IV. Time-independent scattering formalism

More useful than Time-Dependent approach for setting up calculations.

A. The Green's Operator

We define the "Green's Operator" or "Green's Function" associated with H as

$$G(z) = (z-H)^{-1}$$

This is defined for $z \in \rho(H)$, the resolvent set of H

$G(z)$ is an analytic function of z, i.e. has a convergent power series expansion, for $z \in \rho(H)$ (This means that the matrix elements of $G(z)$ are analytic functions)

This follows from the fact that $G(z)$ is <u>bounded</u> for $z \in \rho(H)$

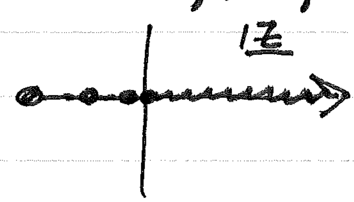
$$G(z) = \frac{1}{z-z_0+z_0-H} = \frac{1}{z_0-H}\left[1-\frac{z_0-z}{z_0-H}\right]^{-1} = G(z_0)\sum_{n=0}^{\infty}(z_0-z)^n[G(z_0)]^n$$

But if $\sup \frac{\|G(z_0)\psi\|}{\|\psi\|} = \|G(z_0)\| < \infty$ ($G(z_0)$ bounded)

This series converges uniformly for $|z_0-z| < \frac{1}{\|G(z_0)\|}$;

i.e. $G(z)$ is analytic in a disc about z_0

The "spectrum" of H consists of the points of nonanalyticity of $G(z)$



Recalling the spectral representation

$$f(H) = \int_{\mathbb{R}} f(z)\, dP_z$$

we see that $G(z)$ may be written

$$G(z) = \frac{1}{z-H} = \sum_n \frac{|n\rangle\langle n|}{z-E_n} + \int dE \sum_m \frac{|E,m\rangle\langle E,m|}{z-E}$$

bound states (point spectrum)

scattering states (continuous spectrum)

$G(z)$ is ambiguous (undefined) for $z \in \sigma(H)$

For the point spectrum, this just reflects the fact that the solution to

$$(z-H)\,\psi = \varphi$$

has a solution ψ which is determined only up to the kernel of $z-H$. $G(z)$ has a pole at z = eigenvalue of $H = E_n$. The residue of the pole is the projection operator $|n\rangle\langle n|$, i.e., $\langle \psi | G(z) | \varphi\rangle$ is analytic with a pole at $z = E_n$ and residue $\langle \psi | n\rangle\langle n|\varphi\rangle$

Thus knowing the poles of $G(z)$ is equivalent to knowing all the bound states of H.

For z in the continuous spectrum $[0,\infty]$ (if scattering threshold is at $E=0$) $G(z)$ has a branch cut.

The discontinuity across this cut is a projection operator

$$G(E_0+i\epsilon) - G(E_0-i\epsilon) = \int dE \sum_m |E,m\rangle\langle m,E| \left(\frac{1}{E_0+i\epsilon-E} - \frac{1}{E_0-i\epsilon-E}\right)$$

If it makes sense to continue $|E,m\rangle\langle m,E|$ as an analytic function of E, this is equivalent to $\oint dE \sum_m |E,m\rangle\langle E,m| \frac{1}{E_0-E}$ around $\odot$.

$$= -2\pi i \sum_m |E,m\rangle\langle m,E|$$

(i.e. $\lim_{\epsilon \to 0} \frac{\epsilon/\pi}{x^2+\epsilon^2}$ is a representation of $\delta(x)$) 4.3

of course, this projection operator makes sense only when smeared with suitable functions

The operator $(z-H)^{-1}$ fails to exist for z in the continuous spectrum because $(z-H)\psi=0$ has solutions, although the solutions are not normalizable

For the free Hamiltonian $H^0 = \frac{1}{2}p^2$, the Green's operator can be explicitly found - It is the solution to $(z-\frac{1}{2}p^2)G^0(z) = \mathbb{1}$, or, in the momentum representation

$$\langle \vec{p}\,'| G^0(z) | \vec{p}\rangle = \frac{1}{z-\frac{1}{2}p^2}\,\delta^3(\vec{p}-\vec{p}\,')$$

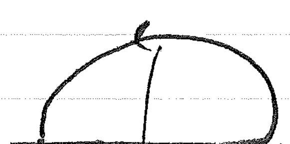
To find G^0 in the ~~momentum~~ position representation, we may Fourier transform

$$\langle \vec{x}\,'|G^0(z)|\vec{x}\rangle = \int d^3p\, d^3p'\, \langle \vec{x}\,'|\vec{p}\,'\rangle\langle \vec{p}\,'|G^0(z)|\vec{p}\rangle\langle \vec{p}\,|\vec{x}\rangle$$

$$= \frac{1}{(2\pi)^3}\int d^3p\, \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{x}\,')}}{z-\frac{1}{2}p^2}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty p^2 dp \int_{-1}^{1} d\cos\theta\, \frac{e^{ip|x-x'|\cos\theta}}{z-\frac{1}{2}p^2}$$

$$= \frac{-i}{4\pi^2}\int_0^\infty \frac{p\,dp}{|x-x'|}\,\frac{e^{ip|x-x'|}-e^{-ip|x-x'|}}{z-\frac{1}{2}p^2} = \frac{i}{2\pi^2|x-x'|}\int_{-\infty}^{\infty} dp\,p\,\frac{e^{ip|x-x'|}}{p^2-2z}$$



close contour in upper half plane $\Rightarrow$

$$= \frac{i}{2\pi^2|x-x'|}\,\frac{2\pi i}{2\sqrt{2z}}\sqrt{2z}\; e^{i(2z)^{\frac{1}{2}}|x-x'|}$$

where $\sqrt{z}$ means root with positive imaginary part

$$\boxed{\langle \vec{x}\,'|G^0(z)|\vec{x}\rangle = -\frac{1}{2\pi}\,\frac{\exp[i\sqrt{2z}\,|\vec{x}-\vec{x}\,'|]}{|\vec{x}-\vec{x}\,'|}}$$ or $-\frac{1}{2\pi}\frac{e^{\pm ipr}}{r}$

↓ + square root

↑ − square root

This has a discontinuity at $z = E_0 = \frac{1}{2}p_0^2$

$$\langle \vec{x}' | G^0(z+i\epsilon) - G^0(z-i\epsilon) | \vec{x}\rangle = \frac{-1}{2\pi|x-x'|}\left(e^{ip_0|x-x'|} - e^{-ip_0|x-x'|}\right)$$

$$= \frac{-i}{\pi|x-x'|}\sin(p_0|x-x'|) = -2\pi i \int d\Omega \langle \vec{x}'|p\Omega\rangle \langle p\Omega|\vec{x}\rangle,$$

as on p.4.2

Very simple analytic structure:
only two sheets, and a square root branch cut

B. Relation of Green's Operator to Wave Operators

The significance of $G(z)$ is clarified if we establish a connection with $\Omega_\pm$

Recall $\Omega_\pm = \lim_{t\to\mp\infty} e^{iHt}e^{-iH^0t}$

which can be written as the integral of its derivative

$$\Omega_\pm = \mathbb{1} + i\int_0^{\mp\infty} e^{iHt}Ve^{-iH^0t} \qquad V = H - H^0.$$

Now, let's consider $\Omega_\pm$ acting on momentum eigenstates.

$$\Omega_\pm|\vec{p}\rangle = |\vec{p}\rangle + i\lim_{\epsilon\to0^+}\int_0^{\mp\infty} dt\, e^{\pm\epsilon t} e^{iHt}Ve^{-iH^0t}|\vec{p}\rangle$$

As usual, this must be understood as a statement which makes sense only acting on momentum components of a normalizable wave function. (In fact $\Omega_\pm$ don't exist acting on momentum eigenstates, because momentum eigenstates do not spread)
If $\Omega_\pm|\vec{p}\rangle$, suitably interpreted, converges anyway, then there is no harm in inserting the $e^{\pm\epsilon t}$ — an "adiabatic turning-on-and-off function."

