IX. Multichannel Scattering

There are a few complications in scattering theory which we have not yet encountered in this course, which has been concerned only with potential scattering.

1) Effects of spin (Taylor, Chapters 5, 7)
2) Effects of (Bose or Fermi) statistics (Taylor, Chapter 22)

These are simple generalizations of what we have already done. (But important.) I will leave to you the responsibility of reading the appropriate chapters

3) Multichannel (inelastic) scattering (Taylor Chapters 16–21)

This is also a fairly straightforward generalization, but there are some new physical phenomena we will encounter. I will try to emphasize the new features, and will also try to follow the book closely.

A. Channels

We have considered a particle bouncing off a potential. This is a description in cm frame of

$$A + B \to A + B \qquad \underline{\text{Elastic Scattering}}$$

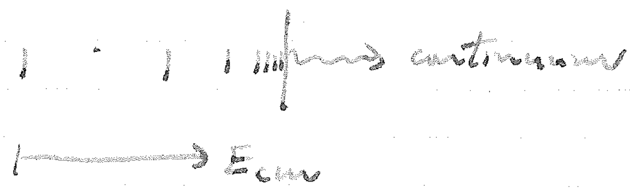
But in general the particles in the final state need not be the same as in the initial state.

E.g.
$$e + H \to e + H \qquad \text{(elastic)}$$
$$e + H \to e + H^* \qquad \text{(excitation)}$$
$$e + H \to e + e + H^+ \qquad \text{(ionization)}$$

or
$$e + p \to e + p \qquad \text{(Elastic scattering determines charge distrib.)}$$
$$e + p \to e + p + \pi^+ + \pi^- \qquad \text{("Deep inelastic" scattering determines inner structure)}$$
etc.

Each possible final state is called a channel, and each channel has a threshold energy at which it becomes kinematically allowed.

E.g. eH scattering is necessarily elastic if $E_{cm} < (1-\frac{1}{4})$ Rydberg, the energy needed to reach 1st excited state.



there is a threshold associated with each hydrogenic energy level

Does it make sense to regard eH^* as an asymptotic scattering state? Rigorously no, because H^* deexcites by γ emission with some characteristic lifetime. The scattering process is really

$$e+H \to e+H+\gamma.$$

But it makes some physical sense, because H^* is typically quite long lived compared to typical atomic physics time scale (10^{-8} sec compared to 10^{-16} sec). An even more extreme example is the neutron. To be more mathematically rigorous, we may imagine turning off coupling to radiation field, or weak interactions, so that H^* and n become stable.

We may also wish to consider unstable particle in initial state. E.g. n scattering off nuclei.

Channel Hamiltonian

How do we generalize the definitions of asymptotic states, wave operators, S-matrix, etc. to the multichannel case?

Consider for concreteness a system of three elementary particles a, b, c which interact in pairs. The Hamiltonian is

$$H = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + \frac{p_c^2}{2m_c} + V_{ab}(\vec{x}_{ab}) + V_{bc}(\vec{x}_{bc}) + V_{ac}(\vec{x}_{ac})$$

By the "channel Hamiltonian" we mean that part of H which we must retain to describe asymptotic states in a given channel

E.g. in 3 particle channel: $a+b+c$ we have

$$H \to H^0 = \sum p_i^2/2m_i$$

the potentials become irrelevant as particles separate

But if we consider channel $a + (bc)$ where b and c are bound, we have

$$H \to H^2 = \sum p_i^2/2m_i + V_{bc}$$

and

$$e^{-iHt}|\Psi\rangle \xrightarrow[t\to-\infty]{} e^{-iH^2 t}|\Psi_{in}\rangle$$

defines an asymptotic in state

the wave function $|\Psi_{in}\rangle$ has the form

$$\langle x|\Psi_{in}\rangle = \chi(\vec{x}_a\, \vec{X}_{cm})\, \phi_{(bc)}(\vec{x}_{bc})$$

x represents the positions $(\vec{x}_a, \vec{x}_b, \vec{x}_c)$
$\phi_{(bc)}$ is the bound state wave function

In general, associated with the channel α is the channel Hamiltonian and wave operators

$$|\Psi\rangle = \underset{\sim}{\Omega}{}^{\alpha}_{\pm}|\Psi_{in}\rangle = \lim_{t\to\mp\infty} e^{iHt}e^{-iH^\alpha t}|\Psi^\alpha_{in}\rangle$$

Note that $\underset{\sim}{\Omega}{}^\alpha_\pm$ acts not on $\mathcal{H}$, but on a subspace $\mathcal{S}^\alpha$ of in/out asymptotes in channel α.

(The limit exists by a wave-packet spreading argument, as before, with $H - H^\alpha$ as the "potential")

Each state approaches a sum of asymptotic states in different channels

$$e^{-iHt}|\psi\rangle \xrightarrow[t\to\mp\infty]{} e^{-iH^0 t}|\psi^0_{\substack{in\\out}}\rangle + e^{-iH^1 t}|\psi^1_{\substack{in\\out}}\rangle + \cdots$$

Asymptotic completeness becomes the statement that the full Hilbert space is the direct sum of scattering channels and bound states. $R^\alpha_\pm$ is Range $\Omega^\alpha_\pm$

$$\mathcal{H} = \mathcal{B} + \mathcal{R} \quad \text{and} \quad \mathcal{R} = \bigoplus_\alpha \mathcal{R}^\alpha_\pm$$

where $\mathcal{R}^\alpha_{(\pm)}$ is space of states that originated as (or will become) an asymptotic state in channel α

Multichannel S-Matrix

The probability amplitude for a state asymptotic to ψ^α_{in} to evolve into one asymptotic to $\psi^{\alpha'}_{out}$ is the "overlap" of $\Omega^\alpha_+|\psi^\alpha_{in}\rangle$ and $\Omega^{\alpha'}_-|\psi^{\alpha'}_{out}\rangle$ which is

$$\langle\psi^{\alpha'}_{out}|S^{\alpha'\alpha}|\psi^\alpha_{in}\rangle \text{ where } S^{\alpha'\alpha} = \Omega^{\alpha'\dagger}_- \Omega_+$$

This defines the multichannel S-Matrix. It is unitary,

$$S^{\dagger\alpha\beta} S^{\beta\alpha'} = \mathbb{1}^\alpha \delta^{\alpha\alpha'}$$

which expresses conservation of probability.

