VII. Complex Angular Momentum + Regge Poles

A. Analytic Continuation in ℓ

We have previously considered the radial equation

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\lambda V + p^2\right] \phi_{\ell p}(r) = 0$$

with λ and p complex numbers, and saw that $\phi_{\ell p}(r;\lambda)$ could be continued as an analytic function of λ and p, as well as r. Now we wish to study the properties of the regular solution $\phi_{\ell p}$ (and hence $f_\ell(p)$ and $S_\ell(p)$) when ℓ is a continuous (in fact, complex) variable.

This sounds a little crazy, because we know that only integer values of ℓ are physical. But the property ℓ=integer is a property of spherical harmonics, and nothing prevents us from studying the radial equation when ℓ is not an integer.

What is the motivation? So far we have considered each partial wave separately – but the $S_\ell(p)$'s must surely be related if they are all derived from the same potential; we can establish such relations by continuing S_ℓ as an analytic function of ℓ. In particular, the full scattering amplitude

$$f(p,\theta) = \Sigma(2\ell+1) f_\ell(p) P_\ell(\cos\theta)$$

may have properties not shared by the individual terms.

Let's first argue that $\phi_{\ell p}$ solving the above radial eqn defines an analytic function of ℓ. Our strategy is the same as that used to show $\phi_{\ell p}$ is an analytic function of p: we show that the iterative solution to the radial equation is a uniformly convergent series of analytic functions of ℓ

The simplest thing to do is to construe $2V - p^2$ as the potential, so that we may use as our Green's function

$$\lim_{p\to 0} g_{\ell p}(r,r') = \lim_{p\to 0} \begin{cases} \frac{1}{p}\, \hat{j}_\ell(pr)\hat{n}_\ell(pr') - \hat{n}_\ell(pr)\hat{j}_\ell(pr') & ; r > r' \\ 0 & ; r < r' \end{cases}$$

$$= \frac{1}{2\ell+1}\left[\left(\frac{r}{r'}\right)^\lambda - \left(\frac{r'}{r}\right)^\lambda\right](rr')^{1/2} \quad \text{where } \boxed{\lambda = \ell + \tfrac{1}{2}}$$

This is the appropriate Green's function only for $\mathrm{Re}\,\lambda > 0$.

The boundary condition satisfied by the regular solution is

$$\tilde{\varphi}_{\ell p} \xrightarrow[r\to 0]{} r^{\ell+1}$$

This B.C. is not adequate as $\ell \to -\frac{1}{2}$, because does not fix the coeff of the more regular solution

(I write $\tilde{\varphi}$ because this is a different normalization condition than that satisfied by $\varphi_{\ell p}$)

Note the ~~the~~ B.C., as well as the eqn, depends on ℓ

If ℓ is not an integer, $\tilde{\varphi}_{\ell p}(r)$ has a branch point at $r=0$.

Now, the integral equation satisfied by $\tilde{\varphi}_{\ell p}$ is

$$\boxed{\tilde{\varphi}_{\ell p}(r) = r^{\ell+1} + \frac{1}{2\ell+1}\int_0^r dr' \left[\left(\frac{r}{r'}\right)^\lambda - \left(\frac{r'}{r}\right)^\lambda\right](rr')^{\frac{1}{2}}(2V(r') - p^2)\,\tilde{\varphi}_{\ell p}(r')}$$

Let us assume $\boxed{\mathrm{Re}\,\lambda > 0}$. Then the bound

$$\left|\left(\frac{r}{r'}\right)^\lambda - \left(\frac{r'}{r}\right)^\lambda\right| \le 2\left(\frac{r}{r'}\right)^{\mathrm{Re}\,\lambda}$$

is satisfied, because $r > r'$

Consider the interative solution

$$\tilde{\varphi}_{\ell p}(r) = \sum_{n=0}^{\infty} \tilde{\varphi}^{(n)} \quad \text{where } \varphi^{(0)} = r^{\ell+1}$$

and

$$\tilde{\varphi}^{(n+1)} = \frac{1}{(2\ell+1)}\int_0^r dr' \left[\left(\frac{r}{r'}\right)^\lambda - \left(\frac{r'}{r}\right)^\lambda\right](rr')^{\frac{1}{2}}(2V - p^2)\,\tilde{\varphi}^{(n)}$$

$\tilde{\varphi}^{n+1}$ is bounded by:

$$|\tilde{\varphi}^{(n+1)}| \le \frac{r^{\mathrm{Re}\,\lambda + \frac{1}{2}}}{|\lambda|}\int_0^r \frac{dr'}{r'^{\,\mathrm{Re}\,\lambda - \frac{1}{2}}}\,|2V - p^2|\,|\varphi^{(n)}|$$

We may continue to apply this bound iteratively (Compare p. 6.15) until we obtain

$$|\tilde{\varphi}^{(n+1)}| \le \frac{r^{\mathrm{Re}\lambda+\frac{1}{2}}}{(n+1)!\,|\lambda|^{n+1}} \left[\int_0^r dr'\, r'\, |2V-p^2|\right]^{n+1}$$

And now the series can be summed

$$\boxed{|\tilde{\varphi}_{\ell p}| \le r^{\mathrm{Re}\lambda+\frac{1}{2}} \exp\left[\frac{1}{|\lambda|}\int_0^r dr'\, r'\, |2V-p^2|\right]}$$

provided $\boxed{\mathrm{Re}\,\lambda = \mathrm{Re}(\ell+\tfrac{1}{2}) > 0}$

Each $\tilde{\varphi}^{(n)}$ is an analytic function of ℓ (with r, p^2 fixed) because it is an integral with finite limits of an analytic function. The bound shows that the series converges uniformly if $\mathrm{Re}\,\lambda > 0$. Therefore $\tilde{\varphi}_{\ell p}(r)$ defines an analytic function of ℓ.

(In particular, if $V=0$, we have shown that $\hat{j}_\ell(z)$ is analytic as a function of ℓ.)

The case $\mathrm{Re}\,\lambda \le 0$ requires separate attention. The problem is that the regular ($r^{\ell+1}$) and irregular ($r^{-\ell}$) solutions "cross over" when $\ell = -\frac{1}{2}$. By making appropriate assumptions about $V(r)$, we can continue to $\mathrm{Re}\,\ell < -\frac{1}{2}$ (just as we continued to $\mathrm{Im}\, p < 0$).

The Jost function can be defined as the <u>Wronskian</u>

$f_\ell(p) = \frac{1}{p}\, W(\chi^+_{\ell p}, \varphi_{\ell p})$ (page 6.23) where $\chi^+_{\ell p} \xrightarrow[r\to\infty]{} \hat{h}^{\pm}_\ell(pr)$

By a similar argument to that above, we may show that $\chi^+_{\ell p}$ is analytic in ℓ for $\mathrm{Re}\,\ell > -\frac{1}{2}$, and hence $f_\ell(p)$ and $S_\ell(p) = \frac{f_\ell(-p)}{f_\ell(p)}$ are analytic functions of ℓ

B. Regge Poles

The S-matrix $S_\ell(E)$ is an analytic function of ℓ and E. The poles of $S_\ell(E)$ occur at a location in the E plane $E = \bar{E}_n(\ell)$ which (for each integer n) depends analytically on ℓ, or at a location in the ℓ plane, $\ell = \alpha_n(E)$, which varies analytically with E.

A pole in the ℓ plane at $\ell = \alpha_n(E)$ is called a Regge Pole (After Tullio Regge). and the trajectory of α_n as E varies is called a Regge Trajectory

Let's determine some of the properties of these trajectories: the radial eqn satisfied by $\varphi_{\ell p}$ is

$$\left[\frac{d^2}{dr^2} - \frac{\lambda^2 - \frac{1}{4}}{r^2} - 2V + p^2\right]\varphi_{\ell p} = 0 \qquad \lambda = \ell + \tfrac{1}{2}$$

If p, V are real (We will study how α_n moves as E ranges over real values) then φ^* obeys

$$\left[\frac{d^2}{dr^2} - \frac{\lambda^{*2} - \frac{1}{4}}{r^2} - 2V + p^2\right]\varphi_{\ell p}^* = 0$$

and
$$\varphi^*\varphi'' - \varphi\varphi^{*\prime\prime} = \frac{d}{dr}(\varphi^*\varphi' - \varphi\varphi^{*\prime}) = \frac{(\lambda^2 - \lambda^{*2})}{r^2}\varphi^*\varphi$$

$$\Rightarrow \boxed{\varphi^*\varphi' - \varphi\varphi^{*\prime}\Big|_0^\infty = 2i\,\mathrm{Im}\,\lambda^2 \int_0^\infty dr\, \frac{|\varphi|^2}{r^2}}$$

Now, suppose that $S_\ell(E)$ has a pole below threshold, at $E<0$. We know that then

$$\varphi \underset{r\to\infty}{\to} e^{-pr} \quad \text{as well as} \quad \varphi \underset{r\to 0}{\to} r^{\ell+1}$$

Thus, provided $\mathrm{Re}\,\ell > -\frac{1}{2}$, the RHS of the above eqn vanishes, and

$$0 = 2i\,\mathrm{Im}\,\lambda^2 \int_0^\infty dr\, \frac{|\varphi|^2}{r^2} \Rightarrow \mathrm{Im}\,\lambda^2 = 0 \text{ or } \boxed{\ell \text{ is real}}$$
(assuming $\ell > -\frac{1}{2}$)

Below threshold, Regge poles occur only on the real axis in the ℓ plane.

