Symmetry (and Conservation Laws)

We constructed explicitly the operators $U(\Lambda)$, $U(x_0)$ representing the Poincare group, by specifying their action on a basis for the states. But there is also a systematic procedure for constructing these operators (and not just for Poincare group, but for any continuous symmetry) that can be applied to interacting theories as well as free field theory

Symmetry in Quantum Mechanics:

There is a correspondence:

(continuous) Symmetry $\longleftrightarrow$ Conservation Law

First what is a symmetry (in classical or quantum mechanics)? Operation ("rotation") that leaves dynamics invariant:

$$\begin{tikzcd}
\boxed{\text{initial}} \arrow[r, "\text{dynamics}"] \arrow[d, "\text{rotation}"] & \boxed{\text{final}} \arrow[d, "\text{rotation}"] \\
\boxed{\text{new initial}} \arrow[r, "\text{dynamics}"] & \boxed{\text{new final}}
\end{tikzcd}$$

"Commutative Diagram"

unitary, to preserve probability amplitudes

In quantum mechanics, symmetry is a unitary operator (or antiunitary in case of a discrete sym; see later) that represents the rotation (up to a phase):

$$U(R_1)\,U(R_2) = U(R_1 R_2)$$

(Being unitary, it preserves $[q,p]=i$ -- a "Canonical transformation") In nonrelativistic quantum mechanics, U is a symmetry if it commutes with Hamiltonian $H(p,q)$.

or..

$$U(R)\, H\, U(R)^{-1} = H$$

For then U commutes with e^{-iHt}, which generates time evolution. E.g., in Schrödinger picture

$$\psi_{final} = e^{-iHt}\psi_{initial}$$
$$\psi_{new} = U(R)\,\psi_{old}$$

So "commutative diagram" $\Rightarrow$

$$e^{-iHt}\,U(R)\,\psi_0 = U(R)\,e^{-iHt}\,\psi_0$$

(or $U(R)\,H\,U(R)^{-1} = H$ to linear order in t, since ψ_0 is arbitrary)

If symmetry is continuous, we can consider infinitesimal rotation $R = I + \epsilon T$ and

$$U(1+\epsilon T) = \mathbb{1} + i\epsilon Q + O(\epsilon^2)$$

(U unitary $\Rightarrow$ Q hermitian)
So, to order ϵ, $UHU^{-1} = H$ becomes

$$[Q, H] = 0$$

Associated with a continuous symmetry is a hermitian operator that commutes with H
Q is called the "generator" of a rotation
A finite rotation can be built out of infinitesimals

$$R = \left(I + \tfrac{T}{N}a\right)^N$$

For N large enough T/N is infinitesimal, and

$$U(R) = \left(\mathbb{1} + \tfrac{iQ}{N}a\right)^N \longrightarrow e^{iaQ}$$

(up to a correction of order $1/N$)

Now Q also commutes with e^{-iHt}, which means that it is a conserved quantity (if it has no explicit time dependence) In Schrodinger picture,

$$Q e^{-iHt} \psi_0 = e^{-iHt} Q \psi_0$$

-- Time evolution preserves eigenvalues of the observable Q. In Heisenberg picture,

$$\dot{Q} = -i[Q, H] + \frac{\partial}{\partial t} Q = 0,$$

if Q is a symmetry generator (and H has no explicit t-dependence).

Conversely, if Q is (hermitian and) conserved, we can construct the associated symmetry operator $\exp(iaQ)$

(To discuss Poincare invariance in quantum field theory, we need to generalize the above discussion! Since H is a 4-vector component, it doesn't commute with Lorentz boost generator. Also, generator Q must be in algebra of observables.)

Symmetry in Classical Mechanics

In classical mechanics, there is a connection parallel to that above, in Hamiltonian formalism

Symmetry $\longleftrightarrow$ Canonical transformation Preserves H
$\longleftrightarrow$ Conservation Law

But it is convenient to discuss symmetry in the Lagrangian formalism. Here is where it is easiest to identify symmetries of a classical system, which become quantum symmetries upon canonical quantization.

A classical symmetry (in the Lagrangian formalism) is a transformation acting on $q^a, \dot{q}^a$ that leaves the action invariant (given trajectory with $\delta S = 0$ -- classical soln -- transformed trajectory also has $\delta S = 0$)

$$\text{Action} \quad S = \int_{t_1}^{t_2} L(q^a, \dot{q}^a, t)\, dt$$

up to an irrelevant constant [So action is invariant if L changes by a total time derivative. Symmetry has the form

$$q'^a = q'^a(q^a, \dot{q}^a)$$

If it is continuous, we can consider infinitesimal form

$$q'^a = q^a + \epsilon\, \delta q^a(q^a, \dot{q}^a)$$

The corresponding change in L to order ϵ is

$$\epsilon\, \delta L = \epsilon \left[\frac{\partial L}{\partial q^a} \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a \right] = \epsilon \frac{dF}{dt}$$

-- symmetry of action

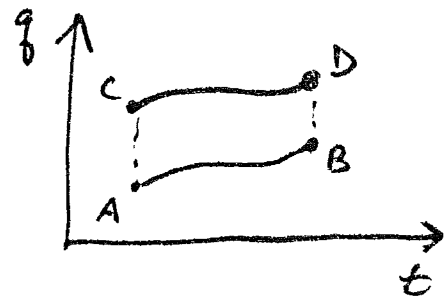
Now, use the Euler-Lagrange equation

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^a}\right) \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a = \frac{dF}{dt}$$

$$\Rightarrow \quad \frac{d}{dt} Q = 0, \quad \boxed{Q = \frac{\partial L}{\partial \dot{q}^a} \delta q^a - F}$$

-- Conservation Law

In general, symmetries of differential equations do not imply conservation laws; rather it is a symmetry and a variational principle that are needed:



Suppose a symmetry takes the trajectory AB shown to a trajectory CD; i.e.

$$S_{AB} = S_{CD}$$

(ignoring possible change by integral of total derivative)

But also, least action principle tells us that

$$S_{AB} = S_{ACDB} = S_{AC} + S_{CD} + S_{DB}$$

$$\Rightarrow \quad S_{AC} + S_{DB} = 0 \text{ or } S_{AC} = S_{BD}$$

This is the conservation law. (The conserved quantity is "a piece of the action".) When time interval is infinitesimal, but change in q is finite, finite part of action comes from kinetic term. S_{AC} is change in action when $AB \to ACD$, so

$$S_{AC} = \delta S = \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a dt, \text{ and } \delta \dot{q}^a = \frac{\delta q^a}{dt}$$

So

$$S_{AC} = \frac{\partial L}{\partial \dot{q}^a} \delta q^a$$

(Similarly, we can find the right conserved quantity if there is a total derivative term.)

Upon canonical quantization, we'll have conserved operator

$$Q = p^a \delta q^a - F$$

(up to ordering problems) and if F and δq^a are independent of p,

$$(\mathbb{1}+i\epsilon Q)\, q^a (\mathbb{1}-i\epsilon Q) = q^a + i\epsilon [Q, q^a]$$
$$= q^a + \epsilon\, \delta q^a$$

-- the operator Q generates a transformation that is the same as the classical symmetry.

