_{\ell,p} \xrightarrow[r\to\infty]{} \frac{1}{2i}\left[ f_\ell(-p)\, e^{i(pr - \frac{1}{2}\ell\pi)} - f_\ell(p)\, e^{-i(pr - \frac{1}{2}\ell\pi)} \right]$$

is exponentially damped, $\varphi_{\ell,p} \sim e^{-(\operatorname{Im} p) r}$

But then $\varphi_{\ell,p}$ is a bound solution to the radial eqn

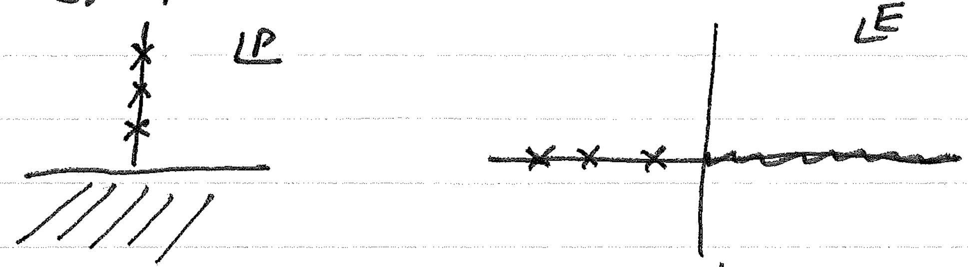
that is, $\underline{E = \frac{1}{2}p^2 \text{ is a bound state energy}}$

Thus, $p$ must be pure imaginary ($H$ is hermitian).

(Moreover, each zero of $f_\ell$ is a pole of $S$, for if $f_\ell(-p) = 0$ we would have $\phi \equiv 0$)

This is a remarkable connection between bound states and scattering behavior, established by analytic continuation. It is one of the major results of scattering theory.

Im $p > 0$ defines the "physical sheet" of the energy plane




The only singularities in the physical sheet of the cut energy plane are poles along the negative real axis.

Now let us consider the conditions on the potential $V$ which must be satisfied for $S_\ell$ to have an analytic continuation off the real axis.

First, why is this a problem at all, if we know that $G = (E-H)^{-1}$ is analytic, by the spectral theory. The problem is S matrix elements depend on $E$ through $(E-H)^{-1}$ and the states $\langle \hat{p}'|$, $|\hat{p}\rangle$. The plane wave functions tend to grow exponentially for complex energies. (In jargon, the difficultly is that we must "go on shell" to define S-matrix elements.)

We use

$$f_\ell(p) = 1 + \frac{1}{p}\int_0^\infty dr'\, \hat{h}_\ell^+(pr')\, 2V(r')\, \varphi_{\ell p}(r')$$

to continue $f_\ell$, recalling that

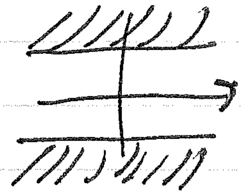
$$\left|\hat{h}_\ell^+(pr')\, \varphi_{\ell p}(r')\right| \le C e^{a|\lambda|} \frac{|pr'|}{1+|pr'|} e^{(|\mathrm{Im}\,p| - \mathrm{Im}\,p)r}$$

i) If $V(r)$ has a finite range ($V=0$ for $r>a$) and $\int_0 r'|V(r')| < \infty$, then $f_\ell$ is an entire function of $p$, and $S_\ell$ is meromorphic

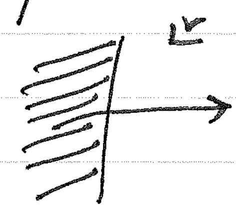
ii) Suppose $V(r)$ is exponentially bounded

$$\int^{\infty} dr\, |V| e^{2ar} < \infty, \quad a > 0$$

Then $f_\ell(p)$ is analytic for $\mathrm{Im}\, p > -a$
and $S_\ell$ is analytic in the strip $-a < \mathrm{Im}\, p < a$

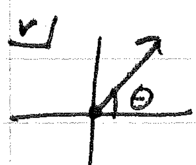


iii) we say potential is an analytic potential, if
$V(r)$ is analytic in $\mathrm{Re}\, r > 0$ and



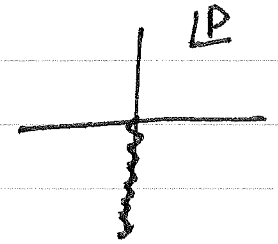

e.g. Yukawa potential $\left\{ \int_0^{e^{i\theta}\times\infty} dr'\, |2V| \frac{|pr'|}{1+|pr'|} < \infty, \right.$ where integration is along the ray $r' = e^{i\theta}\rho$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

This means it is possible
to continue $\varphi_{\ell p}(r)$ into $\mathrm{Re}\, r > 0$, and it still
satisfies our bound $|\varphi_{\ell p}| \le \mathrm{const} \left(\frac{|pr|}{1+|pr|}\right)^{\ell+1} e^{|\mathrm{Im}\, pr|}$

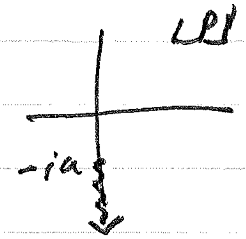



This mean we can distort
contour in $f_\ell(p) = 1 + \frac{1}{p}\int dr'\, \hat{h}_\ell^{+}(pr')\, 2V(r')\, \varphi_{\ell p}(r')$
to $\mathrm{Arg}\, r = \theta$

Now $\int dr\, e^{|\mathrm{Im}\, pr| - \mathrm{Im}\, pr}$
converges for $\boxed{\mathrm{Im}\, p e^{i\theta} > 0}$




This means $f_\ell(p)$ can be
continued into whole $p$ plane
except for cut on negative imag axis




Finally, if $V$ is also
exponentially bounded, the cut
begins at $p = -ia$

We have worked hard to show that $f_\ell(p)$ can be continued into lower half plane. But we should note that all singularities for $\mathrm{Im}\, p < 0$ must disappear if we truncate the potential — these singularities are determined solely by the $r \to \infty$ asymptotic behavior of the potential. Therefore, these singularities clearly can be of little relevance physically.