Momentum Representation

Asymptotic states can be expanded in terms of plane waves (momentum eigenstates) where one momentum is needed for each body (bound or elementary) in the state

- Three free particles a, b, c (channel 0): the momentum eigenstates are

$$\langle x | p,\alpha\rangle = \prod_i (2\pi)^{-3/2} e^{i\vec{p}_i\cdot\vec{x}_i}$$

- $a + (bc)$ — (channel 1):
  Expand in states of the form

$$\langle x | p, 1\rangle = \frac{1}{(2\pi)^3} e^{i\vec{p}_a\cdot\vec{x}_a} e^{i\vec{P}_{bc}\cdot\vec{x}_{cm}} \phi_{(bc)}(\vec{x}_{bc})$$

- In any channel we have normalized momentum eigenstates normalized so that

$$\langle p',\alpha | p,\alpha\rangle = \delta(p-p')$$ — one δ function for each mom. component needed to label the states

<u>Energy Conservation and Momentum Conservation</u> still holds. $\langle p',\alpha' | S^{\alpha'\alpha} | p,\alpha\rangle$ vanishes unless initial and final states have same value of "channel energy" H^{α} ($H^{\alpha'}$). This follows from intertwining relations

$$H\underset{\sim}{\Omega}^{\alpha}_{\pm} = \underset{\sim}{\Omega}^{\alpha}_{\pm} H^{\alpha}$$, as in one channel case

Now $\underset{\sim}{S}$ also commutes with total momentum (In the one-channel case it did not, only because we separated out center of mass motion and described scattering as interaction between particle and potential.) Potentials depend on <u>relative</u> positions $\Longrightarrow$ translation invariance

We can separate out cm momentum (which is conserved) in our momentum basis states from relative momenta. $\underset{\sim}{S}$ elements will have momentum conserving, as well as energy conserving δ function. We thus will define on-shell t matrix as

$$\boxed{\begin{aligned}\langle p',\alpha'|\, \underset{\sim}{S}-\mathbb{1}\,|p,\alpha\rangle &= \\ &-2\pi i\,\delta(E_T'-E_T)\,\delta^3(\vec{P}_T'-\vec{P}_T)\,\underbrace{t(p',\alpha';p,\alpha)}_{\text{depends only on relative momenta}}\end{aligned}}$$

Because of energy conservation, typically for a given initial state, not all final state channels are <u>open</u>. Enough energy is need to reach <u>threshold</u> of the channel

E.g at low energy only elastic scattering is possible and S is "one dimensional" $S(\vec{p}',\vec{p})$
But as channels open it becomes 2x2, 3x3 etc

<u>Cross Sections</u>

How to we obtain cross-sections (which are experimentally observed) from transition probabilities
$$|\langle \psi_{out}|S|\psi_{in}\rangle|^2$$

We consider the experimentally relevant case of a 2-body in state, but we wish to consider n-body out states

The initial state is a two particle state with relative momentum $\vec{p}$ and reduced mass m

How will our calculation of $2\to n$ cross section be different from calculation of $2\to 2$ cross sections? The difference is in the density of final states. Before we stuck in factors of $(2\pi)^3/V$ to relate continuum normalization to states to normalization of eigenstates of momentum in a box. But we then divided by $(2\pi)^3/V$ - the density of states factor - to relate volume of momentum space and probability of transition to count rate. The same cancellation will occur in the n-body case

We will consider the cross section in the CM frame. And we specify only relative momentum. By factoring out CM motion we have removed the momentum conserving δ function from consideration.

Doing our world-in-a-box calculation exactly as on page 2.16 we find that the no. of transitions (per unit time) to a final state in the momentum space volume element $dp' = \prod_{i=1}^{n-1} d^3p_i'$ where the $\vec{p}_i$ are relative momenta, is

$$\frac{dN}{dt} = \frac{1}{V}\,\delta(E'-E)\,(2\pi)^4\,\left|t_{\ell}(p',\alpha';\vec{p},\alpha)\right|^2 \prod_{i=1}^{n-1} d^3p_i'$$

(we did not have the $(2\pi)^4$ before because we expressed this in terms of f rather than t.)

To get a cross section, I divide the event rate by a flux, which in my box is

$$\frac{p}{m}\,\frac{1}{V}$$

(I won't set reduced mass equal to one any more, because there may be other masses describing final state particles)

Associated with a volume Δ' of final state momentum space is a cross section

$$\boxed{\sigma(\Delta',\alpha';\vec{p},\alpha) = (2\pi)^4\,\frac{m}{p}\int_{\Delta'} dp'\,\delta(E'-E)\,\left|t(p',\alpha';\vec{p},\alpha)\right|^2}$$

This is the general formula, the special cases of greatest interest are the two-body and three-body final states.

<u>Two-body phase space</u>

We integrate d^3p' over the one relative momentum, and the energy δ-function is

$$\delta\left(p'^2/2m' + W_{\alpha'} - p^2/2m - W_\alpha\right)$$

where W_α, are threshold energies

and $\delta(p'^2/2m' - C) = \frac{m'}{p'}\,\delta(p' - \ldots)$

so $\sigma = (2\pi)^4 \dfrac{mm'}{p} \int p'^2 dp' \, d\Omega \; |t|^2$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}(\vec{p}\,'\alpha'; \vec{p}\,\alpha) = mm' \frac{p'}{p} \left|t(\vec{p}\,'\alpha'; \vec{p}\,\alpha)\right|^2 \times (2\pi)^4}$$

Of course, this reduces to our old formula in the one channel case.

<u>Three-body phase space</u>

If we have $a + b + c$ in the final state, we have some freedom in choice of relative momentum. Let us imagine that c goes undetected, and only a and b are observed.

If we specify $\vec{p}_a$ and $\vec{p}_b$, $\vec{p}_c$ is determined, and therefore so is the total energy. A non-redundant labeling of final state in CM frame is

$$\vec{p}_b \text{ and } \hat{p}_a$$

Integrating out the magnitude of a in

$$\int d^3p_a' \, d^3p_b' \; \delta(E' - E)$$

is in general rather complicated, because

$$E' = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + \frac{(\vec{p}_a + \vec{p}_b)^2}{2m_c} + W'$$

But it is typical of $m_c \gg m_a, m_b$ so that we can ignore the K.E. of particle c. Then we have simply

$$d\sigma = (2\pi)^4 \frac{m}{p} \; \frac{m_a}{p_a} \; p_a^2 \, d\Omega_a \, p_b^2 \, dp_b \, d\Omega_b \;\; |t|^2$$

and $\quad p_b\, dp_b = m_b\, dE_b$, so we may write

$$\boxed{\frac{d^3\sigma}{d\Omega_a\, d\Omega_b\, dE_b} = (2\pi)^4 \frac{m}{p}\, m_a m_b p_a p_b \left| t(\vec{p}_a, \vec{p}_b, \alpha'; \vec{p}, \alpha) \right|^2 \qquad m_c \gg m_a, m_b}$$