Classical Field Theory

Now we have $S = \int d^4x\, \mathcal{L}(\phi^a, \partial_\mu \phi^a)$

$a = 1 \ldots, N$

(N real scalar fields)

A symmetry changes $\mathcal{L}$ by a total derivative (or changes S by a surface term)

$$\phi'^a = \phi^a + \epsilon\, \delta\phi^a$$

$$\epsilon\, \delta \mathcal{L} = \epsilon \left[\frac{\partial \mathcal{L}}{\partial \phi^a} \delta\phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \partial_\mu \delta \phi^a \right] = \epsilon\, \partial_\mu F^\mu$$

Of course, in finding F^μ, we must not use the eqn of motion -- For ϕ satisfying Euler-Lagrange eqn, all variations leave S invariant. But having determined F, we may now use eqn of motion

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \right) = \frac{\partial \mathcal{L}}{\partial \phi^a},$$

or $$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \delta\phi^a \right] = \partial_\mu F^\mu$$

$$\Rightarrow \partial_\mu J^\mu = 0, \quad \boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \delta\phi^a - F^\mu}$$

This is a "continuity equation" (Noether's Thm.)

$$\frac{\partial}{\partial t} \rho = \vec{\nabla} \cdot \vec{J} \qquad \text{where } J^\mu = (\rho, \vec{J})$$

And, if we ignore the surface term, there is a conserved quantity:

$$Q(t) = \int d^3x \, J^0(\vec{x}, t) \Rightarrow$$

$$\dot{Q} = \int d^3x \, \frac{\partial}{\partial t} J^0 = \int d^3x \, \vec{\nabla} \cdot \vec{J} = 0$$

$$Q = \int d^3x \left[\pi^a \delta q^a - F^0 \right]$$

How does Q transform under Lorentz transformations?
E.g., suppose J^μ is a 4-vector
Let's try to make Q look covariant

$$Q(t=0) = \int \underbrace{d^4x}_{\text{invariant measure}} \, \delta(x^0) \, n \cdot J(x) \qquad n^\mu = (1, \vec{0})$$

$$= \int d^4x \, \partial_\mu \Theta(n \cdot x) \, J^\mu(x)$$

$$\left(\frac{d}{ds} \Theta(s) = \delta(s) \Rightarrow \partial_\mu \Theta = n_\mu \delta(n \cdot x) \right)$$

We cannot integrate by parts in this expression (and ignore the surface term), because for $t \to \infty$ (where $\Theta = 1$) we still have $\int d^3x \, J^0(\vec{x}, t)$)

Now, change frame $x \to x' = \Lambda x$
this has only the effect of changing n

$$Q' - Q = \int d^4x \, \partial_\mu \left[\Theta(n' \cdot x) - \Theta(n \cdot x) \right] J^\mu$$

Now we can throw away surface term $\Rightarrow$
$Q' = Q$

obvious generalization: J_μ = tensor rank n,
Q = tensor rank $n-1$.

Examples:

Spacetime translation

If action is translation invariant, there are four symmetries

$$\phi'^a(x) = \phi^a(x+\epsilon e^\mu) = \phi^a(x) + \epsilon\,\partial^\mu \phi^a(x)$$

and four associated conserved currents. The Lagrange density changes by

$$\mathcal{L}'(x) = \mathcal{L}(x+\epsilon e^\mu) = \mathcal{L}(x) + \epsilon\,\partial^\mu \mathcal{L}(x)$$

(this is implicit x dependence, through x dependence of fields)

$$\text{or } F^{\mu\nu} = \eta^{\nu\mu}\mathcal{L}$$

$$J^\nu = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^a}\,\delta\phi^a - F^\nu$$

$$= \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^a}\,\partial^\mu \phi^a - \eta^{\nu\mu}\mathcal{L}$$

The four currents assemble into a rank-two tensor

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^a}\,\partial^\mu \phi^a - \eta^{\mu\nu}\mathcal{L}}$$

This is the "energy-momentum tensor", satisfying

$$\partial_\nu T^{\mu\nu} = 0$$

There are four conserved quantities

$$\underline{P}^\mu = \int d^3x\; T^{\mu 0}$$

-- transforms as a four-vector, the energy-momentum

One component is just the Hamiltonian

$$P^0 = H = \int d^3x \left[\pi^a \dot{\phi}^a - \mathcal{L} \right]$$

and the other components are

$$\vec{P} = -\int d^3x \left[\pi^a \vec{\nabla} \phi^a \right] \qquad \left(\partial^i \phi = -\tfrac{\partial}{\partial x^i} \phi \right)$$

So far the discussion is general. Now specialize to the free scalar field

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left[e^{-ik\cdot x} a(\vec{k}) + e^{ik\cdot x} a(\vec{k})^* \right]$$

$$\pi(x) = \dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left[-i a(\vec{k}) e^{-ik\cdot x} + i a(\vec{k})^+ e^{ik\cdot x} \right]$$

$$\vec{P}(t) = -\int d^3x \; \pi(x) \vec{\nabla} \phi(x)$$

$$= -\frac{1}{2} \int d^3k \left[a(\vec{k}) a(-\vec{k}) \vec{k} - a(\vec{k}) a(\vec{k})^+ \vec{k} - a(\vec{k})^+ a(\vec{k}) \vec{k} + a(\vec{k})^+ a(-\vec{k})^+ \vec{k} \right]$$

odd terms integrate to zero

$$\vec{P} = \frac{1}{2} \int d^3k \; \vec{k} \left[a(\vec{k})^+ a(\vec{k}) + a(\vec{k}) a(\vec{k})^+ \right]$$

Or, if we adopt normal ordering prescription

$$:\vec{P}: = \int d^3k \; \vec{k} \, a(\vec{k})^+ a(\vec{k})$$

and $:P^\mu: = \int d^3k \; k^\mu a(\vec{k})^+ a(\vec{k})$

Acting on the plane wave states, we have

$$P^\mu |\vec{k}\rangle = k^\mu |\vec{k}\rangle$$

-- confirming our earlier observation that these states are plane waves.

Also

$$[P^\mu, \phi(x)] = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[-k^\mu e^{-ik\cdot x} a(\vec{k}) + k^\mu e^{ik\cdot x} a(\vec{k})^\dagger \right]$$

$$= -i\partial^\mu \phi(x)$$

Thus $(\mathbb{1} + i\epsilon_\mu P^\mu)\phi(x)(\mathbb{1} - i\epsilon_\mu P^\mu)$

$$= \phi(x) + \epsilon_\mu \partial^\mu \phi(x) = \phi(x+\epsilon)$$

Thus $U(a) = e^{ia_\mu P^\mu}$ is the representation of translations:

$$U(a)\,\phi(x)\,U(a)^{-1} = \phi(x+a)$$

<u>Lorentz Transformations</u>

A Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ must satisfy

$$\eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho = \eta_{\lambda\rho}$$

For infinitesimal: $\Lambda^\mu{}_\lambda = \delta^\mu{}_\lambda + \epsilon^\mu{}_\lambda$

$$\eta_{\mu\nu}(\delta^\mu{}_\lambda + \epsilon^\mu{}_\lambda)(\delta^\nu{}_\rho + \epsilon^\nu{}_\rho) = \eta_{\lambda\rho}$$

$$\Rightarrow \quad \epsilon_{\lambda\rho} + \epsilon_{\rho\lambda} = 0 \qquad (6 \text{ generators})$$

We recall that, for a Lorentz invariant theory of scalar fields

$$\phi'^a(x) = \phi^a(\Lambda x)$$

is a symmetry of the action. In infinitesimal form

$$\phi'^a(x) = \phi^a(x^\mu + \epsilon^{\mu\nu}x_\nu)$$
$$= \phi^a(x) + \epsilon^{\mu\nu}x_\nu \partial_\mu \phi^a(x),$$

and the Lagrange density changes by (because it is a scalar)

$$\mathcal{L}'(x) = \mathcal{L}(x+\epsilon^{\mu\nu}x_\nu) = \mathcal{L}(x) + \underbrace{\epsilon^{\mu\nu}x_\nu \partial_\mu \mathcal{L}}_{= \partial_\mu(\epsilon^{\mu\nu}x_\nu \mathcal{L})}$$

(since ϵ is antisymmetric)

And so we have a conserved current

$$J^\lambda = \frac{\partial \mathcal{L}}{\partial \partial_\lambda \phi^a} \epsilon_{\mu\nu} x^\nu \partial^\mu \phi^a$$
$$- \epsilon_{\mu\nu} \eta^{\mu\lambda} x^\nu \mathcal{L}$$
$$= \underbrace{\epsilon_{\mu\nu}}_{\text{picks out antisymmetric part}} x^\nu T^{\mu\lambda}$$

We have 6 conserved currents

$$\mathcal{M}^{\lambda\mu\nu} = x^\nu T^{\mu\lambda} - x^\mu T^{\nu\lambda}$$
$$\partial_\lambda \mathcal{M}^{\lambda\mu\nu} = 0 \quad (= T^{\mu\nu} - T^{\nu\mu}$$

-- so e-m tensor is symmetric)

And 6 conserved quantities are

$$J^{\mu\nu} = \int d^3x \left(x^\nu T^{\mu 0} - x^\mu T^{\nu 0} \right)$$

-- In the quantum theory, these are the generators of Lorentz transformations.

