J. PRESKILL

2. Interacting Scalar Fields

2. INTERACTING SCALAR FIELDS

We have studied intensively the quantum theory of a free scalar field. This study has had some interesting features, but there has not been much physics in it; nothing but noninteracting particles!

In our discussion of the "Axioms", we considered some of the qualitative features that a more interesting interacting field theory should have. Now we want to look more closely at particular examples, and make the study of these examples more quantitative.

In a scalar theory, we will obtain an interacting field theory by adding to the quadratic action discussed earlier an interaction term that is higher order in the fields:

$$S = S_{free} + S_{int}$$

Correspondingly, the Hamiltonian becomes

$$H = H_0 + H',$$

where H_0 is the Hamiltonian of the free theory, and H' is the new term. How shall we analyze this theory?

- We could try to diagonalize H exactly. There are a few interesting theories in 2 spacetime dimensions for which that has actually been done. But usually (e.g., in 4 dimensions) it is much too hard.

- We can do perturbation theory, expanding in powers of H' (regarding the new term H' as "small.")

Perturbation theory has disadvantages. It won't reveal profound qualitative changes in the theory induced by H' (Example: quark confinement) But --

- It allows us to extract quantitative information, and can be the basis of very accurate calculations:

Anomalous magnetic moment:

$$g_e = 2(1+a_e)$$

$$(a_e)_{\text{theory}} = 1.159\,652\,359\,(282) \times 10^{-3}$$
$$(a_e)_{\text{exp}} = 1.159\,652\,410\,(200) \times 10^{-3}$$

- It has provided much insight into the nature of interacting relativistic quantum field theory.

So, we want to formulate a perturbation expansion. Furthermore, since the form of the outcome of any perturbative calculation is constrained by the requirement of relativistic covariance, it turns out to be extremely convenient to set up a manifestly covariant expansion. (It makes calculations simpler)

This was the achievement of Schwinger, Tomonaga, and Feynman.

For one thing, we should ask the right questions, questions that treat space and time on an equal footing, questions about processes in spacetime. An example of a quantity with the desired relativistic invariance properties is

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$$

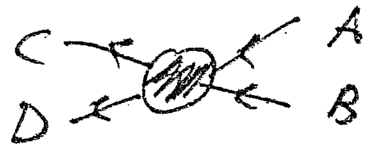
-- the "Wightman function" encountered in our discussion of axiomatic field theory.

Even more useful is the Green function:

$$\langle 0|\,{}^{\prime}T^{\prime}[\phi(x_1)\cdots\phi(x_n)]|0\rangle,$$

where 'T' denotes time ordering. (A time-ordered product is invariant under restricted Lorentz transformations if the fields commute for spacelike separation.)

A related physical process is the scattering of two particles:



The probability amplitude for such a scattering process is an example of a quantity that can be conveniently calculated using covariant perturbation theory.

Time-Dependent Perturbation Theory (Interaction Picture)

To get started, we'll review a method for doing perturbative calculations in nonrelativistic quantum mechanics. Implicit in this method is a choice of frame (it is formulated in terms of the Hamiltonian), but we will eventually obtain results that can be stated in a frame-independent way.

The idea underlying the method is that interacting fields (or any operators in the interacting theory) are "close to" the corresponding free fields in perturbation theory, and can be expanded in pert. theory, the leading term being a free field.

If $\phi(x)$ is a field of the interacting theory, we may define an "interaction picture" field:

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)}.$$

It is equated with the (Heisenberg picture) field $\phi(x)$ at time t_0, but evolves in time according to the <u>free</u> dynamics generated by H_0. Comparing fields at the same time (and suppressing the $\vec{x}$ dependence),

$$\phi_I(t,\vec{x}) = U(t,t_0)\,\phi(t)\,U(t,t_0)^{-1}$$

where $$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$

Here U is a unitary change of basis that "rotates away" the motion <u>not</u> caused by the perturbation H'; it propagates (Schroedinger picture) states forward from t_0 to t with <u>full</u> dynamics generated by $H = H_0 + H'$, then propagates back from t to t_0 with the free dynamics generated by H_0.

U obeys the boundary condition $U(t_0,t_0)$, so it can be completely determined by a first order differential equation:

$$\frac{d}{dt}U(t,t_0) = e^{iH_0(t-t_0)}(-i)(H-H_0)e^{-iH(t-t_0)}$$

or $$\frac{d}{dt} U(t,t_0) = -i\, e^{iH_0(t-t_0)} H' e^{-iH_0(t-t_0)} \times e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$= -iH_I(t)\, U(t,t_0)$$

where $$H_I(t) = e^{iH_0(t-t_0)} H' e^{-iH_0(t-t_0)}$$

Here H' is a Schrodinger picture operator, and H_I is a "free" Heisenberg operator. We can also write

$$H_I(t) = U(t,t_0)\, H'_{Heisenberg}\, U(t,t_0)^{-1}$$

E.g., if $H'_{Heisenberg}$ is a polynomial in Heisenberg picture fields

$$H' = H'(\phi),$$

we have

$$H_I = H'(\phi_I)$$

-- the corresponding function of <u>free</u> interaction picture fields

Now, we wish to construct a general solution to

$$\frac{d}{dt} U(t,t_0) = -iH_I(t)\, U(t,t_0),$$

satisfying $$U(t_0,t_0) = 1,$$

as an expansion in powers of H_I. It can be converted to an integral equation

$$U(t,t_0) = 1 + \int_{t_0}^{t} -iH_I(t')\, U(t',t_0)\, dt,$$

that can be solved by iteration ---

$$U(t,t_0) = 1 + \int_{t_0}^{t} dt_1 \left(-iH_I(t_1)\right)$$
$$+ \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \left(-iH_I(t_1)\right)\left(-iH_I(t_2)\right) + \cdots$$

That is, the nth term in the expansion is:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n$$
$$\left(-iH_I(t_1)\right) \cdots \left(-iH_I(t_n)\right).$$

This expression can be written in a more compact form if we notice that the times $t_1 \cdots t_n$ are ordered so that

$$t > t_1 > t_2 > \cdots > t_n > t_0$$

If we extend each t_K integral from t_0 to t, we obtain the same expression $n!$ times (all possible orderings of n times), except that

$$(-iH_I) \cdots (-iH_I)$$

must always be arranged in <u>temporal order</u>; i.e., with later times to the left.
So the nth term is

$$\frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \, T\left[\left(-iH_I(t_1)\right) \cdots \left(-iH_I(t_n)\right)\right]$$

where T denotes <u>time ordering</u>.

Now we have

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} T\left[\left\{\int_{t_0}^{t} dt_1 \left(-iH_I(t_1)\right)\right\}^n\right]$$

So the series exponentiates:

$$U(t,t_0) = \text{`}T\text{'} \exp\left[\int_{t_0}^{t} dt' \, -iH_I(t')\right] \quad (\text{"Dyson's Formula"})$$

(Formally, although $H_I(t)$'s at different times do not commute, differentiating wrt t brings down $H_I(t)$ in front of the exponential, because of the time ordering.)

Some properties of $U(t,t_0)$:

- Group Property: $U(t,t')U(t',t_0) = U(t,t_0)$

  This property is not so obvious from the expression $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$, since H and H_0 need not commute. But it follows easily from the time-ordered expression for U, if $t \geq t' \geq t_0$.

  $$\text{`}T\text{'}\exp\left(\int_{t'}^{t} \sim\right) \text{`}T\text{'}\exp\left(\int_{t_0}^{t'} \sim\right) = \text{`}T\text{'}\exp\left(\int_{t_0}^{t} \sim\right)$$

  would be true if integrands were c-nos. But it is true for operators too, because time ordering on both sides is the same.

- Inverse: $U(t,t_0)^{-1} = U(t_0,t)$

  To see this, recall that U is unitary, and H_I is hermitian:

  $$U(t,t_0)^{-1} = U(t,t_0)^{+} = \text{`}T\text{'}\exp\left[\int_{t_0}^{t} dt' \, iH_I(t')\right]$$

  $$= \text{`}\bar{T}\text{'}\exp\left[\int_{t}^{t_0} -iH_I(t')dt'\right]$$

the $\bar{T}$ signifies that time ordering is reversed.