## G. Levinson's Theorem

Our next task will be to examine the properties of the zeros of $f_\ell(p)$ for $\mathrm{Im}\, p \geq 0$. In particular, we wish to establish

1) zeros of $f_\ell(p)$ for $\mathrm{Im}\, p > 0$ are simple zeros.
2) zeros at $p = 0$ are simple if $\ell = 0$, but double if $\ell > 0$.

To show this, we first introduce an irregular solution, defined by the B.C.

$$\boxed{\chi^{\pm}_{\ell p}(r) \xrightarrow[r\to\infty]{} \hat{h}^{\pm}_{\ell}(pr)}$$

These are the solutions to the radial eqn which are asymptotically outgoing, incoming waves

Then

$$\phi_{\ell p} \equiv \frac{1}{2i}\left[f_\ell(-p)\,\chi^{+}_{\ell p}(r) - f_\ell(p)\,\chi^{-}_{\ell p}\right]$$

The Wronskian is

$$W(\chi^{+}_{\ell p}, \chi^{-}_{\ell p}) = \chi^{+}_{\ell p}\,\chi^{-\prime}_{\ell p} - \chi^{+\prime}_{\ell p}\,\chi^{-}_{\ell p} = -2ip$$

and hence

$$\boxed{f_\ell(p) = \frac{1}{p}\, W(\chi^{+}_{\ell p}, \phi_{\ell p})}$$

Suppose $f_\ell(p_0) = 0 \implies \phi_{\ell p} = \lambda \chi^+_{\ell p}$

$$\frac{d}{dp} f_\ell(p_0) = \frac{1}{p_0}\left[W(\dot{\chi}^+_{\ell p}, \varphi_{\ell p}) + W(\chi^+_{\ell p}, \dot{\varphi}_{\ell p})\right]\Big|_{p=p_0}$$

To evaluate this:

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2V + p^2\right)\varphi_{\ell p} = 0$$

$$\left( \qquad\qquad + p'^2\right)\chi^+_{\ell p'} = 0$$

$$\chi^+_{\ell p'}\varphi_{\ell p}'' - \varphi_{\ell p}\chi^{+\prime\prime}_{\ell p'} = (p'^2 - p^2)\chi_{\ell p'}\varphi_{\ell p}$$

$$= \frac{d}{dr} W(\chi^+_{\ell p'}, \varphi_{\ell p})$$

$\cancel{W(\chi^+_{\ell p'}, \varphi_{\ell p}) = (p'^2-p^2)\int_0^r \chi^+_{\ell p'}\varphi_{\ell p}}$

$\cancel{W(\dot{\chi}^+, \varphi) = 2p_0\int_0^r \chi^+_{\ell p_0}\varphi_{\ell p_0}}$ ~~set $p' = p_0$ ... regular at origin, differentiate~~

$$W(\chi^+_{\ell p_0}, \varphi_{\ell p}) = (p_0^2 - p^2)\int_0^r \chi^+_{\ell p_0}\varphi_{\ell p}$$

(because $\chi_{\ell p_0}$ is regular at origin)

$$W(\chi^+_{\ell p_0}, \dot{\varphi}_{\ell p_0}) = -2p_0\int_0^r \chi^+_{\ell p_0}\varphi_{\ell p_0}$$

$$W(\chi_{\ell p}, \varphi_{\ell p_0}) = (p_0^2 - p^2)\int_r^\infty \chi^+_{\ell p}\varphi_{\ell p_0}$$

(because $\varphi_{\ell p_0}$ is exponenentially decreasing.)

$$W(\dot{\chi}^+_{\ell p_0}, \varphi_{\ell p_0}) = -2p_0\int_r^\infty \chi^+_{\ell p_0}\varphi_{\ell p_0}$$

Finally
$$\boxed{\frac{d}{dp} f_\ell(p_0) = -2\int_0^\infty \lambda\left(\chi^+_{\ell p}\right)^2 \neq 0}$$

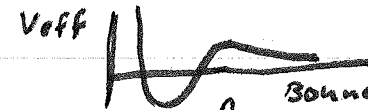
because $\chi_{\ell p}^+ \sim (-i)^\ell \times$(Real Solution)

<u>The zero is simple</u>

On the real axis, zeros may occur only at $p=0$.
(Because $f_\ell(p) = f_\ell(-p)^* = 0$ requires $\varphi_{\ell p} \equiv 0$ –
For $p=0$, the solutions to the radial eqn for $r \to 0, \infty$ are $r^{\ell+1}, r^{-\ell}$.) The regular solution is identically zero for $p=0$ since B.C. is $\varphi \to \hat{j}(pr) = 0$.

There may be a solution which behaves like $r^{\ell+1}$ at origin, $r^{-\ell}$ at $r \to \infty$ $\left[\text{solve} \left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right)\varphi_\ell = 0\right]$

For $\ell > 0$, this is a <u>bound state</u>.




Bound by coulomb barrier

Do such zero energy bound states correspond to zeros of $f_\ell(p)$? We have

$$f_\ell(p) = \frac{1}{p} \mathcal{W}(\chi^+, \varphi_\ell) = \mathcal{W}(p^\ell \chi^+, p^{-(\ell+1)} \varphi_\ell)$$

and
$$\left.\begin{array}{l} \frac{\varphi_\ell}{p^{\ell+1}} \to r^{\ell+1} \text{ at origin} \\ p^\ell \chi^+ \to r^{-\ell} \text{ at } \infty \end{array}\right\} \text{when } p \to 0$$
$\leftarrow$ approaches $\hat{j}_\ell$
$\leftarrow$ approaches $\hat{h}^+_\ell$

Thus
$$f_\ell(0) = 0 \iff \begin{array}{l} \text{there is a solution to radial eqn} \\ \left.\begin{array}{l} \sim r^{\ell+1} \text{ at origin} \\ \sim r^{-\ell} \text{ at } \infty \end{array}\right\} \text{ i.e. a zero energy bound state} \end{array}$$