Three components of $J^{\mu\nu}$ are the angular momentum: $\mu, \nu = 1, 2, 3$. The other three conservation laws carry what information?

$$J^{i0} = \int d^3x \; x^0 T^{i0} - x^i T^{00}$$
$$= x^0 \underline{P}^i - \underbrace{\int d^3x \, x^i \, T^{00}}_{\text{I-component of "center of mass"}}$$

So "center of mass" moves with constant velocity, and all energy density contributes to "mass."

Do these operators really generate Lorentz transformations? Consider the rotations:

$$J^{ij} = \int d^3x \left[x^j \pi(x) \partial^i \phi(x) - x^i \pi(x) \partial^j \phi(x) \right]$$

From $[\phi(x), \pi(x')]_{e.t.} = i\delta^3(\vec{x}-\vec{x}')$, we have

$$[J^{ij}(t), \phi(\vec{x},t)] = -i(\; x^j \partial^i \phi - x^i \partial^j \phi)$$

(and J^{ij} is t-independent). So

$$(\mathbb{1} + \tfrac{i}{2}\epsilon_{ij} J^{ij})\, \phi(x)\, (\mathbb{1} - \tfrac{i}{2}\epsilon_{ij} J^{ij})$$
$$= \phi(x) + \epsilon_{ij}(x^j \partial^i \phi(x))$$
$$= \phi(\; (1+\underline{\epsilon})\vec{x}, t) \quad \text{-- An infinitesimal rotation of } \vec{x}$$

Internal Symmetry

The symmetries considered so far are "spacetime symmetries": they act on the x dependence of the fields. But in a theory with more than one field, another type of symmetry is possible: internal symmetry, that mixes up the different fields.

For example, consider a theory with two real scalar fields

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^1\partial^\mu\phi^1 + \tfrac{1}{2}\partial_\mu\phi^2\partial^\mu\phi^2$$
$$-\tfrac{1}{2}m_1^2(\phi^1)^2 - \tfrac{1}{2}m_2^2(\phi^2)^2$$

In the case $m_1 = m_2$, the two fields $\phi^{1,2}$ are equivalent, and the action is left invariant by a rotation in the ϕ^1, ϕ^2 plane

$$\phi^1 \to (\phi^1)' = \cos\theta\,\phi^1 + \sin\theta\,\phi^2$$
$$\phi^2 \to (\phi^1)^2 = -\sin\theta\,\phi^1 + \cos\theta\,\phi^2$$

These rotations form the group $SO(2)$

What is the associated conserved current? In infinitesimal form

$$\phi^1 \to \phi^1 + \epsilon\phi^2$$
$$\phi^2 \to \phi^2 - \epsilon\phi^1$$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^1}\,\delta\phi^1 + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^2}\,\delta\phi^2$$

$$= \partial^\mu\phi^1\,\phi^2 - \partial^\mu\phi^2\,\phi^1$$

And the conserved charge is

$$Q = \int d^3x\,(\phi^2\dot{\phi}^1 - \phi^1\dot{\phi}^2)$$

we may express $\phi^{1,2}$ in terms of $a^{1,2}, (a^{1,2})^\dagger$

$$\phi^{1,2} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[e^{-ik\cdot x}a^{1,2}(\vec{k}) + e^{ik\cdot x}a^{1,2}(\vec{k})^\dagger\right]$$

$$\begin{aligned} Q = -i\int d^3k\, \tfrac{1}{2}\big[\, & a^1(\vec{k})\,a^2(-\vec{k}) - a^1(\vec{k})^\dagger a^2(\vec{k}) \\ & + a^1(\vec{k})\,a^2(\vec{k})^\dagger - a^1(\vec{k})^\dagger a^2(-\vec{k})^\dagger \\ & - a^1(\vec{k})\,a^2(-\vec{k}) - a^1(\vec{k})^\dagger a^2(\vec{k}) \\ & + a^1(\vec{k})\,a^2(\vec{k})^\dagger + a^1(\vec{k})^\dagger a^2(-\vec{k})^\dagger \,\big] \end{aligned}$$

of course a^1 and $(a^2)^\dagger$ commute, so

$$Q = -i\int d^3k\left[\,a^2(\vec{k})^\dagger a^1(\vec{k}) - a^1(\vec{k})^\dagger a^2(\vec{k})\right]$$

The Hamiltonian of this theory is just the sum of two:

$$:H: = \int d^3k\ \omega_k\left[a^1(\vec{k})^\dagger a^1(\vec{k}) + a^2(\vec{k})^\dagger a^2(\vec{k})\right]$$

It is easy to see that Q commutes with H. It destroys a particle of type (1) and creates one of type (2) with the same energy (since $m_1 = m_2$). (In fact, $[P^\mu, Q] = 0$.)

It is convenient to redefine the oscillators (i.e. choose a different basis for particles of energy ω_k) so that Q is <u>diagonal</u> in the new basis. So define

$$b(\vec{k}) = \frac{1}{\sqrt{2}}\left[a^1(\vec{k}) + i a^2(\vec{k})\right]$$

$$c(\vec{k}) = \frac{1}{\sqrt{2}}\left[a^1(\vec{k}) - i a^2(\vec{k})\right]$$

$$\Rightarrow\quad \begin{aligned} a^1(\vec{k}) &= \frac{1}{\sqrt{2}}\left[b(\vec{k}) + c(\vec{k})\right] \\ a^2(\vec{k}) &= \frac{1}{i\sqrt{2}}\left[b(\vec{k}) - c(\vec{k})\right] \end{aligned}$$

These new oscillators obey

$$[b(\vec{k}), b(\vec{k}')] = 0 \qquad [b(\vec{k}), b(\vec{k}')^+] = \delta^3(\vec{k}-\vec{k}')$$
$$[c(\vec{k}), c(\vec{k}')] = 0 \qquad [c(\vec{k}), c(\vec{k}')^+] = \delta^3(\vec{k}-\vec{k}')$$
$$[b(\vec{k}), c(\vec{k}')] = 0 \qquad [b(\vec{k}), c(\vec{k}')^+] = 0$$

In terms of the new oscillators

$$H = \int d^3k \; \omega_k \left[b(\vec{k})^+ b(\vec{k}) + c(\vec{k})^+ c(\vec{k}) \right]$$

and

$$Q = \int d^3k \; \tfrac{1}{2} \Big[(b(\vec{k})^+ - c(\vec{k})^+)(b(\vec{k}) + c(\vec{k})) + (b(\vec{k})^+ + c(\vec{k})^+)(b(\vec{k}) - c(\vec{k})) \Big]$$

$$= \int d^3k \left[b(\vec{k})^+ b(\vec{k}) - c(\vec{k})^+ c(\vec{k}) \right]$$

In the new basis, the one-particle states have definite charge, either +1 or -1.