Thus the identity $U(t,t_0)^{-1} = U(t_0,t)$ holds if time-ordering is interpretted as path ordering. That is, the time ordering is reversed for a path running backwards in time.

$$\overset{\longleftarrow}{\underset{t}{\bullet}\!\!\longrightarrow\!\!\underset{t_0}{\bullet}}$$

Now the group property can be extended to all intermediate time t'. E.g. if $t' > t > t_0$.

$$U(t,t')\,U(t',t_0)$$

$$= U(t,t')U(t',t)U(t,t_0) = U(t,t_0)$$

- Lorentz invariance

Suppose the perturbation involves no time derivatives

$$S = S_0 + S' \qquad S' = \int d^4x\, \mathcal{L}'(\phi(x))$$

Then $\quad H' = -\int d^3x\, \mathcal{L}'(\phi(x)), \qquad (\mathcal{L}' = -\mathcal{H}')$

where $\mathcal{L}'$ is a Lorentz scalar.

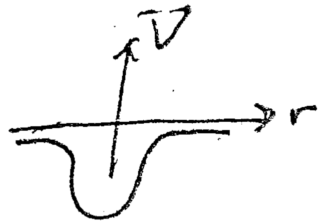
$$U(t,t_0) = \text{'}T\text{'} \exp\left[i\int_{t_0}^{t} dt' \int d^3x'\, \mathcal{L}(\phi_I(x'))\right]$$

The only source of frame dependence in this expression is the limit of the time integration. The integrand is Lorentz invariant, as is the measure $\int d^4x'$. (As noted earlier, the T ordering doesn't spoil Lorentz invariance, because T order is frame dependent for only spacelike separation, and fields commute for spacelike separation.)

Scattering Theory

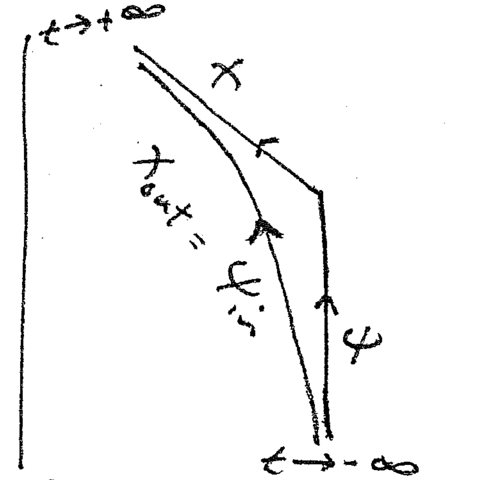
To continue with our program of formulating a perturbative expansion that is not dependent on the choice of reference frame, we need to avoid the dependence on t, t_0. This is just a matter of studying the right quantities, quantities that do not require the specification of an initial and final time. Such quantities are <u>scattering</u> <u>amplitudes</u>

The main idea of scattering theory is that interactions "turn off" at large negative and positive times, and all states evolve like free states. E.g., for a particle scattering off a potential in non-relativistic quantum mechanics, this will be true if the potential has compact support.



In a relativistic theory, we note that wave-packet states tend to separate and spread, so that they cease to interact in the remote past and future.

Drawing some imagery from classical scattering theory, we may say that every state approaches an "asymptote" in the far past and future: its time evolution becomes arbitrarily well approximated by the free (unperturbed) time evolution.



If ψ denotes an "asymptotic state", we will denote by ψ_{in} the corresponding state that evolves according to the exact dynamics, and approaches the asymptote ψ as $t \to -\infty$. Similarly, ψ_{out} is the state that approaches the asymptote ψ as $t \to +\infty$. In equations:

$$e^{-iHt}\psi_{in}(0) \longrightarrow e^{-iH_0 t}\psi(0) \quad \text{as } t \to -\infty$$

$$e^{-iHt}\psi_{out}(0) \longrightarrow e^{-iH_0 t}\psi(0) \quad \text{as } t \to +\infty$$

(These are Schrödinger picture states)

Or --

$$\psi_{in} = \lim_{t \to -\infty} e^{iHt} e^{-iH_0 t}\psi$$

$$\psi_{out} = \lim_{t \to +\infty} e^{iHt} e^{-iH_0 t}\psi$$

(which can be regarded as a relation between the fixed Heisenberg states of the two dynamics).

Now, the central question of scattering theory concerns the relation between the in and out asymptotes. What is the probability amplitude for a state that approaches ψ in the remote past to evolve into a state that approaches χ in the remote future? It is

$$(\chi_{out}, \psi_{in}) = \lim_{t \to \infty} \lim_{t' \to -\infty} (\chi, e^{iH_0 t} e^{-iHt} \times e^{iHt'} e^{-iH_0 t'}\psi)$$

Recall that $U(t,0) = e^{iH_0 t} e^{-iHt}$,
so this can be written

$$(\chi_{out}, \psi_{in}) = \lim_{\substack{t \to \infty \\ t' \to -\infty}} (\chi, U(t,0)\, U(t',0)^{-1} \psi)$$

But $U(t',0)^{-1} = U(0,t')$, and then the group property gives

$$(\chi_{out}, \psi_{in}) = \lim_{\substack{t \to \infty \\ t' \to -\infty}} (\chi, U(t,t')\psi)$$

or, finally

$$(\chi_{out}, \psi_{in}) = (\chi, S\psi)$$

where $S \equiv U(\infty, -\infty)$

is the <u>S-matrix</u>. It is the linear operator that takes in asymptotes to out asymptotes.

Some properties of S

- S is <u>unitary</u> $\quad S^+ S = S S^+ = \mathbb{1}$

(Showing this is actually subtle, even though $U(t,t_0)$ is unitary, since the limit of a unitary operator need not be unitary, unless its range is the whole Hilbert space.)
This is "conservation of <u>probability</u>":

$$1 = (\psi, \psi) = (\psi, S^+ S \psi) = \sum_n |(n, S\psi)|^2$$

"what comes in, goes out"

- $[S, H_0] = 0$ -- Thus, scattering preserves the unperturbed energy.

To see this, recall $S = \lim_{\substack{t \to \infty \\ t' \to -\infty}} e^{iH_0 t} e^{-iHt} e^{iHt'} e^{-iH_0 t'}$

Thus
$$e^{i\omega H_0} S = \lim e^{iH_0(t+\omega)} e^{-iHt} e^{iHt'} e^{-iH_0 t'}$$
$$= \lim e^{iH_0 t} e^{-iH(t-\omega)} e^{iHt'} e^{-iH_0 t'}$$
$$= \lim e^{iH_0 t} e^{-iHt} e^{iH(t'+\omega)} e^{-iH_0 t'}$$
$$= \lim e^{iH_0 t} e^{-iHt} e^{iHt'} e^{-iH_0(t'-\omega)}$$
$$= S e^{iH_0 \omega}$$

so $[e^{i\omega H_0}, S] = 0$ and $[H_0, S] = 0$ follows by differentiating wrt ω and setting $\omega = 0$.

The S matrix encodes all the information about the scattering behavior of the theory. In fact, it also tells us about the complete spectrum (e.g. bound states) through the magic of analytic continuation. If we knew the S matrix exactly, we would know <u>everything</u> about the physics of the theory.

Furthermore,

$$S = T \exp\left[i \int d^4x \, \mathcal{L}(\Phi_I(x))\right]$$

where integral is over all spacetime. By eliminating a boundary to the region of integration, we have constructed a Lorentz-invariant object.

Wick's Theorem

Now we want to evaluate

$$S = T \exp\left[-i\int d^4x\, \mathcal{H}'(\phi_I(x))\right]$$

perturbatively in powers of $\mathcal{H}'$. Since $S = \lim_{\substack{t\to\infty \\ t_0\to-\infty}} U(t,t_0)$, and recalling that

$\phi_I(x)$ is a free field that coincides with the field $\phi(x)$ at the time $t_0 \to -\infty$, we have

$$\phi_I(\vec{x},t) \longrightarrow \phi(\vec{x},t) \text{ as } t\to-\infty.$$

ϕ_I is the free field that coincides with the exact interacting field ϕ in the remote past; it is sometimes called the "in-field" ϕ_{in}. Like any free field ϕ_I can be expanded in terms of a and $a^\dagger$. Now we understand that the states created by $a^\dagger$ are the asymptotic incoming plane wave states.

What about the normalization of the field ϕ_I? The normalization of a free field is fixed by the canonical commutation relations — $\partial_0\phi(x) = \pi(x)$ must be the momentum conjugate to ϕ, if ϕ is a conventionally normalized free field. But ϕ_I and π_I differ from the exact ϕ and π by a unitary change of basis, that preserves the commutators. So ϕ_I is conventionally normalized.

(This argument is actually wrong, because it treats the nature of the limit $\phi_I \to \phi$ too naively. But we won't get around to correcting it for a few more lectures.)