Now the integral becomes

$$i\lim_{\epsilon\to0^+}\int_0^{\mp\infty} dt\, e^{-i(E_p\pm i\epsilon - H)t} = i\,\frac{-1}{-i}\,\frac{1}{(E_p\pm i\epsilon - H)}$$

(The ϵ kills upper limit of integration)

$$= G(E_p \pm i\epsilon)$$

Finally, we have derived

$$\boxed{\Omega_\pm |\vec{p}\rangle = |\vec{p}\rangle + G(E_p \pm i\epsilon) V |\vec{p}\rangle}$$ (lim $\epsilon \to 0^+$ understood)

and we understand at last the reason for the $\Omega_\pm$ sign convention.

The improper states defined by

$|\vec{p}\pm\rangle = \Omega_\pm |\vec{p}\rangle$ are sometimes called the <u>stationary scattering states</u>. From the intertwining relations $H\Omega_\pm = \Omega_\pm H_0$, we see that these are "improper eigenstates" of H

$$H|\vec{p}\pm\rangle = \Omega_\pm E_p |\vec{p}\rangle = E_p |\vec{p}\pm\rangle$$

We will discuss the further properties of these states in a short while. (They are unnormalizable solutions to the Schrodinger Eqn.)

C. The Lippman-Schwinger Equation

From the simple operator equation

$$A^{-1} = B^{-1} + B^{-1}(B-A)A^{-1} \qquad A^{-1} = \frac{1}{z-H}$$

It follows that $(= B^{-1} + A^{-1}(B-A)B^{-1})$ $\qquad B^{-1} = \frac{1}{z-H_0}$

$$\boxed{\begin{aligned} G(z) &= G^0(z) + G^0(z) V G(z) \\ G(z) &= G^0(z) + G(z) V G^0(z) \end{aligned}}$$ Lippmann-Schwinger Eqn

This trivial equation is the basis of perturbation theory since it is solved by

$$G(z) = (\mathbb{1} - G^0 V)^{-1} G^0(z)$$
$$= G^0(z) + G^0 V G^0 + (G^0 V)^2 G^0 + \cdots$$

We will discuss the properties of this series (the Born series or Neumann series) later.

D. Relation of Green's operator to the S-Matrix

It is convenient to define the T operator

$$T(z) = V + V G(z) V$$

it obeys the identities:

$$G^0 T = G V \qquad \text{which follow from the}$$
$$T G^0 = V G \qquad \text{Lippmann-Schwinger Eqn.}$$

T is related to the S-matrix $\quad S = \Omega_-^{+} \Omega_+$

$$S = \lim_{t \to \infty} e^{iH_0 t} e^{-2iHt} e^{iH_0 t}$$

$$\left(= \lim_{t', t \to \infty} e^{iH_0 t'} e^{-iHt'} e^{-iHt} e^{iH_0 t}\right)$$

$$S = \mathbb{1} - i \int_0^\infty dt \, e^{iH_0 t} V e^{-2iHt} e^{iH_0 t} + e^{iH_0 t} e^{-2iHt} V e^{iH_0 t}$$

Now we consider matrix elements of S in the momentum representation, and repeat the trick of inserting $e^{-\epsilon t}$:

$$\langle \vec{p}' | S - 1 | \vec{p} \rangle = -i \lim_{\epsilon \to 0^+} \int_0^\infty dt \, \langle \vec{p}' | \left(V e^{i(E_{p'} + E_p + i\epsilon - 2H)t} + e^{i(E_{p'} + E_p + i\epsilon - 2H)t} V \right) | \vec{p} \rangle$$

$$= \frac{1}{2} \langle \vec{p}' | \left[V G\left(\tfrac{1}{2}E_p' + \tfrac{1}{2}E_p + i\epsilon\right) + G\left(\tfrac{1}{2}E_p' + \tfrac{1}{2}E_p + i\epsilon\right) V \right] | \vec{p} \rangle$$

and using the identities $\quad V G = T \frac{1}{z - H^0}$

$$G V = \frac{1}{z - H^0} T$$

$$= \frac{1}{2} \left[\underbrace{\frac{1}{\frac{1}{2}E_{p'} - \frac{1}{2}E_p + i\epsilon} + \frac{1}{\frac{1}{2}E_p - \frac{1}{2}E_{p'} + i\epsilon}}_{} \right] \langle \vec{p}' | T\left(\tfrac{1}{2}E_p + \tfrac{1}{2}E_p' + i\epsilon\right) | \vec{p} \rangle$$

$$\frac{1}{z + i\epsilon} - \frac{1}{z - i\epsilon} = \frac{-2i\epsilon}{z^2 + \epsilon^2} = -2\pi i \, \delta(z)$$
$$z = E_p - E_{p'}$$

Hence $\quad \boxed{\langle \vec{p}' | (S - 1) | \vec{p} \rangle = -2\pi i \, \delta(E_p - E_{p'}) \langle \vec{p}' | T(E_p + i\epsilon) | \vec{p} \rangle}$

The (on-shell) t matrix defined on page (2.13) is related to the T operator by

$$t(\vec{p}\,';\vec{p}) = \lim_{\epsilon\to 0^+} \langle \vec{p}\,'| T(E_p + i\epsilon) |\vec{p}\rangle \qquad (E_p = E_{p'})$$

but T has matrix elements between plane wave states of different energies, and can be continued into the complex energy plane

The T operator is sometimes called the "off-shell" T-matrix because it generalizes t to unphysical momenta, for which $E_p \neq E_{p'}$. This generalization is useful because the (off-shell) T-matrix satisfies a Lippmann-Schwinger eqn of its own.

$$T(z) = V + VG^0(z)\,T(z)$$

(which we derive by inserting

$$T = V + VGV = V + V(G^0 + G^0VG)V$$)

and this eqn also has a perturbative expansion

$$T = (1 - VG^0)^{-1}V = V + VG^0V + \cdots$$

(Born Series)

E. Stationary Scattering States

These are obtained by acting on plane waves with the wave operators.

$$|\vec{p}\pm\rangle = \Omega_\pm |\vec{p}\rangle \qquad \text{stationary \underline{scattering states}}$$

By the intertwining relation $H\Omega_\pm = \Omega_\pm H_0$ we see that these are "improper eigenstates" of the exact Hamiltonian H

$$H|\vec{p}\pm\rangle = \Omega_\pm H_0 |\vec{p}\rangle = E_p |\vec{p}\pm\rangle$$

The waveoperators take plane waves to (nonnormalizable) solutions to the time-independent Schrodinger eqn $\quad (H-E)|\vec{p}\pm\rangle = 0$

(The wave packet $\int d^3p\, \tilde{\psi}(p)|\vec{p}\rangle$ becomes $\int d^3p\, \tilde{\psi}(p)|\vec{p}\pm\rangle$)

The states $|\vec{p}\pm\rangle$ obey a Lippmann-Schwinger Eqn of their own

$$|\vec{p}\pm\rangle = |\vec{p}\rangle + GV|\vec{p}\rangle$$
$$= |\vec{p}\rangle + (G^0 + G^0VG)V|\vec{p}\rangle$$
$$\boxed{|\vec{p}\pm\rangle = |\vec{p}\rangle + G^0 V|\vec{p}\pm\rangle}$$

i.e. $G^0(E_p \pm i\epsilon)$

These solutions are the focal point of the standard textbook discussion of scattering theory, e.g., in Merzbacher. To establish the connection with the standard treatment, we must derive the asymptotic form at spatial infinity of the solutions $|\vec{p}\pm\rangle$.