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Multichannel Optical Theorem

Unitarity of multichannel S-matrix $\Rightarrow$

$$\sum_\beta S^{\alpha\beta} S^{\dagger\beta\alpha} = \mathbb{1}$$

in the α channel

Defining $\langle \vec{p}' | S^{\alpha\alpha} - \mathbb{1}^\alpha | \vec{p} \rangle = \frac{i}{2\pi m}\, \delta(E-E')\, f^\alpha(\vec{p}', \vec{p})$

in a two body channel α, with reduced mass m

we find $\quad S^{\alpha\beta} = \mathbb{1} + R^{\alpha\beta} \Rightarrow \quad R^{\alpha\alpha} + R^{\dagger\alpha\alpha} = -\sum_\beta R^{\dagger\alpha\beta} R^{\alpha\beta}$

($\mathbb{1}^\alpha$ has no cross-channel components)

$$\Rightarrow \left[f^\alpha(\vec{p}', \vec{p}) - f^\alpha(\vec{p}, \vec{p}')^* \right] \frac{i}{2\pi m} = \sum_\beta \int d^3 p_i''\, \delta(E-E')\, (2\pi)^2$$

$$t(p''; \beta, \vec{p}'; \alpha)^*\, t(p''; \beta, \vec{p}, \alpha)$$

Now consider $\vec{p} = \vec{p}'$ and compare to expression for σ in p(9.7)

$$2i\, \mathrm{Im} f^\alpha(\vec{p}, \vec{p}) = \frac{ip}{2\pi}\, \sigma_T^\alpha(\vec{p}) \quad \text{or} \quad \boxed{\mathrm{Im} f^\alpha(\vec{p}, \vec{p}) = \frac{p}{4\pi}\, \sigma_T^\alpha(\vec{p})}$$

The optical theorem is unchanged, except that σ_T now includes a sum over inelastic channels.

Partial Waves

For rotationally invariant systems, it remains convenient to use an angular momentum basis. This is particularly simple if the only open channels are two-body, so that we may use the basis

$$|E, l, m, \alpha\rangle$$

The S-matrix becomes $S_l^{\alpha'\alpha}$ – diagonal in l, and the multichannel partial wave amplitude may be defined by

$$f_l^{\alpha'\alpha}(E) = \frac{S_l^{\alpha'\alpha} - \delta^{\alpha'\alpha}}{2i\;(p'p)^{1/2}}$$

so defined that

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{p'}{p}|f|^2}$$

$$f(\vec{p}', \alpha'; \vec{p}, \alpha) = -(2\pi)^2 (m'm)^{\frac{1}{2}}\, t(\vec{p}', \alpha'; \vec{p}, \alpha)$$

$$= \sum_l (2l+1)\, f_l^{\alpha'\alpha}(E)\, P_l(\cos\theta)$$

Now $S_l^{\alpha'\alpha}$ is a unitary $n \times n$ matrix if there are n channels. Thus, if S_l^{00} represents the elastic matrix element, we see that

$$|S_l^{00}(E)| < 1$$

as soon as inelastic channels open up.

we have $\sum_\alpha |S_l^{0\alpha}(E)|^2 = 1$

The phase shift given by

$$S_l^{00} = e^{2i\delta_l(E)}$$

$$\left.\begin{array}{l}\text{Flux is being lost} \\ \text{into the new channels}\end{array}\right\} \text{ is no longer real}$$

This conclusion about S_l^{00} remains true as many body resonances open up.

Shadow Scattering

The elastic cross section is still given by

$$\sigma_{el}^{\alpha}(p) = 4\pi \sum_l (2l+1) |f_l^{\alpha}|^2$$

and the optical theorem says that

$$\sigma_T^{\alpha}(p) = \frac{4\pi}{p} \sum_l (2l+1)\, \operatorname{Im} f_l^{\alpha}$$

What happens if $S^{\alpha\alpha}$ is zero, so that all of the incident flux is "absorbed" by inelastic channels
Then

$$f_\ell^\alpha = \frac{1}{2ip} \Longrightarrow \qquad \sigma_{el}^\alpha(\ell) = \frac{\pi}{p^2}(2\ell+1)$$

$$\sigma_T^\alpha(\ell) = \frac{2\pi}{p^2}(2\ell+1)$$

Even if there is total absorption, the elastic cross section equals the inelastic cross section. If absorption occurs up to $\ell \sim pR$, we have

$$\sigma_{el} \sim \pi R^2$$
$$\sigma_T \sim 2\pi R^2$$

This is the shadow scattering phenomenon. The elastic scattering is peaked in the forward direction with $\Delta\Theta \sim (pR)^{-1}$ – and is thus consistent with the existence of a classical limit.

The wave reaches its asymptotic form, with the shadow filled in by diffraction effects only a distance

$$R/(\Delta\Theta) \sim R^2 p \quad \text{or} \quad R\left(\frac{R}{\lambda}\right)$$

This is a very large distance if the objects being scattered are macroscopic. In the classical limit, the elastically scattered wave is detected only at very small angles and very large distances from the scattering center.

B. Time - Independent Formalism

Stationary Scattering States and off-shell T-matrix

Associated with each channel Hamiltonian H^α is a Green's function $G^\alpha(z) = \frac{1}{z - H^\alpha}$

If $G(z) = \frac{1}{z - H}$, then the Lippmann-Schwinger eqn

$$\left[G = G^\alpha + G^\alpha V^\alpha G, \qquad V^\alpha = H - H^\alpha \right]$$

is a trivial identity

As in one-channel case, the action of wave operators on eigenstates of H^α can be represented

$$\boxed{|p, \alpha \pm\rangle = \Omega^\alpha_\pm |p\,\alpha\rangle = |p, \alpha\rangle + G(E \pm i\epsilon) V^\alpha |p, \alpha\rangle}$$

(Here dependence on cm motion has been factored out, and p represents all relative momenta)