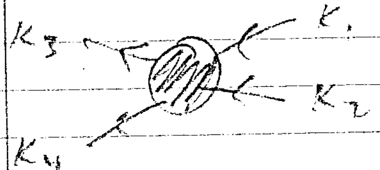
Now we've seen that the S-matrix S can be expanded in terms of operators a and $a^\dagger$ that destroy and create the asymptotic scattering states:

$$\phi_I(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[a(\vec{k}) e^{-ik\cdot x} + a(\vec{k})^\dagger e^{ik\cdot x} \right]$$

where e.g.,

$$|\vec{k}\rangle = a(\vec{k})^\dagger |0\rangle$$

is an asymptotic plane wave state, and $|0\rangle$ is the asymptotic vacuum. This expansion can be carried out systematically by using Wick's theorem.



For example, we'll want to compute scattering amplitudes:

$$\langle k_3, k_4 | k_1, k_2 \rangle_{\substack{\\ \text{out}}\ \text{in}}$$

$$= \langle k_3, k_4 | S | k_1, k_2 \rangle .$$

For this purpose, it is very convenient to normal order each term in S. (Then only terms with two a's and two $a^\dagger$'s can contribute to nonforward scattering.) We want to be able to reexpress a term such as

$$(-i)^n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, T\left[H_I(x_1) \cdots H_I(x_n) \right]$$

as a normal ordered product. How are time-ordered products related to normal-ordered products of free fields?

To begin at the beginning, consider a product of two free scalar fields:

$$T'[\phi^a(x)\,\phi^b(y)] \equiv \,:\phi^a(x)\,\phi^b(y): + \overbrace{\phi^a(x)\,\phi^b(y)}$$

This expression defines the "contraction" of two fields --

$$\overbrace{\phi^a(x)\,\phi^b(y)}.$$

The contraction is evidently a c-no. For suppose $x^0 > y^0$. Then

$$T'[\phi^a(x)\,\phi^b(y)] = \phi^a(x)\;\phi^b(y)$$
$$= [\phi^{a(+)}(x) + \phi^{a(-)}(x)][\phi^{b(+)}(y) + \phi^{b(-)}(y)]$$

(where + and - denote pos. and neg frequency parts)

$$= \,:\phi^a(x)\,\phi^b(y): + [\phi^{a(-)}(x), \phi^{b(+)}(y)]$$

so $$\overbrace{\phi^a(x)\,\phi^b(y)} = [\phi^{a(-)}(x), \phi^{b(+)}(y)] = \delta^{ab}\Delta_+(x-y)$$

(where Δ_+ is the c-no. function defined previously).

Similarly

$$\overbrace{\phi^a(x)\,\phi^b(y)} = \delta^{ab}\Delta_+(y-x) \quad \text{for} \quad x^0 < y^0$$

Since we know that $\overbrace{\phi\,\phi}$ is a c-no., we evaluate it by taking matrix element between vacuum states of the above identity:

$$\overbrace{\phi^a(x)\,\phi^b(y)} = \langle 0|\,T'[\phi^a(x)\,\phi^b(y)]\,|0\rangle$$
$$= \delta^{ab}\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\,\frac{i}{k^2 - m_a^2 + i\epsilon}$$

(using the result of a homework exercise).

Now consider a product of many free fields. Let ϕ_k denote $\phi^{a_k}(x_k)$.

Then: (<u>Wick's</u> <u>Theorem</u>)

$T[\phi_1 \cdots \phi_n]$

$$= :\phi_1 \cdots \phi_n:$$

$$+ \left(\phi_1^{\sqcap} \phi_2^{\sqcap} : \phi_3 \cdots \phi_n : + \text{all other terms with one contraction} \right)$$

$$+ \left(\phi_1^{\sqcap} \phi_2^{\sqcap}\ \phi_3^{\sqcap} \phi_4^{\sqcap} : \phi_5 \cdots \phi_n : + \text{all other terms with two contractions} \right)$$

$$+ \text{etc.}$$

We can prove the theorem by induction on n:

It is trivially true for $n=1$, and true for $n=2$ by definition of the contraction.

Now show that, if true for $n-2$, $n-1$, then true for n. Suppose $x_1^0 \geq x_2^0 \geq \cdots x_n^0$. (Argument works the same for any time ordering

Then LHS: $T[\phi_1 \cdots \phi_n] = \phi_1 \phi_2 \cdots \phi_n = (\phi_1^{(+)} + \phi_1^{(-)})\phi_2 \cdots \phi_n$
(where + and − denote positive and negative frequency components)

RHS = (Terms with ϕ_1 <u>not</u> contracted)
+ (Terms with ϕ_1 contracted)

But by the induction hypothesis for $n-1$ fields,
(Terms with ϕ_1 not contracted)

$$= \phi_1^{(+)}(\phi_2 \cdots \phi_n) + (\phi_2 \cdots \phi_n)\phi_1^{(-)}$$

Now consider terms with ϕ_1 contracted. Because x_1^0 is the latest time

$$\overbrace{\phi_1 \phi_K} = [\phi_1^{(-)}, \phi_K^{(+)}] = [\phi_1^{(-)}, \phi_K] \text{ for each } K$$

so by the induction hypothesis for n-2 fields ---

(terms with ϕ_1 contracted)

$$= \overbrace{\phi_1 \phi_2}\, \phi_3 \cdots \phi_n + \overbrace{\phi_1 \phi_3}\, (\phi_2 \phi_4 \cdots \phi_n) + \cdots$$

$$= [\phi_1^{(-)}, \phi_2 \cdots \phi_n]$$

Adding together the two contributions gives

$$RHS = (\phi_1^{(+)} + \phi_1^{(-)})\, \phi_2 \cdots \phi_n$$

This proves Wick's theorem.

The External Field Problem

As an example that illustrates the utility of Wick's theorem, we will consider a scalar field (mass m) coupled to a classical source ("external field")

The interaction is

$$\mathcal{H}' = \lambda\, \rho(x)\, \phi(x)$$

where λ is a (real) coupling constant, assumed small, and $\rho(x)$ is a real function of spacetime position x. So that our scattering theory will apply to this problem, let us suppose that the source $\rho(x)$ "turns off" for $t \to \pm\infty$, so that dynamics returns to free dynamics in remote past and future.

The classical field eqn in the presence of the source is

$$(\partial^\mu \partial_\mu + m^2)\phi(x) = -\lambda \rho(x),$$

which is linear and hence exactly solvable. The quantum problem can also be solved exactly. The solution is most easily obtained by summing up our covariant perturbation expansion to all orders in λ

We have derived

$$S = U(\infty, -\infty) = \text{'}T\text{'} \exp\left[-i\lambda \int d^4x\, \rho(x)\, \phi(x)\right].$$

In order to evaluate matrix elements of S readily, we use Wick's theorem to reexpress it in terms of normal-ordered products. (Although not explicit in the above notation, it is understood that ϕ is the conventionally normalized free field ϕ_I.)

A generic term in expansion of S, of order λ^n and containing p contractions, is --

$$\frac{1}{n!}(-i\lambda)^n \left[\int d^4x_1\, d^4x_2\, \rho(x_1)\rho(x_2)\, \overbrace{\phi(x_1)\,\phi(x_2)}\right]^p$$

$$\times :\left[\int d^4x_3\, \rho(x_3)\, \phi(x_3)\right]^{n-2p}: \times \left(\begin{matrix}\text{combinatoric}\\ \text{factor}\end{matrix}\right)$$

The combinatoric factor is the number of ways of choosing p pairs from n objects, namely

$$\frac{n!}{2^p\, p!\,(n-2p)!}$$

Thus, this term is

$$\frac{1}{p!}\left[\tfrac{1}{2}(-i\lambda)^2\int d^4x_1\, d^4x_2\ \rho(x_1)\rho(x_2)\ \overbrace{\phi(x_1)\,\phi(x_2)}\right]^p$$

$$\times\ \frac{(-i\lambda)^k}{k!} : \left[\int d^4x_3\ \rho(x_3)\,\phi(x_3)\right]^k : \ ,$$

where p is the number of contractions, and k is the number of uncontracted fields.
We can now obtain the full S by summing both p and k from 0 to ∞, obtaining

$$S = A : \exp\left[-i\lambda\int d^4x\ \rho(x)\,\phi(x)\right]:$$

where A is the c no. factor...

$$A = \exp\left[\tfrac{1}{2}(-i\lambda)^2\int d^4x_1\, d^4x_2\ \rho(x_1)\rho(x_2)\ \overbrace{\phi(x_1)\,\phi(x_2)}\right]$$

A source varying in time and space can create particles, so we can expect a nonvanishing amplitude for the asymptotic vacuum to evolve into an n particle state:

$$_{out}\langle k_1 \cdots k_n | 0\rangle_{in} = \langle k_1 \cdots k_n | S | 0\rangle$$

To evaluate this amplitude, we first Fourier transform

$$\rho(x) = \int \frac{d^4k}{(2\pi)^4}\ \tilde{\rho}(k)\, e^{-ik\cdot x}$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[\alpha(k)e^{-ik\cdot x} + \alpha(k)^{\dagger}e^{ik\cdot x}\right]$$

(relativistic normalization)

Thus,

$$\int d^4x\, \rho(x)\phi(x) = \int d^4k\, \tilde{\rho}(k) \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \times \left[\alpha(k')\delta^4(k+k') + \alpha(k')^\dagger \delta^4(k-k')\right]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\tilde{\rho}(-k)\alpha(k) + \tilde{\rho}(k)\alpha(k)^\dagger\right]$$

-- the Fourier component $\tilde{\rho}(k^0, \vec{k})$ creates a particle of momentum $\vec{k}$, and the component $\tilde{\rho}(-k^0, -\vec{k})$ destroys a particle of momentum $\vec{k}$.