How does $f_\ell(p)$ behave near $p=0$? We should evaluate $f_\ell$ again, but instead we give a heuristic argument.
We assume $V$ is exponentially bounded so that $f_\ell$ is analytic there

$$f_\ell(p) = 1 + \frac{1}{p}\int \hat{h}^+ 2V\phi \qquad \hat{h}^+ = \hat{n} + i\hat{j}$$

$$\text{and} \quad \left.\begin{array}{l} \hat{j} \sim p^{\ell+1}(1+O(p^2)) \\ \hat{n} \sim p^{-\ell}(1+O(p^2)) \end{array}\right\} \text{even/odd properties on p.6.19}$$
$$\hat{\phi} \sim p^{\ell+1}(1+O(p^2))$$

$$\Rightarrow f_\ell(p) = 1 + (a_\ell + b_\ell p^2 + \cdots) + i(c_\ell p^{2\ell+1} + \cdots)$$

$f_\ell(0) = 0$ iff $a_\ell = -1$ and

$$\boxed{f_\ell(0) = 0 \Rightarrow f_\ell \sim p^2 \text{ for } \ell > 0\,; \quad f_\ell \sim p \text{ for } \ell = 0}$$

(This argument is not quite complete, because we did not show $b_\ell, c_\ell \neq 0$. This can be shown explicitly by differentiating Wronskian, as before)

Another important property of the zeros of $f_\ell(p)$ is

3) <u>There are a finite number of zeros of $f_\ell$ in $\mathrm{Im}\, p > 0$</u>.

This is because

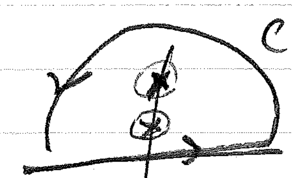
a) $f_\ell \to 1$ as $|p| \to \infty$, so there are no zeros outside some finite radius

b) the zeros cannot accumulate at any $p \neq 0$, because $f_\ell$ is analytic in $\mathrm{Im}\, p > 0$

c) the zeros cannot accumulate at $p = 0$, because $f_\ell$ is at least <u>continuous</u> at $p=0$ (from integral rep + bnds.) and $f_\ell \sim \begin{cases} p \\ p^2 \end{cases}$ means that zero at $p=0$ is isolated

Therefore, there are a finite number of bound states of angular momentum $\ell$

Now we can relate no. of bound states to phase shift at threshold.




$n_\ell$ = no. of simple zeros $= \frac{1}{2\pi i} \int dp \, \frac{\dot{f}_\ell}{f_\ell}$

(assume $f_\ell(0) \neq 0$, so $f_\ell$ has no zeros on contour)

We can close contour (i.e. ignore semicircle)

because $|f_\ell - 1| \lesssim \frac{1}{|p|}$

$\left( f_\ell \sim a_0(p - p_0) \Rightarrow \int_{C_0} \frac{dp}{p - p_0} = 2\pi i \right)$

$$S_\ell = e^{2i\delta_\ell} = \frac{f_\ell(p)^*}{f_\ell(p)} \Rightarrow f_\ell(p) = |f_\ell(p)|\, e^{-i\delta_\ell}$$

$$\ln f_\ell = \ln|f_\ell| - i\delta_\ell$$

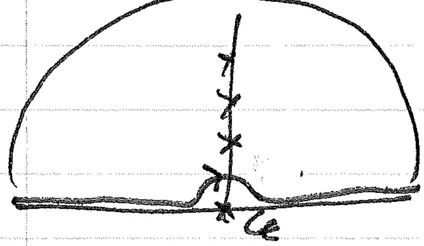
Since $f_\ell(-p) = f_\ell(p)^*$, $\ln f_\ell(-p) = \ln|f_\ell| + i\delta_\ell$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\ln f_\ell = \frac{1}{2\pi i}\,(-2i)\int_0^{\infty} d\delta_\ell = -\frac{1}{\pi}\left[\delta_\ell(\infty) - \delta_\ell(0)\right]$$

Hence $\boxed{\pi n_\ell = \delta_\ell(0) - \delta_\ell(\infty)}$ Levinson's Theorem

Now, what if $f_\ell(0) = 0$?

We distort contour as shown, to avoid singularity




$\ell = 0$ $f \sim a_0 p$

$\ell > 0$ $f \sim a_\ell p^2$

integral over semicircle of radius $\epsilon$ is

$$\int_{C_\epsilon} dp \begin{Bmatrix} 1/p & \ell=0 \\ 2/p & \ell>0 \end{Bmatrix} = -\int_0^{\pi} i\,d\theta \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{matrix} -i\pi & \ell=0 \\ -2i\pi & \ell>0 \end{matrix}$$

$$\boxed{\text{For } \ell=0 \quad n_0 = \frac{1}{\pi}\left[\delta_\ell(0) - \delta_\ell(\infty)\right] - \frac{1}{2}}$$

For $\ell > 0$, the zero at $p=0$ is a bound state, so

$$n - 1 = \frac{1}{\pi}\left(\delta_\ell(0) - \delta_\ell(\infty)\right) - 1$$

or $\boxed{n_\ell = \frac{1}{\pi}\delta_\ell(0) - \delta_\ell(\infty), \quad \ell > 0}$

The only exceptional case is

$$\pi\left(n_0 + \tfrac{1}{2}\right) = \delta_0(0) - \delta_0(\infty) \text{ if } f_0(0) = 0$$

Thus, we have found a very concrete consequence of the abstract result that the analytically continued S-matrix has poles at bound state energies. Studying analytic properties is not a complete waste of time.

## Alternative Proof of Levinson's Theorem

Put the world in a spherical box  with B.C. $\psi = 0$. Regular solution is real, and behaves asymptotically like

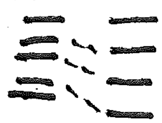
$$\phi \sim e^{-i\delta_\ell} e^{-ikr} + e^{i\delta_\ell} e^{ikr} \sim \sin(kr + \delta_\ell(k))$$

The condition for a stationary scattering state is

$$kR + \delta_\ell(k) = n\pi \quad \text{if "box" has side } R$$

Now consider "turning on" the potential $\lambda V$ adiabatically taking $\lambda = 0$ to $\lambda = 1$. Quantum states do not appear or disappear; they are merely shifted in energy. But some scattering states may "go below threshold" and become bound states.