S-matrix elements are

$$\langle p'\beta | S^{\beta\alpha} | p, \alpha\rangle = \langle p' \beta - | p, \alpha + \rangle \quad \text{and}$$

$$|p, \alpha +\rangle = |p\,\alpha -\rangle + \left[G(E + i\epsilon) - G(E - i\epsilon) \right] V^\alpha |p\,\alpha\rangle$$

and when quantity in brackets "acts back" on $\langle p', \beta - |$, we see that it becomes

$$\frac{1}{E - E' + i\epsilon} - \frac{1}{E - E' - i\epsilon} = -2\pi i \delta(E - E')$$

$$\boxed{\begin{aligned} \langle p' \beta | (S - \mathbb{1})^{\beta\alpha} | p, \alpha\rangle &= -2\pi i\, \delta(E' - E) \langle p' \beta - | V^\alpha | p\,\alpha\rangle \\ \text{or} \qquad &= -2\pi i\, \delta(E' - E) \langle p' \beta | V^\beta | p\,\alpha +\rangle \end{aligned}}$$

Thus, we have two alternative ways of continuing the t-matrix "off shell"

E.g. $|p\alpha+\rangle = (\mathbb{1} + G(E+i\epsilon)V^\alpha)|p\alpha\rangle$

$\Rightarrow \quad t(p';\beta;p,\alpha) = \langle|(V^\beta + V^\beta G(E+i\epsilon)V^\alpha)|\rangle$

or $\quad = \langle|V^\alpha + V^\beta G(E+i\epsilon)V^\alpha|\rangle$

These agree "on-shell" because

$$V^\beta - V^\alpha = H^\beta - H^\alpha \to E' - E$$

But we have two extensions which are not equivalent on shell:

$$\boxed{\begin{aligned} T^{\beta\alpha}(z) &= V^\alpha + V^\beta G(z) V^\alpha \\ \tilde{T}^{\beta\alpha}(z) &= V^\beta + V^\beta G(z) V^\alpha \end{aligned}}$$

C. Born Approximation

Becomes

$$t(p'\beta,p,\alpha)\big|_{Born} = \langle p'\beta|V^\alpha|p\alpha\rangle \quad \text{"prior form"}$$
$$= \langle p'\beta|V^\beta|p\alpha\rangle \quad \text{"post form"}$$

Elastic Scattering:

Consider scattering of a off (bc) bound state

For simplicity assume

$m_c \gg m_a, m_b$ (so c just stays fixed at origin)

$V_{ac} = 0$ (so a interacts only with b)

our exact Hamiltonian is

$$H = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_c} + V_{ab}(\vec{r}_{ab}) + V_{bc}(\vec{r}_b)$$

while the relevant channel Hamiltonian is

$$H' = p_a^2/2m_a + p_b^2/2m_b + V_{bc}(\vec{r}_b)$$

$$V' = V_{ab}(\vec{r}_{ab})$$

The wave channels in this channel are

$$\langle x_a x_b | \vec{p}, 1\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{x}_a} \phi_1(\vec{x}_b)$$

and the born amplitude is

$$t_{born} = \langle \vec{p}' 1 | V' | \vec{p}\, 1\rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3x_a\, d^3x_b\; e^{-i\vec{p}'\cdot x_a} \phi_1(x_b)^* V_{ab}(\vec{x}_a - \vec{x}_b) e^{i\vec{p}\cdot\vec{x}_a} \phi_1(\vec{x}_b)$$

Now make the change of integration variable $\vec{x}_a \to \vec{x}_a - \vec{x}_b$

$$t_{born}(\vec{p}', \vec{p}) = \left[\frac{1}{(2\pi)^3} \int d^3x\, V_{ab}(\vec{x}) e^{-i\vec{q}\cdot\vec{x}}\right] \int d^3x_b\, e^{-i\vec{q}\cdot\vec{x}_b} |\phi(x_b)|^2$$

$$\boxed{t_{born} = t_{born}(\vec{p}', \vec{p})\Big|_{pointlike} \times g(\vec{q})}$$

$t_{born}|_{pointlike}$ is born amplitude for scattering of a off free b

$$\boxed{g(\vec{q}) = \int d^3x_b\, e^{-i\vec{q}\cdot x_b} |\phi(x_b)|^2 \quad \text{is elastic form factor}}$$

$g(\vec{q})$ is fourier transform of the probability distribution of particle b.

This method is used to determine e.g. charge distribution of proton in ep scattering

Inelastic Born Approximation - Excitation

Now we consider $a + (bc) \to a + (bc)^*$

Again, assume $m_c \gg m_b, m_a$,

but the assumption $V_{ac} = 0$ is superfluous:

$$t_{born} = \langle \vec{p}', 2 | V' | \vec{p}, 1\rangle =$$

$$\frac{1}{(2\pi)^3} \int d^3x_a\, d^3x_b\; e^{-i\vec{p}'\cdot\vec{x}_a} \phi_2(x_b)^* \left[V_{ab}(\vec{x}_{ab}) + V_{ac}(\vec{x}_a)\right] e^{i\vec{p}\cdot\vec{x}_a} \phi_1(\vec{x}_b)$$

The Vac term vanishes because ϕ_1 and ϕ_2 are orthogonal. The interaction of a with c can't cause excitation if c is infinitely heavy.

$$= t_{born}(\vec{p}\,';\vec{p})|_{pointlike} \times g_{12}(\vec{q})$$

t_{born} is "off the energy shell", $p \neq p'$

and

$$\boxed{g_{12}(\vec{q}) = \int d^3x_b\, e^{-i\vec{q}\cdot\vec{x}_b}\, \phi_2(\vec{x}_b)^* \phi_1(\vec{x}_b) \quad \text{- Inelastic form factor}}$$

E.g. for excitation of hydrogenic atom by electron, inelastic form factor can be explicitly calculated, since wave functions are known, and integrals can be done in closed form.

D. Target-State Expansion and Coupled Channels

What is the asymptotic form of stationary scattering states in the multichannel case?

From L-S eqn we have

$$|p\,1+\rangle = |p\,1\rangle + G^1(E+i\epsilon)V^1|p\,1+\rangle$$

and we can find G^1 by summing over spectrum of the channel Hamiltonian H^1. Assume that channel 1 is $a+(bc)$ channel, with $m_c \gg m_a, m_b$

then
$$H^1 = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} + V_{bc}(\vec{x}_b)$$

and

$$\langle x_a x_b | G^1(z) | x_a' x_b' \rangle = \frac{1}{(2\pi)^3}\int d^3p \sum_\alpha \frac{e^{i\vec{p}\cdot(x_a - x_a')}\, \phi_\alpha(\vec{x}_b)\phi_\alpha(\vec{x}_b')^*}{z - p^2/2m_a - E_\alpha}$$

where $\sum_\alpha$ includes bound and continuum states of b

The integral d^3p can be done explicitly, as in the one-channel case

$$\boxed{\begin{aligned}\langle x_a x_b | G^1(z) | x_a' x_b' \rangle &= \frac{-m_a}{2\pi} \sum_\alpha \frac{e^{ip_\alpha |x_a - x_a'|}}{|x_a - x_a'|} \phi_\alpha(x_b) \phi_\alpha(x_b')^* \\ p_\alpha &= [2m_a(E - E_\alpha)]^{\frac{1}{2}}\end{aligned}}$$