Now

$$S = A \exp\left[-i\lambda \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tilde{\rho}(k)\alpha(k)^\dagger\right]$$

+ terms with $\alpha(k)$'s,

and recalling

$$\langle k' | \alpha(k) | 0\rangle = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}'),$$

we see that

$$\langle k_1 | S | 0\rangle = (-i\lambda) A \tilde{\rho}(k_1)$$

-- amplitude for source to create one particle. Similarly

$$\langle k_1, \ldots, k_n | S | 0\rangle = (-i\lambda)^n A\, \tilde{\rho}(k_1) \cdots \tilde{\rho}(k_n).$$

The $\frac{1}{n!}$ gets cancelled, for there are $n!$ ways for n α's to create n particles.

2.21

Let's check that S is unitary.
Unitarity requires

$$1 = \langle 0|0\rangle = \langle 0|S^\dagger S|0\rangle = \sum_n |\langle n|S|0\rangle|^2$$

or
$$1 = \sum_{n=0}^{\infty}\int\frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}}\cdots\int\frac{d^3k_n}{(2\pi)^3 2\omega_{k_n}}|\langle k_1\cdots k_n|S|0\rangle|^2\left(\frac{1}{n!}\right)$$

(because particles are identical.)

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{2n}|A|^2\left[\int\frac{d^3k}{(2\pi)^3 2\omega_k}|\tilde{\rho}(k)|^2\right]^n$$

$$= |A|^2\exp\left[\lambda^2\int\frac{d^3k}{(2\pi)^3 2\omega_k}|\tilde{\rho}(k)|^2\right]$$

So it works if
$$|A|^2 = \exp\left[-\lambda^2\int\frac{d^3k}{(2\pi)^3 2\omega_k}|\tilde{\rho}(k)|^2\right]$$

We know
$$A = \exp(\tfrac{1}{2}\alpha), \quad |A|^2 = \exp(\mathrm{Re}\,\alpha)$$
where
$$\alpha = (-i\lambda)^2\int d^4x_1 d^4x_2\,\rho(x_1)\rho(x_2)\,\overline{\phi(x_1)\phi(x_2)}$$

So evaluate α by Fourier transforming:
$$\alpha = (-i\lambda)^2\int\frac{d^4k}{(2\pi)^4}\tilde{\rho}(k)\tilde{\rho}(-k)\frac{i}{k^2-m^2+i\epsilon},$$

and
$$\mathrm{Re}\,\alpha = -i\lambda^2\int\frac{d^4k}{(2\pi)^4}|\tilde{\rho}(k)|^2\frac{1}{2}\left[\frac{1}{k^2-m^2+i\epsilon}-\frac{1}{k^2-m^2-i\epsilon}\right].$$

Now use
$$\lim_{\epsilon\to 0^+}\left[\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon}\right] = -2\pi i\,\delta(x)$$

$$Re\,\alpha = -\frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^3} |\tilde{\rho}(k)|^2 \, \delta(k^2 - m^2)$$

We can do the k^0 integral; δ function has its support at $k^0 = \pm\omega_k$, so

$$Re\,\alpha = -\lambda^2 \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\tilde{\rho}(k)|^2$$

— the check of unitarity works

Interpretation:

the nth term in the sum is probability that n particles are created by the source:

Poisson statistics — different particles are independent

$$P(0\to n) = \frac{1}{n!} e^{-\beta} \beta^n \qquad (\beta = Re\,\alpha)$$

We have verified that $\sum_{n=0}^{\infty} P(0\to n) = 1$ ("conservation of probability"). We can also compute expected number of emitted quanta

$$\langle n \rangle = \sum_{n=0}^{\infty} n\, P(0\to n) = \beta.$$

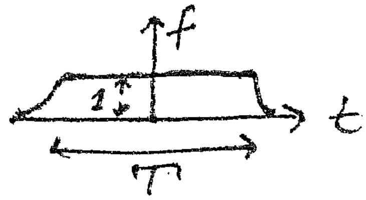
Notice that, if $\rho(x) = \delta^4(x)$ is localized in spacetime, then $\tilde{\rho}(k) = 1$, and

$$\langle n \rangle = -\lambda^2 \int \frac{d^3k}{(2\pi)^3 2\omega_k} = \infty$$

this is an (ultraviolet) divergence related to one we visited before, when discussing the fluctuations of a free field. It was noted then that an apparatus capable of measuring $\phi(x)$ with perfect resolution would create an indefinite number of particles. That remark is reflected here: the number of particles created by a localized source is infinite.

Energy of a Static Source

Next, we wish to consider how the vacuum of the theory responds to a static source $\rho(\vec{x})$.



But, in order that our asymptotic theory apply, we will have to imagine that the source turns on/off slowly at large negative and positive times. That is, consider a source term

$$\mathcal{H}' = \lambda f(t) \rho(\vec{x}) \phi(t, \vec{x})$$

where f is a function that turns on slowly at time $-T/2$, persists at $f=1$ for time T, then turns off slowly. If we can choose f so that it has no fourier components with $|k^0| \geq m$, then we know from our previous calculation that no particles can be created or destroyed. The S-matrix is uninteresting. But an interesting quantity that we can compute is the ground state energy in the presence of the source.

The S-matrix element

$$\langle 0|0\rangle_{in}^{out} = \langle 0|S|0\rangle = A$$

$$= \exp\left[\left(\frac{-i\lambda}{2}\right)^2 \int \frac{d^4K}{(2\pi)^4} |\tilde{f}(K^0)|^2 |\tilde{\rho}(\vec{K})|^2 \frac{i}{K^2-m^2+i\epsilon}\right]$$

is a pure phase. What is the interpretation of this phase?

We know that

$$U(t,-t') = U(t,0)\,U(0,-t) \quad \text{where } U(t,0) = e^{iH_0 t}e^{-iHt}$$
$$= U(t,0)\,U(-t,0)^{-1}$$
$$= e^{iH_0 t}e^{-iHt}e^{-iHt'}e^{iH_0 t'}$$

Therefore $\langle 0|U(t,-t')|0\rangle = \langle 0|e^{-iH(t+t')}|0\rangle$.