We derived on p. (4.3) that

$$\langle \vec{x} | G^0(E_p \pm i\epsilon) | \vec{x}' \rangle = -\frac{1}{2\pi} \frac{e^{\pm ip|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|},$$

and therefore

$$\langle \vec{x} | \vec{p}\pm \rangle = \langle \vec{x} | \vec{p} \rangle - \frac{1}{2\pi} \int d^3x' \, \frac{e^{\pm ip|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} V(\vec{x}') \langle \vec{x}' | p\pm \rangle$$

Let's suppose that V is "localized" in a region with a characteristic size of order a, so that we can expand in a/r for $r = |\vec{x}| \to \infty$

$$|\vec{x}-\vec{x}'| = (x^2 - 2\vec{x}\cdot\vec{x}' + x'^2)^{\frac{1}{2}} = r\left(1 - \frac{\vec{x}\cdot\vec{x}'}{r^2} + \cdots\right)$$

$$\Rightarrow \langle \vec{x} | \vec{p}\pm \rangle = \langle \vec{x} | \vec{p} \rangle - \frac{e^{\pm ipr}}{2\pi r} \int d^3x' \, e^{\mp ip\hat{x}\cdot\vec{x}'} V(\vec{x}') \langle \vec{x}' | \vec{p}\pm \rangle$$

$$= \frac{1}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{x}} - (2\pi)^{\frac{1}{2}} \langle \pm p\hat{x} | V | \vec{p}\pm \rangle \frac{e^{\pm ipr}}{r} + \cdots$$

and $\langle \pm p\hat{x} | V | \vec{p}\pm \rangle = \langle \pm p\hat{x} | V(1 + GV) | \vec{p} \rangle$

$$= \langle \pm p\hat{x} | T(E_p \pm i\epsilon) | \vec{p} \rangle$$

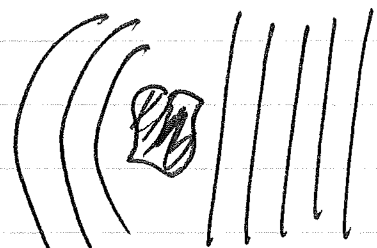
or

$$\boxed{\langle p\hat{x} | V | \vec{p}+ \rangle = -\frac{1}{(2\pi)^2} f(p\hat{x}, \vec{p})}$$

$$\Rightarrow \boxed{\langle x | \vec{p}+ \rangle = \frac{1}{(2\pi)^{3/2}} \left(e^{i\vec{p}\cdot\vec{x}} + f(p\hat{x}, \vec{p}) \frac{e^{ipr}}{r} + \cdots \right)}$$

Thus, the solution to the Schrodinger eqn which becomes asymptotically in the far past a plane wave has a piece which is an <u>outgoing</u> spherical wave. Taking $E_p \pm i\epsilon$

corresponds to imposing outgoing/incoming boundary conditions on the greens functions and solutions



Note: $|\vec{p}+\rangle$ rather than $|\vec{p}-\rangle$ is the solution which is experimentally relevant, because in an experiment, we control the momentum of the beam, not of the final state ($|\vec{p}-\rangle$ is a collapsing spherical wave.)

V Methods of Approximation

A. The Born Series

As physicists, we are chiefly interested in scattering theory because it enables us to compute quantities (scattering cross sections) which can be observed experimentally. E.g. Given a potential $V(\vec{x})$, we would like to compute $d\sigma/d\Omega$. But except in exceptional cases, we cannot solve this problem exactly, we must resort to approximate methods.

The 'T' operator obeys a Lippmann-Schwinger Eqn:

$$T = V + VGV$$
$$= V + V(G^0 + G^0VG)V$$
$$\boxed{T = V + VG^0T}$$

This has the solution $T = (1 - VG^0)^{-1}V$ if the inverse exists, or the formal series solution

$$T = V + VG^0V + VG^0VG^0V + \cdots$$

This the <u>Born Series</u> (or Neumann Series) solution

If we replace V by λV where λ is a real parameter (a "coupling constant") the Born Series is a power series in λ. Whether it converges will be determined by the analytic structure of T as a function of λ (see below)

B. The Born Approximation

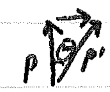
If λ is sufficiently small, it should be a reasonable approximation to keep only the first term in the Born Series:

$$T^{(1)} \equiv V$$ This is called the "(first) Born approximation"

If V is a potential function, then...

$$-\frac{1}{(2\pi)^2} f^{(1)}(\vec{p}\,';\vec{p}) = t^{(1)}(\vec{p}\,';\vec{p}) = \langle \vec{p}\,' | T(E_p + i\epsilon) | \vec{p} \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3x \, e^{-i(\vec{p}\,' - \vec{p})\cdot x} \, V(\vec{x})$$



$$\boxed{t^{(1)}(\vec{p}\,';\vec{p}) = \frac{1}{(2\pi)^3} \tilde{V}(\vec{q})}$$ where $\vec{q} = \vec{p}\,' - \vec{p}$ is the momentum transfer

This is just the Fourier transform of the potential. In Born approximation t depends only on $\vec{q}$, and not seperately on E_p. E.g. at $\vec{q}=0$ (or any other fixed $\vec{q}$), the amplitude is independent of energy. (It is also real, the optical theorem

$$\mathrm{Im}\, t(\vec{p},\vec{p}) = -\pi p \int d\Omega \, |t|^2$$

fails because $|t|^2$ is order λ^2)

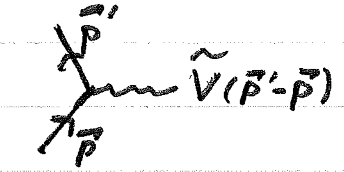
(scattering angle $|\vec{q}| = 2|\vec{p}|\sin\frac{\theta}{2}$: scattering peaked toward forward at high energy)

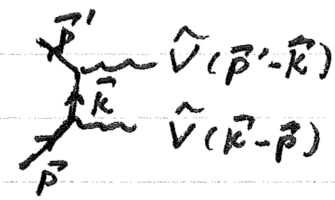
Higher order corrections can be systematically calculated:

$$T^{(2)} = V + V G^0 V$$

$$t^{(2)}(\vec{p}\,',\vec{p}) = \int d^3k \, \langle \vec{p}\,' | V | \vec{k} \rangle \frac{1}{E_p - E_k + i\epsilon} \langle \vec{k} | V | \vec{p} \rangle$$

The terms in the Born series can be conveniently represented by diagrams:

$t^{(1)} =$ 

$t^{(2)} =$ 

The generalization to higher orders is obvious

represents a "freely propagating" particle

represents action of the interaction V

These diagrams are called Feynman Diagrams. The Born series is a particularly simple example of a Feynman diagram expansion.

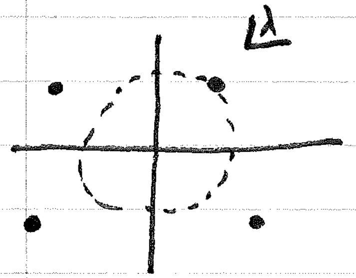
C. Convergence of the Born Series

Under what conditions does the born series converge? It is an expansion of

$$(1-\lambda K)^{-1}V \quad (\text{where } K = VG^{0})$$

i.e., it is essentially the resolvent of K

$$\propto \left(\frac{1}{\lambda} - K\right)^{-1}$$



The radius of convergence of an expansion about $\lambda = 0$ is determined by the closest singularity of $(1-\lambda K)^{-1}$ to the origin.

The radius of convergence is zero if K is an unbounded operator, but it is finite if K is bounded, and determined by the sup of the spectrum of K, i.e.

$$\lambda_{max} = \frac{1}{\|K\|} \quad \text{is radius of convergence}$$

Therefore, we are interested in the properties of the spectrum of K.