If $S_\ell(E)$ has a pole above threshold, $E>0$, then we know $f_\ell(p)=0 \Rightarrow \varphi \sim Ce^{ipr}$ and the eqn becomes

$$2ip|C|^2 = 2i(\mathrm{Im}\,\lambda^2)\int_0^\infty dr\,\frac{|\varphi|^2}{r^2}$$

and hence $\mathrm{Im}\,\lambda^2 = 2(\mathrm{Re}\,\ell+\tfrac{1}{2})(\mathrm{Im}\,\ell) > 0$

and if $\mathrm{Re}\,\ell > -\tfrac{1}{2}$, this means $\boxed{\mathrm{Im}\,\ell > 0}$

Above threshold, the Regge poles are in the upper half of the ℓ plane. (This is reasonable, because φ is an outgoing wave if $f_\ell(p)$ vanishes; the loss of flux is compensated by the complex source term $i\,\frac{\mathrm{Im}\,\lambda^2}{r^2}$. This would be a sink, rather than a source of flux, if $\mathrm{Im}\,\lambda^2 < 0$.)

Below threshold ($E<0$), an argument similar to that which we used to show that bound state poles are simple as functions of p shows that Regge poles are simple as functions of ℓ. We can also say how the Regge poles move as E increases.

For $\lambda = \lambda(E)$ the position of a Regge pole, we differentiate radial eqn wrt E.

$$\left(\frac{d^2}{dr^2} - \frac{\lambda^2-\frac{1}{4}}{r^2} - 2V + 2E\right)\frac{d\varphi}{dE} = \left[\frac{d\lambda^2/dE}{r^2} - 2\right]\varphi$$

$$\Rightarrow \frac{d}{dr}\left(\varphi\frac{d\varphi'}{dE} - \varphi'\frac{d\varphi}{dE}\right) = \left[\frac{d\lambda^2}{dE}\frac{1}{r^2} - 2\right]\varphi^2$$

For bound states the LHS integrated from zero to ∞ will be zero (if $\mathrm{Re}\, l > -\frac{1}{2}$) and we have

$$\boxed{\frac{1}{2}\frac{d\lambda^2}{dE} = \frac{\int_0^\infty dr\, u_l^2}{\int_0^\infty dr\, u_l^2/r^2} = \frac{1}{\langle r^{-2}\rangle} > 0}$$

(u_l is real)

Thus for $\lambda = l + \frac{1}{2} > 0$, the location of the Regge pole moves to the right as E increases.

This is to be expected, of course. As l increases the centrifugal barrier increases so bound states will be pushed closer to threshold.

The equality is $\quad (\lambda^2 - \frac{1}{4})\frac{dE}{d\lambda^2} = \left\langle \frac{\lambda^2 - \frac{1}{4}}{2r^2} \right\rangle$

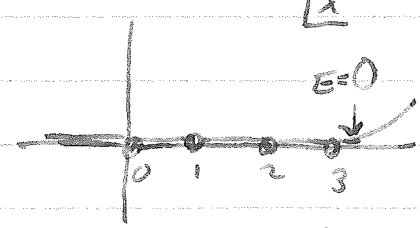
- i.e. the rate of increase of E with λ is directly related to the expectation value of the centrifugal term. (Which by the way, is actually attractive for $-\frac{1}{2} < l < 0$. As $E \to -\infty$, trajectories slip off to negative values of l, where they can take advantage of attractive centrifugal potential.)

The usual pattern of bound states in a central potential is...

$\longrightarrow$ Energy increasing

Energy increasing $\downarrow$	0 radial nodes	$l = 0, 1, 2, \ldots$
	1 radial node	$l = 0, 1, 2, \ldots$
	2 radial node	$l = 0, 1, 2, \ldots$
	$\vdots$	

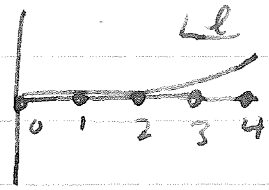
Regge theory offers a new perspective on this series. For each fixed number of radial nodes, the states of different orbital angular momentum l are linked by a trajectory in the l plane, a <u>Regge trajectory</u>.



Each time the Regge pole crosses an integer value of ℓ, it corresponds to a physical bound state

Since we know that there cannot be bound states for arbitrarily large values of ℓ (if $\int dr\, r |V| < \infty$, by the Bargmann inequality) the pole reaches some finite value of ℓ at $E=0$, and then for $E>0$ wanders into u.h.p.

The Regge trajectory links not only bound states of different values of ℓ; it can also associate a bound state at one value of ℓ with a <u>resonance</u> at another (larger) value of ℓ.



This is because if $S_\ell(E)$ has a pole for E real at ℓ very close to an integer, but with $\operatorname{Im} \ell > 0$, then it also has a pole at ℓ = integer and $\operatorname{Im} E$ small, i.e. a resonance pole.

Thus if the trajectory skims by close to an integer after leaving the real axis, it causes a resonance. Let us consider, therefore, the threshold behavior of the trajectory; i.e. the way it behaves in the vicinity of $E=0$.

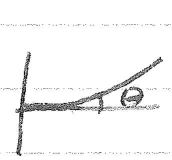
We recall the behavior of the Jost function near threshold

$$f_\ell \sim a + bp^2 + icp^{2\ell+1} + \cdots$$

and $a \propto (\ell - \ell_0)$ where ℓ_0 is the position of the Regge pole at threshold

Hence, for small positive values of $E = \frac{1}{2}p^2$, we have

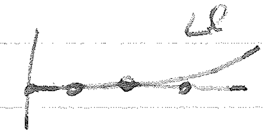
$$Re(\ell-\ell_0) \sim E$$
$$Im(\ell-\ell_0) \sim E^{\ell_0+\frac{1}{2}}$$



The angle that the Regge trajectory makes with real axis is

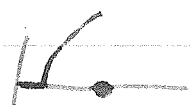
$$\tan\theta = \frac{d\, Im(\ell-\ell_0)}{d\, Re(\ell-\ell_0)} \propto (\ell_0+\tfrac{1}{2})\, E^{\ell_0-\frac{1}{2}}$$

Thus, if $\ell_0 > \frac{1}{2}$



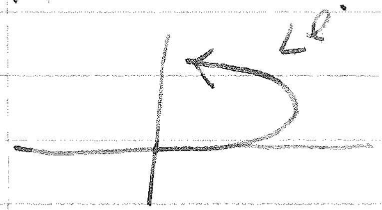
The trajectory hugs the real axis, and is likely to cause resonances near threshold

But if $-\frac{1}{2} < \ell_0 < \frac{1}{2}$



The trajectory leaves the real axis sharply, and is unlikely to cause a low energy resonance

What becomes of the trajectory after it leaves the real axis? For analytic potentials, we can show



that the trajectory cannot go to arbitrarily large values of Re ℓ. It always turns around and heads left. (This is actually not necessarily true for finite-range potentials, like a square well.)

Suppose that the potential $V(r)$ has an analytic continuation to the imaginary axis which satisfies

$$|V(ix)| < \mathcal{M}/x^2$$

we will show that

$$\boxed{Im(\ell+\tfrac{1}{2})^2 < \mathcal{M}}$$

By a similar but slightly more complicated argument which I will not give here (see, if you are interested, Potential Scattering by de Alfaro + Regge, section 8.3.) we see that if

$$V(ix) < N/x \text{ , then positions of poles satisfy}$$

$$\mathrm{Re}(\ell+\tfrac{1}{2}) < N/p$$

(At high energy, the Regge poles move back toward the LHP.)

The radial eqn for imaginary values of r becomes

$$\varphi'' + \left[2V(ix) - p^2 - \frac{\lambda^2 - \frac{1}{4}}{x^2}\right]\varphi = 0 \quad , \lambda = \ell + \tfrac{1}{2}$$

and

$$\varphi^{*\prime\prime} + \left[2V^*(ix) - p^2 - \frac{\lambda^{*2} - \frac{1}{4}}{x^2}\right]\varphi^* = 0 \text{; if } p \text{ is real}$$

Therefore,

$$\varphi^*\varphi'' - \varphi^{*\prime\prime}\varphi = \frac{d}{dx}(\varphi^*\varphi' - \varphi^{*\prime}\varphi) = 2i\left(\frac{\mathrm{Im}\,\lambda^2}{x^2} - \mathrm{Im}\,V(ix)\right)\varphi^*\varphi.$$

At the location of a Regge pole, the Jost function vanishes, and

$$\varphi_{\ell p}(r) \xrightarrow[r\to\infty]{} \text{const } e^{ipr} \propto e^{-px}$$

Thus, for $\mathrm{Re}\,\ell > -\frac{1}{2}$, integrating the RHS of the above eqn from 0 to ∞ gives zero and we have

$$\mathrm{Im}\,\lambda^2\left(\int_0^\infty dx \frac{|\varphi|^2}{x^2}\right) = \int_0^\infty V(ix)\,|\varphi|^2\,dx$$

$$\text{But } V < \frac{M}{x^2} \implies < \left(\int_0^\infty dx \frac{|\varphi|^2}{x^2}\right) M$$

$$\implies \boxed{\mathrm{Im}\,\lambda^2 < M}$$

This is $(\mathrm{Im}\,\ell)(\mathrm{Re}\,\ell + \frac{1}{2}) < \frac{M}{2}$,

so Regge pole can have large real part only if imaginary part becomes small.