Because $\delta_\ell(k) \to 0$ as $k \to \infty$, it is easy to match up corresponding states for $k$ large




The total no. of scattering states to "disappear" is the number of "level crossings" $n$, given by

$$n_\ell = \frac{\delta_\ell(0) - \delta_\ell(\infty)}{\pi}$$

which is therefore also the number of bound states.

**Problem:**

Find an alternative argument for Levinson's formula

$$n_\ell = \delta_\ell(0) - \delta_\ell(\infty) / \pi$$

(where $n$ is the number of bound states) reasoning as follows. Impose the boundary condition $\psi = 0$ on the wave function $\psi$ on the sphere $r = R$, where $R$ is very large but finite. (With these B.C., the spectrum of the Hamiltonian $H$ is discrete.) Now imagine adiabatically turning on the potential $\lambda V$, i.e. let $\lambda$ vary smoothly from $\lambda = 0$ to $\lambda = 1$. Use the facts that $\delta_\ell(p) \to 0$ as $p \to \infty$ and that stationary states of $H$ cannot spontaneously appear or disappear to show that the no. $n_\ell$ of bound states of the potential $V$ of angular momentum $\ell$ is given by Levinson's formula.

What happens as bound state pops through threshold

## H. Threshold Behavior

We gave a sloppy discussion of the behavior of the partial wave amplitude $f_\ell(p)$ back on p. 6.11. We return to this subject now, armed with our more sophisticated formalism.

Recall that we showed then that

$$f_\ell = -\frac{1}{p^2}\int_0^\infty dr\, \hat{j}_\ell(pr)\, 2V(r)\, \psi_{\ell p}(r)$$

where $$f(\vec{p}\,';\vec{p}) = \sum_\ell (2\ell+1) f_\ell(p) P_\ell(\hat{p}\cdot\hat{p}')$$

(by considering asymptotic behavior of stationary scattering states.)

Now we know $\psi_{\ell p}(r) = \dfrac{\phi_{\ell p}(r)}{f_\ell(p)}$

and we have derived bounds (for $p$ real)

$$|\hat{j}_\ell(pr)| \leq B_\ell\left(\frac{pr}{1+pr}\right)^{\ell+1}$$

$$|\phi_{\ell p}(r)| \leq \Delta_\ell\left(\frac{pr}{1+pr}\right)^{\ell+1}\exp\left[\Delta_\ell\int_0^\infty dr'|2V(r')|\right]$$

(assuming integral exists)

Hence, we have the bound

$$\boxed{|f_\ell| \leq \frac{(\text{const})}{|f_\ell(0)|}\frac{1}{p^2}\int_0^\infty dr\, |2V|\left(\frac{pr}{1+pr}\right)^{2\ell+2}}$$

This holds in a neighborhood of $p=0$ (we have replaced $f_\ell(p)$ by $f_\ell(0)$)

Let us first suppose $f_\ell(0) \neq 0$. Then

$$\boxed{|f_\ell(p)| \leq \frac{\text{const}}{|f_\ell(0)|} \left(\int_0^\infty dr\, 2|V| r^{2\ell+2}\right) p^{2\ell}}$$

if the integral exists. This confirms our earlier result that

$$f_\ell(p) \to -a_\ell p^{2\ell} \quad \text{as } p \to 0$$

If, however, $V \sim \frac{1}{r^n}$ for $r \to \infty$,
then the bound does not hold for

$$2\ell + 3 - n \geq 0 \quad \text{or} \quad \ell \geq \frac{n-3}{2}$$

Instead we may use

$$\boxed{|f_\ell(p)| \leq \frac{\text{const}}{f_\ell(0)} \frac{1}{p^2} \int dr\, 2|V| (pr)^{n-1-\epsilon} \leq \text{const}'\, p^{n-3-\epsilon}}$$

i.e. $f_\ell = O(p^{n-3})$

If, however, $f_\ell(0) = 0$,
we recall that

$$f_\ell \sim p^2 \text{ for } p \to 0 \text{ if } \ell \neq 0$$

Hence $\boxed{f_\ell(p) \sim p^{2\ell-2} \text{ if } f_\ell(0) \neq 0,\ \ell \neq 0}$

E.g. p-wave amplitude fails to vanish at threshold

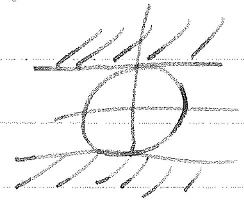
Also.. $f_\ell \sim p$ if $\ell = 0$

Hence $\boxed{f_0 \sim \frac{1}{p} \text{ if } f_0(0) \neq 0}$

This means that s-wave cross section is $\infty$ at threshold!
In practice if ~~has~~ $f_0$ has a zero near $p=0$, $\sigma_0 = 4\pi |f_0|^2$
is anomalously large. (E.g. $\sigma_0 = 70$ barns for
n-p scattering.)

$\ast \xleftarrow{\ell p}$ Deuteron ($\ell=2$)
$\ast \leftarrow$ virtual state ($\ell=0$)

Let us suppose now that $V$ is exponentially bounded, $\int_0^\infty dr\, |V|\, e^{2ar} < \infty$, so that $f_\ell$ is analytic in strip $|\mathrm{Im}\, p| < a$



This means that $f_\ell$ has a convergent power series expansion about $p = 0$

An often applied form of this power series is the following: First note that

$$S_\ell(p) = \frac{f_\ell(-p)}{f_\ell(p)} = e^{2i\delta_\ell(p)}$$

Clearly $S_\ell(-p) = \dfrac{1}{S_\ell(p)}$ and $\delta_\ell(-p) = -\delta_\ell(p) + n\pi$

$$\Rightarrow \tan\delta_\ell(-p) = -\tan\delta_\ell(p)$$

<u>odd function</u>

$$e^{2i\delta_\ell} = 1 + 2ipf_\ell \Rightarrow f_\ell(p) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{p} \sim -a_\ell p^{2\ell-1}$$

$$\Rightarrow \tan\delta_\ell(p) = -a_\ell p^{2\ell+1}\,[1 + O(p^2))$$

Hence

$$\boxed{p^{2\ell+1}\cot\delta_\ell(p) = -\frac{1}{a_\ell} + \frac{r_\ell}{2}p^2 + O(p^4)}$$

This is called the "effective range approximation"
These two parameter fits tend to work quite well at low energy in nuclear scattering processes.