We may also take linear combinations of the fields ϕ_1 and ϕ_2, to obtain fields that create and destroy particles of definite charge

$$\psi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[e^{-ik\cdot x} b(\vec{k}) + e^{ik\cdot x} c(\vec{k})^+ \right]$$

$$\psi^+ = \frac{1}{\sqrt{2}}(\phi^1 - i\phi^2) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[e^{-ik\cdot x} c(\vec{k}) + e^{ik\cdot x} b(\vec{k})^+ \right]$$

Thus ψ and ψ^+ each transform irreducibly under the SO(2) symmetry

$$\psi \to \cos\theta\, \psi - \sin\theta (i\phi^1 + \phi^2) = e^{-i\theta}\psi$$
$$\psi^+ \to e^{+i\theta}\psi$$

We can express the action in terms of these new fields: $\psi^\dagger\psi = \frac{1}{2}(\phi^a\phi^a)$,

so

$$\mathcal{L} = \partial^\mu\psi^\dagger\partial_\mu\psi - m^2\psi^\dagger\psi$$

and the commutators are...

$$[\psi,\psi]_{et} = [\psi^\dagger,\psi^\dagger]_{et} = 0$$

$$[\psi,\dot\psi^\dagger]_{e.t.} = i\,\delta^3(\vec{x}-\vec{x}')$$

This is just what we would have obtained by canonically quantizing $\mathcal{L}(\psi,\partial\psi,\psi^\dagger,\partial\psi^\dagger)$, treating ψ and $\psi^\dagger$ as <u>independent</u> variables.

Similarly, the observation that $\mathcal{L}$ is invariant under the symmetry with

$$\delta\psi = -i\,\psi$$
$$\delta\psi^\dagger = i\,\psi^\dagger$$

we may construct the current

$$J^\mu = -i\,\psi\partial^\mu\psi^\dagger + i\,\psi^\dagger\partial^\mu\psi$$

-- which is the same as the current we had before, except for the ordering of the factors (it's the same if we normal order)

So, in a roundabout way, we have arrived at the canonical quantization of a single <u>complex</u> scalar field. This theory has a conserved (internal) charge Q, and there are degenerate particles with $Q = \pm 1$. Aside from their opposite charge, these particles are identical. We may speak of a particle and antiparticle

This is our first encounter with a general phenomenon -- the existence of antiparticles in relativistic quantum field theory. Whenever a particle carries some internal quantum number, there is another particle with the opposite quantum number.

Why is this necessary? It is (again!) a consequence of causality.

Suppose we try to construct a theory with particle $(Q=1)$, but no antiparticle $(Q=-1)$. We could introduce field

$$\psi^{(-)} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\, e^{-ik\cdot x}\, b(\vec{k})$$

that destroys this particle; $\psi^{(-)+}$ creates it. But $\psi^{(-)}$ is not a local object:

$$\begin{aligned} \Delta_+(x-y) &= [\psi^{(-)}(x), \psi^{(-)+}(y)] \\ &= \langle 0|\, \psi^{(-)}(x)\, \psi^{(-)+}(y)\, |0\rangle \end{aligned}$$

vanishes outside the light cone, as we've seen.

To construct a local object, we can take

$$\phi(x) = \psi^{(-)}(x) + \psi^{(-)}(x)^+,$$

as we did before. But this is a real field that does not carry definite charge, and we have again a theory of a real scalar field

If we want a complex scalar field which is also local, we take

$$\psi(x) = \psi^{(-)}(x) + \psi^{(+)}(x)$$
$$\psi^{(+)}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\, e^{ik\cdot x}\, c(\vec{k})^{+}$$

If ψ has definite charge, then $\psi^{(+)}$ creates a particle with $Q=-1$, the antiparticle. We achieve causality by cancelling the amplitude

for particle $(Q=1)$ to propagate from x to y (outside light cone) with amplitude for antiparticle $(Q=-1)$ to propagate from y to x. (Heuristically, propagation outside the light cone from x to y looks like propagation from y to x in some frame -- if charge flows from x to y, there is a frame in which propagating particle must have $Q=-1$)

(Of course, ψ is not hermitian, but if it is local, the hermitian operators constructed from it are local observables.)

Charge Conjugation

So far, we've been considering only continuous symmetries, but the action might also be invariant under a discrete transformation.

For example, in the case of the theory of two scalar fields,

$$\phi^1 \to \phi^1 \qquad (\text{or } \psi \leftrightarrow \psi^+)$$
$$\phi^2 \to -\phi^2$$

leaves the action invariant; the symmetry is not just $SO(2)$, but $O(2)$.

This discrete symmetry can also be represented in the Hilbert space. That is, we can construct U such that

$$U\phi^1U^+ = \phi^1$$
$$U\phi^2U^+ = -\phi^2$$

and thus U should commute with H. In terms of oscillators

$$U a^1 U^+ = a^1$$
$$U a^2 U^+ = -a^2$$

and

$$U b U^+ = c$$
$$U c U^+ = b$$

This symmetry operation interchanges particle and antiparticle (and is thus easily defined acting on states). We will call it charge conjugation, and denote it by U_C. It evidently has the property

$$U_C^2 = \mathbb{1}$$

Obviously U_c commutes with H -- particle and antiparticle have the same energy. But it anticommutes with Q

$$U_c Q U_c = -Q,$$

since particle and antiparticle have opposite charge.

Since $U_c^2 = 1$, its eigenvalues are ± 1, and it is diagonal in the $a^{1,2}$ basis

Discrete Spacetime Symmetries

Having brought up the subject of discrete symmetries, let us consider more carefully the structure of the Lorentz group.

The defining property of the Lorentz group is

$$\eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho = \eta_{\lambda\rho} \quad \text{or} \quad \Lambda^T \eta \Lambda = \eta$$

The 4x4 matrices that satisfy this condition do not form a connected set, but actually comprise 4 disconnected components.

First, note that $(\det \Lambda)^2 = 1$. But we can have $\det \Lambda = +1$ or -1, and obviously these two sectors cannot be continuously connected.

A further division arises when we consider the sign of $\Lambda^0{}_0$

$$\Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^i{}_0 \Lambda^i{}_0 = 1 \Rightarrow (\Lambda^0{}_0)^2 \geq 1$$

(boosts dilate the time). But we can have $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq 1$, and these components are clearly disconnected. (Conv. preserve forward and backward light cone.)

The Lorentz transformations with $\det \Lambda = 1$ form a subgroup (obviously). So do those with $\Lambda^0{}_0 \geq 1$, since

$$\Lambda_1{}^0{}_0 \Lambda_2{}^0{}_0 + \Lambda_1{}^0{}_i \Lambda_2{}^i{}_0 = (\Lambda_1 \Lambda_2)^0{}_0$$

But $\sum_i (\Lambda^i{}_0)^2 = (\Lambda^0{}_0)^2 - 1$

And Schwarz $\Rightarrow \sum_i \Lambda_1{}^i{}_0 \Lambda_2{}^i{}_0 \leq (\Lambda_1{}^0{}_0{}^2 - 1)^{1/2} (\Lambda_2{}^0{}_0{}^2 - 1)^{1/2} \leq \Lambda_1{}^0{}_0 \Lambda_2{}^0{}_0$

So if $\Lambda_1{}^0{}_0 \Lambda_2{}^0{}_0$ is ≥ 1, we have $(\Lambda_1 \Lambda_2)^0{}_0 \geq 0$ and hence ≥ 1

We can also take the <u>intersection</u> of these two subgroups, $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. This is the connected (identity) component of the Lorentz group -- the subgroup consisting of transformation that can be smoothly deformed to the identity.