C. Regge Poles for the Coulomb Potential

We have remarked several times that our general scattering formalism does not apply to the Coulomb ~~potential~~, because of its long distance tail. The partial wave S-matrix does not exist for the Coulomb potential in general because solutions to the radial equation behave asymptotically like

$$\psi_{\ell p} \longrightarrow \sin\left(pr + \frac{e^2}{p}\ln pr + \text{const}\right)$$

i.e. the influence of the potential $V = -\frac{e^2}{r}$ does not "turn off"

Nonetheless, we can find values of ℓ, E such that $\psi \sim e^{ipr}$(polynomial) and we define this to be location of a pole $\ell(E)$. The nice thing about the Coulomb potential is that we can solve for locations of Regge pole exactly. This is just a simple generalization of finding bound state energies.

The radial equation

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2V - p^2\right]\psi_{\ell p} = 0$$

becomes, if I make the substitution $\psi = \phi e^{ipr}$,

$$\phi'' + 2ip\phi' - \left[\frac{\ell(\ell+1)}{r^2} + 2V\right]\phi = 0$$

Now suppose $V = -e^2/r$, and lets solve by power series:

$$\phi = \sum_{m=0}^{\infty} a_m r^{\ell+1+m}$$

coefficient of $r^{\ell+m-1}$ in above equation becomes

$$\left[(\ell+1+m)(\ell+m) - \ell(\ell+1)\right]a_m + \left[2ip(\ell+m) + 2e^2\right]a_{m-1} = 0$$

or

$$\boxed{m(2\ell+1+m)\,a_m + \left[2ip(\ell+m) + 2e^2\right]a_{m-1} = 0}$$

This power series "ordinarily" generates the expected $\phi \sim e^{-2ipr}$

i.e. $$\frac{a_m}{a_{m-1}} = -\frac{2ip(l+m)+e^2}{m(2l+1+m)} \to -\frac{2ip}{m} \quad \text{for } m \to \infty$$

But if
$$2ip(l+m)+e^2 = 0 \quad \text{for some } m = 1, 2, 3, \ldots$$
then ϕ is a polynomial, and $\psi \sim$ polynomial $\times\, e^{ipr}$

this defines location of a Regge pole

$$\boxed{\alpha_m = l = -m - \frac{e^2}{2ip}}$$

Below threshold, $p = i\sqrt{2E}$, so we have $\frac{e^2}{\sqrt{-2E}} = l+m \Rightarrow E = \frac{e^4}{2(l+m)^2}$

(Take square root w/ $\operatorname{Im} p > 0$ for physical sheet.)

(Balmer Formula)

Below threshold, $$\boxed{\alpha_m = -m + \frac{e^2}{\sqrt{-2E}}, \quad E < 0}$$

Above threshold $$\boxed{\alpha_m = -m + i\frac{e^2}{\sqrt{2E}}, \quad E > 0}$$

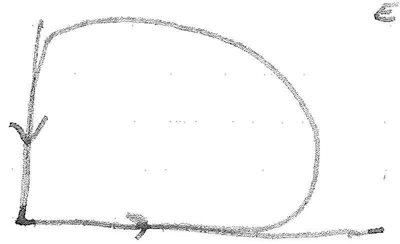
$m = 1, 2, 3, \ldots$

For both $E \to +\infty$, $E \to -\infty$, poles approach the same negative integer. For $e^2 \to -e^2$ potential is repulsive and there are no bound states

Special features of Coulomb potential:

- Poles spaced by integers $\Rightarrow$ accidental degeneracy of bound states
- Long range potential overcomes centrifugal term, so bound states of arbitrarily large l occur (Bargmann $\oint$ offers no constraint) Trajectories go off to $+\infty$ on real axis as E increases to zero
- 
  trajectory is pathological at threshold. It jumps around the circle at ∞ from real to imag axis. Hence, no resonances.

But the $E \to \pm\infty$ behavior of the Coulomb trajectories is more typical



e.g. For Yukawa potential $\alpha_m \to -m$ as $E \to \pm\infty$, but trajectory tracks smoothly from real axis to imaginary axis

D. The Sommerfeld-Watson Transform

Now we want to see what Regge pole theory can tell us about the full amplitude, the sum of the partial wave series.

$$f(E,\Theta) = \sum_{\ell}(2\ell+1) f_\ell(E) P_\ell(\cos\Theta).$$

We have already allowed E and ℓ to become complex; now, we wish to consider $z = \cos\Theta$ to be a complex variable, and study $f(E,z)$ as an analytic function of z

In particular, we will consider the asymptotic behavior of f for large z. This sounds very unphysical. Why is it interesting? For two reasons:

1) We will soon consider the dispersion relations satisfied by $f(E,z)$, and for this purpose we see that "physical" values of $f(E,z)$ are related to the unphysical values, which we must therefore also study.

2) The primary physical motivation comes from relativistic scattering theory. Although the Schrodinger eqn fails, we believe ~~that~~ there is a Regge theory for relativistic scattering, although it is more complicated (e.g. "Regge cuts" in the ℓ plane as well as poles.)
But in relativistic scattering $\cos\Theta \to \infty$ can be interesting, since
$$q^2 = +2p^2(1-\cos\Theta) = \text{momentum transfer}$$

corresponds to actual high energy scattering in the crossed channel.

If we are interested in $f(E,\theta)$ as $|\cos\theta| \to \infty$, then we might expect Regge theory to be applicable. Because θ and l are in a sense conjugate canonical variables, and just as we saw singularities of $V(r)$ determine asymptotic behavior of $\tilde{V}(q)$, we might expect that asymptotic $(z) \to \infty$ behavior of $f(E,z)$ is determined by singularities in the l plane of $f_l(E)$

(In the following discussion, some of the mathematical results will be stated without proof, because the proofs are tedious and unenlightening. For details see
R. Newton, The Complex j-Plane
V. de Alfaro + T. Regge, Potential Scattering)

Let us (following Regge's original paper and the two references above) consider potentials of the form

These have a field-theoretic motivation. $\left\{ V(r) = \int_\mu^\infty da \, \frac{e^{-ar}}{r} \rho(a) \right.$ where $\rho(a)$ is some distribution

"Generalized Yukawa" or "Yukawian" potential
[E.g. exponentially bounded analytic potentials

$$\int dr \, |V| e^{2ar} < \infty \qquad \text{and} \int_0 x|V(x)|\,dx < \infty \,]$$

For such potentials, it can be shown that (Assume that E is real)

Difficult to penetrate Coulomb barrier as $l \to \infty$

$$\boxed{|f_l(E)| < h(E)\, l^{-\frac{1}{2}} e^{-\alpha l} \; ; \; \cosh\alpha = 1 + \frac{\mu^2}{2p^2}}$$

A similar bound is satisfied by Legendre polynomials

$$P_l(\cos\theta) \leq g(\theta)\, l^{-\frac{1}{2}} e^{+|\operatorname{Im}\theta| l}$$

Hence if we wish to continue to complex values of θ, The series $\quad f(E,\theta) = \sum_\ell (2\ell+1)\, f_\ell(E)\, P_\ell(\cos\theta)$

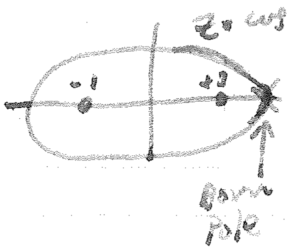
converges uniformly in θ, and defines an analytic function of θ only if

$$\boxed{|\mathrm{Im}\,\theta| < \alpha}$$

The boundary of this region of analyticity is called the (small) Lehmann ellipse, because in the $z = \cos\theta$ plane, it is an ellipse.

$$\cos(\theta_R + i\theta_I) = \cos\theta_R \cosh\theta_I - i \sin\theta_R \sinh\theta_I$$

$$\Rightarrow \boxed{\frac{(\mathrm{Re}\,z)^2}{\cosh\theta_I^2} + \frac{(\mathrm{Im}\,z)^2}{\sinh\theta_I^2} = 1} \qquad \cosh^2\theta_I = 1 + \frac{\mu^2}{2p^2}$$



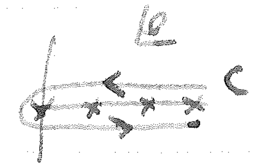
Foci are at $\cosh^2\theta_I - \sinh^2\theta_I = 1$ and -1

A Yukawa potential has the "Born pole" at $q^2 + \mu^2 = 0$ or $-\mu^2 = 2p^2(1-\cos\theta)$ or $\cos\theta = 1 + \frac{\mu^2}{2p^2}$

which prevents us from enlarging the ellipse

How can we continue around the pole and outside the ellipse?
Regge's idea:

$$\boxed{f(E,z) = \frac{1}{2i}\oint_C d\ell\, \frac{(2\ell+1)\, f_\ell(E)\, P_\ell(-z)}{\sin\pi\ell}}$$



(along contour shown in ℓ plane)

$$\sin\pi\ell \sim \pi(-1)^n(\ell-n) \quad \text{for } \ell \sim n, \text{ an integer}$$

So, evaluating residues gives

$$\frac{2\pi i}{2i}\sum_\ell \frac{(2\ell+1)}{\pi(-1)^\ell} f_\ell(E)\, P_\ell(-z) = \sum_\ell (2\ell+1)\, f_\ell(E)\, P_\ell(z)$$

This is the <u>Sommerfeld-Watson transform</u>

The integral as evaluated above converges only in the Lehmann ellipse, but we can analytically continue $f_\ell(E,z)$ outside the Lehmann ellipse by distorting the contour

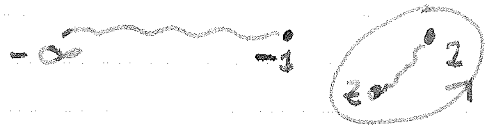
Note first that the integral makes sense; we have seen that $f_\ell(E)$ is analytic in $\mathrm{Re}\,\ell > -\frac{1}{2}$ except for poles and that the poles are all off the real axis for $E > 0$.