Now we recall Lippmann-Schwinger equation

$$\langle x_a x_b | p\, 1+ \rangle = \langle x_a x_b | p\, 1 \rangle + \int d^3x_a' d^3x_b' \langle x_a x_b | G^1(E+i\epsilon) | x_a' x_b' \rangle V^1(x_a' x_b') \langle x_a' x_b' | p\, 1+ \rangle$$

and suppose, as in the one channel case, that V^1 is localized. We take $x_a \to \infty$ limit, which is appropriate if a is unbound in the final state (collision without rearrangement). We may take $x_a \gg x_a'$ and obtain

$$\boxed{\begin{aligned}\langle x_a x_b | p\, 1+ \rangle \xrightarrow[x_a \to \infty]{} &\frac{1}{(2\pi)^{3/2}} \Big[e^{i\vec{p}\cdot\vec{x}_a} \phi_1(\vec{x}_b) \\ &+ \sum_\alpha \frac{e^{ip_\alpha r_a}}{r_a} \phi_\alpha(\vec{x}_b) \int d^3x_a' d^3x_b' \phi_\alpha(\vec{x}_b')^* e^{ip_\alpha \hat{x}_a \cdot \vec{x}_a'} V^1 \langle x_a' x_b' | p\, 1+ \rangle \Big] \\ &= \frac{1}{(2\pi)^{3/2}} \Big[e^{i\vec{p}\cdot\vec{x}_a} \phi_1(\vec{x}_b) + \sum_\alpha f(p_\alpha \hat{x}_a, \alpha; p, 1) \frac{e^{ip_\alpha r_a}}{r_a} \phi_\alpha(\vec{x}_b) \Big] \\ \text{where} \quad f &= -(2\pi)^2 m_a \langle p_\alpha \hat{x}_a, \alpha | V^1 | p, 1+ \rangle\end{aligned}}$$

This is the obvious generalization of one-channel case. For closed channels, $e^{ip_\alpha r_a} \to 0$, but in general we have discrete sum and integral over continuum.

there may, in general, also be "rearrangement" collisions in which a becomes captured such as

$$a + (bc)_s \rightarrow b + (ac)_\beta$$

then we must consider asymptotic form of $\langle x_a x_b | p\, + \rangle$ for $x_b \to \infty$. It is easy to guess that

$$\langle x_a x_b | p\, + \rangle \xrightarrow[x_b \to \infty]{} \frac{1}{(2\pi)^{3/2}} \Big[-(2\pi)^2 m_b \sum_\beta \frac{e^{ip_\beta r_b}}{r_b} \chi_\beta(\vec{x}_{ac}) \langle p_\beta \hat{x}_b, \beta N' | p\, \ldots$$

(no incident wave term: χ_β is ac bound state)

If $f = -(2\pi)^2 (m_a m_b)^{\frac{1}{2}}\, t$, then

$$\boxed{\langle x_a x_b | p\, + \rangle \xrightarrow[x_b \to \infty]{} \frac{1}{(2\pi)^{3/2}} \left(\frac{m_b}{m_a}\right)^{\frac{1}{2}} \sum_\beta f(p_\beta \hat{x}_b, \beta; \vec{p}\, s) \frac{e^{ip_\beta r_b}}{r_b} \chi_\beta(\vec{x}_{ac})}$$

<u>Target-State Expansion</u>

Is there a general way of expressing the multiple ~~scattering~~ channel scattering problem in terms of (approximately) solvable differential equations? What is our recourse when Born approx fails?

Consider a projectile (coordinate $\vec{x}$) incident on some target (coordinates x_{tar}). We know that

$$\langle \vec{x}, x_{tar} | p\, + \rangle \xrightarrow[x \to \infty]{} \frac{1}{(2\pi)^{3/2}} \Big[e^{i\vec{p}\cdot\vec{x}} \phi(x_{tar}) + \sum_\alpha f(p_\alpha \hat{x}, \alpha; \vec{p}, s) \frac{e^{ip_\alpha r}}{r} \phi_\alpha(x_{tar}) \Big]$$

In general, the exact wave function, as well as its asymptotic form has such an expansion in "target states."

$$\langle \vec{x}, x_{tar} | \vec{p}\, 1 + \rangle = \sum_\alpha \eta_\alpha(\vec{x})\, \phi_\alpha(x_{tar})$$

where

$$\eta_\alpha \xrightarrow[x\to\infty]{} \frac{1}{(2\pi)^{3/2}} \left[e^{i\vec{p}\cdot\vec{x}} \delta_{\alpha 1} + f \frac{e^{ip_\alpha r}}{r} \right]$$

Roughly speaking, the η_α's are one-particle "wave function" for the projectile. What eqns do they satisfy?

Hamiltonian is $\quad H = H' + V' = \frac{P^2}{2m} + H_{target} + V_1$

$$\text{Let} \quad \psi = \sum \eta_\alpha(\vec{x})\, \phi_\alpha(x_{tar}), \quad H\psi = E\psi \Rightarrow$$

$$\sum_\alpha \left[\frac{-\nabla^2}{2m} + E_\alpha + V'(x, x_{tar}) \right] \eta_\alpha \phi_\alpha = E \sum_\alpha \eta_\alpha \phi_\alpha$$

Now we can multiply by $\phi^*_{\alpha'}(x_{tar})$ and integrate dx_{tar} to obtain:

$$\boxed{\begin{aligned} &-\frac{\nabla^2}{2m} \eta_\alpha(\vec{x}) + \sum_{\alpha'} \tilde{V}_{\alpha\alpha'}\, \eta_{\alpha'}(\vec{x}) = (E - E_\alpha)\, \eta_\alpha(\vec{x}) \\ &\qquad \tilde{V}_{\alpha\alpha'} = \int dx_{tar}\, \phi_\alpha(x_{tar})^* V'(x_{tar}, \vec{x})\, \phi_{\alpha'}(x_{tar}) \end{aligned}}$$

This is an infinite number of coupled one-particle equations. In general these are intractable, but sometimes it is a reasonable approximation to truncate the system by retaining only N equations. This is the <u>(N-state) Coupled Channel Approximation</u>, which can then be solved with the $x \to \infty$ boundary condition satisfied by η_α.