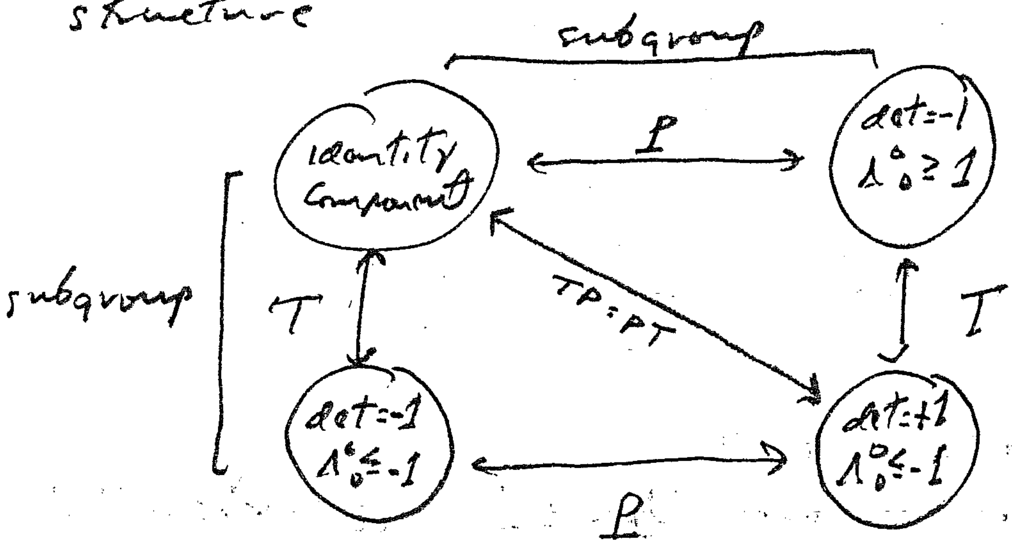
To see that the other components are not empty, note that

$\underline{P} = \mathrm{diag}(1,-1,-1,-1)$ has $\det \underline{P} = -1$, $\underline{P}^0{}_0 = 1$

$T = \mathrm{diag}(-1,1,1,1)$ has $\det T = -1$ $T^0{}_0 = -1$

$PT = \mathrm{diag}(-1,-1,-1,-1)$ has $\det \underline{P}T = 1$ $(PT)^0{}_0 = -1$

Schematically, the Lorentz group has the structure



(identity component and TP also generate a subgroup)

By constructing the generators of infinitesimal Lorentz transformations, we determined how the identity component of the Lorentz group is represented, but it remains to find representations of the discrete transformations P and T.

The operator representing parity, U_P should have the property

$$U_P\, \phi(\vec{x},t)\, U_P^{\dagger} = \phi(-\vec{x},t)$$

(Actually, the property $U_P^2 = I$ allows a minus sign here -- "pseudoscalar" instead of scalar. But in the free scalar field theory, whether we include the minus sign or not is purely conventional, since we also have the internal discrete symmetry $C: \phi \to -\phi$)

On the oscillators in

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[e^{-ik\cdot x} a(\vec{k}) + e^{ik\cdot x} a(\vec{k})^{\dagger} \right]$$

this symmetry acts as

$$U_P\, a(\vec{k})\, U_P^{\dagger} = a(-\vec{k})$$

And, on states $U_P\,|\vec{k}\rangle = |-\vec{k}\rangle$ (if $U_P|0\rangle = |0\rangle$)

This clearly commutes with H; these two states are degenerate. The parity symmetry tells us that the particles have a degeneracy larger than that obtained from the action of the rotation group $SO(3)$. Flipping the 3-momentum also produces a degenerate state not obtained by any rotation.

Determining the transformation properties under T involves a subtlety, namely that T cannot be represented in QM by a linear operator; rather, operator must be antilinear.

Time inversion means

$$e^{-iHt} \to e^{iHt}$$

But there is no linear operator that commutes with H such that

$$U_T e^{-iHt} U_T^{-1} = e^{iHt}$$

(we didn't run into this problem with parity, because $\vec{I}$ anticommutes with U_P.)
What we need instead is an antiunitary operator.

(Antilinearity: $U(a\psi + b\chi) = a^* U\psi + b^* U\chi$)
Symmetry represented by an antiunitary operator is just as good as symmetry represented by a unitary operator. Point is, symmetry means ~~probability~~ (amplitudes)^2 are preserved by the transformation. For a unitary operator

$$(U\psi, U\chi) = (\psi, \chi)$$

for an antiunitary operator

$$(U\psi,\ U\chi) = (\chi,\psi) = (\psi,\chi)^*$$

-- To see this, expand in a basis

$$(U(a_i e^i),\ U(b_i e^i)) = a_i b_i^* = (a_i e^i,\ b_i e^i)^*$$

In either case, the probability $|(\psi,\chi)|^2$ is preserved.

In fact, there is a general theorem (E. Wigner, Group Theory, Chapter 20) that says that symmetry:

$$\psi \to \psi' \text{ with } |(\psi,\chi)|^2 = |(\psi',\chi')|^2$$

can always be chosen to be either unitary or antiunitary (not both).

Here is another way to understand why T must be represented antilinearly. T changes the sign of time derivatives, and hence of canonical momenta

$$U_T\, q(t)\, U_T^{-1} = q(-t)$$

$$U_T\, p(t)\, U_T^{-1} = -p(-t)$$

But then

$$[q(t), p(t)] = i$$

$$\Rightarrow [q(-t), p(-t)] = -U_T\, i\, U_T^{-1}$$

If U_T were unitary, this would be inconsistent with canonical commutation relation!

Note that the antiunitary alternative can arise only for a discrete symmetry. There is no infinitesimal form for an antiunitary operator; in particular, the product of two antilinear operators is linear.

How do we represent 'T' in theory of a single free real scalar field? It is actually simpler to consider PT. We want

$$U_{PT}\,\phi(x)\,U_{PT}^{-1} = \phi(-x)$$

But, since U_{PT} is <u>anti</u>linear, this is realized by

$$U_{PT}\,a(\vec{k})\,U_{PT}^{-1} = a(\vec{k})$$

This means that all Fock space states are actually PT <u>invariant</u> (if the vacuum is)

$$U_{PT}\,|\vec{k}_1, - \vec{k}_n\rangle = |\vec{k}_1, - \vec{k}_n\rangle$$

The case of a complex scalar field is slightly more interesting:

$$b(\vec{k}) = \frac{1}{\sqrt{2}}[a_1(\vec{k}) + i a_2(\vec{k})]$$
$$c(\vec{k}) = \frac{1}{\sqrt{2}}[a_1(\vec{k}) - i a_2(\vec{k})]$$

Thus and
$$\left.\begin{aligned} U_{PT}\, b\, U_{PT}^{-1} &= b \\ U_{PT}\, c\, U_{PT}^{-1} &= c \end{aligned}\right\} \text{ so } U_{PT}\,\psi(x)\,U_{PT}^{-1} = \psi(-x)$$

In the case of a complex scalar field $\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, we have

$$\psi(x) = \int \frac{d^3k}{(2\pi)^{3}\,2\omega_k}\left[e^{-ik\cdot x}\, b(\vec{k}) + e^{ik\cdot x}\, c(\vec{k})^\dagger\right]$$

where
$$b(\vec{k}) = \tfrac{1}{\sqrt{2}}[a_1(\vec{k}) + i\, a_2(\vec{k})]$$
$$c(\vec{k}) = \tfrac{1}{\sqrt{2}}[a_1(\vec{k}) - i\, a_2(\vec{k})]$$

So, if we define an antilinear operator U such that
$$U a_1 U^{-1} = a_1$$
$$U a_2 U^{-1} = a_2,$$
we have
$$U b\, U^{-1} = c$$
$$U c\, U^{-1} = b,$$
and therefore,
$$U \psi(x) U^{-1} = \psi(-x)^\dagger$$

We now identify this operator as CPT; it complex conjugates as well as taking $x \to -x$

For the charged scalar field, we define PT by

$$\left.\begin{aligned} U_{PT}\, b\, U_{PT}^{-1} &= b \\ U_{PT}\, c\, U_{PT}^{-1} &= c \end{aligned}\right\} \Rightarrow U_{PT}\,\psi(x)\, U_{PT}^{-1} = \psi(-x).$$

Or, in other words, ϕ_1 is PT even and ϕ_2 is PT odd.

Note: unlike U_C and U_P, U_{PT} is not an observable.

e.g. $U_{PT}\, e^{i\theta}\psi = e^{-i\theta} U_{PT}\,\psi$

-- so eigenvalues of U_{PT} are not characteristic of (a well-defined property of) rays in Hilbert space.

But U_{PT} invariance still places restrictions on matrix elements.