While the source is turning on the Hamiltonian H is time-dependent, but according to the adiabatic theorem of quantum mechanics, if it turns on sufficiently slowly, no excitation occurs, and $|0\rangle$ evolves into the vacuum state $|0\rangle_{phys}$ -- the ground state in the presence of the source. Therefore

$$\langle 0|S|0\rangle = \exp\left[-i(\underbrace{\gamma_{on} + \gamma_{off}}_{\text{phases (independent of } T\text{) from turning on and off}} + \underbrace{E_0 T}_{\text{ground state energy in presence of static source}})\right]$$

So we can find E_0 by extracting the term in the logarithm of $\langle 0|S|0\rangle$ that is linear in T.

$$E_0 = i \lim_{T\to\infty} \frac{1}{T} \ln(\langle 0|S|0\rangle)$$

Here

$$\ln\langle 0|S|0\rangle = -i\frac{\lambda^2}{2}\int\frac{d^4K}{(2\pi)^4}\,|\tilde{f}(K^0)|^2\,|\hat{\rho}(\vec{K})|^2\,\frac{1}{K^2-m^2+i\epsilon}\,.$$

As the support T of $f(t)$ gets large $\tilde{f}(K^0)$ approaches $2\pi\delta(K^0)$ and

$$\int\frac{dK_0}{2\pi}|\tilde{f}(K^0)|^2 \to T$$

$$\text{So}\quad E_0 = -\frac{\lambda^2}{2}\int\frac{d^3K}{(2\pi)^3}\,|\tilde{\rho}(\vec{K})|^2\,\frac{1}{\vec{K}^2+m^2}$$

$$= \frac{\lambda^2}{2}\int d^3x_1\,d^3x_2\;\rho(\vec{x}_1)\,\rho(\vec{x}_2)\;V(\vec{x}_1-\vec{x}_2)$$

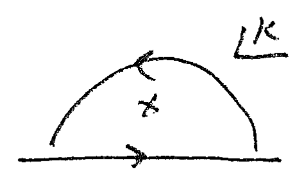
where
$$V(\vec{x}) = -\int\frac{d^3K}{(2\pi)^3}\,e^{i\vec{K}\cdot\vec{x}}\,\frac{1}{\vec{K}^2+m^2}$$

Evidently, $V(\vec{x})$ may be interpreted as the potential energy of two static sources separated by $\vec{x}$. Evaluating the Fourier transform ---

$$V(r) = -\int_0^\infty\frac{2\pi K^2\,dK}{(2\pi)^3}\,\frac{1}{iKr}\left(e^{iKr}-e^{-iKr}\right)\frac{1}{K^2+m^2}$$

(performing angular integral)

$$= \frac{-1}{(2\pi)^2\,ir}\int_{-\infty}^{\infty}\frac{K\,dK}{K^2+m^2}\,e^{iKr}$$



$$= \frac{-1}{2\pi r}\,\frac{im}{2im}\,e^{-mr}\qquad\text{(by contour integration)}$$

$$= \frac{-1}{4\pi r}\,e^{-mr}$$ -- An (attractive) Yukawa potential

this becomes a Coulomb potential in $m \to 0$ limit.

A Nonlinear Field Theory

More interesting, and much more complicated, than a free field coupled to an external field are nonlinear field theories, with interaction terms in the action higher order than quadratic in the fields. The perturbation expansion is an essential tool for unraveling the properties of such theories.

As an explicit example, consider three scalar fields $\phi_1(x), \phi_2(x), \phi_3(x)$, with masses m_1, m_2, m_3, coupled together by the interaction

$$\mathcal{H}' = \lambda\, \phi_1(x)\, \phi_2(x)\, \phi_3(x)$$

(This will give a Hamiltonian that is unbounded from below, and hence a sick theory, but it will still serve to demonstrate some properties of the perturbative expansion.)

Now, the S-matrix

$$S = T \exp\left[i \int d^4x\, \mathcal{H}'(\phi_j(x))\right)$$

(free "interaction picture" field)

can be expanded in powers of λ:

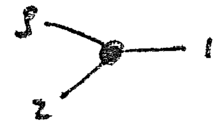
$$S = \sum_{n=0}^{\infty} S^{(n)}$$

order λ^n

where $S^{(0)} = 1,$

and
$$S^{(1)} = -i\lambda \int d^4x \; \phi_1(x)\phi_2(x)\phi_3(x)$$

(It is understood that these are free fields; we'll drop the subscript to indicate interaction picture.) We do not need Wick's theorem to normal order $S^{(1)}$ -- it is already normal ordered, since the fields ϕ_1, ϕ_2, ϕ_3 are distinct. This operator can be represented by a diagram



Each line is a field. That the lines meet at a single vertex means that they have the same spacetime argument x. It is understood that x is integrated over all spacetime, and that the "vertex" is accompanied by a factor $-i\lambda$.

$S^{(1)}$ can contribute to the matrix element of the S-matrix

$${}_{out}\langle k_2, k_3 | k_1 \rangle_{in} = \langle k_2, k_3 | S^{(1)} | k_1 \rangle + \cdots$$

Recall that, for a relativistically normalized one particle state $|k\rangle$, and conventionally normalized free scalar field $\phi(x)$

$$\langle 0 | \phi(x) | k \rangle = e^{-ik\cdot x}$$

Thus, $\langle k_2, k_3 | S^{(1)} | k_1 \rangle$

$$= -i\lambda \int d^4x \; e^{-ik_1\cdot x} e^{ik_2\cdot x} e^{ik_3\cdot x} = (-i\lambda)(2\pi)^4 \delta^4(k_1 - k_2 - k_3)$$

As we noted earlier, if H' is Lorentz invariant and translation invariant, then so is S; it commutes with the Poincare generators acting on the asymptotic states. So, in a basis of momentum eigenstates

$$\langle f|S|i\rangle = \langle f|U^{-1}(a)\, S\, U(a)|i\rangle$$
$$= e^{ia\cdot(p_i - p_f)}\langle f|S|i\rangle$$
$$\Longrightarrow \quad p_i = p_f$$

Therefore, we can write

$$\langle f|S-1|i\rangle = (2\pi)^4 \delta^{(4)}(p_f - p_i)\; i \quad \langle f|A|i\rangle,$$

which defines the "scattering amplitude" $\langle f|A|i\rangle \equiv A_{fi}$. Further, from Lorentz invariance,

$$\langle f|S|i\rangle = \langle f|U(\Lambda)^{-1} S\, U(\Lambda)|i\rangle = \langle \Lambda f|S|\Lambda i\rangle.$$

So A_{fi} is a Lorentz invariant function of the momenta. We have verified these properties explicitly to order λ, and found

$$iA = -i\lambda$$

Note that the energy-momentum conserving δ function ensures that $S_{fi}^{(1)} = 0$ unless the decay $1 \to 2+3$ is kinematically allowed ($m_1 \geq m_2 + m_3$).

In the next order:

$$S^{(2)} = \tfrac{1}{2}(-i\lambda)^2 \int d^4x_1\, d^4x_2\; T\left[\phi_1(x_1)\phi_2(x_1)\phi_3(x_1)\,\phi_1(x_2)\phi_2(x_2)\phi_3(x_2)\right]$$

Now we can reexpress this in terms of normal ordered products, and classify terms according to the number of contractions.

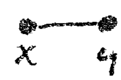
- <u>Zero contractions</u>

The term with no contractions can be represented by

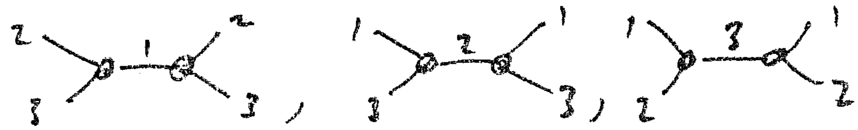
The diagrammatic rules are the same as before, with the understanding that the product of fields represented by lines is a <u>normal ordered</u> product. Topologically, this diagram is <u>disconnected</u>. The significance of that is, if we take any matrix element, it will have <u>two</u> e.m. conserving $\delta^{(4)}$'s, one for each connected component.

- <u>One Contraction</u>

There are three ways of making one contraction. If we represent a contraction by a line that connects two vertices, the three terms obtained can be represented by the diagrams--



$= \overline{\phi(x)\,\phi}(y)$

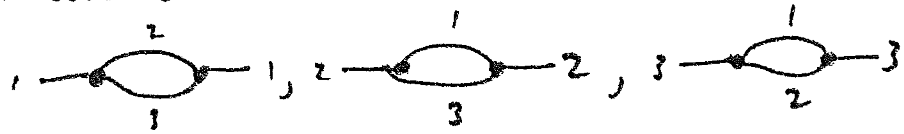


-- we can contract a pair of ϕ_1's, ϕ_2's, or ϕ_3's

Note! these diagrams are not Feynman diagrams. Feynman diagrams, which we'll come to shortly, represent contributions to <u>matrix elements</u> of S. But these diagrams represent <u>operators</u>, terms in the Wick expansion of S.

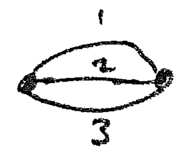
- <u>Two Contractions</u>

Similarly, there are three ways of making two contractions:



- <u>Three Contractions</u>

And one way of making three contractions:



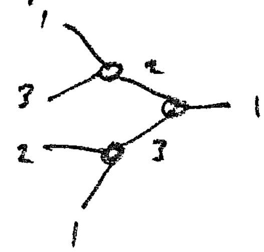
<u>Symmetry Factors</u>

Are there combinatoric considerations (as for the external field problem) in associating operators with the diagrams? In all the above diagrams, there is just a single way of doing the contractions that gives the operator shown. Hence, the factor of $\frac{1}{2}$ in front of $S^{(2)}$ is not canceled.