For $\ell = 0$, $r_\ell$ turns out to be the range of e.g. a square well ~~potential~~ hence the name "effective range". There is an effective depth also, since $f_0 \sim \dfrac{mVR^3}{\hbar^2} = a_0$

S-wave scattering can be well approximated by scattering off a square well at low energy

## I. Resonance Poles and the Second Sheet

Let us restrict our attention now to exponentially bounded potentials, for which $f_\ell(p)$ is analytic for $\mathrm{Im}\,p > -a$ and $S_\ell(p)$ is analytic in the strip $|\mathrm{Im}\,p| < a$

We may also consider $S_\ell(E)$ on the cut energy plane.

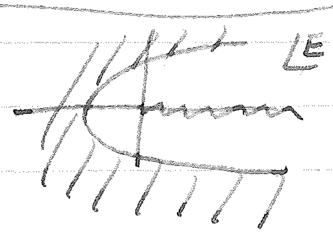
$$E = \tfrac{1}{2}p^2 = \tfrac{1}{2}(p_1^2 - p_2^2) + i\,p_1 p_2 \quad \text{if } p = p_1 + i p_2$$

$$p_2^2 < a^2 \text{ iff } (p_1^2 + a^2)\,p_2^2 < (p_1^2 + a^2)\,a^2 \text{ or}$$

$$(p_1 p_2)^2 < a^2(p_1^2 - p_2^2) + a^4$$

Thus, $S_\ell(E)$ is analytic inside a parabola

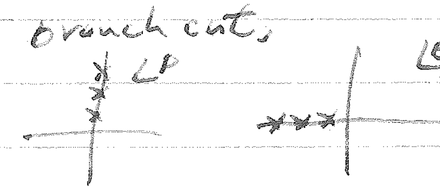
$$\boxed{(\mathrm{Im}\,E)^2 < 2a^2(\mathrm{Re}\,E) + a^4}$$




The E plane has two sheets with a cut along real axis

$\mathrm{Im}\,p > 0$ becomes "physical sheet"
$\mathrm{Im}\,p < 0$ becomes "second sheet"

Inside the parabola, the only singularities of $S_\ell(E)$, other than the square root branch cut, are poles. We have already seen that the only poles for $\mathrm{Im}\,p > 0$ are simple poles on the positive imaginary axis corresponding to bound states. These become poles at bound state energies on physical sheet in the energy plane.




Although we have not paid much attention to them up to now, there may also be zeros of the Jost function for $\mathrm{Im}\,p < 0$, which become poles on the second sheet of $S_\ell(E)$.
(Since $S_\ell(p) = \dfrac{f_\ell(-p)}{f_\ell(p)}$, it is equivalent to consider zeros of $S_\ell$ on first sheet)

i.e. we recall that the analytically continued Jost function obeys
$$f_\ell(p) = f_\ell(-p^*)^* \quad \text{(Reflection Principle)}$$

And therefore, the S-Matrix, which is

$S_\ell = \frac{f_\ell(p)^*}{f_\ell(p)}$ for $p$ real, becomes $\frac{f_\ell(p^*)^*}{f_\ell(p)} = \frac{f_\ell(-p)}{f_\ell(p)}$

What is the significance of a zero of $f_\ell(p)$ for $\mathrm{Im}\, p < 0$? We recall that the asymptotic form of the analytically continued regular solution is

$$\varphi_{\ell p}(r) \underset{r\to\infty}{\longrightarrow} \frac{1}{2i}\left[ f_\ell(p^*)^* e^{i(pr - \frac{1}{2}\ell\pi)} - f_\ell(p) e^{-i(pr - \frac{1}{2}\ell\pi)} \right]$$

If $f_\ell(p) = 0$, the solution still grows exponentially for $r \to \infty$ if $\mathrm{Im}\, p < 0$. It is not normalizable, and there is no reason why the energy $E = \frac{1}{2}p^2$ at which zero occurs cannot be complex

But, in the vicinity of a zero,
$$f_\ell(p) = A(p - p_1 + ip_2)$$
$$f_\ell(-p) = f_\ell(p^*)^* = A^*(p - p_1 - ip_2)$$

<u>If the imaginary part $p_2$ is small</u>, both $f_\ell(p), f_\ell(-p)$ are small when $p$ is near $p_1$, compared to the free regular solution $\hat{j}_\ell(pr)$. A zero with $\mathrm{Im}\, p < 0$ <u>near the real axis</u> is a value of $p$ at which potential tends to "hold the wave function in."
For $p_2$ small, this pole corresponds to a long-lived metastable state.

Another way to understand the connection with metastability is to consider time dependent Schrodinger eqn, which has solution
$$\psi(\vec{r}, t) = \frac{u_\ell(r)}{r} Y_{\ell m}(\Omega) e^{-iEt}$$

If $E = E_R - i\Gamma/2$, then $|\psi(r,t)|^2 \propto e^{-\Gamma t}$
Hence probability flux is leaking out radially, but only slowly if $\Gamma$ is small.

Thus, the zeros of $f_\ell(p)$ for $\mathrm{Im}\, p < 0$ or poles of $S_\ell$ on second sheet of energy plane are of interest for two (related) reasons:

1) If Imag. part is small, these poles are <u>resonance poles</u>. They can either cause phase shift $\delta_\ell$ to <u>rapidly</u> change by $\pi$, a phenomenon often observed experimentally, or metastability, a state obeying an exponential decay law.