Thus, the convergence properties of the Born series for

$$T = (1-\lambda K)^{-1} V \quad , \quad \lambda K = \lambda V G^0$$
$$\text{or} \quad = V(1-\lambda K)^{-1} \qquad \lambda K = G^0(\lambda V)$$

are determined by the analytic properties of the resolvent of K, or $(1-\lambda K)^{-1}$

The problem of inverting $I-\lambda K$ is equivalent to solving the integral equation

$$\boxed{\psi = \varphi + \lambda K \psi}$$

(E.g. Lippman-Schwinger eqn for $|p+\rangle$ has this form.)

where $\varphi, \psi \in \mathcal{H}$

we are interested in K which can be represented as an integral Kernel acting on $L^2(M)$, M = measure space

$$\langle \bar{x} | \psi \rangle = \langle \bar{x} | \varphi \rangle + \lambda \int d^3x' \langle \bar{x} | K | \bar{x}' \rangle \langle \bar{x}' | \psi \rangle$$

or, less pendantically $\boxed{\psi(\bar{x}) = \varphi(\bar{x}) + \lambda \int d^3x' \, K(\bar{x}, \bar{x}') \, \psi(\bar{x}')}$

("Inhomogeneous Fredholm Eqn of the second Kind")

A particularly important class of kernels is the "L^2" kernels, i.e. those such that $K(x,x')$ is an $L^2(M\times M)$ function,

$$\int dx\,dx'\,|K(x,x')|^2 < \infty$$

Such kernels are said to be "L^2", "Hilbert-Schmidt", or "Fredholm"

If $L^2(M)$ has the countable orthonormal basis $\{\varphi_n\}$, then $L^2(M\times M)$ has the basis $\varphi_n(x)\varphi_m^*(x')$. Then

$$K = \sum_{nm} a_{nm}\,\varphi_n(x)\,\varphi_m^*(x').$$

(Series converges uniformly in $L^2(M)$ norm

i.e. $\int dx\,dx'\,|K - \Sigma|^2 < \epsilon$)

Because K is L^2, we have

$$\int dx\,dx'\,|K(x,x')|^2 = \sum_{\substack{nm\\ n'm'}} a_{nm}a_{n'm'}^* \int dx\,\varphi_n(x)\varphi_{n'}^*(x) \int dx'\,\varphi_m^*(x')\varphi_{m'}(x')$$

$$= \sum_{nm} |a_{nm}|^2 < \infty$$

This is the same thing as

$$\mathrm{tr}\,K^\dagger K = \sum_m (\varphi_m, K^\dagger K\varphi_m) = \sum_{n,m} \|a_{nm}\varphi_m\|^2 = \sum_{n,m} |a_{nm}|^2$$

So an abstract way of defining the Hilbert-Schmidt property is $\boxed{\mathrm{tr}\,K^\dagger K < \infty}$

(The trace can be defined with respect to any orthonormal basis, and is independent of basis)

The Hilbert-Schmidt property says that the operator K can be represented by a "matrix"

$$K = \sum_{n,m} a_{nm} \, |n\rangle\langle m|$$

where the sum of absolute values squared of the "matrix elements" is finite. Hence this matrix is "almost" finite dimensional. (and certainly bounded) In fact K can be uniformly approximated by a finite rank operator (an operator with finite dimensional range) in the sup norm.

This is clear, because $\mathrm{tr}\, K^{+}K = \sum |K_n|^2 > \|K\|^2_{sup}$ $\left[\begin{array}{l} \text{i.e. } (\varphi, K\psi) \\ \leq \|\varphi\| \, \|K\psi\| \\ \leq \|\varphi\| (\psi K^{+}K)^{1/2} \|\psi\| \end{array}\right.$

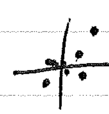
and we can choose N, M so that

$$\Big\| K - \sum_{n,m}^{N,M} a_{nm} |n\rangle\langle m| \Big\|_{sup} < \Big\| K - \sum_{n,m}^{N,M} a_{n,m} |n\rangle\langle m| \Big\|_2 < \epsilon$$

An operator which can be uniformly approximated by finite rank operators is said to be <u>compact</u>. Hilbert-Schmidt operators are compact, but not all compact operators are Hilbert-Schmidt

The spectrum of a compact operator resembles that of a finite rank operator:

Th^{m} If K is compact, then every $z \in \sigma(K)$ (except 0) is an eigenvalue of finite multiplicity.
$\sigma(K)$ is bounded and has no limit points except (possibly) zero. (Because K = finite rank plus remainder with $|z| < \epsilon$)



- Zero is always in $\sigma(K)$ if K is compact but not finite rank (K^{-1} does not exist)

- Either $(K - z)\psi = 0$ has a solution, or else $(K-z)^{-1}$ exists (If $z \neq 0$)
This is the <u>Fredholm alternative</u>

If $\psi = \varphi + \lambda K \psi$, K compact has at most one solution, then a solution must exist (E.g. for soln to Laplace's eqn with given B.C., uniqueness is easy to prove, existence follows.)

Compactness of the Kernel

Consider $K = G^0 V$, or $K(\vec{x},\vec{x}') = \left(-\frac{1}{2\pi}\right) \frac{e^{ip|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} V(\vec{x}')$

Is this a Hilbert-Schmidt operator?

$$\mathrm{tr}\, K^+K = \frac{1}{(2\pi)^2}\int d^3x\, d^3x' \; \frac{e^{-2(\mathrm{Im}\,p)|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|^2} |V(\vec{x}')|^2$$

$$= \frac{1}{(2\pi)^2}\left[\int d^3x' |V(\vec{x}')|^2\right]\left[\int d^3x \; \frac{e^{-2(\mathrm{Im}\,p)|\vec{x}|}}{|\vec{x}|^2}\right]$$

If $\boxed{\begin{matrix} V \in L^2(\mathbb{R}^3) \\ \mathrm{Im}\,p > 0 \end{matrix}}$ Then $\mathrm{tr}\, K^+K < \infty$ and K is compact

$\mathrm{Im}\,p > 0$ is "physical halfplane", but...

We are really interested in $\mathrm{Im}\,p = \epsilon \to 0$.
How do we control $\mathrm{tr}\,K^+K$ as $\epsilon \to 0$?

We introduce a simple trick (due to Schwinger)

Write $V(\vec{x}) = |V(\vec{x})|\,\sigma(\vec{x})$ where $\sigma(\vec{x}) = \pm 1$

solving

$$|V|^{\frac{1}{2}}\psi = |V|^{\frac{1}{2}}\varphi + \lambda |V|^{\frac{1}{2}} K |V|^{-\frac{1}{2}} |V|^{\frac{1}{2}}\psi$$

for $|V|^{\frac{1}{2}}\psi$ is equivalent to solving for ψ, so we may consider the modified kernel

(At least, for every soln to $\psi = \varphi + \lambda K\psi$ $\exists$ soln to $\psi' = \varphi' + \lambda K'\psi'$)

$$K' = |V|^{\frac{1}{2}} K |V|^{-\frac{1}{2}} = |V|^{\frac{1}{2}} G^0 |V|^{\frac{1}{2}}\sigma$$

if $K = G^0|V|\sigma$

$$\mathrm{tr}\,K'^+K' = \frac{1}{(2\pi)^2}\int d^3x\, d^3x' |V(\vec{x})||V(\vec{x}')| \; \frac{e^{-2(\mathrm{Im}\,p)|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|^2}$$

This remains finite as $\mathrm{Im}\,p \to 0$, if

$$\boxed{\begin{matrix} \displaystyle\int d^3x \frac{|V(\vec{x})|}{|\vec{x}-\vec{x}'|^2} < M \;\; (\text{independent of } \vec{x}') \\[2ex] \displaystyle\int d^3x' \; |V(\vec{x}')| < \infty \end{matrix}}$$

Sufficient conditions for Kernel to be compact (not necessary)

$$G^0 V \psi = k(E)\psi$$
$$\Rightarrow |V|^{\frac{1}{2}} G^0 |V|^{\frac{1}{2}} \upsilon \, |V|^{\frac{1}{2}}\psi = k(E)|V|^{\frac{1}{2}}\psi$$
$$\Rightarrow k \in \sigma(K) \Rightarrow \sigma(E)\; k \in \sigma(K')$$

Spectrum of K

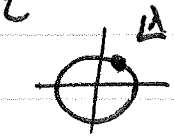
$K = G^0(E)V$ (or $|V|^{\frac{1}{2}} G^0 |V|^{\frac{1}{2}}\upsilon$) is compact

$\Rightarrow$ $\sigma(K) =$ isolated (bounded) eigenvalues plus zero

Eigenvalues $k(E)$ have trajectories in complex k plane as functions of E

If $K(E)\psi_E = k(E)\psi_E$ then $(I - \lambda K)^{-1}$ has a pole at $\lambda = \frac{1}{k(E)}$