Generically, we might expect the $\frac{1}{n!}$ in $S^{(n)}$ to get canceled when we carry out the Wick expansion, because we can permute the vertices to find a different contraction that gives the same normal ordered operator. But this counting can change if the diagrams have <u>symmetries</u> -- permutations of the vertices that do <u>not</u> give rise to a new contraction.

In fact, <u>all</u> of the above diagrams have a two-fold symmetry; they are unchanged by an interchange of the two vertices. We may understand this symmetry as the origin of the factor of $\frac{1}{2}$ that accompanies them

Such symmetries need not be present in higher order diagrams. For example, one of the diagrams appearing in the Wick expansion of $S^{(3)}$ is



In this diagram, the three vertices are distinguishable. Thus, there are $3!$ contractions associated with this diagram, and

$$\frac{1}{3!}\,(\phi_1\phi_2\phi_3)(\phi_1\phi_2\phi_3)(\phi_1\phi_2\phi_3)$$

the $\frac{1}{3!}$ in front of $S^{(3)}$ gets completely canceled.

To our list of rules for associating operators with diagrams, we may add one more

- Divide by the "symmetry factor" of the graph

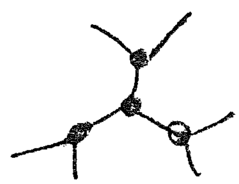
symmetry factor = number of permutations of the vertices that leave the contraction unchanged

In the theory considered here, the considerations above give the complete symmetry factor. But, in general, another contribution to the symmetry factor of a graph occurs if there are identical fields at a single vertex.

To illustrate this, consider a theory of a single scalar field $\phi(x)$ with interaction

$$\mathcal{H}' = \frac{1}{3!}\lambda\,\phi(x)^3$$

The factor $\frac{1}{3!}$ is included in H', because there are typically $3!$ different ways of contracting the fields of a vertex with the fields of neighboring vertices, so the $\frac{1}{3!}$ gets cancelled in the diagrams contributing to S, if



permutations of the lines ending on a given vertex give rise to new contractions.

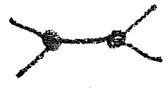
But the $\frac{1}{3!}$ does not always get completely cancelled, because permutations of the lines may not give a new contraction. We must include in the symmetry factor of a graph the number of permutations of lines that leave the contraction unchanged.

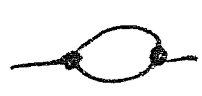
symmetry factor = (perms of vertices) × (perms of lines) that leave contraction unchanged

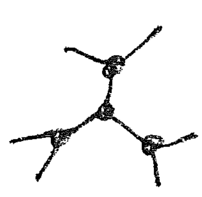
Denote $S(G)^{-1}$ = symmetry factor of graph G

Examples (in $\frac{1}{3!}\lambda\phi^3$ theory):

 $S = \frac{1}{3!}$ -- three lines are identical

 $S = \frac{1}{8}$ -- two identical vertices, two pairs of identical external lines

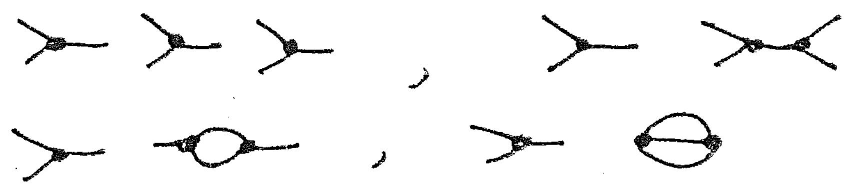
 $S = \frac{1}{4}$ -- two identical vertices, two identical internal lines

 $S = \frac{1}{3!\,2^3}$ -- three identical vertices, three pairs of identical external lines

Disconnected Graphs

In order λ^2 we encounted the disconnected diagram

In higher orders, there are more such diagrams E.g., in $O(\lambda^3)$



These always factorize into products of connected graphs encountered in lower orders.

What is the symmetry factor of a disconnected graph? It is not exactly the product of the symmetry factors of its connected components, because identical connected components may appear. The permutations of the vertices that leave the contraction unchanged include those that interchange identical connected components.

Therefore, we can express the sum of all graphs in terms of the connected graphs as

$$S = \sum \text{graphs} = \sum_{n_r=0}^{\infty} \prod_{r=1}^{\infty} \frac{1}{n_r!} \left(G_r^{(c)}\right)^{n_r}$$

where $\quad G_r^{(c)} \quad r = 1, 2, \ldots$

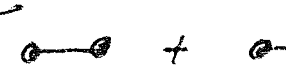
are all possible connected graphs. This exponentiates ---

$$S = \prod_{r=1}^{\infty} \exp G_r^{(c)} = \exp\left[\sum_{r=1}^{\infty} G_r^{(c)}\right] = e^{S^{(c)}}$$

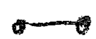
where $S^{(c)}$ is the sum of all connected graphs that appear in the Wick expansion.

Note that this combinatoric analysis generalizes that performed before for the special case of the external field problem. We had found

$$S = \exp\Big[\ \underbrace{\ \text{[diagram 1]} + \text{[diagram 2]}\ }\ \Big]$$



these are the only two connected diagrams.

Also  has symmetry factor $S = \frac{1}{2}$, since the vertices are identical.

We've reduced the problem of computing S to that of summing up all connected diagrams. But in a nonlinear theory, this is very difficult. It becomes very complicated to enumerate all the diagrams in high orders of perturbation theory. (Furthermore, the expansion may not converge, so even if we could enumerate all the graphs, it might not be enough.)

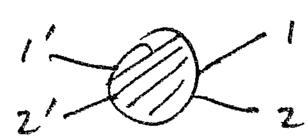
Scattering

In the nonlinear field theory with interaction

$$\mathcal{H}' = \frac{1}{3!}\lambda\,\phi(x)^3,$$

let us compute the amplitude for two-body elastic scattering, to order λ^2

The amplitude we are interested in is

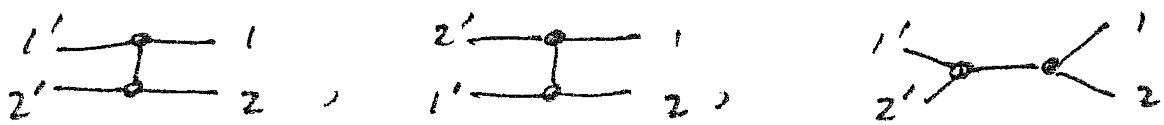


$$\,_{\text{out}}\langle k_1' k_2' | k_1 k_2 \rangle_{\text{in}} = \langle k_1' k_2' | S | k_1 k_2 \rangle$$

The piece of S that contributes to (the connected part of) the amplitude involves a normal ordered product of four fields. In order λ^2,

$\langle k_1' k_2' | S^{(2)} | k_1 k_2 \rangle = \langle k_1' k_2' |$  $| k_1 k_2 \rangle$

By matching up external lines to incoming and outgoing particles, we obtain diagrams that represent contributions to this matrix element. These are Feynman Diagrams



By convention, we draw the initial state on the right, and the final state on the left.

Each of these three diagrams represents 8 different ways of assigning external lines to incoming and outgoing particles, so the symmetry factor in  gets completely cancelled. We can state rules for assigning symmetry factors to Feynman diagrams directly, if we regard the assignment of an external line to a particle as a "contraction". Then

$S^{-1}(G)$ = (no. of perms of vertices) × (no. of perms of internal lines) that leave contraction unchanged

For example 
this Feynman Diagram
has symmetry factor $\frac{1}{2}$, because of the two identical internal lines.

Recalling that $\langle 0|\phi(x)|k\rangle = e^{-ik\cdot x}$ (for relativistically normalized states), we see that

the Feynman diagram has factors $e^{-ik_1\cdot x}\ e^{-ik_2\cdot y}$ associated with initial states and factors $e^{ik_1'\cdot x'}e^{ik_2'\cdot y'}$ associated with final states, where x, y, x', y' are positions of vertices with which initial and final particles are contracted. We can now state a simple set of rules for assigning to each Feynman diagram a <u>number</u>

<u>Feynman Rules (in Position Space)</u>

For each topologically distinct diagram:

- Label each vertex with a spacetime position x_k, and include for each vertex the "factor"
$$(-i\lambda)\int d^4x_k$$

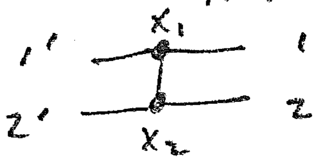
- For each <u>internal line</u> connecting vertex j to vertex k, include the factor
$$\overline{\phi(x_j)\,\phi(x_k)} = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_j-x_k)} \frac{i}{k^2-m^2+i\epsilon}$$

- For each "external" line --
  Contracting vertex j to incoming particle with momentum k_α, include factor $\exp(-i k_\alpha \cdot x_j)$
  Contracting vertex j to outgoing particle with momentum k_α', include factor $\exp(i k_\alpha' \cdot x_j)$

- Divide by symmetry factor (if any).