2) These poles are (sometimes) related to bound state poles — they are "almost" bound states which would become bound if potential were stronger.

We will first consider the resonance phenomenon, which is easy to understand mathematically.
If S matrix has a pole at $E = E_R - i\Gamma/2$ (on the second sheet) then since $S_\ell$ has modulus 1 for E real, in the vicinity of the pole it has the form

$$S(E) = e^{2i\delta(E)} = e^{2i\delta_{bg}} \frac{E - E_R - i\Gamma/2}{E - E_R + i\Gamma/2}$$

(We have suppressed $\ell$, but we are considering partial wave S-matrix, of course)
Therefore

$$\delta(E) = \delta_{b.g.} + \delta_{res}$$

where
$$\sin 2\delta_{res} = \frac{-\Gamma/(E-E_R)}{[(E-E_R)^2 + (\Gamma/2)^2]} \qquad \cos 2\delta_{res} = \frac{(E-E_R)^2 - (\Gamma/2)^2}{[(E-E_R)^2 + (\Gamma/2)^2]}$$

$$\Longrightarrow \sin^2 \delta_{res} = \frac{1}{2}(1 - \cos 2\delta_{res}) = \frac{(\Gamma/2)^2}{(E-E_R)^2 + (\Gamma/2)^2}$$

In general, then, we have $\sin\delta = \sin\delta_{bg}\cos\delta_{res} + \cos\delta_{bg}\sin\delta_{res}$

or
$$\boxed{\sin\delta = \frac{(E-E_R)\sin\delta_{bg} + \frac{\Gamma}{2}\cos\delta_{bg}}{\left[(E-E_R)^2+(\Gamma/2)^2\right]^{\frac{1}{2}}}}$$

If $\Gamma \ll E_R$, then $\delta$ rapidly increases from $-\delta_{bg}$ to $\delta_{bg}+\pi$ as $E$ increases through $E_R$

An important special case is $\delta_{bg} \sim 0$, which is typically true if resonance occurs close to threshold.

$$\sigma_\ell(E) = \frac{4\pi}{p^2}(2\ell+1)\sin^2\delta_\ell$$

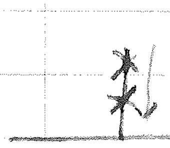
$$\boxed{\sin^2\delta_\ell = \frac{(\Gamma/2)^2}{(E-E_R)^2+(\Gamma/2)^2}}$$

Breit-Wigner resonance formula

$\delta_\ell$ always attains its unitarity limit near the resonance, but also goes through zero nearby (if $\delta_{bg}\neq 0$). $\sigma_\ell$ may not actually be peaked at $E=E_R$. The characteristic property of a resonance is the increase of $\delta_\ell$ by $\pi$ in vicinity of $E_R$.

This increase of $\delta_\ell$ is associated with a phase delay - a tendency for projectile to resonate with target, as we will see below

But first we wish to understand better the connection between resonances and bound states. To do so, we consider the trajectories of poles in cut $E$ plane (or zeros of $f_\ell$ in $p$ plane) as we change strength of potential: $V \to \lambda V$




As potential weakens, all bound states approach threshold, and eventually disappear from uhp. Where do the zeros of $f_\ell$ go?

we can be a little more definite about when bound states disappear, by deriving bound on $n_\ell$ - no. of bound states of angular momentum $\ell$. We recall the integral eqn satisfied by $\psi_{\ell p}$ (p 6.8)

$$\psi_{\ell,p}(r) = \hat{j}_\ell(pr) + \int_0^\infty G^0_{\ell p}(r,r')\, 2V(r')\, \psi_{\ell p}(r')$$

where
$$G^0_{\ell p}(r,r') = -\frac{1}{p}\hat{j}_\ell(pr_<)\hat{h}^+_\ell(pr_>)$$

this Green's function is symmetric, $G^0_{\ell p}(r,r') = G(r',r)$ and we may consider the hermitian kernel

$$K = -|2V|^{\frac{1}{2}} G^0_{\ell p} |2V|^{\frac{1}{2}} \quad \Big\} \begin{matrix}\text{i.e. hermitian for} \\ E<0,\ p \text{ imaginary}\end{matrix}$$

appropriate for a strictly attractive potential $-|V|$

As in our discussion on p(5.8), the number of bound states in the potential $-|V|$ is the no. of eigenvalues of $K$ which are greater than 1 when $E=0$

Because the potential $V$ has no more bound states than.

$$\tilde{V} = \begin{cases} V & \text{for } V \le 0 \\ 0 & \text{for } V > 0 \end{cases}$$

we have the bound

$$n_\ell \le \operatorname{tr} K = \int_0^\infty dr\, -|\tilde{V}|\, G^0_{\ell,p=0}(r,r)$$

$$\text{and} \quad G^0_{\ell p}(r,r)\Big|_{p=0} = -\frac{1}{p}\frac{(pr)^{\ell+1}}{(2\ell+1)!!}(pr)^{-\ell}(2\ell-1)!! = -\frac{r}{2\ell+1}$$

$$\Rightarrow \boxed{n_\ell \le \frac{\int_0^\infty dr\ r|2\tilde{V}|}{2\ell+1}} \qquad \text{Bargmann Inequality}$$

Hence, there are no bound states for

$$\lambda < \frac{2\ell+1}{\int_0^\infty r|2\tilde{V}|}$$

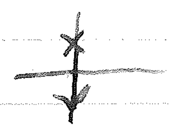
and also, no bound states for sufficiently large $\ell$

The trajectories of poles of $S$ as we vary $\lambda$ must be considered separately in the cases $\ell = 0$ and $\ell \neq 0$

## $\ell = 0$

We know that a zero at threshold, $f_\ell(0) = 0$, is simple and also that $f_\ell(-p^*) = f_\ell(p)^*$ (Reflection principle)

(A zero which is not on the Imag axis has an image zero, reflected in imaginary axis.)