~~So U_{CPT} acts exactly as U_C does. It is the antiunitary operator~~

$$\cancel{U_{CPT} = U_C U_{PT}}$$

~~that acts trivially in Fock space. In this case, the PT symmetry does not really add anything to our knowledge of the theory.~~

The discrete symmetries P and T become more interesting in interacting theories. In particular, they are not automatic. (We've already seen that C, an internal symmetry, need not be satisfied -- consider $m_1 \neq m_2$ in the theory of two scalar fields.)

Example - A theory of four real scalars, ϕ^a, $a = 1, \ldots, 4$

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \left[\sum_a \lambda^a(\phi^a)^3\right]$$

$$- \eta\,\epsilon^{\mu\nu\lambda\sigma}\partial_\mu\phi_1(x)\partial_\nu\phi_2(x)\partial_\lambda\phi_3(x)\partial_\sigma\phi_4(x)$$

"Invariant" means $\mathcal{L}(x) \to \mathcal{L}(-x)$

[The ϕ^3 term is invariant under P, T only if ϕ^a are P, T even

$$P: \phi^a(\vec{x},t) \to \phi^a(-\vec{x},t)$$
$$T: \phi^a(\vec{x},t) \to \phi^a(\vec{x},-t)$$

But then the ϵ term is odd under P, T. So no definition of P or T leaves $\mathcal{L}$ invariant. However PT is a symmetry: $U_{PT}\mathcal{L}(x)U_{PT}^{-1} = \mathcal{L}(-x)^*$
$= \mathcal{L}(-x)$

(Note: λ, η real required by Hermiticity.)

This is a special case of a very general phenomenon. A theory of many hermitian scalar fields is always PT-invariant if it is Lorentz invariant. This is simply because a Lorentz-invariant interaction always involves an even number of derivatives.

But it is actually possible to formulate an argument for PT-invariance in a "non-Lagrangian" framework. For one thing this argument will ensure that the classical PT symmetry argued for above survives in quantum theory (no "anomaly"). And, furthermore, it is very instructive to see what we can say about quantum field theory without appealing to the canonical-quantization-of-classical-theory strategy.

Of course, in order to prove a "PT theorem", we will need precisely stated assumptions. But our assumptions are so general and well-motivated, that we will be pedantic and call them "axioms".

(The following section goes beyond "the free scalar field", but is included in Chapter 1 anyway.)

The Axioms of Relativistic Quantum Field Theory

(for scalar fields)

A relativistic quantum field theory is a Hilbert space $\mathcal{H}$ (with states corresponding to rays in $\mathcal{H}$), and a set of fields $\phi^a(x)$. The fields are operator-valued distributions defined on $\mathbb{R}^4$ (for a field theory in 4 spacetime dimensions); that is, when smeared with smooth functions on $\mathbb{R}^4$, they are linear operators acting in $\mathcal{H}$.

fields may be chosen to be hermitian

The theory is assumed to have the following properties:

(1) The Relativistic Transformation Law

There is a unitary representation of the Poincare group acting on $\mathcal{H}$ such that

$$U(a,\Lambda)\,\phi^a(x)\,U(a,\Lambda)^{-1} = \phi^a(\Lambda x + a)$$

(in a theory of scalar fields).
The generator of translations is called P^μ;

$$U(a,1) = e^{i a_\mu P^\mu},$$

and the eigenvalues of P^μ are in the "forward light cone"

$$P^\mu P_\mu \equiv m^2 \geq 0$$
$$P^0 \geq 0$$

[Note: Here "Poincare group" means restricted Poincare group; $\det\Lambda = 1$, $\Lambda^0{}_0 \geq 1$

② <u>The Vacuum</u>

There exists a Poincare-invariant state

$$U(a,\Lambda)|0\rangle = |0\rangle$$

This state is called the "vacuum". It is unique (up to a phase).

③ <u>Completeness</u>

Polynomials in (smeared) fields acting on the vacuum are a basis for (are "dense" in) $\mathcal{H}$.

(This axiom allows us to avoid reference to the canonical formalism, but carries the same basic idea -- the fields are a <u>complete</u> set of observables.)

④ <u>Causality</u>

$$[\phi^a(x), \phi^b(y)] = 0 \quad \text{for } (x-y)^2 < 0$$

(the fields form a <u>local</u> algebra of observables, in the sense we've discussed)

The axioms ① - ④ are based on sound physics principles. But to <u>prove</u> things we need an additional assumption, one that is "merely" technical:

(5) Polynomial Boundedness

The matrix elements of ϕ^a are "tempered distributions"; that is

$$|\langle A|\,\tilde{\phi}^a(p)\,|B\rangle| \leq \text{polynomial in } p$$

(The fields are no more singular as distributions than a finite derivative of a δ-function. This assumption is important, because it allows us to "analytically continue" the fields $\phi^a(x)$ to complex values of x, and thus apply the powerful machinery of analytic function theory to the study of quantized fields.)

These 5 axioms are the properties that, by consensus, any relativistic quantum field theory ought to have. (They are called "Wightman Axioms.") Why are they important?

- For one thing, they serve as the goal of any attempt to "construct" an (interacting) RQFT.

- They might serve as the basis of a non-Lagrangian approach to constructing and studying RQFT. (Hasn't happened yet, for the most part.)

- They have interesting consequences.

Irreducibility of the Field Algebra

The Axiom ③ could have been replaced by the equivalent axiom:

③' The fields $\phi^a(x)$ are an irreducible set of operators in $\mathcal{H}$: the only operator that commutes with all $\phi^a(x)$ is a multiple of the identity

Let's prove ③' from axioms ①-⑤.

We want to show that, if $[V, \phi^a(x)] = 0$, then $V = \alpha \mathbb{1}$. (For notational simplicity, suppose there is just one field $\phi(x)$.) So, assume $[V, \phi(x)] = 0$. Then

$$\langle 0| V \phi(x_1+a) \cdots \phi(x_n+a) |0\rangle$$
$$= \langle 0| \phi(x_1+a) \cdots \phi(x_n+a) V |0\rangle$$

From the transformation law ① and invariance of the vacuum ②, we have

$$\langle 0| V\, U(a,1)\, \phi(x_1) \cdots \phi(x_n) |0\rangle$$
$$= \langle 0| \phi(x_1) \cdots \phi(x_n)\, U(-a,1)\, V |0\rangle$$

Now, Fourier transform in a:

$$U(a,1) = \sum_n e^{ip_n \cdot a} |n\rangle\langle n|$$
$$= \int d^4K\, e^{iK\cdot a} \sum_n \delta^4(K - p_n) |n\rangle\langle n|$$
$$= \int d^4K\, e^{iK\cdot a} E(K),$$

where $E(K)$ projects onto subspace of $\mathcal{H}$ on which P^μ has eigenvalue K^μ. So we have

$$\langle 0| V\, E(K)\, \phi(x_1) \cdots \phi(x_n) |0\rangle$$
$$= \langle 0| \phi(x_1) \cdots \phi(x_n)\, E(-K)\, V |0\rangle$$

But the spectrum of states satisfies $P^0 \geq 0$, so $E(K) = 0$ for $K^0 < 0$, and for $K^0 > 0$ we have

$$0 = \langle 0 | V E(K) \phi(x_1) \cdots \phi(x_n) | 0 \rangle \,, \quad K^0 > 0 .$$

Now we use the completeness axiom (3). Since any state can be expanded in (smeared) fields acting on $|0\rangle$, we conclude that every state with $K^0 > 0$ is orthogonal to $V|0\rangle$. The only state to which $V|0\rangle$ is not orthogonal is the state with $K^\mu = 0$, the vacuum. Thus

$$V|0\rangle = \alpha |0\rangle .$$

Furthermore, since $[V, \phi] = 0$,

$$V \phi(x_1) \cdots \phi(x_n) |0\rangle = \alpha \, \phi(x_1) \cdots \phi(x_n) |0\rangle .$$

And, since any state can be expanded in fields acting on $|0\rangle$

$$V \Psi = \alpha \Psi \quad \text{for } \Psi \in \mathcal{H} ;$$

that is,

$$V = \alpha \mathbb{1}$$

Actually, we needed only Axioms (1)-(5) in the proof. Conversely, if fields are irreducible, they are complete; i.e., (1) (2) (3') $\Rightarrow$ (3). So we could choose either (3) or (3') as an axiom. We choose (3) because it appears weaker, and is more easily motivated.