Sometimes, in nuclear physics scattering is dominated by a few nearby strongly coupled states, so approx. is not bad

The coupled channel approx does not satisfy unitarity, because probability is lost into the channels we ignore. However there is an effective (optical) potential for which eqns hold exactly. E.g. for just elastic scattering

$$\left(-\frac{\nabla^2}{2m} + V_{opt}\right)\eta_1(\vec{x}) = (E - E_1)\,\eta_1(\vec{x})$$

where V_{opt} is $\begin{cases} \text{nonlocal} \\ \text{energy dependent} \\ \text{non Hermitian} \end{cases}$

Still, it can be used in phenomenological analyses of e.g. nuclear scattering.

E. Analyticity of the Multichannel S-Matrix

Not much is known rigorously about the analytic properties of the S-Matrix in the multichannel case, but it is believed that much of what we learned about the one-channel case survives intact.

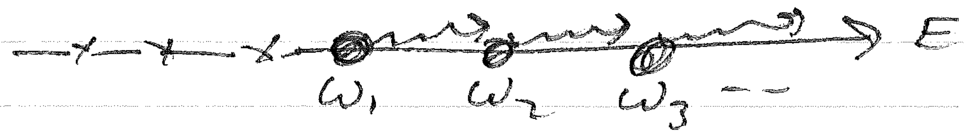
In the Text there is a detailed discussion of a special case: the partial wave S-matrix with N two body channels, in which all potentials have finite range. The discussion uses the coupled channel formalism above, but is mainly a transcription of what we already know into a matrix notation.

What are the main new features?

1) thresholds
2) Resonance Theory

Thresholds

Consider all channels to be 2 body, so only E or $p_\alpha = [2m_\alpha(E - W_\alpha)]^{\frac{1}{2}}$ kinematically labels the state. There are, in general, bound state



poles below first threshold and square root branch cuts at each threshold.

For $W_n < E < W_{n+1}$

S is an $n \times n$ matrix in channel space

At W_{n+1} it graduates to $(n+1) \times (n+1)$. The topology of the Riemann surface is quite complicated.

"Physics" lives on top of all the cuts. The "physical sheet" is defined by requiring all p_α to be in upper half plane. This causes contributions of all closed channels to asymptotic form of wave function to decay exponentially, and prevents asymptotic incoming spherical waves from appearing.

Threshold behavior of cross section

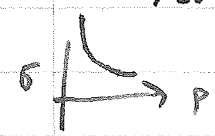
Recall the formula for $2 \to 2$ cross section

$$\frac{d\sigma}{d\Omega}(\vec{p}'\alpha';\vec{p}\,\alpha) = m m' \frac{p'}{p} \left| t(\vec{p}'\alpha';\vec{p}\,\alpha) \right|^2 (2\pi)^4$$

The matrix element of T is expected to behave smoothly in the vicinity of the threshold, so the behavior of the cross section at threshold is determined by the kinematic factor p'/p

1) Exothermic Reaction

For an exothermic reaction $p \to 0$ with p' fixed



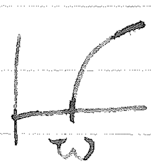
$$\frac{d\sigma}{d\Omega} \sim \frac{1}{p}$$

This is because incident flux is very small. The "$1/v$ law"

Cross section diverges at threshold. Important in nuclear physics - e.g. neutron-induced fission and reactor design.

2) Endothermic reaction

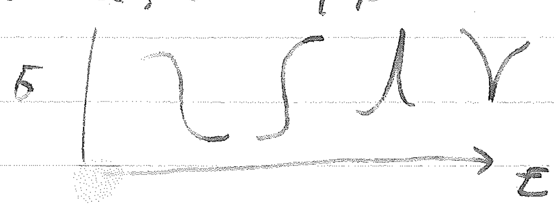
Here $p' \to 0$ with p fixed so $\frac{d\sigma}{d\Omega} \propto \sqrt{E-W}$



This has infinite slope at threshold

By unitarity (i.e., the optical theorem), the infinite slope of $\sigma(E)$ is felt by amplitude and therefore also has an impact on other channels, where slopes also become infinite (e.g. in channel coupled to the new threshold which is already open)

We can get "rounded step" or "cusp"

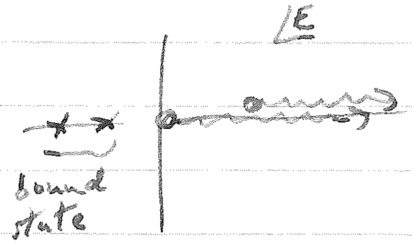


We'll say a little more about the connection between unitarity and threshold behavior in the next lecture.

F. Resonances

In the multichannel case, we will find some new elements in the theory of resonance poles which did not turn up in the one channel case.

Let's imagine that n two body channels are open, so that $S = S_{\alpha'\alpha}$ is an $n \times n$ matrix (It may or may not be projected onto a particular partial wave)



The S matrix has many right hand cuts, and may have bound state poles, otherwise it is analytic on the physical sheet.

In the one-channel case, S was real below threshold; now its eigenvalues are real; i.e., it is Hermitian. Thus in addition to the unitarity condition

this makes sense only below 3 particle threshold

$$\boxed{S^\dagger(E)S(E) = \mathbb{1}, \quad E \text{ physical}}$$

we have the Schwarz reflection principle

$$\boxed{S(z) = S^\dagger(z^*)}$$

From these two equations, we see immediately how to continue S through the cut. We need an analytic function $\tilde{S}(z)$ such that

$$\tilde{S}(E-i\epsilon) = S(E+i\epsilon)$$
$$= S^\dagger(E-i\epsilon) = S^{-1}(E-i\epsilon)$$

(Unitarity can be used below cut, by Schwarz reflection principle)

Thus, we may simply choose the function $\boxed{\tilde{S}(z) = S^{-1}(z)}$

S^{-1} is analytic in z if it exists. The "zeros" of S on the first sheet have become poles on the 2nd sheet

There are no essential singularities or cuts on the second sheet. If there were, they would have to appear on the first sheet too, because the inverse of a cut or essential singularity is a cut or essential singularity.