$P_\ell(z)$ also has an analytic continuation to complex ℓ

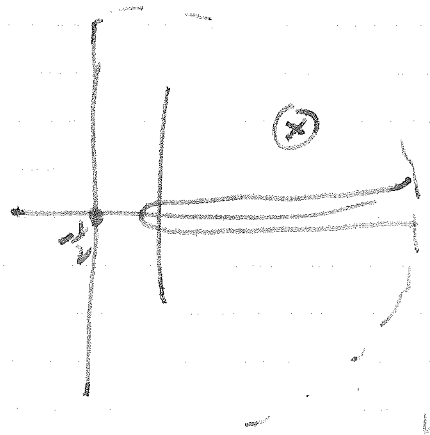
$$P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell = \frac{1}{2\pi i\, 2^\ell}\int_C dz'\,\frac{(z'^2-1)^\ell}{(z'-z)^{\ell+1}}$$

(where contour C enclosed pole at $z'=z$

This has branch points at $z' = \pm 1, z$ for $\ell \neq$ integer



If contour is chosen as shown $P_\ell(z)$ has cut running from -1 to $-\infty$ in ℓ plane



Let us try to distort contour as shown. $(\sin\pi\ell)^{-1}$ and $(2\ell+1)P_\ell(-z)$ have no singularities encountered by contour.

Vertical line is at $\mathrm{Re}\,\ell = -\frac{1}{2}$, the boundary of the region of analyticity which we established.

Is there a contribution from quarter circles at ∞?
For Yukawian potentials, Regge showed that it vanishes

$$\boxed{\,|S_\ell(E) - 1| \le C(E)\,e^{-\alpha\,\mathrm{Re}(\ell+\frac{1}{2})} \qquad \text{where } \cosh\alpha = 1 + \frac{\mu^2}{2p^2}\,}$$

(For square well, however, it does not vanish)

The integral $-\displaystyle\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\frac{d\ell}{2i}\cdot\frac{(2\ell+1)f_\ell(E)P_\ell(-z)}{\sin\pi\ell}$ converges in general (It is called the "Background integral")

Provided $\mathrm{Re}\,\Theta > 0$ this excludes real axis $z > 1$ (where $P_\ell(-z)$ has a cut) — the background integral is analytic in ℓ plane except for cut from $z = 1$ to $z = \infty$ [illegible]

[this conclusion follows from the bound
$$|f_\ell(E)| < \frac{h(E)}{|\ell|^{\frac{1}{2}}} \qquad \frac{P_\ell(-z)}{(\sin\pi\ell)} \lessgtr g|\ell|^{-\frac{1}{2}} e^{-\mathrm{Re}\,\Theta\,|\mathrm{Im}\,\ell|}$$]

Finally, we may encounter Regge poles when we distort contours, so

$$\boxed{f(E,z) = -\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{d\ell}{2i} \frac{(2\ell+1) f_\ell(E) P_\ell(-z)}{\sin\pi\ell} + \beta_i(E) P_{\alpha_i(E)}(-z)}$$

where $\beta_i(E)$ is residue of $\frac{1}{2i}\frac{(2\ell+1) f_\ell(E)}{\sin\pi\ell}$ at $\ell = \alpha_i(E)$

(Note: the interpolation of $f_\ell(E)$ to noninteger ℓ is not unique, I can add a function which vanishes at the integers without changing the sum; but it is unique if I insist that the contour at ∞ not contribute when I distort the contour)

Asymptotic Behavior

Now we conveniently find the asymptotic behavior as $|z| \to \infty$ of $f(E,z)$. The key is the behavior

$$P_\ell(z) \sim z^{\ell_R + i\ell_I} \text{ or } |P_\ell| \xrightarrow[|z|\to\infty]{} \mathrm{const}\,|z|^{\ell_R}$$

Thus, for $|z| \to \infty$, the background integral vanishes, and

$$\boxed{f(E,z) \xrightarrow[|z|\to\infty]{} \beta(E)\,(-z)^{\alpha(E)}}$$

this is the central result of Regge theory: the asymptotic $|z| \to \infty$ behavior is controlled by the Regge pole furthest to the right.

Resonances

Let's briefly consider again how Regge poles are related to resonances, in light of the Sommerfeld-Watson transform. Near a pole

$$f(E,z) \sim \beta(E)\, P_{\alpha(E)}(-z)$$

Now, let's project out a partial wave, using

$$f_\ell(E) = \frac{1}{2}\int_{-1}^{1} f(E,z)\, P_\ell(z)\, dz$$

and $\frac{1}{2}\int_{-1}^{1} P_\ell(z) P_\alpha(-z)\,dz = \dfrac{\sin\pi\alpha}{\pi(\alpha-\ell)(\alpha+\ell+1)}$ [goes to usual $(-1)^\ell/2\ell+1$ or zero for $\alpha\to\ell$, $\alpha\to$ other integer]

Hence

$$f_\ell(E) \sim \frac{\beta(E)\sin\pi\alpha(E)}{\pi}\ \frac{1}{[\alpha(E)-\ell][\alpha(E)+\ell+1]}$$

If α is close to ℓ, this is a resonance

$$\alpha \sim \ell + (E-E_R)\frac{d\alpha_R}{dE} + i\alpha_I(E)$$

$$\Rightarrow \boxed{f_\ell \sim \frac{\beta\sin\pi\alpha}{\pi(2\ell+1)}\ \frac{1}{\frac{d\alpha_R}{dE}\left[E-E_R+\frac{i\Gamma}{2}\right]}}$$

Note that $\beta\sin\pi\alpha$ has no pole at $\alpha\sim\ell$ – see p. 7.16

where $\boxed{\Gamma/2 = \dfrac{\alpha_I(E_R)}{d\alpha_R/dE}}$ This is a Breit Wigner resonance

If we measure several resonances, spacing in energy gives estimate of $d\alpha_R/dE$ and α_I can be inferred from widths

Interpretation of $\alpha_I(E_R)$

As Γ is an energy width, α_I is an "angular momentum width" i.e. $\Delta\theta = 1/\alpha_I$ is angle through which projectile orbits during lifetime of the resonance

Recall $\dfrac{d}{dE}\lambda^2 = \langle r^{-2}\rangle^{-1} \sim R^2$ $\quad R$ = size of "classical orbit"

$\lambda = \ell+\frac{1}{2}$ was shown for ~~...~~ (p. 7.6)

If $\mathrm{Im}\,\lambda$ small, we still expect

$$\frac{d\lambda}{dE} \sim \frac{R^2}{2\lambda} \quad \text{and} \quad \frac{\Gamma}{2} \sim \frac{\alpha_I}{d\alpha_R/dE} \sim \frac{2\lambda}{R^2}\alpha_I$$

classical speed is v such that $vR \sim \hbar\lambda$

Angle of orbit is

$$\Delta\theta = \frac{v}{R}\left(\frac{\Gamma}{2}\right)^{-1} = \frac{\hbar\lambda}{R}\,\frac{1}{R}\,\frac{R^2}{2\lambda\alpha_I} = \frac{\hbar}{2\alpha_I}$$

so $\boxed{\alpha_I \,\Delta\theta \sim \hbar}$, as expected
if α_I is an angular momentum "width"

E. Regge Poles in Relativistic Scattering

The "practical" applications of Regge Theory are in particle physics. We quickly realize that the Regge treatment of high energy asymptotic behavior, while mathematically consistent, is not physically consistent. At high energies, nonrelativistic quantum mechanics breaks down. Does Regge theory survive in a relativistic context?

A form of Regge theory survives, but it is more complicated. E.g., there are cuts in the l plane (Regge cuts) and poles do not necessarily dominate asymptotic behavior. Also, it is hard to prove general theorems.

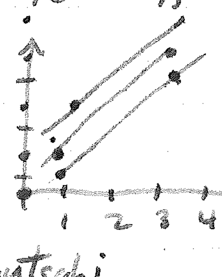
What contact can we make between particle physics data and Regge theory?

Regge Trajectories

The meson and baryon resonances seem to be naturally classified in Regge families. One finds

$$\boxed{M^2 = K^{-1}(\mathrm{Re}\,\alpha) + M_0^2}$$

where K^{-1} is a universal slope for mesons and baryons (but the intercept M_0^2 varies). $\underline{K^{-1} \approx 1 \text{ GeV}^2}$.

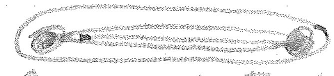


Chew-Frautschi
Plot

A peculiarity - Trajectories have a definite signature, and Re α is physical only for values of j differing by 2.

all particles are composite
(Even Nucleon is on a Regge trajectory)

Why linear trajectories?



This is explained by QCD, which predicts that widely separated quarks are connected by a string. Orbitally excited mesons consist of two quarks widely separated. When ℓ increases, the string stretches, and the increase in M^2 is due to extra string.

Orbitally excited baryon is diquark and quark

Same slope because string energy is the same

In QCD, we can actually calculate K^{-1} in terms of other strong interaction parameters, like m_{proton}, and such calculations are now being done.

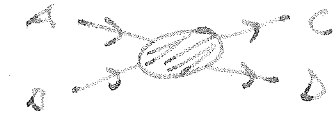
High Energy Scattering

In relativistic scattering, we have no central potential, and no Schrodinger equation. But we may still define partial wave amplitudes by projecting with Legendre polynomials

$$f_\ell(E) = \frac{1}{2}\int_{-1}^{1} dz\, P_\ell(z)\, f(E,z); \quad f = \sum (2\ell+1) f_\ell(E) P_\ell(z)$$

It is convenient to express f in terms of Lorentz-invariant quantities.