More mathematical detail concerning the axioms and their consequences can be found in

R. Streater and A. Wightman, PCT, Spin + Statistics, and all That.
R. Jost, The General Theory of Quantized Fields.

Here, I'll sketch a proof of a fundamental result, the PT theorem, skipping over a few mathematical technicalities.

Symmetry in Axiomatic Field Theory

The axiom (2) is a bit less general than the others. (that is, the uniqueness part.) With it, we can prove a strong statement about symmetries of the vacuum:

Suppose V is a unitary operator that commutes with the Poincare representation

$$U(a,\Lambda)\, V = V\, U(a,\Lambda)$$

Then

$$U(a,\Lambda)\, V|0\rangle = V|0\rangle$$

Thus $V|0\rangle$ is a "vacuum", and uniqueness implies

$$V|0\rangle = e^{i\alpha}|0\rangle$$

(With a suitable phase convention for V, the vacuum is invariant.)

This result may bother you if you have heard about "spontaneous symmetry breakdown," a phenomenon in which a field theory respects a symmetry (that commutes with Poincare) but the vacuum is not invariant. In fact, the contrapositive of what we've proved is: If the vacuum is not invariant, then it is not unique, a correct statement.

Nonetheless, we'll stick with (2) as originally stated (uniqueness of vacuum) for the ensuing discussion.

What exactly do we mean by "symmetry" in quantum field theory? In a sense, every unitary operator in $\mathcal{H}$ is a symmetry (antiunitary operators, too), in that it is a transformation, a change of basis, that preserves probabilities:

$$(U\psi, U\chi) = (\psi, \chi)$$

But clearly some unitary operators are more interesting than others -- those that, in our previous language, are symmetries of the dynamics. The useful symmetries are those that have a simple action on the fields (and thus have simple commutation relations with the Poincare generators.)

For example, consider an internal symmetry that acts linearly on the fields:

assume V linear, but not necessarily unitary

$$V \phi^a(x) V^{-1} = M^{ab} \phi^b(x)$$

(satisfied for all a and x) Claim: V commutes with the Poincaré generators, and in particular with P^0 -- it is a symmetry of the dynamics in our old sense (otherwise, if we invoked this equation at one time, it could not be true at subsequent times.)

Proof: It follows from above that

$$V\, U(a,\Lambda)\, \phi^a(x)\, U(a,\Lambda)^{-1} V^{-1} = M^{ab} \phi^b(\Lambda x + a)$$

$$= U(a,\Lambda) V \phi^a(x) V^{-1} U(a,\Lambda)^{-1}$$

or $\quad \phi^a(x) = \Omega\, \phi^a(x)\, \Omega^{-1}$

where $\Omega = U^{-1}(a,\Lambda) V^{-1} U(a,\Lambda) V$.

Ω commutes with <u>all</u> smeared fields. But the smeared fields are a complete set of observables (all operators are functions of fields) -- the only operator that commutes with <u>all</u> is a multiple of $\mathbb{1}$. (This can be proved from the axioms; see Streater-Wightman, Section 4.2, or P1.75ab)

Thus

$$\Omega = e^{i\alpha} \mathbb{1}, \text{ or}$$

$$U(a,\Lambda) V = V\, U(a,\Lambda)\, e^{i\alpha}$$

It follows as above that

$$V|0\rangle = \beta |0\rangle$$

(β need not have $|\beta| = 1$ if V is not unitary)

And now applying $UV = VUe^{i\alpha}$ to $|0\rangle$, we have $e^{i\alpha} = 1$. So

$$[U(a,\Lambda), V] = 0$$

Its action on all states is then determined, up to phase, $V|0\rangle = \beta|0\rangle$

A general unitary operator will not act simply on the fields. Conversely, we can always *define* a linear operator by specifying, for example,

$$V\phi^a(x)V^{-1} = M^{ab}\phi^b(x).$$

But such an operator is not necessarily *unitary*, and it cannot be regarded as a symmetry operator unless it is unitary (or antiunitary).

The PT Theorem

(for hermitian scalar fields)

In view of the above remark, I can always *define* a PT operator Θ in a scalar field theory by specifying that Θ is antilinear, and

$$\Theta\phi^a(x)\Theta^{-1} = \phi^a(-x).$$

To show that the theory has PT symmetry, we must prove that Θ so defined is antiunitary.

But this actually follows from just the Axioms ①-⑤ (the "PT theorem")!
(we may need to adjust normalization of Θ, $\Theta \to c\Theta$.)

The crucial ingredient in the proof of this theorem is easy to understand: it is the "complexification" of the Lorentz group:

The real Lorentz group $L(\mathbb{R})$ consists of the real 4x4 matrices Λ satisfying $\Lambda^T \eta \Lambda = \eta$. The connectedness structure of $L(\mathbb{R})$ was indicated schematically on page 1.66

$L(\mathbb{R})$ may be extended to $L(\mathbb{C})$ -- the complex Lorentz group consisting of <u>complex</u> 4x4 matrices obeying $\Lambda^T \eta \Lambda = \eta$.
As for the $L(\mathbb{R})$ transformations $\Lambda \in L(\mathbb{C})$ we have $\det \Lambda = \pm 1$, so $L(\mathbb{C})$ has two components. But it has only two -- $\Lambda^0_0 \leq -1$ is connected to $\Lambda^0_0 \geq 1$ in $L(\mathbb{C})$; the forward and backward light cones can be <u>smoothly</u> interchanged.

E.g. consider, acting in the t-z plane

$$U(\theta) = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} = \cos\theta + i\sigma_1 \sin\theta$$

Evidently $U(\theta)^T \sigma_3 \, U(\theta)$

$$= (\cos\theta + i\sin\theta\,\sigma_1)\,\sigma_3\,(\cos\theta + i\sin\theta\,\sigma_1) = \sigma_3$$

so $U(\theta) \in L(\mathbb{C})$. But

$$U(0) = \mathbb{1} \qquad U(\pi) = -\mathbb{1}$$

this connects PT to identity in $L(\mathbb{C})$.

On the other hand, P and T have det -1, and so are not connected to identity in $L(\mathbb{C})$.

This is the key respect in which PT is different from P and T. The idea of the PT theorem is that Lorentz invariance should imply PT invariance, if we can legitimately continue $\phi(x)$ to complex x, and argue that the resulting field theory has $L(\mathbb{C})$ symmetry (under infinitesimal transformations)

Theorem (PT theorem)

If, in a theory of a single hermitian scalar field $\phi(x)$ (obeying Axioms ①-⑤) the antilinear operator Θ satisfies

$$\Theta\,\phi(x)\,\Theta^{-1} = \phi(-x),$$

then Θ is antiunitary (when appropriately rescaled).