Suppose the (e.g. partial wave) S-Matrix has a pole at $E = E_R - i\Gamma/2$. We may write

$$S = S_B + \frac{B}{E - E_R + i\Gamma/2} \quad \text{— An } n\times n \text{ matrix}$$

$\xrightarrow{x} E$ We suppose that the pole is near the real axis, so that there is a physical values of the energy near which the energy dependence of S is dominated by the pole. Then we approximate S_B by a constant, and unitarity (independent of E) requires S_B to be unitary. (Actually we have by analytic continuation $S(E)S^\dagger(E^*) = \mathbb{1}$ so these "smoothness" assumptions are not yet needed)

Wanted to know we are talking about a region where $S_B \sim$ constant

Let's rewrite this as

$$S = S_B\left(\mathbb{1} - \frac{i\Gamma P}{E - E_R + i\Gamma/2}\right)$$

From $S(E)S^\dagger(E^*) = \mathbb{1} \Rightarrow \left(\mathbb{1} - \frac{i\Gamma P}{E - E_R}\right)\left(\mathbb{1} + \frac{i\Gamma P^\dagger}{E - E_R^*}\right) = \mathbb{1}$

The residues of the poles on the LHS must vanish

$$\Rightarrow \quad (\mathbb{1} - P)P^\dagger = 0$$
$$P(\mathbb{1} - P^\dagger) = 0 \quad \Rightarrow \boxed{P = P^\dagger, \quad P^2 = P}$$

Hence P is an orthogonal projection

Such a projection can be written

$$P = \sum_{\nu} e^{(\nu)} e^{(\nu)+}$$

Suppose first that there is only one e, a single ground state.

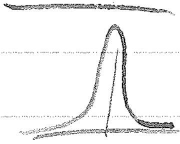
$$\boxed{S = S_B \left(\mathbb{1} - \frac{i\Gamma e e^{+}}{E - E_R + i\Gamma/2} \right)}$$

First, we ignore background scattering, $S_B = \mathbb{1}$

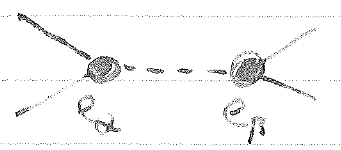
then

$$\boxed{(S - \mathbb{1})_{\alpha\beta} = \frac{-i\Gamma e_\alpha e^{*}_\beta}{E - E_R + i\Gamma/2}}$$

This is a Breit-Wigner resonance



A graphical representation of the scattering process is



the resonance "couples" to channel β with strength $\gamma_\beta = \sqrt{\Gamma}\, e_\beta$, "propagates" with propagator $\frac{-i}{E - E_R + i\Gamma/2}$ and then decays through coupling of strength $\gamma_\alpha = \sqrt{\Gamma}\, e_\alpha$ to the channel α.

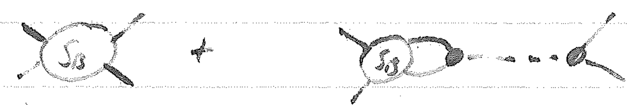
Production and decay are two independent events. This is the key physical feature of multichannel resonance theory.

The quantities $|\gamma_\alpha|^2 = \Gamma |e_\alpha|^2$ are called partial widths of the resonance; they must all sum up to the total width Γ

If we do not ignore background scattering, but still assume a single resonance, we have

$$(S-\mathbb{1})_{\alpha\beta} = (S_B-\mathbb{1})_{\alpha\beta} - S_B \frac{i\Gamma e_\alpha e_\beta^*}{E-E_R+i\Gamma/2}$$

which becomes diagramatically



After decay of the resonance, the final state particles experience background scattering. (That the background scattering is at the end instead of the beginning is a matter of convention.)

$|\sqrt{\Gamma}(S_B e_\alpha)|^2 = |\gamma_\alpha|^2$ is partial width to leave resonance for channel
$\Gamma|e_\beta|^2 = |\gamma_\beta|^2$ " " " " enter " from "

In general, we must make the distinction between partial widths for entering and leaving, if resonance is not Breit Wigner, and T is not a good symmetry.
(T commutes with H and is antiunitary $\Rightarrow$ $S = T^+S^+T \Rightarrow \langle \chi|S|\phi\rangle = \langle T\chi|S^+|T\phi\rangle^* = \langle T\phi|S|T\chi\rangle$ $\Rightarrow$ S is symmetric)
S_B just multiplies by a phase.

What do we see when a resonance dominates a given partial wave?

$$\sigma_\alpha^{elastic}(E,l) = \frac{\pi}{p^2}(2l_\alpha+1)|S_l^{\alpha\alpha}-1|^2$$

$$\sigma_\alpha^{total}(E,l) = \frac{2\pi}{p^2}(2l_\alpha+1)\,\mathrm{Im}\left(\frac{1}{i}\right)(S_l^{\alpha\alpha}-1)$$

or

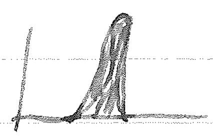
$$\sigma_\alpha^{elastic}(E,l) = \frac{\pi}{p^2}(2l_\alpha+1)\frac{\Gamma^2|\tilde{e}_\alpha|^2|e_\alpha|^2}{(E-E_R)^2+\Gamma^2/4}$$

$$\sigma_\alpha^{total}(E,l) = \frac{\pi}{p^2}(2l_\alpha+1)\frac{\Gamma^2\,\mathrm{Re}(\tilde{e}_\alpha e_\alpha) - \Gamma(E-E_R)\,\mathrm{Im}\,\tilde{e}_\alpha e_\alpha}{(E-E_R)^2+\Gamma^2/4}$$

In the Breit-Wigner case, $\underbrace{\tilde{e}_\alpha = e_\alpha}_{\text{or just by time reversal!}}$, so we have

$$\frac{\sigma^{elastic}}{\sigma_{Total}} = |e_\alpha|^2$$ – As expected, the probability of resonance decay into entrance channel.

The "area under" the total cross section curve – i.e. the resonance peak – is



$$\int dE\, \sigma_T(E) \propto \frac{\Gamma}{m}|e_\alpha|^2$$ – partial width into entrance channel

By measuring height of σ_T resonance, we measure partial width (as well as total width)! E.g. J/ψ

Now suppose P is a multidimensional resonance projector $P = e^{(r)}e^{(r)+}$ where $e^{(r)+}e^{(s)} = \delta^{rs}$. Unitarity enforces the form

$$S = S_B\left(1 - \frac{i\Gamma P}{E - E_R + i\Gamma/2}\right) = S_B \prod_r \left(1 - \frac{i\Gamma e^{(r)}e^{(r)+}}{E - E_R + i\Gamma/2}\right)$$

which suggests the form of S when there are many nearby poles, in region where S_B is "constant"

$$S = S_B \prod_i \left(1 - \frac{i\Gamma_i e^{(i)}e^{(i)+}}{z - z_i}\right)$$ – which is the general form.