The radius of convergence of the born series is determined by smallest $|\lambda(E)| = \frac{1}{|k(E)|}$ i.e., by largest eigenvalue of K



The trajectory $k(E)$

1) First, consider $E<0$, real $\Rightarrow$ K' is hermitian, $k(E)$ real

$$K(E)\psi_E = \frac{1}{E-H^0} V\psi_E = k(E)\psi_E \Rightarrow \left(H^0 + \frac{1}{k}V\right)\psi_E = E\psi_E$$

(Thus the potential λV where $\lambda = \frac{1}{k(E)}$ has bound state of energy E)

Differentiate $\left(\psi_E, [(E-H^0)k(E) - V]\psi_E\right) = 0$

(Don't differentiate ψ_E, because of variational principle i.e. $A\psi = a\psi \Rightarrow \delta\left[\frac{(\psi, A\psi)}{(\psi,\psi)}\right] = 0$)

$$k(E) + \frac{dk}{dE}\langle E - H^0\rangle = 0 \text{ or } \frac{1}{k}\frac{dk}{dE} = \frac{1}{\langle H^0 - E\rangle} > 0 \quad \text{for } E<0$$

For $E \to -\infty$ $\quad K = \frac{1}{E-H^0}V \to 0$

i.e. each $k(E)$ approaches zero as $E \to -\infty$

(And they approach uniformly, since

$$\operatorname{tr} K^\dagger K = C\int d^3x\, \frac{e^{-2\,\mathrm{Im}\,p|\vec{x}|}}{|x|^2} = 4\pi C\int_0^\infty dr\, e^{-r(2\,\mathrm{Im}\,p)} = \frac{4\pi C}{2\,\mathrm{Im}\,p}$$

and $\quad \mathrm{Im}\,p = \sqrt{-2E} \to \infty$)

Thus, all $R(E)$ start out close to zero on real axis for $E \to -\infty$

As E increases ...

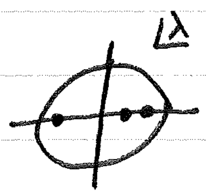
- Positive $R(E)$ increase
- Negative $R(E)$ decrease

} i.e. they move away from the origin

As $R(E)$ passes through ...

$R(E) = \frac{1}{\lambda}$ $\quad H^0 + \lambda V$ has bound state of energy E (attractive eigenvalue)

$-\frac{1}{\lambda}$ $\quad H^0 - \lambda V$ has bound state of energy E (repulsive eigenvalue)



As $E \to 0^-$,
Number of positive (negative) eigenvalues
$R(E) > \frac{1}{\lambda}$ $\quad (R(E) < -\frac{1}{\lambda})$ is total no.
of bound states in potential $H^0 + \lambda V$ $(H^0 - \lambda V)$

For $E < 0$, the Born series converges only if both $H^0 + \lambda V$ and $H^0 - \lambda V$ have no bound states of energy less than E. This is always true for λ sufficiently small, because K is compact (has a maximum eigenvalue)

Note that attractive and repulsive eigenvalues are equally dangerous; i.e., both interfere with convergence of Born series.

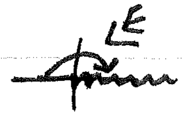
2) Now consider $E > 0$

we still need all eigenvalues satisfying $|R(E)| < \frac{1}{\lambda}$ for Born series to converge. Always true for λ sufficiently small

If $H \pm \lambda V$ has any bound states, the series fails to converge for E sufficiently small

But the situation is more complicated than in $E < 0$ case, because we have no monotonicity properties on $R(E)$, and they are free to wander in complex plane

$|V|^{\frac{1}{2}} \frac{1}{E-H^0+i\epsilon} |V|^{\frac{1}{2}}$ has an antihermitian part for $E>0$ (there is a discontinuity across the cut)



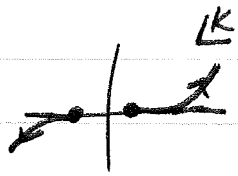
We are interested in physical half plane $E+i\epsilon$

Because $\frac{1}{R}\frac{dR}{dE}$ holds for $E<0$, adding a small positive imag part to E when $\mathrm{Re}\,E<0$ causes

$R(E) \longrightarrow$ upper half plane if attractive (>0)
lower " " " repulsive (<0) $\quad E \to E+i\epsilon$

R must remain complex as E does, otherwise $\left(H^0 + \frac{V}{R(E)}\right)\psi_E = E\psi_E \Rightarrow$ hermitian H has complex eigenvalue

Thus, as E approaches cut from above, attractive, repulsive eigenvalues remain in natural half planes



It is possible for convergence to fail even if there are no bound states, if some $R(E)$ pierces the $|R| = \frac{1}{\lambda}$ circle somewhere in cx plane

- But Newton, p283, shows that Born series converges uniformly at all energy if the potential $-\lambda|V|$ has no bound states

High Energy Bound

Under suitable assumptions, it can be shown that all eigenvalues $R(E)$ return uniformly to the origin as $E \to \infty$, so Born series converges at sufficiently high energy.

The idea is that the phase $e^{ip|\vec{x}-\vec{x}'|}$ oscillates rapidly as $\mathrm{Re}\,p \to \infty$, so that $\|K\|_{sup} \to 0$

We will actually show that

$\|K^2\|^2_{sup} \leq \mathrm{Tr}(K^2)(K^{2\dagger}) < \epsilon$ For E sufficiently large consider

$$K'(\vec{x},\vec{x}') = -\frac{1}{2\pi}|V(\vec{x})|^{\frac{1}{2}} \frac{e^{ip|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}|V(\vec{x}')|^{\frac{1}{2}} v(\vec{x}')$$

$$K'(\vec{x},\vec{y})^2 = \frac{1}{(2\pi)^2}|V(\vec{x})|^{\frac{1}{2}}|V(y)|^{\frac{1}{2}}v(y)\int d^3z'\, V(\vec{z}')\, \frac{e^{ip|\vec{x}-\vec{z}|}e^{ip|\vec{z}-\vec{y}|}}{|\vec{x}-\vec{z}||\vec{z}-\vec{y}|}$$

Let's write the integral as

$$I_Q = \int d^3z\, e^{ip\omega(\vec{z})}Q(\vec{z}) \qquad \omega(\vec{z}) = |\vec{x}-\vec{z}|+|\vec{z}-\vec{y}| \qquad Q(\vec{z}) = \frac{V(\vec{z})}{|\vec{x}-\vec{z}||\vec{z}-\vec{y}|}$$

The idea is, for p real and $\to\infty$, the oscillations of the exponential kill the integral

Proof/ Our earlier assumptions on Q imply $\int d^3z\, |Q(\vec{z})| < \infty$

therefore, we can approximate Q by a C_0^∞ function Q'

$$\int |Q-Q'|\, d^3z < \epsilon$$

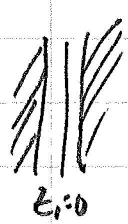
and $I_Q = I_{Q'} + I_{Q-Q'} = I_{Q'} + \epsilon$ (since $|e^{ip\omega}|=1$ for p real)

so it suffices to consider C_0^∞ Q (so we can integrate by parts)

Now, let z_1, z_2, z_3 be Cartesian coordinates

Let $\omega_1 = \frac{\partial\omega}{\partial z_1} = \frac{z_1}{|\vec{x}-\vec{z}|} + \frac{z_1}{|\vec{z}-\vec{y}|}$

$$I_Q = \int dz_1 dz_2 dz_3\, e^{ip\omega(\vec{z})}\omega_1\left(\frac{Q}{\omega_1}\right)$$

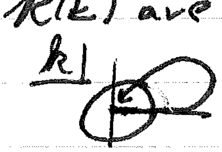


$$= \frac{1}{ip}\iiint_V e^{ip\omega(\vec{z})}\frac{\partial}{\partial z_1}\left(\frac{Q}{\omega_1}\right) + \iint_S \frac{e^{ip\omega(\vec{z})}}{ip}\frac{Q(z)}{\omega}$$

$$+ \iiint_{V'} e^{i\omega} Q \leftarrow \text{(this cannot be made small if } Q \text{ is a } \delta\text{-func.)}$$

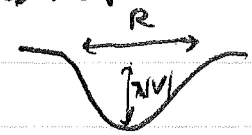
$$|I_Q| \leq \frac{C}{p} + \epsilon < \epsilon' \text{ for } p \text{ sufficiently large}$$

Thus $\|K\|_{\sup} < \epsilon$ and all $k(E)$ are within ϵ of the origin



(They always return home)

In summary, the Born approx converges for



1) $\lambda \to 0$ (Weak Potential)

If the potential has a range R we need $\boxed{\lambda V \ll \frac{1}{mR^2}}$ (A potential with no bound states)

It follows that the series converges for

2) $R \to 0$ (Because this squeezes out bound states)

3) $E \to \infty$ But, how high must E be to gaurentee convergence?