Hence, after the zero passes through threshold it stays on negative imaginary axis — It becomes a "virtual state"

A virtual state is not a resonance; it does not cause the phase shift to rapidly advance by $\pi$. Nor does it have any clear physical interpretation — It is a soln to radial eqn in which coef of decreasing exponential vanishes, but the decreasing exponential is asymptotically negligible anyhow.

$f_\ell$ is analytic in $p$ and $\lambda$, so it has a double power series expansion. Recalling the expansion on p(6.25) we see that, if $f_0$ has a threshold zero for $\lambda = \lambda_0$, then

$$f_0(\lambda, p) = a(\lambda - \lambda_0) + ibp + O(p^2)$$

where $a, b$ are real

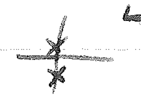
or $\boxed{f_0 \sim ib\left[p - ia'(\lambda - \lambda_0)\right]}$ and $a'$ must be $> 0$, so that the zero is in uhp for $\lambda > \lambda_0$

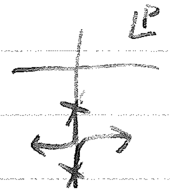
The zero at $\bar{p} = ia'(\lambda - \lambda_0)$ slides along Im axis linearly with $\lambda$

$$S_0 = \frac{f_0(p^*)^*}{f_0(p)} = -\frac{p + ia'(\lambda - \lambda_0)}{p - ia'(\lambda - \lambda_0)} \Rightarrow$$

$$f_0(p) = \frac{S_0 - 1}{2ip} = i\,\frac{1}{p - ia'(\lambda - \lambda_0)} \Rightarrow \boxed{\sigma_0 = 4\pi |f_0|^2 = \frac{4\pi}{a'^2(\lambda - \lambda_0)^2 + p^2}}$$

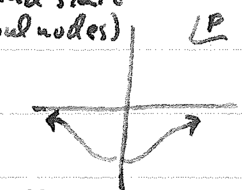
Thus, the cross section does not know the difference between a virtual state and a bound state. A pole on either the pos. or neg imaginary axis will cause $\sigma_0$ to be large at low energy, if it is close to the origin



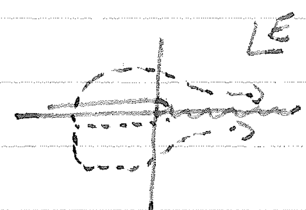


If two virtual states collide as we change $\lambda$, coalescing to form a double zero on neg imag axis, they may subsequently leave imag axis, and venture into cx plane symmetrically (consistent with $f_\ell(-p^*) = f_\ell(+p)^*$).

- Such double zeros may not occur in uhp – bound states "repel" each other, but they are not forbidden in lhp. (i.e. nth bound state always has n-1 radial nodes)



- The zero may then approach real axis, and become a resonance – Nothing forbids S wave resonance, but they are not associated with threshold effects, as for $\ell \neq 0$



- In energy plane bound state ducks under the cut and moves back along neg real axis on second sheet

If pole collides with another and leaves neg real axis, they move symmetrically into upper/lower half plane <u>on second sheet</u>

If they then approach positive real axis, the pole in lhp is near physical region and becomes a resonance, but the pole in uhp is far from physical region, and has no physical effect.

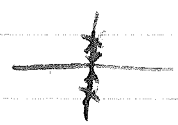
$\ell > 0$

Now the asymptotic form as $p \to 0$ is

$$f_\ell(\lambda, p) \approx a(\lambda-\lambda_0) + \qquad b p^2 \quad ; \quad a, b \text{ real}$$

There is a double zero at threshold for $\lambda = \lambda_0$.

$$= b\,[p^2 + a'^2(\lambda-\lambda_0)] \qquad a'^2 > 0$$



As bound state approaches threshold, a virtual state comes up to meet it; zeros are at $\bar{p} = i a'(\lambda-\lambda_0)^{1/2}$

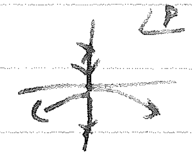
At threshold, the zeros turn $90^\circ$ and head along $\pm$ Real axis

since leading Imag part of $f_\ell$ is $O(p^{2\ell+1})$ zero actually solves

$$\bar{p}^2 - a'^2(\lambda_0-\lambda) \sim i\bar{p}^{2\ell+1}, \text{ and } \bar{p}^{2\ell} \sim (\lambda_0-\lambda)^\ell$$

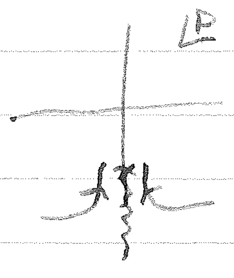
$$\Rightarrow \operatorname{Re}\bar{p}\, \operatorname{Im}\bar{p} \sim (\operatorname{Re}\bar{p})^{2\ell+1} \Rightarrow \boxed{\operatorname{Im}\bar{p} \sim (\lambda_0-\lambda)^\ell}$$

The larger $\ell$ is, the longer the resonance



hugs the real axis, and the sharper the resonance (Projectile resonates longer in a thicker centrifugal barrier.)

If we consider exponentially bounded



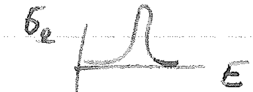
analytic potentials, $f_\ell(p)$ has cut on negative real axis. Zeros approaching the cut symmetrically may either slip under the cut or move along it.

If they reach the branch point, it is possible for one zero to survive as a virtual state, and the other to disappear.

# The resonance phenomenon and time delay

Resonances – poles on the second sheet – give rise to two quite distinct physical phenomena:

1) Bumps in the cross section, as we have already discussed.




This phenomenon is presumably a consequence of metastability – i.e. an "almost stationary" state which decays at the rate $\Gamma$ – but we observe the width $\Gamma$ of the bump, rather than directly measure a lifetime

2) Long-lived metastable state – e.g. a radioactive nucleus – we measure directly the $e^{-\Gamma t}$ attenuation of signal. This also corresponds to a pole on second sheet, but we don't measure resonance directly.


What determines which kind of measurement we make?