Proof:

(i) First, it follows from ②, Uniqueness of the vacuum, that the vacuum is PT-invariant up to a phase. The argument is virtually identical to that on page 1.78:

$$\Theta\, U(a,\Lambda)\,\phi(x)\,U(a,\Lambda)^{-1}\Theta^{-1} = \phi(-\Lambda x - a)$$

$$U(-a,\Lambda)\,\Theta\,\phi(x)\,\Theta^{-1}\,U(-a,\Lambda)^{-1} = \phi(-\Lambda x - a)$$

Thus $\Theta U(a,\Lambda)\Theta^{-1} = U(-a,\Lambda)$
(from the irreducibility of the field algebra)

$\Rightarrow \quad U(-a,\Lambda)\Theta|0\rangle = \Theta|0\rangle$

or $\quad \Theta|0\rangle = \alpha|0\rangle$, if $|0\rangle$ is the unique Poincare invariant state. We may adjust the phase of Θ (and normalization) so that $\Theta|0\rangle = |0\rangle$

(ii) Next, we can use <u>completeness</u> of the fields to relate antiunitarity to a statement about vacuum expectation values of strings of fields

Antiunitarity means $(\Theta\Psi, \Theta X) = (X, \Psi)$
for $X, \Psi \in \mathcal{H}$, and —

any state can be expanded in smeared fields acting on the vacuum:

$$\Psi[A] = \sum_n \int dx_1 \cdots dx_n \, A^{(n)}(x_1 \cdots x_n)\, \phi(x_1) \cdots \phi(x_n)|0\rangle$$

$$\Theta\Psi[A] = \Psi[\bar{A}]$$

$$\bar{A}^{(n)}(x_1 \cdots x_n) = A^{(n)*}(-x_1 \cdots -x_n)$$

we want to show $(\Psi[B], \Psi[A]) = (\Psi[\bar{A}], \Psi[\bar{B}])$
where

$$(\Psi[B], \Psi[A]) = \sum_{n,m} \int dx_1 \cdots dx_n \, dy_1 \cdots dy_m$$

$$B^*(x_1 \cdots x_n)\, A(y_1 \cdots y_m) \langle 0|\phi(x_n) \cdots \phi(x_1)\phi(y_1) \cdots \phi(y_m)|0\rangle$$

and $(\Psi[\bar{A}], \Psi[\bar{B}]) = \sum_{n,m} \int dx\,dy\; B^*(-x_1, \ldots, -x_n)\, A(-y_1, \ldots, -y_m)$

$$\langle 0|\phi(y_m) \cdots \phi(y_1)\, \phi(x_1) \cdots \phi(x_n)|0\rangle$$

Therefore, it will suffice to show

$$\langle 0|\phi(x_1) \cdots \phi(x_n)|0\rangle = \langle 0|\phi(-x_n) \cdots \phi(-x_1)|0\rangle$$

(iii) Now, by translation invariance of the vacuum, these vacuum expectation values are functions of only <u>difference</u> <u>variables</u>:

$$\langle 0|\phi(x_1) \cdots \phi(x_n)|0\rangle = W_n(z_1, \ldots z_{n-1})$$

$z_1 = x_1 - x_2$
$z_2 = x_2 - x_3$
etc

(Wightman functions)

What we need to show is:

$$W_n(z_1, \ldots, z_{n-1}) = W_n(z_{n-1}, \ldots, z_1)$$

(iv) Next, we infer a property of W that follows from <u>causality</u>. If all differences $x_i - x_j$ are spacelike, then the fields commute. So

$$\langle 0|\phi(x_1) \cdots \phi(x_n)|0\rangle = \langle 0|\phi(x_n) \cdots \phi(x_1)|0\rangle$$

$$\Rightarrow W_n(z_1, \ldots z_{n-1}) = W_n(-z_{n-1}, \ldots, -z_1)$$

This is not quite what we want to show, and, in any case, it is true only when all separations are spacelike.

(Evidently, to complete the proof, we need to be able to replace z by $-z$ on the RHS of above eqn, <u>and</u> show that its validity extends beyond the region of spacelike differences.)

(v) Only now does the proof get technical (and polynomial boundedness begins to play a crucial role). Because the fields are tempered distributions, it can be shown that...

- W is a boundary value of a <u>unique</u> analytic function in the region $\operatorname{Im} z_j^0 < 0$.

- In its region of analyticity, W is invariant under ($\det = 1$) <u>complex</u> Lorentz transformations

$$W(z_1, \ldots z_{n-1}) = W(\Lambda z_1, \ldots \Lambda z_{n-1})$$

(<u>analytic</u> continuation of $L(\mathbb{R})$ invariance on the boundary).

- The region of analyticity extends to real values of z, when all $(x_i - x_j)$ are spacelike.

(No singular field commutators to spoil analyticity.)

The proof that W is an analytic function is in Streater and Wightman, and will not be reproduced here, but the basic idea is simple:

Consider for example the two point function

$$\langle 0|\phi(z)\phi(0)|0\rangle = \langle 0|\phi(0)e^{-iP\cdot z}\phi(0)|0\rangle$$

$$= \sum_n e^{-iP_n\cdot z}\,|\langle 0|\phi(0)|n\rangle|^2$$

(using translation invariance, and inserting a sum over states. Σ is schematic as states are of course a continuum.)

This expression is a sum of analytic functions $e^{iP_n\cdot z}$, but the sum may not be convergent. But $P_n^0 \geq 0$, and, if $\mathrm{Im}\, z^0 < 0$, then the sum converges if $|\langle 0|\phi|n\rangle|^2$ does not grow too fast with P_n (ensured by "polynomial boundedness")

(vi) Finally, since $\Lambda = -\mathbb{1}$ is a complex Lorentz transformation:

$$W_n(z_1,\dots,z_{n-1}) = W_n(z_{n-1},\dots,z_1)$$

throughout the region of analyticity, and also in the limit $\mathrm{Im}\, z_j^0 \to 0$.

(The identity (iv) extends to whole region of analyticity, by uniqueness of analytic continuation.)

This completes the proof

Important Point:

It is not correct to conclude that the identity

$$W_n(z_1, \ldots z_{n-1}) = W_n(-z_{n-1}, \ldots, -z_1)$$

extends to the boundary ($\mathrm{Im}\, z_j^0 = 0$) of the region of analyticity. That would mean, for example, that

$$\langle 0|\, \phi(x)\, \phi(0)\, |0\rangle = \langle 0|\, \phi(0)\, \phi(x)\, |0\rangle$$

even when $x^2 > 0$, which is false even in free field theory.

In free field theory we have

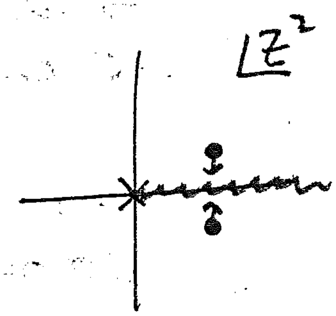
$$\Delta_+(x) = \langle 0|\, \phi(x)\, \phi(0)\, |0\rangle \quad \text{(--actually a function of } x^2\text{; a scalar)}$$

and

$$i\Delta(x) = \Delta_+(x) - \Delta_+(x)^* = \langle 0|\, [\phi(x), \phi(0)]\, |0\rangle$$
$$= 2i\, \mathrm{Im}\, \Delta_+(x)$$

For $x^2 < 0$, $\Delta(x) = 0$, so $\Delta_+(x)$ is real. Thus, the analytic function $\Delta_+(x^2)$ obeys the "Schwarz reflection principle":

$f(z) = f(z^*)^*$ throughout region of analyticity, if it is real in some real neighborhood



So $i\Delta(z) = \mathrm{Disc}\, \Delta_+(z)$
-- the discontinuity of Δ_+ across the positive real axis in the cut z^2 plane. In the identity

$$W_2(z) = W_2(-z)$$

the cut is approached from opposite sides as real z is approached, and we wind up with

$$W_2(x) = W_2(-x)^*$$

-- i.e. that $\Delta(x)$ is real. (The "physical" W's are obtained by approaching cut from below.)

But the complex Lorentz group allows us to continue around the cut, and relate two values of z with $\mathrm{Im}\, z^0 < 0$.

The "CPT Theorem"

- The proof of the PT theorem generalizes trivially to the case of <u>many</u> real scalar fields $\phi^a(x)$!
- In the case of complex scalar fields

$$\varphi^a(x) = \phi_1^a(x) + i\,\phi_2^a(x)$$

$$\Theta\, \varphi^a(x)\, \Theta^{-1} = \varphi^a(-x)^\dagger$$

if

$$\Theta\, \phi_1^a(x)\, \Theta^{-1} = \phi_1^a(-x)$$
$$\Theta\, \phi_2^a(x)\, \Theta^{-1} = \phi_2^a(-x) \qquad (\text{for } \Theta \text{ antiunitary})$$

Thus is the <u>CPT</u> operator, and the previous argument shows that Θ is <u>antiunitary</u>.

<u>Spin-Statistics Connection</u>

-- An exercise