Consider $A + B \to C + D$



Four momenta
$p_A = (E_A, \vec{p}_A)$ $\quad E_A^2 = p_A^2 + m_A^2$
etc.

Let $s = (p_A + p_B)^2 = (E_A + E_B)^2$ in cm frame $= E_{cm}^2$

$m_A = m_C$

$t = (p_A - p_C)^2$ In case of elastic scattering $p_A^2 = p_C^2$

$\Rightarrow t = -2\vec{p}_A^{\,2}(1-\cos\theta)$
$= -\vec{q}^{\,2}$, the momentum transfer

$f = f(s,t)$ – a function of Lorentz invariants ??

In fact, relativistic Regge theory is applied not to f, but to the relativistically-invariant amplitude

$$A(s,t) = 8\pi s^{\frac{1}{2}} \; ; f(E,z) = 8\pi s^{\frac{1}{2}} f(s,t)$$

We know that the Regge asymptotic behavior is

$$A(s,t) \sim s^{\frac{1}{2}} B(E) \; z^{\alpha(E)} \quad \text{as } z \to \infty, \text{ fixed } E$$
$$\text{or } \sim g(s)\, t^{\alpha(s)} \quad \text{as } t \to \infty, \text{ fixed } s$$

But the great virtue of $A(s,t)$ is that it is crossing symmetric. i.e.

$$\boxed{A(t,s) \to g(t)\, s^{\alpha(t)} \quad \text{as } s \to \infty,\ t \text{ fixed}}$$

This describes scattering

$A + \bar{C} \to \bar{B} + D$ – i.e. t, s have physical values. (changing sign of 4-momentum changes the particle to an outparticle)

- The Regge Poles in s channel are bound states
- The Regge Poles in t channel determine physical asymptotic behavior in crossed channel.

In relativistic scattering, Regge poles play a dual role

The same amplitude A describes scattering

$A + B \to C + D$

and $A + \bar{C} \to \bar{B} + D$

It is just an analytic continuation to get from one to another!

Regge Poles and High Energy Experiments

Is the Regge asymptotic behavior observed?

There is much literature on this subject. See e.g.
Collins + Squires, Regge Poles in Particle Physics
Frautschi, Regge Poles and S-Matrix Theory
Barger and Cline,

The applications to elastic scattering are important, because this involves soft processes not treatable in QCD perturbation theory. (Also total cross sections)

a) Total Cross Section:
We have not yet discussed inelastic scattering, but optical theorem still applies - Imag part of forward elastic scattering is related to total cross section

$$\sigma_T = \frac{4\pi}{p} \operatorname{Im} f(E,0) = \frac{1}{2p s^{1/2}} \operatorname{Im} A(s)$$

$$\text{and } \operatorname{Im} A(s) \sim s^{\alpha(0)} \text{ for } s \to \infty$$

$$\text{Hence } \sigma_T \sim s^{\alpha(0)-1}$$

We should expect all scattering processes with the same quantum nos. in the crossed channel to behave like same power of s - i.e. be dominated by some Regge pole. This applies in particular to forward amplitude $A+B \to A+B$, where q nos are those of $A\bar{A}$, $B\bar{B}$; i.e., vacuum q. nos.

It is observed that all total cross sections (pp, $\bar{p}p$, πp, Kp, etc) are roughly constant for large p, suggesting

$$\alpha(0) \sim 1$$

(Actually, asymptotic behavior looks like a log, which could be due to contributions from Regge cuts, or other trajectories.)

This pole at $t=0$, $\alpha=1$ is called the pomeron. It is not a physical particle because its trajectory has even signature

b) <u>Forward Peak</u>

For elastic scattering

$$\frac{d\sigma}{dt} = \frac{d\Omega}{dt}|f|^2 \sim \frac{1}{s^2}|A|^2 \sim g(t) \quad s^{2\alpha(t)-2}$$

This is dominated by the pomeron, with $\alpha(t) \sim 1 + \alpha' t$

$$\Rightarrow \frac{d\sigma}{dt} \sim g(t)\, s^{\alpha' t} = g(t)\, e^{-\alpha'(\ln s)|t|}$$

If $g(t)$ is slowly varying, this is an exponential diffractive peak (which is seen) with a width which decreases logarithmically with s

The decreasing width suggests that the "optical size" of the target increases as energy increases

The logarithmic shrinking is not seen. Perhaps more than one trajectory is contributing, or $g(t)$ is not really slowly varying.

F. The Froissart Bound

At high energy (short wavelength compared to size of target) we expect cross sections to be nearly geometrical. This is the limit of "geometrical optics", and we expect the target to cast a sharp shadow (And at sufficiently high energy, we are out of the "resonance region")

A heuristic argument for constant cross sections (or cross sections bounded by constants) at high energy goes as follows:
In partial-wave expansion

$$f(E,\theta) = \sum_\ell (2\ell+1) f_\ell(E)\, P_\ell(\cos\theta),$$

we expect, by semiclassical reasoning, that little scattering occurs for

$$\ell \gtrsim pR \qquad \text{if } R \text{ is range of potential}$$

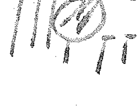
Since $f_\ell(p) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{p}$, the unitarity bound is $|f_\ell| \lesssim \frac{1}{p}$

and we therefore expect, by optical theorem

$$\sigma_T = \frac{4\pi}{p}|\mathrm{Im}\, f(E,0)| \lesssim \frac{4\pi}{p^2}\sum_{\ell=0}^{pR}(2\ell+1) \sim 4\pi R^2$$

i.e. $\boxed{\sigma_T \lesssim 4\pi R^2}$

This differs by the infamous factor of 4 from the geometrical cross section. We understand the discrepancy –



To create a shadow behind the sphere there must be a scattered wave 180° out of phase with incident wave.

But the short wavelength limit should be the classical limit, and how can the classical cross section be 4 times geometrical?

The explanation is that the "extra flux" is all very close to the forward direction. Terms in $\sum_\ell (2\ell+1)P_\ell(\cos\theta)$ interfere for $\theta \neq 0$ and sum to a term concentrated to within $\frac{1}{pR}$ of $\theta = 0$
(Again, we see a connection between an angular momentum width and an angular width.)

Now we will present a more careful argument, showing that total cross section can actually grow like $\log^2 E$ at high energies:

Let consider total amplitude f as a function of $E = \frac{1}{2}p^2$ and $q^2 = 2p^2(1-\cos\theta)$

$$f(E,q^2) = \sum_\ell (2\ell+1) f_\ell(E) P_\ell(\cos\theta) = \sum_\ell (2\ell+1) f_\ell(E) P_\ell(1 - q^2/4E)$$

We will also use a property which we showed is satisfied by relativistic amplitudes; namely, "polynomial boundedness"

$\boxed{f(E,q^2) \lesssim E^{2\alpha}}$ for fixed q^2, $E \to \infty$

(This followed from Regge theory and crossing)

Now, we consider the amplitude for large E (real, positive) and $q^2 < 0$ ($\cos\Theta > 1$) chosen so that the partial-wave series converges. (q^2 inside the Lehmann ellipse)

Legendre polynomials obey $P_\ell(z) > 0$ for $z > 1$ and also $\mathrm{Im} f_\ell(E) = \mathrm{Im}\, \frac{e^{i\delta_\ell}\sin\delta_\ell}{p} = \frac{\sin^2\delta_\ell}{p} > 0$ for physical E

Hence, in

$$\mathrm{Im} f = \sum_\ell (2\ell+1)\,\mathrm{Im} f_\ell(E)\, P_\ell(1 - q^2/4E) \lesssim \frac{1}{p} E^{2\alpha}$$

each term is real and positive, and therefore satisfies the bound separately (For $E > 0$, $q^2 < 0$)

$$\boxed{(2\ell+1)\,\mathrm{Im} f_\ell(E)\, P_\ell(1 - q^2/4E) \lesssim \frac{1}{p} E^{2\alpha}} \quad \boxed{\begin{matrix} E > 0 \\ q^2 < 0 \end{matrix}}$$

Now we invoke a bound on P_ℓ:

$$P_\ell(1 - q^2/4E) > \frac{\mathrm{Const}}{(2\ell+1)^{1/2}}\, e^{\ell\sqrt{-q^2/2E}}$$

which I state without proof. Therefore

$$\mathrm{Im} f_\ell(E) \lesssim \frac{\mathrm{const}'}{(2\ell+1)^{1/2}} \frac{E^{2\alpha}}{p}\, e^{-\ell\sqrt{-q^2/2E}}$$

$$\le \frac{\mathrm{const}'}{p} \exp\left[(2\alpha)\ln E - \ell\sqrt{-q^2/2E}\,\right]$$

We now bound the partial-wave series for $\mathrm{Im} f$ by dividing it up into two pieces. We first sum up to

$$\ell = L(s) = \frac{(2\alpha)\ln E}{\sqrt{-q^2/2E}}$$ where argument of exponential is zero

and then we sum the rest

In the first part of the sum, it is best to use the unitarity bound

$$\sum_{\ell=0}^{L(s)} \mathrm{Im} f_\ell (2\ell+1) \lesssim \frac{1}{p}(L(s)+1)^2 \sim \frac{4\alpha^2 \ln^2 E}{p}\left(\frac{2E}{-q^2}\right)$$

the rest of the sum is

$$\sum_{\ell=L(s)}^{\infty}(2\ell+1)\,\mathrm{Im} f_\ell < \frac{\text{const}'}{p}\sum_{\ell=1}^{\infty}(2\ell+1)\, e^{-\sqrt{-q^2/2E}\,\ell}$$

and this sum can be bounded by

$$\int dx\, x\, e^{-ax} = \frac{1}{a^2} = \frac{2E}{-q^2}$$

which is asymptotically small compared to the other term.