- To see bumps, we need energy resolution. i.e. we need to use wave packets which have an energy width small compared to $\Gamma$

$$\Delta E \ll \Gamma$$

- To measure lifetime, we need to produce the state at a definite time – we need time resolution

But $\Delta t = \frac{\Delta x}{p} \sim \frac{1}{p\Delta p} = \frac{1}{\Delta E}$, and we require $\Delta t \ll \frac{1}{\Gamma}$, or

$$\Delta E \gg \Gamma$$

1) Time Delay

Consider an incident wave packet

$$\psi_{in} = \int d^3p\, \phi(\vec{p})\, |\vec{p}\rangle$$ which collides with target at $t \sim 0$

At positive times, the wave packet becomes a superposition of stationary scattering states

$$\psi(\vec{x},t) = \int d^3p\, \phi(p)\, e^{-iE_p t} \langle \vec{x} | \vec{p}+ \rangle$$

which has the asymptotic form

$$\psi(\vec{x},t) \xrightarrow[r\to\infty]{} \frac{1}{(2\pi)^{3/2}} \int d^3p\, \phi(\vec{p})\, e^{-iE_p t} \left[ e^{i\vec{p}\cdot\vec{x}} + f(p\hat{x},\vec{p}) \frac{e^{ipr}}{r} \right]$$

Now suppose that $f$ is dominated by a resonance in a single partial wave

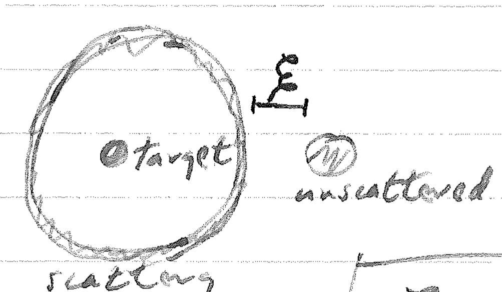
$$f_\ell(p\hat{x},\hat{p}) = (2\ell+1) \frac{e^{i\delta_\ell}\sin\delta_\ell}{p} P_\ell(\hat{x}\cdot\hat{p})$$

and that $\Delta E \ll \Gamma$, so that $f$ changes slowly over range in $\vec{p}$ that are important

we pull $f$ outside the integral, except that we write $\delta_\ell(p) = \delta_\ell(p_R) + \delta_\ell'(p_R)(p-p_R)$
the scattered wave is

$$\Rightarrow \psi_{scat}(\vec{x},t) \sim \frac{f_\ell}{r} \int d^3p\, \phi(\vec{p})\, e^{-iE_p t} e^{ip(r+\delta_\ell')}$$

But this integral gives simply $\psi_{in}(\vec{x}',t)$ at the argument $(r+\delta_\ell')\hat{p}$




This means that the scattered wave lags behind the unscattered wave packet by a distance

$$\boxed{\xi = \frac{d\delta_\ell}{dp}(p_R)}$$

and the scattered wave is delayed in time by

$$\boxed{\tau = \frac{\xi}{v_0} = \frac{dp}{dE}\frac{d\delta_\ell}{dp} = \frac{d\delta_\ell}{dE}(E_R)}$$

For a resonance $\frac{d\delta_\ell}{dE}$ is positive (delay rather than advance in phase) and of order $\Gamma$!

2) Exponential Decay

Production and decay of any metastable state must be described by scattering theory, because the state did not exist infinitely far in past; it was produced in a scattering event. This is true even for e.g. a $U^{238}$ nucleus, which may have been produced $5\times10^9$ yrs ago in a supernova.

But if lifetime can be measured directly, it must be that

$$\Delta E \gg \Gamma$$

The scattered wave is

$$\psi_{scat}(\vec{x},t) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\int d^3p\, \phi(\vec{p})\, e^{-iE_p t}\, \frac{f(p\hat{x},\vec{p})}{r} e^{ipr}$$

For simplicity, suppose $f$ is dominated by a Breit-Wigner resonance in $l$th partial wave

$$f = (2l+1) f_l\, P_l(\hat{x}\cdot\hat{p}) \text{ and } S_l = 1+2ipf_l \approx \frac{E-E_R-i\Gamma/2}{E-E_R+i\Gamma/2}$$

$$\Rightarrow f_l \sim \frac{1}{p}\,\frac{-\Gamma/2}{E-E_R+i\Gamma/2}$$

To do the angular integral, we note that $\phi$ can be expanded in $Y_l^{m=0}$ spherical harmonics

and $P_l(\hat{x}\cdot\hat{p}) = \frac{4\pi}{2l+1}\sum_m Y_{lm}(\hat{x})\, Y_{lm}(\hat{p})^*$

so $\int d\Omega$ is proportional to $Y_l^0(\hat{x})$,

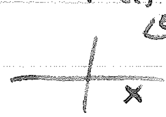
And we can pull $\phi_l(E)$ outside integral, so

$$\psi_{scat} \sim \frac{Y_{l0}(\hat{x})}{r}\int_0^\infty dp\, \frac{e^{i(pr-Et)}}{E-E_R+i\Gamma/2}$$

and $p \sim p_R + \frac{E-E_R}{p_R}$, and integral can be extended to $-\infty$, making a small error

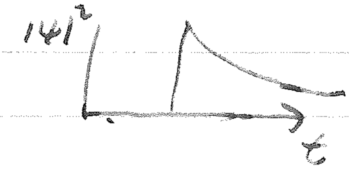
$$\sim \frac{Y_{l0}(\hat{x})}{r}\int_{-\infty}^{\infty} dE\, \frac{e^{-i(E-E_R)(t-r/p_R)}}{E-E_R+i\Gamma/2}$$

This integral can be done by completing a contour




$$\sim \begin{cases} 0 \; ; & t < r/p_R \\ -2\pi i \, e^{-\Gamma(t - r/p_R)/2} \; ; & t > r/p_R \end{cases}$$

At any fixed value of $r$, the scattered wave appears




at $t \sim r/p_R$, and then decays exponentially with lifetime $\Gamma^{-1}$

Note: No general principle forbids resonance "multipoles". For multipoles, the exponential decay law becomes modified. (Cf. the critically damped harmonic oscillator.) This modification is left as an exercise.