Carrying out the x_k integrations gives the corresponding contribution to $\langle f | S | i \rangle$

Let's apply these rules to the diagram:



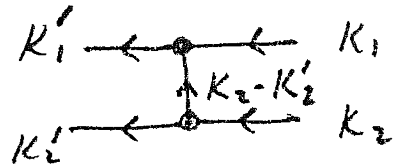
The symmetry factor is 1, so we have

$$\text{Graph} = (-i\lambda)^2 \int d^4x_1\, d^4x_2\, e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_1'\cdot x_1} e^{ik_2'\cdot x_2} \times \phi(x_1)\phi(x_2)$$

$$= (-i\lambda)^2 \int d^4x_1\, e^{-i(k_1-k_1')\cdot x_1} e^{-i(k_2-k_2')\cdot x_2} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \frac{i}{k^2-m^2+i\epsilon}$$

Now, we can do $x_{1,2}$ integrals trivially:

$$\text{Graph} = (-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(k_1-k_1'+k)(2\pi)^4 \delta^{(4)}(k_2-k_2'-k) \times \frac{i}{k^2-m^2+i\epsilon}$$



The effect of the x-integrations is to impose "momentum conservation" at the vertices. We can visualize momentum "flowing" through the graph, with momentum

$$k = k_2 - k_2' = k_1' - k_1$$

being transferred accross the internal line (or "propagator"). The arrows in the picture indicate the direction of momentum flow. Finally, we have

$$\text{Graph} = (2\pi)^4 \delta^4(k_1 + k_2 - k_1' - k_2')(-i\lambda)^2 \frac{i}{(k_1 - k_1')^2 - m^2 + i\epsilon}$$

So, recalling $(S-1)_{f,i} = (2\pi)^4 \delta^4(p_{in} - p_{out}) i A_{f,i}$,

the corresponding contribution to A is

$$iA = (-i\lambda)^2 \frac{i}{(k_1 - k_1')^2 - m^2 + i\epsilon} + \text{other graphs}$$

Note that, as is required in general, A is a Lorentz-invariant function of the momenta.

This example illustrates that the Feynman rules give expressions that are simpler to write down in momentum space than in position space. Using our position space rules, we can, by expressing $\phi(x), \phi(y)$ in terms of its Fourier transforms, readily obtain a formulation of the Feynman rules that can be expressed directly in momentum space.

Feynman Rules (in momentum space)

For each topologically distinct diagram:

- Assign to each internal line a momentum k_j, and an arrow (direction of momentum flow).

- Assign to each external line the momentum k_α (k_α') of the corresponding incoming (outgoing) particle, and an arrow flowing into (out of) the vertex.

- For each vertex include the factor

$$(-i\lambda)(2\pi)^4\,\delta^4(k_{in}-k_{out}),$$

where $k_{(in,\,out)}$ is momentum flowing (into, out of) vertex.

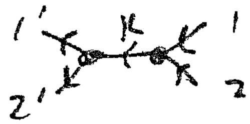
- For each internal line (momentum k_j) include

$$\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}.$$

- Divide by symmetry factor if any.

Of course, many of the k integrations will be trivial, because of the momentum-conserving δ functions at the vertices.

For example, we can apply the momentum space Feynman rules to the diagram



we have
$$(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} (2\pi)^4 \delta^{(4)}(k-k_1-k_2) \times (2\pi)^4 \delta^4(k_1'+k_2'-k)$$
$$= (2\pi)^4 \delta^4(k_1'+k_2'-k_1-k_2)\,(-i\lambda)^2 \frac{i}{(k_1+k_2)^2-m^2+i\epsilon}$$

Or, the contribution to iA is
$$iA = (-i\lambda)^2 \frac{i}{(k_1+k_2)^2-m^2+i\epsilon} + \cdots$$

It is pretty obvious that we can bypass the step of introducing $\int d^4k$ that is then killed by a δ-function. In their simplest form, the Feynman rules become --

Feynman Rules (momentum space -- simplified)

For each connected diagram:

- Assign to each external line the momentum k_α (k_α') of the corresponding incoming (outgoing) particle, and an arrow flowing into (out of) the vertex.
- Assign to each internal line the most general momentum k_j consistent with momentum conservation at the vertices (and an arbitrary direction of flow).

- For each vertex, include the factor $(-i\lambda)$
- For each internal line, include the factor $\frac{i}{k_j^2 - m^2 + i\epsilon}$
- Integrate $\int \frac{d^4k}{(2\pi)^4}$ for each momentum that is left undetermined by momentum conservation at the vertices.
- Divide by the symmetry factor.

-- These rules give an expression for the contribution to iA, with the $(2\pi)^4 \delta^4(\vec{K}_{in} - \vec{K}_{out})$ divided out.

To understand these rules, consider the diagram

First, apply the original version of the Feynman rules

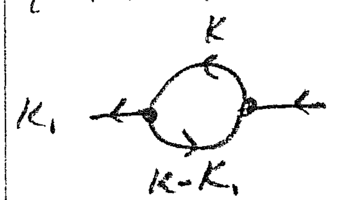
$$\text{Graph} = \left(\frac{-i\lambda}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{k'^2 - m^2 + i\epsilon}$$

(symmetry)

$$\times (2\pi)^4 \delta^{(4)}(k - k' - k_1)(2\pi)^4 \delta^4(k - k_1' - k')$$

$$= (2\pi)^4 \delta^4(k_1' - k_1) \quad \left(\frac{-i\lambda}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k - k_1)^2 - m^2 + i\epsilon}$$

Applying the new rules, we first label the momenta of the internal lines, then write down



$$\Rightarrow \left(\frac{-i\lambda}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k - k_1)^2 - m^2 + i\epsilon}$$

-- all in one step.

Graph "Topology"

The graph  , unlike  has one undetermined momentum carried by the internal lines, and therefore one nontrivial momentum integration must be carried out to evaluate the graph. In terms of the topology of the graphs, it is clear that the key distinction is that the first contains a closed loop, while the second does not. In fact, we can see that this connection between graph topology and the number of momentum integrals holds in general.

For an arbitrary connected Feynman graph, let

V: no. of vertices

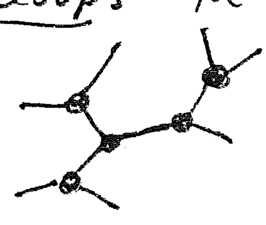
I: no. of internal lines

then there are V δ functions and I momentum integrals. Since the graph is connected, one δ function is the total energy-momentum conserving δ function, which factors out. So the number of integrations to perform is

$$(\text{no. of integrals}) = I-(V-1) = I - V + 1$$

But
$$L = I - V + 1,$$

is just the number of loops in the graph. We can easily see this by induction: For a graph with no loops ("tree graph") we always have



$I = V - 1$ (each time we add a vertex, we also add an internal line)

But to add a loop to a diagram, we take two external lines and connect them:

or

This leaves V unchanged, but increases I by one for each loop added. So

$$L = I - V + 1$$

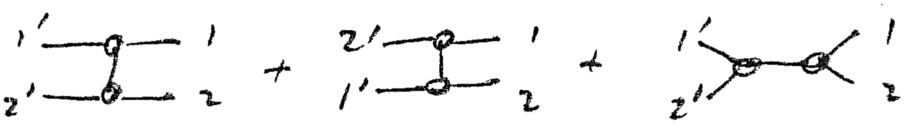
Tree graphs are essentially trivial to evaluate, but graphs with *many loops* are hard.

Scattering (Again)

Let us return now to the evaluation of the two-body scattering amplitude in the theory with

$$H' = \frac{1}{3!}\lambda\phi^3,$$

to order λ^2. We had the three Feynman diagrams

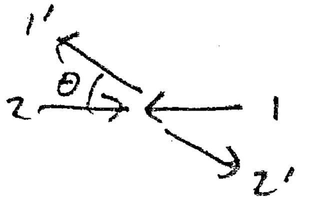


Applying the simplified Feynman rules, we can write down by inspection:

$$iA = (-i\lambda)^2 i \left[\frac{1}{(k_1-k_1')^2-m^2+i\epsilon} + \frac{1}{(k_1-k_2')^2-m^2+i\epsilon} + \frac{1}{(k_1+k_2)^2-m^2+i\epsilon} \right]$$

$+$ higher order

As we expected, the amplitude is a Lorentz-invariant function of the external momentum.