Hence $|\mathrm{Im} f(E,0)| \lesssim \frac{4\alpha^2}{p}\left(\frac{2E}{-q^2}\right)\ln^2 E$, and

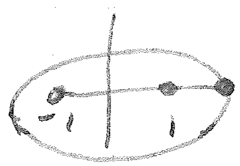
Finally, we have

$$\sigma_{total} = \frac{4\pi}{p}\,\mathrm{Im} f(E,0) \lesssim \frac{16\pi}{p^2}\alpha^2 \ln^2 E \left(\frac{2E}{-q^2}\right)$$

or

$$\boxed{\sigma_{Total} \lesssim \frac{16\pi\,\alpha^2 \ln^2 E}{-q^2}}$$

This is true as long as q^2 is chosen so that the series for $\mathrm{Im} f(E,0)$ converges. We should choose $-q^2$ as large as possible to optimize the bound.



We recall that for Yukawa potentials

$$V(r) = \int_{\mu}^{\infty} da\, \frac{e^{-ar}}{r}\,\rho(a)$$

the partial-wave series converges in the Lehmann ellipse, which touches the Born singularity at

$$\cos\theta = 1 + \frac{\mu^2}{2p^2} \quad \text{or} \quad q^2 = -\mu^2$$

provided E is real and positive.

Actually we can do a bit better if we consider $f - f_{Born}$ (since f_{Born} has no imaginary part), for which series converges if $-q^2 < 4\mu^2$. Thus, if $-q^2 = 4\mu^2$, series for $\mathrm{Im} f$ converges for any E. We have

$$\boxed{\sigma_{Total} \lesssim \frac{4\pi}{\mu^2}\alpha^2 \ln^2 E}$$

Froissart Bound

This is remarkably general. We have used only polynomial boundedness in E and convergence of partial-wave series in (enlarged) Lehmann ellipse.

The α which appears here is the one appropriate for the value $q^2 = 4\mu^2$

All of the above manipulations can be carried out in the relativistic case. In strong interaction physics, the Imag part of ~~partial~~-wave ~~amplitude~~ ~~convergent~~ expansion converges for

$$-q^2 < 4m_\pi^2$$

and we expect $\alpha \sim 1$ (the pomeron dominates, and $\alpha \sim 1 + \alpha'(4m_\pi^2) \sim 1$.) Thus we expect

$$\sigma_{Total} \lesssim \frac{\pi}{m_\pi^2} \ln^2\left(\frac{s}{s_0}\right) \sim 62 \text{ millibarns } \ln^2\left(\frac{s}{s_0}\right)$$

At the highest energy, a slow increase in total cross section is seen. In pp and $\bar{p}p$ scattering, the increase sets in at

$\sigma \sim 43$ millibarns

(Has been measured up to $E_{CM} \sim 60$ GeV at ISR)

"The strong interactions are as strong as possible."

Saturation of the Froissart bound suggests that effective target size increases logarithmically with energy. This agrees with Regge prediction of a shrinking diffractive peak.

What is s_0? $\sigma(pp)$ increases from ~ 38.5 to 43.2 between $E_T = 15$ GeV and $E_T = 60$ GeV

$$\frac{\ln \frac{60^2}{s_0}}{\ln 15^2/s_0} \sim \left(\frac{43.2}{38.5}\right)^{\frac{1}{2}} \sim 1.06$$

$$\Rightarrow \frac{60^2}{s_0} \sim \left(\frac{15^2}{s_0}\right)^{1.06} \Rightarrow \sqrt{s_0} \sim .9\text{ GeV}$$

Are we already in asymptotia?

VIII. Dispersion Relations

A. The Forward Amplitude

Dispersion relations for the scattering amplitude are the legacy of particle physics of the 50's and 60's. In those days, there was a dream that the observed dynamics of the strong interactions was the unique solution to a number of requirements, which included analyticity, unitarity, and relativistic Poincare invariance.

It is not popular to believe today that analyticity determines the dynamics, but that analyticity is a property of the S-matrix in Quantum field theory, which does. Nonetheless, dispersion theory methods are still useful in the theory and phenomenology of elementary particles, even though they do not have some prominent a place any more.

First, we'll consider full (not partial-wave) forward amplitude.

$$f(E) = f(E,0).$$

We could study analytic behavior of the summed partial wave series, but it is actually simplest to work directly with the T matrix

$$f(E) = -2\pi^2 \langle \vec{p} | T(E+i\epsilon) | \vec{p} \rangle \quad ; \quad T = V + VGV$$

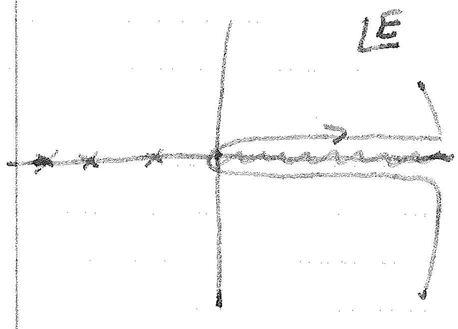
$$= \underbrace{-\frac{1}{2\pi}\int d^3x\, V(r)}_{\substack{\text{Born Amplitude;} \\ \text{independent of } E}} \underbrace{- \frac{1}{2\pi}\int d^3x\, d^3x'\, e^{-i\vec{p}\cdot\vec{x}'} V(r') \langle \vec{x}' | G(E+i\epsilon) | \vec{x} \rangle V(r) e^{i\vec{p}\cdot\vec{x}}}_{\tilde{f}(E)}$$

$$\tilde{f}(E) = -\frac{1}{2\pi}\int d^3x\, d^3x'\, \underbrace{e^{ip|\vec{x}-\vec{x}'|\cos\theta}}_{\substack{\text{analytic in} \\ \text{cut } E\text{ plane}}} \underbrace{\langle \vec{x}' | G(E+i\epsilon) | \vec{x} \rangle}_{\substack{\text{analytic} \\ \text{in cut } E\text{ plane} \\ \text{except for} \\ \text{bound state} \\ \text{poles}}} V(r') V(r)$$

The integrand is an analytic (meromorphic) function in cut E plane; so is $\tilde{f}(E)$ if integral converges uniformly. It does in fact, for V satisfying $\int dr\, r|V(r)| < \infty$, and for all E with $\cos\theta = 1$

(on physical sheet, $\mathrm{Im}\, p > 0$.)

(we'll come back to this point)



Thus $f(E)$ has a simple analytic structure, and we can derive a dispersion relation by considering the contour shown

Other important properties are
1) $\tilde{f}(E) \to 0$ as $E \to \infty$ in any direction.
   This follows from our proof (p. 5.11) that the Born series converges at high energy
   (This also means $G \to G^0$, which allows us to prove uniform convergence)
2) $f(E)$ is real for E real, < 0. (the integrand is real) $\Rightarrow$ $f(E) = f(E^*)^*$ Schwarz Reflection Principle
3) Poles are simple poles: $f \sim \dfrac{\Gamma}{E - E_n}$

Because the circle at ∞ does not contribute, Cauchy's theorem gives us

$$\boxed{f(E) = f_{\text{Born}} + \sum \frac{\Gamma_n}{E - E_n} + \frac{1}{\pi}\int_0^\infty dE' \, \frac{\mathrm{Im}\, f(E')}{E' - E}}$$

that is, $\dfrac{1}{2\pi i}\displaystyle\int_C dE' \frac{\tilde{f}(E')}{E' - E} = \tilde{f}(E) + \frac{\Gamma_n}{E_n - E}$

and the discontinuity of $\tilde{f}(E)$ across cut is $2i\, \mathrm{Im} f(E)$

As E approaches physical region from upper half plane

$$\int_0^\infty dE' \frac{\mathrm{Im} f(E')}{E' - E} = \left[\int_0^{E-\epsilon} + \int_{C_\epsilon} + \int_{E+\epsilon}^\infty\right] \frac{\mathrm{Im} f(E')}{E' - E}$$

$$= i\pi\, \mathrm{Im} f(E') + P\int_0^\infty \frac{\mathrm{Im} f(E')}{E' - E}$$

Thus, this relation may be written

$$\boxed{\mathrm{Re}\,f(E) = f_{Born} + \sum_n \frac{\Gamma_n}{E-E_n} + \frac{1}{\pi}\int_0^\infty dE' \frac{\mathrm{Im}\,f(E')}{E'-E}}$$

understood to be above cut

(Physical Region)

This gives Ref in the physical region in terms of Imf; and

$\mathrm{Im}f(E) = \frac{p}{4\pi}\sigma_{Total}$ – directly observable

These relations are remarkable because

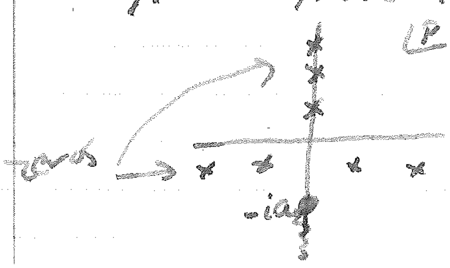
1) they are so general. Only analyticity and bound on asymptotic behavior are needed (we do need to be able to bound $E\to\infty$ behavior in all directions in cx plane, however). Thus, such a relation may continue to hold relativistically, even though Schrodinger eqn no longer applies

2) Involves observable quantity σ_{Total}. In relativistic case, the dispersion relation can be used to find Γ_n, e.g. the π-N coupling constant, which is the residue of a pole

Of course, it is not physically consistent to use NR Quantum mechanics, which breaks down as $E\to\infty$, to bound $f(E)$, but it is mathematically consistent

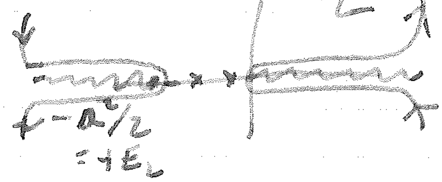
B. Partial-Wave Dispersion Relations

The partial wave amplitudes also obey dispersion relations. We'll consider exponentially bounded analytic potentials. The Jost function $f_\ell(p)$ has been shown to have the structure shown:



It has a cut beginning at $p=-ia$

In the energy plane, on the physical sheet, the partial-wave amplitude $f_\ell(E)$ has both right hand and left-hand cuts.