G. Final State Interactions

How can we study e.g. $\pi\pi$ scattering? there are π beams, but no π targets. what one does is investigate

$$\pi + p \rightarrow \pi + \pi + p$$

and look at e.g. $\pi\pi$ angular distributions in final state. this is a least partly affected by the interactions of the two final pions.

Let's consider in general the division

$$V = V_{I} + V_{II}$$

where

V_{II} = primary interaction which makes a given process occur.

V_{I} = final-state interaction, which modifies the process (and can act in initial state as well as final state)

this perverse labeling follows Taylor; the reason for it emerges below

E.g. Photoelectric effect

$$\gamma + H \longrightarrow e + p$$

We are interested in the effect of the coulomb attraction of e and p on the photoionization process.

V_{II} is interaction of photon with electron
V_I is coulomb interaction

We can treat coulomb interaction exactly, and treat V_{II} as a perturbation.

If we consider a process in which no rearrangements occur, so that $V = V_I + V_{II}$ both before and after collision, we may reason as follows

Lippmann-Schwinger: $G = G_I + G_I V_{II} G_I$
$= G_I + G_I V_{II} G_I + \cdots$

(We will not go beyond lowest order in V_{II})

$$T = V + VGV$$
$$= V_I + V_{II} + (V_I + V_{II})(G_I + G_I V_{II} G_I)(V_I + V_{II}) + \cdots$$

If we keep only terms up to order V_{II}, we have

$$T = T_I + (\mathbb{1} + V_I G_I)\, V_{II}\, (\mathbb{1} + G_I V_I) + \cdots$$

Since $(\mathbb{1} + G_I(E+i\epsilon)V_I)\,|\vec{p}\rangle = |\vec{p}+\rangle_I$
$\langle \vec{p}\,'|\,\mathbb{1} + V_I G_I(E-i\epsilon) = {}_I\langle \vec{p}\,'-|$

we have

$$\boxed{t(\vec{p}\,',\vec{p}) = t_I(\vec{p}\,',\vec{p}) + {}_I\langle \vec{p}\,'-|\, V_{II}\, |\vec{p}+\rangle_I} + \cdots$$

This is the
<u>Distorted Wave Born Approximation</u>

Considering again the photoelectric effect we have, if $V_{II}(\vec{k})$ describes perturbation responsible for absorbtion of photon, and we treat proton as infinitely heavy, then

$$t(\vec{p}, e+p; \vec{k}, \gamma+H) \hat{=} \langle \vec{p}(e+p)- | V_{II}(\vec{k}) | \vec{k}, \gamma+H \rangle$$

V_{II} annihilates the photon, and we are left with a matrix element between a coulomb bound state and a coulomb scattering state

We are not interested in the complications associated with photon polarizations or the long range tail of the coulomb potential. Let us consider a general process in which a quantum of radiation is absorbed an an "atom" is ionized; We have

$$t(\vec{p}, \vec{k}) = \langle \vec{p}- | B_{\vec{k}} | \phi \rangle$$

where ϕ is the bound state, $|\vec{p}-\rangle$ a scattering state, describing an "electron" in field of stationary "proton"

$$t(\vec{p}, \vec{k}) = \int d^3x \quad \langle \vec{p}- | \vec{x} \rangle B_{\vec{k}}(\vec{x}) \, \phi(\vec{x})$$

We now make a further approximation. In atomic physics we are generally interested in the cause in which $pa \ll 1$ where a is size of bound state. This means that $\langle \vec{p}- | \vec{x} \rangle$ varies slowly where integrand is appreciable, and can be approximated by its value at $\vec{x}=0$ (or nearby). Only the $l=0$ partial wave contributes, and we have

$$\langle \vec{p}- | \vec{x} \rangle \cong \frac{1}{(2\pi)^{3/2}} \frac{\psi_{0p}(r)}{pr}$$

(What we actually have on p (6-5) is
$\langle \vec{x} | \vec{p}+ \rangle|_{s\,wave} = \frac{1}{(2\pi)^{3/2}} \frac{\psi_{0p}(r)}{pr}$. The above follows from time reversal invariance.)

Here ψ_{0p} is the exact solution to the radial eqn for the final state potential V_I.

We know $\psi_{0p} = \frac{\phi_{0p}}{f_0(p)} \sim \frac{\sin pr}{f_0(p)}$

Thus we have

$$\boxed{t(\vec{p},\vec{k}) \sim \frac{1}{f_0(p)} \frac{1}{(2\pi)^{3/2}} \int d^3x \, \frac{\sin pr}{pr} B_{\vec{k}}(\vec{x}) \phi(\vec{x})}$$

The effect of the final state interactions is simply to replace $\langle \vec{p} | \vec{x} \rangle$ by $\langle \vec{p}- | \vec{x} \rangle$. Thus we have

$$\boxed{t(\vec{p},\vec{k}) = \frac{t_0(\vec{p},\vec{k})}{f_0(p)}}$$

where t_0 is the amplitude in the absence of final state interaction. We see:

- Attractive final-state interactions <u>enhance</u> the amplitude
- Repulsive final-state interactions <u>suppress</u> the amplitude

Near threshold, $p \sim 0$, $\frac{\sin pr}{pr} \sim 1$, and since R is finite at threshold $B_{\vec{k}} \to$ constant as $p \to 0$. Thus

$$\boxed{t(\vec{p}, \vec{k}) \sim \frac{\text{constant}}{f_0(p)} \quad \text{(Near threshold)}}$$

The threshold behavior of the amplitude is determined solely by the final state interaction, and cross section is

$$\frac{d\sigma}{d\Omega} \propto \frac{p}{k}|t|^2 \sim \text{constant} \frac{p}{|f_0(p)|^2}.$$

$f \sim 1 + iap \Rightarrow$ scattering length can be measured

Many Body final states

For many body final state, the analysis of final state interactions is more complicated, because there are many pairs of particles which can interact

But sometimes the interactions of two particles with each other dominate their interactions with other particles in the final state because

1) Resonance
2) Low relative momentum (they sit on top of each other and have plenty of time to interact)

In this case, distorted wave factorizes

$$\langle x | p - \rangle \simeq \langle x_{cd} | p_{cd} - \rangle \, e^{i \vec{p}_T \cdot \vec{x}_{cd}} \times \text{function of other momenta}$$

scattering state for cd interacting by V_{cd}

$$\Longrightarrow \quad t \sim \frac{1}{f_0(p_{cd})} \times \text{function of other momenta}$$

Thus c-d resonances can be seen in final state by plotting σ vs. p_{cd}
the basis of "bump hunting"

This is how e.g. the ρ resonance is seen in $p\bar{p}$ scattering.