In the zero momentum frame
$E_1 = E_2 = E_1' = E_2'$ (elastic scattering)

where
$\vec{K} = \vec{K}_1 = -\vec{K}_2$

and $(K_1 - K_1')^2 = -(\vec{K}_1 - \vec{K}_1')^2$

$$= -\vec{K}_1^{\,2} - \vec{K}_1'^{\,2} + 2|\vec{K}_1||\vec{K}_2|\cos\theta$$

$$= -2K^2(1-\cos\theta)$$

Similarly, $(K_1 - K_2')^2 = -2K^2(1+\cos\theta)$

These quantities, we note, are just

$-(\vec{K}_1 - \vec{K}_3')^2 = -\vec{q}^{\,2}$ $\quad$ $\vec{q}$ = momentum transfered from 1 to 2 in collision

$-(\vec{K}_1 - K_2')^2 = -\vec{q}_c^{\,2}$ $\quad$ $\vec{q}_c$ = "cross momentum transfer"

Also $(K_1 + K_2)^2 = (E_1 + E_2)^2 = E_T^2$; E_T = total (cm) energy

And so we have ---

$$iA = -i\lambda^2\left[\frac{-1}{\vec{q}^{\,2}+m^2} + \frac{-1}{\vec{q}_c^{\,2}+m^2} + \frac{1}{E_T^2 - m^2}\right]$$

What is the interpretation of the three terms?
We recall that

$$\frac{1}{\vec{q}^{\,2}+m^2} = \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\, V(\vec{x}), \text{ where } V = \frac{e^{-mr}}{4\pi r}$$

Thus $\dfrac{\lambda^2}{\vec{q}^{\,2}+m^2}$ is precisely what we would get in nonrelativistic quantum mechanics in the first Born approximation, for scattering off a Yukawa potential (the potential induced by exchange of a single scalar). The $\frac{1}{\vec{q}_c^{\,2}+m^2}$ term would also be present in nonrelativistic quantum mechanics, since the two final state particles are identical.

The last term, $A \sim \frac{-\lambda^2}{E_T^2 - m^2}$, does not have a nonrelativistic analog; it generates poles in the amplitude at $E_T = \pm m$.

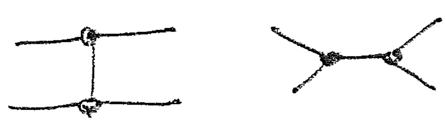
Such poles do occur in the Born series, but not until higher order --

$$A \sim \langle \vec{k}'|V|\vec{k}\rangle + \sum_n \frac{\langle \vec{k}'|V|n\rangle\langle n|V|\vec{k}\rangle}{E_n - E_{\vec{k}}} + \cdots$$

(energy denominators) This is a pole (rather than a cut) only for isolated energy eigenstates $|n\rangle$.

Furthermore, poles at $E_T = mc^2$ go off to ∞ in the nonrelativistic limit $c \to \infty$.

The diagrams invite a metaphoric description of scattering processes. We sometimes say that scattering



is mediated by the "exchange of virtual particles,"

or that 1 and 2 combine to form a "virtual particle" that subsequently decays (after "propagating" for a while) to 1' and 2'. The intermediate states that occur have the same energy and momentum as the initial (and final) state, but they are not on the mass shell. It is in this sense that these particles are merely "virtual."

This language is a useful way of describing the Feynman diagrams, but it is important to recognize that it is only metaphorical; it can be misleading to take the existence of virtual particles too literally.

In nonrelativistic two body scattering, the "intermediate states" that occur in the Born expansion have the same momentum as the initial and final states, but different energy. In covariant perturbation theory, we insist on treating energy and momentum on the same footing, so both energy and momentum are conserved by the "virtual particles" (at the expense of allowing the particles to leave the mass shell).

The presence of poles in scattering amplitudes can be understood without appealing to perturbation theory at all. They are just a consequence of unitarity. More about this later.

Comment on Homework Exercise (Vacuum Energy Divergence)

In homework, we considered model $H' = \frac{1}{2}\lambda \rho(x)\, \phi(x)^2$

This has vac to vac amplitude

$$\langle 0|S|0\rangle = \exp\Big($$  $$+ \cdots\Big)$$

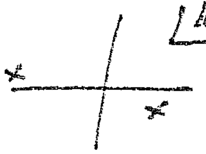
These are both divergent

$\ln \langle 0|S|0\rangle$ has a real part that is constrained by unitarity, and an imaginary part that just contributes a meaningless phase to $\langle 0|S|0\rangle$

We are free to subtract away this annoying phase

E.g.

 $$= -\frac{i\lambda}{2}\int d^4x\, \rho(x)\, \phi(x)\phi(x) = -\frac{i\lambda}{2}\tilde{\rho}(0)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}$$



Do k^0 integral by contour integration

 $$= -\frac{i\lambda}{2}\tilde{\rho}(0)\underbrace{\int\frac{d^3k}{(2\pi)^3\, 2\omega_k}}_{\Delta_+(0) = \infty}$$

In the case of a static source on for a time T, this is

 $$= -iE_0 T = -\frac{i\lambda}{2}T\int d^3x\, \rho(\vec{x})\, \Delta_+(0)$$

or $$E_0 = +\frac{\lambda}{2}\int d^3x\, \rho(\vec{x})\left[\phi^{(-)}(x), \phi^{(+)}(x)\right]$$

this is just the infinite contribution to the vacuum energy that we can remove by normal ordering the interaction

$$H' = \frac{\lambda}{2}\int d^3x\, \rho(\vec{x})\left(:\phi^2(x): + \left[\phi^{(-)}(x), \phi^{(+)}(x)\right]\right)$$

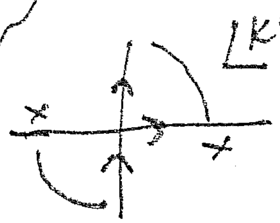
(The effect of normal ordering is to remove from the Wick expansion all contractions of two fields at the <u>same</u> <u>vertex</u>.)

The infinite part of [diagram] is also imaginary, and removing it requires more than mere normal ordering

$$= \frac{1}{4}(i\lambda)^2 \int \frac{d^4K}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} |\tilde{\rho}(p)|^2 \frac{i}{K^2-m^2+i\epsilon}\,\frac{i}{(K+p)^2-m^2+i\epsilon}$$

The K-integral diverges for $K\to\infty$ (ultraviolet)
Expand integrand in powers of p

$$\rightsquigarrow \frac{\lambda^2}{4}\left(\int \frac{d^4p}{(2\pi)^4}|\tilde{\rho}(p)|^2\right)\left(\int\frac{d^4K}{(2\pi)^4}\frac{1}{(K^2-m^2+i\epsilon)^2}\right) + \text{finite}$$



In the K^0 plane, singularities are as shown. Can rotate integration contour $K^0 \to iK^0$

So infinite constant $= i\int \frac{d^4K_E}{(2\pi)^4}\frac{1}{(K^2+m^2-i\epsilon)^2}$

so divergence is imaginary

As for the phase $\exp(\ell)$ we are free to remove this infinite constant by adding a counterterm to the action

$$\text{Action} = S_0 - \int d^4x\, \tfrac{1}{2}\rho(x)\phi(x)^2 \underbrace{-\,c.t.}$$

(a real c-no. added to the action)

And c.t. is chosen so that

 = finite

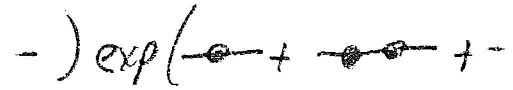
counterterm

Factor $\int \frac{d^4p}{(2\pi)^4}|\tilde{\rho}(p)|^2 = \int d^4x\, \rho(x)^2 = T\int d^3x\, \rho(\vec{x})^2$
in case of a static source

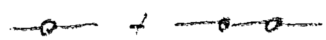
2.49

The exact S-matrix in this problem is

$S = \exp(\Sigma \text{ connected})$

$= \exp($  $+ \cdots) \exp($  $+ \cdots)$

As in the theory with $H' = \lambda\rho(x)\phi(x)$, the first factor is a number whose absolute value is completely determined by unitarity, but its phase is not. (If we add the counterterm that exactly cancels , then

$\langle k' | S | k \rangle =$  to order λ^2.)

We are free to subtract away this irrelevant phase, without in any way modifying the physics of the theory.