If we can "integrate by parts" when $K(x,x')$ acts on $\psi(x')$, we expect that $K \propto \frac{1}{p}$ at high energy. Since $K \sim \int d^3r \frac{e^{ipr}}{r} V$,

By dimensional analysis, we expect that

$$\frac{|V|R}{p} < \frac{1}{\lambda}$$ is required for convergence, or

$$\boxed{\lambda |V| \ll \frac{p}{Rm}}$$ m restored by dimensional analysis

i.e. compare $t^{-1} \sim \frac{|V|}{\hbar}$ with time to traverse potential

(we will discuss how to remedy these convergence problems below.)

The forward Peak

Now we know that the born approximation is a good approximation at sufficiently high energy. Now recall that the born approximation to $t(\bar{p}',\bar{p})$ depends only on $\bar{q} = \bar{p}' - \bar{p}$, and not independently on p^2

Consider a spherical potential $V(r)$

$$t^{(1)} = \langle \vec{p}' | V | \vec{p} \rangle = \frac{1}{(2\pi)^3} \int d^3x \, e^{-i\vec{q}\cdot\vec{x}} V(r)$$

$$= \frac{1}{(2\pi)^2} \int dr \, r^2 \int_{-1}^{1} d\cos\theta \, e^{iqr\cos\theta} V(r)$$

$$\boxed{t^{(1)}(\vec{p}';\vec{p}) = \frac{1}{(2\pi)^2 (iq)} \int_0^\infty dr \, r \, V(r) \left(e^{iqr} - e^{-iqr} \right)}$$

By Riemann-Lebesque, we can expect that the integral goes to zero for q large. Infact, if V is smooth, we can integrate by parts:

$$= \frac{1}{(2\pi)^2 (iq)} \frac{1}{(iq)} \left[2(rV)\Big|_0^\infty - \int_0^\infty dr \frac{d}{dr}(rV)\left(e^{iqr} + e^{-iqr}\right) \right]$$

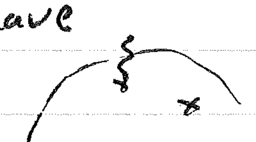
Integrating by parts repeatedly, we have the expansion

$$t^{(1)}(\vec{p}';\vec{p}) = \frac{1}{2\pi^2 q^2} \left[rV - \frac{\frac{d^2}{dr^2}(rV)}{q^2} + \frac{\frac{d^4}{dr^2}(rV)}{q^4} - \cdots \right]_{r=0}$$

(If rV is not singular at the origin)

If V is an even function of r, all terms are zero. $t^{(1)}$ approaches zero more rapidly than any power of $\frac{1}{q^2}$. For V even we have

$$t^{(1)} = \frac{1}{(2\pi)^2 (iq)} \int_{-\infty}^{\infty} dr \, r V e^{iqr}$$



If we can complete contour in upper half plane,
asymptotic behavior determined by r_0 - proximity of nearest singularity to real axis in that half plane

then $t^{(1)} \rightarrow e^{-qr_0}$ $\quad q \to \infty$

(If there are no singularities, $t^{(1)}$ goes to zero more rapidly than an exponential - e.g. Gaussian)

Since $q^2 = (\vec{p}' - \vec{p})^2 = 2p^2 - 2\vec{p}'\cdot\vec{p} = 2p^2(1-\cos\theta) = 4p^2 \sin^2\frac{\theta}{2}$

or $q = 2p\sin\theta/2$,

in this case the scattering amplitude at high energies is

$$f^{(1)} \sim e^{-2pr_0 \sin\theta/2} \sim e^{-pr_0\theta} \text{ at small angles}$$

Hence, if $pr_0 \gg 1$, $d\sigma/d\Omega$ is strongly peaked in the forward direction, and the peaking increases as $p \to \infty$

But if V is not an even function, $V \sim \frac{1}{(q^2)^p}$, and shape of $d\sigma/d\Omega$ as a function of θ becomes independent of p as $p \to \infty$

D. An Example: The Yukawa Potential

We will calculate the first two terms in the Born expansion for $t(\vec{p}';\vec{p})$ in the case

$$V(r) = \frac{1}{r} e^{-ar} \quad \text{– the Yukawa Potential}$$

to illustrate how these calculations are done.

We know that

$$t^{(1)}(\vec{p}';\vec{p}) = \frac{1}{(2\pi)^2 (iq)} \int_0^\infty dr\, r V (e^{iqr} - e^{-iqr})$$

Actually, in this case we prefer to integrate directly:

$$\frac{\lambda}{(2\pi)^2}\frac{1}{i|\vec{p}-\vec{p}'|}\int_0^\infty dr\left(e^{-ar+i|\vec{p}-\vec{p}'|r}-e^{-ar-i|\vec{p}-\vec{p}'|r}\right)$$

$$=\frac{\lambda}{(2\pi)^2}\frac{1}{i|\vec{p}-\vec{p}'|}\left[\frac{1}{a-i|\vec{p}-\vec{p}'|}-\frac{1}{a+i|\vec{p}-\vec{p}'|}\right]=\frac{\lambda}{2\pi^2}\frac{1}{a^2+(\vec{p}-\vec{p}')^2}$$

$$\boxed{t^{(1)}(\vec{p}',\vec{p})=\frac{\lambda}{2\pi^2}\frac{1}{a^2+(\vec{p}-\vec{p}')^2}},$$

a nice, simple answer. (and clearly the sum of the series on p 5.13)

The second Born approximation is

$$t^{(2)}(\vec{p}',\vec{p})=\int d^3K\,\langle\vec{p}'|V|\vec{K}\rangle\frac{1}{E_p-E_K+i\epsilon}\langle\vec{K}|V|\vec{p}\rangle$$

$$=\left(\frac{\lambda}{2\pi^2}\right)^2\int d^3K\,\frac{1}{a^2+(\vec{p}'-\vec{K})^2}\,\frac{1}{\frac{1}{2}p^2-\frac{1}{2}K^2+i\epsilon}\,\frac{1}{a^2+(\vec{K}-\vec{p})^2}$$

To do the angular integral, we first combine denominators using the Feynman trick

$$\frac{1}{ab}=\int_0^1 d\alpha\,\frac{1}{[a\alpha+b(1-\alpha)]^2}$$

$$-2\left(\frac{\lambda}{2\pi^2}\right)^2\int_0^1 d\alpha\int d^3K\,\frac{1}{[K^2+2\alpha\vec{p}'\cdot\vec{K}+\alpha p'^2+2(1-\alpha)\vec{p}\cdot\vec{K}+(1-\alpha)p^2+a^2]^2\,(K^2-p^2_{-i\epsilon})}$$

But $p^2=p'^2$

$$[\quad]\rightarrow(K+\alpha\vec{p}'+(1-\alpha)\vec{p})^2+\cdots$$

$$\text{or}\rightarrow K^2+2\vec{K}\cdot(\alpha\vec{p}'+(1-\alpha)\vec{p})+p^2+a^2$$

$$\int_{-1}^1 d\cos\theta\,\frac{1}{[2KP\cos\theta+A]^2}=\frac{-1}{2KP\,[2KP\cos\theta+A]}\Big|_{-1}^{1}$$

$$=\frac{-1}{2KP}\left(\frac{1}{2KP+A}-\frac{1}{-2KP+A}\right)=\frac{2}{(A+2KP)\times(A-2KP)}$$