We may apply Cauchy's theorem to the contour shown. We recall some properties

1) $f_\ell(p) \to 1$ as $p \to \infty$ along any direction for exponentially bounded potential $\Rightarrow f_\ell \to 0$

2) f_ℓ is real on real axis between cuts in E plane, because
$$f_\ell(E) = \frac{1}{2ip}\,\frac{f_\ell(-p) - f_\ell(p)}{f_\ell(p)}$$
and $f_\ell(p)$ is real when p is imaginary.

3) Poles are simple: $f_\ell(E) \sim \dfrac{\Gamma_n}{E - E_n}$

Thus, contour at $R = \infty$ does not contribute in
$$\frac{1}{2\pi i}\oint_C dE'\,\frac{f_\ell(E')}{E' - E} = f_\ell(E) + \sum_n \frac{\Gamma_n}{E_n - E}$$
we have only integrals of discontinuities across cuts
$$= \frac{1}{2\pi i}\, 2i\left(\int_{-\infty}^{+E_L} + \int_0^{\infty}\right) dE'\,\frac{\operatorname{Im} f_\ell(E')}{E' - E}$$

or

$$\boxed{f_\ell(E) = \sum_n \frac{\Gamma_n}{E - E_n} + \frac{1}{\pi}\left[\left(\int_{-\infty}^{+E_L} + \int_0^{\infty}\right) dE'\,\frac{\operatorname{Im} f_\ell(E')}{E' - E}\right]}$$

Again as E approaches physical region from above cut, we will pick up $\operatorname{Im} f_\ell$ and a principal value:

$$\mathrm{Re}\, f_\ell(E) = \sum \frac{\Gamma_n}{E-E_n} + \frac{1}{\pi}\left(\int_{-\infty}^{+E_L} + P\int_0^\infty\right) dE' \,\frac{\mathrm{Im}\, f_\ell(E')}{E'-E}$$

(Physical Region)

Can $\mathrm{Im} f_\ell$ be expressed in terms of a "physical" quantity?

$$f_\ell = \frac{e^{2i\delta_\ell}-1}{2ip} = \frac{1}{p} e^{i\delta_\ell}\sin\delta_\ell$$

$$\Rightarrow \mathrm{Im} f_\ell = \frac{\sin^2\delta_\ell}{p} = |f_\ell|^2 \leftarrow$$ the measured partial-wave cross section

In relativistic case, the discontinuity is also measurable across LH cut, because it is a cross section for a crossed process. Of course, we will need to consider inelastic channels also.

C. Dispersion Relation at Fixed Non-zero q^2

Using Regge theory, we have shown that the partial-wave amplitudes can be analytically continued in the cut q^2 plane. We also know, that for a fixed value of q^2, $f(E,q^2)$ is analytic in the E plane, with both a RH and a LH cut. These results ~~hold~~ for Yukawian potentials:

$$V(r) = \int_\mu^\infty da \; \frac{e^{-ar}}{r} \rho(a) \; , \quad \mu > 0 .$$

Remarkably, when we consider the full amplitude, the LH cut along the negative real axis disappears! This can be seen for $-q^2 < 4\mu^2$, i.e. inside the Lehmann ellipse (enlarged) where f - f_{Born} has a convergent partial wave expansion.

This is actually most easily established by working with the full amplitude directly. The idea is to show that the integral in

$$F(\vec{p}\,';\vec{p}) = -\frac{1}{2\pi}\int d^3x \; e^{-i(\vec{p}\,'-\vec{p})\cdot\vec{x}} V(r)$$

$$-\frac{1}{2\pi}\int d^3x\, d^3x' \, e^{-i\vec{p}\,'\cdot\vec{x}\,'} V(r') \langle \vec{x}\,'| G(E_p+i\epsilon) |\vec{x}\rangle V(r) e^{i\vec{p}\cdot\vec{x}}$$

converges uniformly and therefore defines an analytic function of E with a cut and bound state poles, but no left cut. This can be shown for $q^2 \le 4\mu^2$. For details see...

Goldberger and Watson, Collision Theory, Chapter 10
Newton, Scattering Theory, Chapter 10

Now we can derive dispersion relation for $f(E, q^2)$ by exactly the same reasoning as for the forward case. For our Yukawa potential, the Born Term is

$$\int_\mu^\infty d\alpha\, \rho(\alpha) \frac{-2}{q^2+\mu^2} = f_{born}(q^2)$$

and we have

$$\boxed{f(E,q^2) = f_{born}(q^2) + \sum_n \frac{\Gamma_n(q^2)}{E-E_n} + \frac{1}{\pi}\int_0^\infty dE' \frac{\mathrm{Im} f(E',q^2)}{E'-E}}$$

This fixed-q^2 dispersion relation is not as useful as the $q^2=0$ relation because
1) $\mathrm{Im} f$ cannot be related to σ_{total}
2) ~~Integration~~ extends over "unphysical" E
that is $0 \le q^2 = 2p^2(1-\cos\theta) \le 4p^2 = 8E$
and $E < \frac{1}{8}q^2$ is not kinematically allowed

It is more useful to consider f as an analytic function of both E and q^2 simultaneously

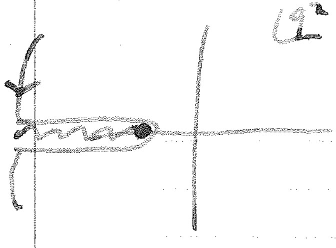
D. Dispersion Relation at Fixed Energy

Let's first consider $f(E,q^2)$ as a function of q^2 with E fixed. We recall the main results of Regge theory, for Yukawian potentials:

1) $\tilde{f}(E,q^2)$ is analytic for any fixed $E>0$ in the q^2 plane except for a cut from $q^2 = 4\mu^2$ to $-\infty$.

2) Asymptotically, $\tilde{f}(E,q^2) \longrightarrow (q^2)^{\alpha(E)}$ as $q^2 \to \infty$

I may choose E so that $\mathrm{Re}\,\alpha < 0$, since we know that $\mathrm{Re}(l+\frac{1}{2}) <$ const for Yukawian potentials. Then we can ignore circle at ∞ in the contour integral...



$$\frac{1}{2\pi i}\int_C dq'^2 \, \frac{\tilde{f}(E,q'^2)}{q'^2-q^2} = \tilde{f}(E,q^2)$$

$$\tilde{f} = f - f_{Born}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{-4\mu^2} dq'^2 \, \frac{\mathrm{Disc}\,\tilde{f}(E,q'^2)}{q'^2-q^2}$$

The discontinuity is not necessarily imaginary here, and $\mathrm{Disc}\, f \neq \mathrm{Disc}\,\tilde{f}$, since Born term also has branch point at $q^2 = -\mu^2$

Anyhow, writing $\rho(E,q'^2) = \frac{1}{2i}\mathrm{Disc}\,\tilde{f}(E,q'^2)$, we have the representation

$$\boxed{f(E,q^2) = f_{Born}(q^2) + \frac{1}{\pi}\int_{-\infty}^{-4\mu^2} dq'^2 \, \frac{\rho(E,q'^2)}{q'^2-q^2}}$$

This looks even less useful than the fixed-q^2 dispersion relation, since all values of q^2 integrated over here are nonphysical. But we will find a use for this representation.

If $f \to$ constant, this is a dispersion relation for $f(E,q^2) - f(E,a_i^2)$, or, in general, a dispersion relation for $f(E,q^2)$ − (polynomial in q^2). Hence "subtracted"

Subtractions:

We can derive a dispersion relation for $f(E,q^2)$ with E fixed even if $\alpha(E) > 0$ by a "subtraction" procedure. If $\alpha < n$, we consider

$$\frac{1}{2\pi i}\int_C dq'^2 \frac{\tilde{f}(E,q'^2)}{(q'^2-q^2)}\prod_{i=1}^{n}\left(\frac{q^2-a_i^2}{q'^2-a_i^2}\right) = \tilde{f}(E,q^2) - \sum_j \tilde{f}(E,a_j^2)\prod_{i\neq j}\left(\frac{q^2-a_i^2}{a_j^2-a_i^2}\right)$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{-4\mu^2} dq'^2 \frac{\mathrm{Disc}\,\tilde{f}(E,q'^2)}{(q'^2-q^2)}\prod\left(\frac{q^2-a_i^2}{q'^2-a_i^2}\right)$$

But the $f(E,a_i^2)$ are arbitrary functions of E which must be specified. "Subtraction constants"

E. Mandelstam Representation

The Mandelstam representation comes closest to realizing the old dream of a method of calculation relying on analyticity and unitarity alone, and no other dynamical principal.