Integral is

$$\int_0^1 d\alpha\,2\pi(2)\int_0^\infty dK\,K^2\,\frac{1}{(K^2+p^2+a^2+2KP)}\,\frac{1}{(K^2+p^2+a^2-2KP)}\,\frac{1}{(K^2-p^2-i\epsilon)}$$

where $\vec{P}=\alpha\vec{p}'+(1-\alpha)\vec{p}=\vec{p}+\alpha\vec{q}$, $\quad\vec{q}=\vec{p}'-\vec{p}$

we now have

$$\int_0^1 d\alpha \, \frac{-2\lambda^2}{\pi^3}\int_0^\infty dK \frac{K^2}{(K+\underline{P}+iA)(K+\underline{P}-iA)(K-\underline{P}+iA)(K-\underline{P}-iA)(K-p-i\epsilon)(K+p+i\epsilon)}$$

where $A = \frac{1}{2}\sqrt{4(p^2+a^2)-4\underline{P}^2}$

$$\underline{P}^2 = p^2 + 2\alpha\vec{p}\cdot\vec{q} + \alpha^2\vec{q}^{\,2}$$

But $q^2 = (\vec{p}\,'-\vec{p})^2 = 2p^2 - 2\vec{p}\,'\cdot\vec{p}$

$\vec{q}\cdot\vec{p} = \vec{p}\,'\cdot\vec{p} - p^2 = \frac{1}{2}(2p^2-q^2) - p^2 = -\frac{1}{2}q^2$

$$= p^2 - \alpha(1-\alpha)q^2$$

$$\Longrightarrow \boxed{\begin{aligned} A &= \sqrt{a^2+\alpha(1-\alpha)q^2} \\ \underline{P} &= \vec{p}+\alpha\vec{q} \end{aligned}}$$

Integrand is even $\Rightarrow$ extend integral to $\int_{-\infty}^{\infty}$ and complete contour in upper half plane. There are poles at

$$K = \pm\underline{P} + iA$$
$$K = p + i\epsilon$$

and we have $t^2(\vec{p}\,',\vec{p}) = \int_0^1 d\alpha \, \frac{-2\lambda^2}{\pi^3}(\pi i)\left(\sum \text{Residues}\right)$

Residue at $K = p+i\epsilon$ is

$$\frac{1}{2}p \Big/ \left[((p+\underline{P})^2+A^2)((p-\underline{P})^2+A^2)\right]$$

~~This is getting complicated, so lets do only forward scattering~~

$$\cancel{\pi^2\left(\frac{\lambda}{2\pi^2}\right)^2 \int d^3k \, \frac{1}{((k-p)^2+a^2)^2} \, \frac{1}{k^2-p^2-i\epsilon}}$$

But $(p+\underline{P})^2 + A^2 = 4p^2 + 4\alpha\hat{p}\cdot\vec{q} + \alpha^2q^2 + a^2 + \alpha(1-\alpha)q^2$

$$= 4p^2 - 2\alpha q^2 + \alpha^2q^2 + a^2 + \alpha q^2 - \alpha^2 q^2$$

$$= 4p^2 - \alpha q^2 + a^2$$

$$(p-\underline{P})^2 + A^2 = \alpha^2q^2 + a^2 + \alpha q^2 - \alpha^2q^2 = \alpha q^2 + a^2$$

$$(4p^2-\alpha q^2+a^2)(\alpha q^2+a^2) = a^4 + 4p^2a^2 - \alpha^2q^4 + 4\alpha q^2p^2$$

Residue at $p+i\epsilon$ is $\frac{1}{2}p(a^4+4a^2p^2)^{-1}$

Let's go through the detailed computation only in the case of forward scattering, $q=0$. Then we have:

Residue at $p+i\epsilon$ is $\frac{1}{2}p\,(a^4+4a^2p^2)^{-1} = \dfrac{p}{2a^2(4p^2+a^2)}$

Residue at $p+ia$ is

$$(p+ia)^2 \Big/ \left[2(p+ia)(2p)(2ia)(ia)(2p+ia)\right]$$

Residue at $-p+ia$ is

$$(-p+ia)^2 \Big/ \left[(2ia)(-2p+2ia)(-2p)(-2p+ia)(ia)\right]$$

The sum of these is:

$$\frac{-\frac{1}{8}(p+ia)}{a^2p\,(2p+ia)} + \frac{\frac{1}{8}(-p+ia)}{a^2p\,(-2p+ia)}$$

$$= \frac{1}{8pa^2}\left[\frac{p-ia}{2p-ia} - \frac{p+ia}{2p+ia}\right] = \frac{1}{8pa^2}\,\frac{-2ipa}{4p^2+a^2} = \frac{-i}{4a(4p^2+a^2)}$$

Finally we have

$$t^{(2)}(\vec{p},\vec{p}) = \frac{-2\lambda^2}{\pi^2}\left(\frac{1/4a + ip/2a^2}{4p^2+a^2}\right)$$

or

$$\boxed{f^{(2)}(\vec{p},\vec{p}) = \frac{2\lambda^2}{(a^2+4p^2)a}\left(1+i\,\frac{2p}{a}\right)}$$

We can now check the optical theorem, recalling

$$\frac{d\sigma}{d\Omega} \sim |f^{(1)}(\vec{p},\vec{p}\,')|^2 = \left|\frac{-2\lambda}{a^2+q^2}\right|^2 \qquad \vec{q} = \vec{p}\,'-\vec{p}$$

$$q^2 = 2p^2 - 2\vec{p}\cdot\vec{p}\,' = 2p^2(1-\cos\theta) = 4p^2\sin^2\frac{\theta}{2}$$

$$\sigma_T = 2\pi\int_{-1}^{1} d\cos\theta \frac{2\lambda^2}{[a^2+2p^2-2p^2\cos\theta]^2}$$

$$= 8\pi\lambda^2 \frac{+1}{2p^2} \frac{1}{-2p^2\cos\theta+a^2+2p^2}\Big|_{-1}^{1} = \frac{4\pi\lambda^2}{p^2}\left(\frac{1}{a^2} - \frac{1}{a^2+4p^2}\right)$$

$$= \frac{16\pi\lambda^2}{a^2(a^2+4p^2)} = \frac{4\pi}{p}\left(\frac{4p\lambda^2}{a^2(a^2+4p^2)}\right)$$

$$= \frac{4\pi}{p}\,\mathrm{Im}\, f(\vec{p},\vec{p})$$

The differential cross section in the forward direction is given by the cross term of $f^{(1)}$ and $f^{(2)}$ to order λ^3

$$\left.\frac{d\sigma}{d\Omega}\right|_{q=0} = \frac{4\lambda^2}{a^4} - \frac{8\lambda^3}{a^3(a^2+4p^2)} + \cdots$$

(Im $f^{(1)}$ does not contribute to this order)

The correction is down compared to the first term by

$$\frac{-2\lambda(m)}{a+4p^2/a}$$

we can use dimensional analysis to restore mass

This is a small correction provided

- $\lambda \to 0$ (weak potential)
- $a \to \infty$ (short range)
- $p^2 \to \infty$ (high energy)

As $a \to 0$, forward cross section blows up – An oddity, as $a \to 0$ second term becomes unimportant (in cross section, but not in imag part, of course)
Because of long range of coulomb potential phase of f "goes crazy", but Born approx gives $d\sigma/d\Omega$ correctly at all angles!

The integration can be done for nonzero q^2, giving

$$\boxed{t^{(2)} = \frac{-i\lambda^2}{2\pi^2 q f} \ln \frac{(f+pq)(2f-iqa)}{(f-pq)(2f+iqa)}}$$

(Newton, p292-3)

$$f = (a^4 + 4a^2p^2 + p^2q^2)^{\frac{1}{2}}$$

E. The Fredholm Method

For K compact, the resolvent $(1-\lambda K)^{-1}$ is meromorphic; its only singularities are poles. We need a method of analytically continuing around the poles which spoil the Born expansion.

A possible approach is to write

$$(1-\lambda K)^{-1} = \frac{N(\lambda)}{D(\lambda)}$$

where N, D are entire functions (have convergent power series for all values of λ)

We must choose D to have simple zeros at each pole of $(1-\lambda K)^{-1}$. E.g. if $\{\frac{1}{\lambda_i}\}$ are the eigenvalues of K, take

$$D(\lambda) = \prod_i \left(1 - \frac{\lambda}{\lambda_i}\right)$$

This infinite product is guaranteed to converge if $\sum_i \left|\frac{\lambda}{\lambda_i}\right| < \infty$ i.e. if $\operatorname{tr}(K^+K)^{\frac{1}{2}} < \infty$ and defines an analytic function for all values of λ (is entire).

the function $D(\lambda)\,(1-\lambda K)^{-1} = N(\lambda)$, rearranged as a power series in λ, is also entire, because all its poles have been cancelled.

Now, how do we find an expansion for $D(\lambda)$ in powers of λ? We observe that

$$D(\lambda) = \det(1-\lambda K) = \exp\left[\operatorname{tr} \ln(1-\lambda K)\right]$$
$$= \exp\left[-\sum_n \frac{\lambda^n}{n} \operatorname{tr} K^n\right]$$

where $\operatorname{tr} K^n = \int dx_1 \cdots dx_n\, K(x_1,x_2) K(x_2,x_3) \cdots K(x_n,x_1)$

The argument of the exponential has a finite radius of convergence (the same as for $(1-\lambda K)^{-1}$) but when the exponential is expanded, we have an entire series in λ.

If $\operatorname{tr} K, \operatorname{tr} K^2, \ldots \operatorname{tr} K^{N-1}$ do not exist, but $\operatorname{tr} K^n$ $n \geq N$ do, we can still use the Fredholm method; noting

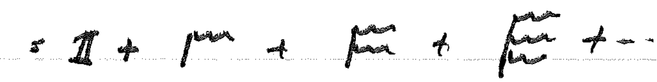
$$1-(\lambda K)^N = \left(1+\lambda K+\cdots+(\lambda K)^{N-1}\right)(1-\lambda K),$$

we can find Fredholm solution for $[1-(\lambda K)^N]^{-1}$

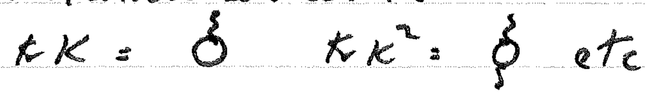
Unfortunately, this method is usually not very useful as an approximation scheme, because when the Born series converges, the Fredholm series for $N(\lambda), D(\lambda)$ tend to converge only very slowly.

The Fredholm solution can be conveniently represented by a diagrammatic expansion. In a notation we have already introduced for the Born series

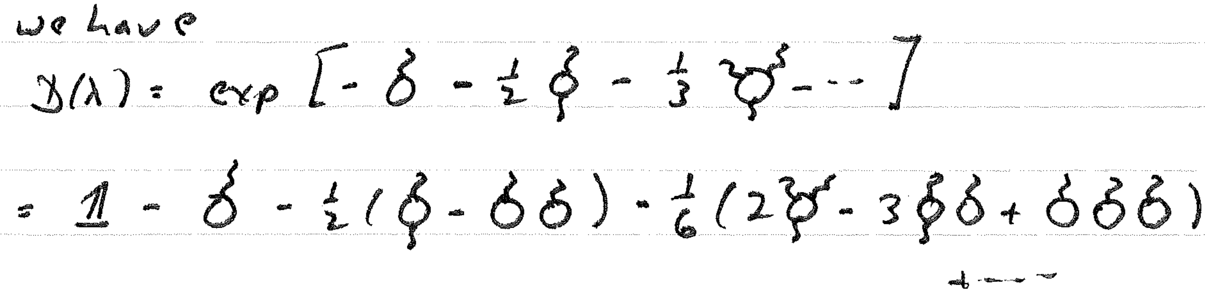
$$(1-\lambda K)^{-1} = \mathbb{1} + G^0\lambda V + G^0\lambda V G^0 \lambda V + \cdots$$



In this notation we write

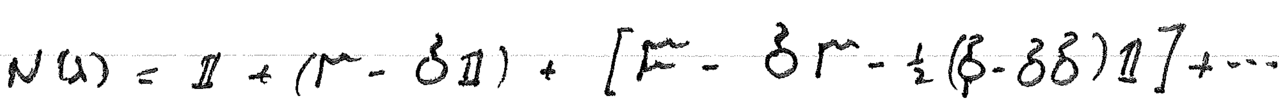


Thus we have



(For the cognoscenti: this is immediately recognized as the Feynman-Dyson expansion for the vac-vac transition amplitude in the presence of an external field, with all Fermi (-1)'s and combinatoric factors in place.)

The numerator is the product of the determinant with the Neumann (Born) series



F. The Quasiparticle Method (Schmidt Method)

Another approximation scheme is based on the observation that a compact kernel can be approximated by a finite rank operator. Let's write

$$K = K_1 + \sum_s |s\rangle\langle\bar{s}| \qquad \left(\langle\bar{s}| \text{ is not necessarily dual to } |s\rangle\right)$$

We can make $\|K_1\|_{\sup}$ small, so that its Born series is rapidly convergent, and then simply invert a finite dimensional matrix to complete the solution.

Problem:

We wish to find $F = (1-K)^{-1}$ satisfying $F = 1 + KF$

Suppose we can find F_1 solving $F_1 = 1 + K_1 F_1$.

How do we express F in terms of F_1

$$K = K_1 + \sum_s |s\rangle\langle\bar{s}| \Rightarrow$$

$$F = F_1 + \qquad F = 1 + \sum_s |s\rangle\langle\bar{s}|F + K_1 F$$

This is an equation of the form

$$F = J + K_1 F \Rightarrow F = (1-K_1)^{-1} J = F_1 J$$

$$\text{or} \quad F = F_1 + \sum_s F_1 |s\rangle\langle\bar{s}|F$$

Now we wish to eliminate $\langle\bar{s}|F$.

$$\langle\bar{t}|F = \langle\bar{t}|F_1 + \sum_s \langle\bar{t}|F_1|s\rangle\langle\bar{s}|F$$

$$\text{or} \quad \sum_s \left(\delta_{ts} - \langle\bar{t}|F_1|s\rangle\right)\langle\bar{s}|F = \langle\bar{t}|F_1$$

Let $\boxed{(\Delta^{-1})_{ts} = \delta_{ts} - \langle\bar{t}|F_1|s\rangle}$ generically invertible

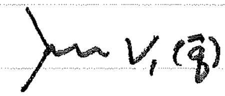
Then the solution is $\langle\bar{s}|F = \sum_t \Delta_{st}\langle\bar{t}|F_1$

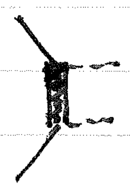
and we finally have $\boxed{F = F_1 + \sum_{s,t} F_1|s\rangle\Delta_{st}\langle\bar{t}|F_1}$

Thus, we have reduced the problem of finding $F = (1-K)^{-1}$ to the problem of finding $F_1 = (1-K_1)^{-1}$ and inverting the matrix Δ^{-1}

By modifying the Kernel, we have "removed" some bound states from the potential V (these are the quasiparticles) and in doing so have "weakened" the potential so that its Born series converges

Now the perturbation expansion includes interactions with the potential V_1 as in the ordinary born series as well as interactions with Quasi-particles





(Particle becomes virtual quasiparticle which has propagator Δ)

Even if we do not know the exact eigenstates of K (and we usually don't) this method is a good approximation scheme. If we make a reasonable guess of what operator $\sum_s |s\rangle\langle\tilde{s}|$ to remove from K, the convergence of the Born series for K_1 can be dramatically improved.