We return to the dispersion relation at fixed-q^2, which we argued could be directly established for $0 \le q^2 < 4\mu^2$, if $V(r)$ is a superposition of Yukawa potentials

$$V(r) = \int_\mu^\infty da\, \frac{e^{-ar}}{r}\,\sigma(a).$$

Let us first consider the case in which there are no bound states. Then

$$f(E,q^2) = f_{Born}(q^2) + \frac{1}{2\pi i}\int_0^\infty \frac{dE'}{E'-E}\left[f(E'+i\epsilon,q^2) - f(E'-i\epsilon,q^2)\right]$$

Now $f(E'\pm i\epsilon, q^2)$ obeys a fixed-E' dispersion relation. Let's suppose that no subtractions are required, i.e. $\alpha(E) < 0$ for all E (which is true for sufficiently weak Yukawa potentials.)

$\frac{1}{2i}[f(E'+i\epsilon, q^2) - f(E'-i\epsilon, q^2)]$ is *real* for physical E and q^2, and therefore obeys the Schwarz reflection principle – and, as a function of q^2 has an *imaginary* discontinuity across the cut, which we found in our Regge theory analysis runs over

$$-\infty < q^2 \leq 4\mu^2$$

This function is given by

$$\frac{1}{\pi}\int_{-\infty}^{-4\mu^2} \frac{dq'^2}{q'^2-q^2}\,\rho(E', q'^2)$$

(assuming no subtractions) where $2i\rho$ is the discontinuity. Thus, we have the double dispersion relation

$$\boxed{f(E, q^2) = f_{Born}(q^2) + \int_0^\infty \frac{dE'}{\pi}\int_{-\infty}^{-4\mu^2}\frac{dq'^2}{\pi}\,\frac{\rho(E', q'^2)}{(E'-E)(q'^2-q^2)}}$$

where ρ is a real function. This is known as the *Mandelstam Representation*

We derived this using fixed-q^2 dispersion relation, which was established for a certain range in q^2, but it is now easy to see that this representation defines an analytic function on the entire cut q^2-plane. Thus, by uniqueness of analytic continuation, it correctly gives $f(E, q^2)$ for all E and q^2 in cut E and q^2 planes

According to the Mandelstam representations, $f(E, q^2)$ is determined completely by its discontinuities (assuming no subtractions or bound states). $f(E, q^2)$ for any complex E, q^2 is determined by the single *real* function $\rho(E, q^2)$ on the cuts, $0 \leq E < \infty$, $-\infty < q^2 \leq -4\mu^2$

But, in fact, we can say much more. The Mandelstam representation can be used to show that $f(E,q^2)$ is uniquely determined, given the Born term $f_{Born}(q^2)$ as a consequence of analyticity and unitarity. Since f_{Born} is the Fourier transform of the potential

$$f_{Born}(q^2) = \frac{1}{(2\pi)^3}\left(-(2\pi)^2\right)\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r),$$

this means that, given the potential, we can determine $f(E,q^2)$ without ever appealing to the Schrodinger equation! We can view the Schrodinger equation as a particularly convenient way of imposing the physical constraints of analyticity and unitarity on the amplitude. But it is a mere convenience, not a necessity.

Let's see how it works, in broad outline. For more detailed discussion consult ...

Goldberger and Watson, Collision Theory
R. Blackenbecler et al., Annals of Physics 10:62 (1960)

We saw back on p 2.17 that unitarity $S^\dagger S = \mathbb{1}$, can be written as

$$\mathrm{Im}\, f(\vec{p}\,';\vec{p}) = \frac{1}{2i}\left[f(\vec{p}\,';\vec{p}) - f(\vec{p};\vec{p}\,')^*\right] = \frac{p}{4\pi}\int d\Omega''\, f(\vec{p}\,'';\vec{p}\,')^*\, f(\vec{p}\,'';\vec{p})$$

— Generalized Optical Theorem

The second equality is completely general. The first equality holds assuming rotational invariance and ~~and time reversal invariance~~. energy conservation ($p'^2 = p^2$).

We may now substitute the Mandelstam Rep into the generalized optical theorem. The integrations are long and tedious, but otherwise elementary. Here we give only the result. Introducing the standard notational conventions

$$s = p^2 = 2E$$
$$t = -q^2$$

and recalling $\quad V(r) = \int_{\mu_0}^{\infty} da \, \frac{e^{-ar}}{r} \sigma(a)$

$$\Rightarrow f_{Born} = \int_{\mu_0}^{\infty} da \, \frac{+2}{t-a^2} \, \sigma(a)$$

we have

$$\boxed{\begin{aligned} \rho(s,t) = & \int_{\mu_0}^{\infty} da \int_{\mu_0}^{\infty} da' \, \sigma(a)\sigma(a') \, F(t,s,a^2,a'^2) \\ & - 2\left(P\int \frac{ds'}{\pi}\right)\int \frac{dt'}{\pi} \int_{\mu_0}^{\infty} da \, \sigma(a) \, \frac{\rho(s',t')}{s'-s} \, F(t,s,a^2,t') \\ & + \int \frac{ds'}{\pi}\int \frac{ds''}{\pi}\int \frac{dt'}{\pi}\int \frac{dt''}{\pi} \, \frac{\rho(s',t')}{s'-s-i\epsilon} \, \frac{\rho(s'',t'')}{s''-s-i\epsilon} \, F(t,s,t',t'') \end{aligned}}$$

The detailed form of $F(t,s,t',t'')$ is not of interest here. (see the references) The important property is that it vanishes provided

$$t < T(s,t',t'')$$

where T is a monotonically increasing function of t' t''. In fact

$$T(s,t',t'') = t' + t'' + 2(t't'')^{\frac{1}{2}}$$

+ positive correction of order $\frac{1}{s}$

(F is an explicitly known algebraic function)

In fact, $\quad T(s,t',t'') = t' + t'' + \frac{t't''}{2s} + 2\left[t't''\left(1+\frac{t'}{4s}\right)\left(1+\frac{t''}{4s}\right)\right]^{\frac{1}{2}}$

we see that

- $\rho(s,t)=0$, for
$$t < T(s,\mu_0^2;\mu_0^2) = T_0(s) \leq 4\mu_0^2$$

- the second and third terms vanish in above expression for $\rho(s,t)$ if
$$t < T(s,\mu_0^2;4\mu_0^2) = T_1(s) \leq 9\mu_0^2$$
and we have simply
$$\rho(s,t) = \int da\, da'\, \sigma(a)\sigma(a')\, F(t,s,a^2;a'^2)$$

- The third term vanishes if
$$t < T(s,4\mu_0^2,4\mu_0^2)$$
$$= T_2(s) \leq 16\mu_0^2$$
and $$\rho = \text{1st term} - 2P\int \frac{ds'}{\pi} \int_{T_0(s')}^{T_1(s')} dt' \int da\, \sigma(a)\, \rho(s's't') \times F(t,s,a^2;t')$$
Here ρ in the integrand is given by our earlier expression, so the integral is $O(\sigma^3)$

- This procedure can be iterated. When we have carried it out to $O(\sigma^n)$, we have determined $\rho(s,t)$ for all s out to
$$t = (n+1)^2\mu_0^2$$

If subtractions are required there are additional functions of s to determine. But this can be done, see references.

No one can claim that the Mandelstam representation is a useful calculational tool in nonrelativistic potential scattering. Moreover, it works only for Yukawa potentials. So why is it important?

Because we find that (at least for Yukawa potentials) the Schrödinger equation is superfluous in nonrelativistic quantum mechanics. The scattering amplitude (which describes all the physics) is determined by analyticity, asymptotic behavior allowing a double dispersion relation, and unitarity.

In the nonrelativistic case, this is a mere curiosity, but the Mandelstam representation had a profound impact on the physics of the sixties, because it was hoped that it could be used as a dynamical principle in the analysis of <u>relativistic problems</u> (where the Schrödinger equation does not apply) and in strong-interaction physics in particular.

This turned out to be wrong in detail, since the Mandelstam representation is not satisfied in Quantum Field Theory. But the basic idea is still important: that the constraints of unitarity and analyticity severely constrain the form of the S-matrix and scattering amplitude, both in nonrelativistic and relativistic scattering.

Note the closing statement in Taylor, <u>Scattering Theory</u> of section on Mandelstam representation, p. 302.

"Certainly a familiarity with these ideas is a prerequisite for an understanding of most of the current literature of high-energy physics."

Perhaps that statement was still true in 1972, but it is no longer even remotely true. What happened?

What happened was the resurgence of quantum field theory, and, in particular, the applications of QFT to strong-interaction physics. Yet, it may still make some sense to regard QFT as a convenient (but not essential) way of imposing on the S-Matrix, the physical constraints which it must obey, such as

Principles for a relativistic S-matrix

1) Unitarity (conservation of probability)
2) Analyticity (Maximal analyticity in the sense of no "unnecessary" singularities. Related to "causality" and "locality")
3) Crossing Symmetry
4) Lorentz-Invariance
5) Connectedness - S-Matrix can be decomposed into interacting "clusters"

e.g. 

It is not even known (rigorously) that these assumptions are consistent in 3+1 dimensions. No one can yet write down any but a trivial (free) S-Matrix satisfying them all.

Further assumptions we might want satisfied by a theory of strong interactions are

6) Regge asymptotic behavior - (in both t and s, by crossing symmetry)

7) Nuclear Democracy - All particles and resonances are composite, i.e. on Regge trajectories

The dream of the '60's was that the solution to these constraints was actually the unique theory of hadrons. (the Bootstrap)

We do not believe this anymore, but we do believe that it is possible to satisfy all these constraints in a quantum field theory of hadrons. (7) is satisfied in a surprising way - there are elementary degrees of freedom (quarks and gluons) but they do not appear at all as physical particles.