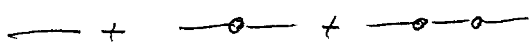
That is, we are entitled to demand that the amplitudes be <u>independent</u> of an arbitrary choice of how we divide up the Lagrangian into a "free" part and an interacting part. E.g.

$$\mathcal{L} = Z\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + \cdots$$
$$= \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + (Z-1)\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \cdots$$

In the first case, the propagator is $\dfrac{i}{Zp^2-m^2}$

In the 2nd case, if we sum up <u>insertions</u> of the interaction, we have



$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\, i(Z-1)p^2\, \frac{i}{p^2-m^2} + \cdots$$

$$= \frac{i}{p^2-m^2+(Z-1)p^2} = \frac{i}{Zp^2-m^2}$$

-- which agrees with the other expression, and checks the Feynman rule.

Coupling Renormalization

At one point in the above discussion, we replaced the coupling λ by

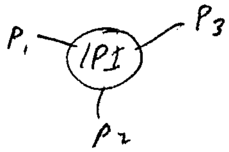
$$\lambda' = Z^{3/2}\lambda,$$

without making special comment. Obviously we are free to do this. λ is a free parameter, and we can always replace it by some function times λ, and express our amplitudes as functions of λ' rather than λ. <u>Physics</u> (e.g. relations among measurable quantities) cannot be affected when we trade in λ for λ' (just as it is unaffected when we choose to compute in terms of m^2 rather than m_0^2.

Having made this observation, we may exploit it further. It may be preferable (more convenient) to use as our expansion parameter in perturbation theory, not λ or λ', but some other quantity that can be directly measured.

For example consider the function Γ defined by

$$-i\Gamma(p_1^2, p_2^2, p_3^2) =$$



$(p_1+p_2+p_3=0)$

-- so e.g. $p_1 \cdot p_2$ is not an independent invariant

We may define a renormalized coupling λ_R by

$$\lambda_R = \Gamma(m^2, m^2, m^2)$$

-- the exact three particle coupling, when all three particles are on-shell. (There is, of course, much arbitrariness in this definition)

If we now wish to compute (properly normalized) amplitudes as functions of renormalized coupling λ_R and physical mass m, we carry out a procedure that is starting to become familiar:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - \tfrac{1}{2} m_0^2 \phi_B^2 - \tfrac{1}{3!}\lambda_0 \phi_B^3$$

-- bare field, mass, coupling

$$= \tfrac{1}{2}\partial_\mu \phi_R \partial^\mu \phi_R - \tfrac{1}{2} m^2 \phi_R^2 - \tfrac{1}{3!}\lambda_R \phi_R^3 + \mathcal{L}_{\text{counterter}}$$

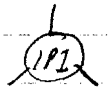
-- renormalized field, mass, coupling

where

$$\mathcal{L}_{\text{counterterm}} = \tfrac{1}{2}(Z-1)[\partial_\mu \phi_R \partial^\mu \phi_R \overset{-m^2\phi_R^2]}{\wedge} - \tfrac{1}{2} Z(\delta m^2)\phi_R^2 - \tfrac{1}{3!}\delta\lambda \phi_R^3$$

and
$$\phi_R = Z^{-\frac{1}{2}}\phi_B$$
$$m_0^2 = m^2 + \delta m^2$$
$$Z^{3/2}\lambda_0 = \lambda_R + \delta\lambda$$

Now order by order in perturbation theory, we may determine Z, δm^2, and $\delta\lambda$, by imposing three renormalization conditions:

$-i\Pi =$  $\qquad -i\Gamma =$ 

$$\Pi(m^2) = \Pi'(m^2) = 0 \qquad \Gamma(m^2, m^2, m^2) = \lambda_R$$

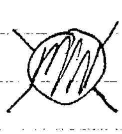
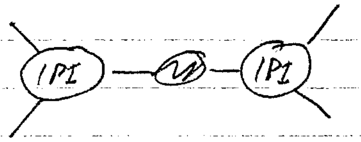
So, for example,

$-i\Gamma =$  $+$  $+$  $+ \; O(\lambda_R^5)$

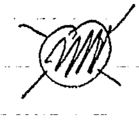
$-i\lambda_R$ $\qquad$ where this is order λ_R^3 expression for $\delta\lambda$, which is chosen to cancel  "on-shell"

This renormalized coupling can be "sort of measured"; it can be extracted from $2\to2$ scattering.

Since

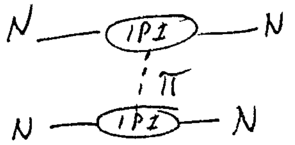
 $=$  $+$ crossed graphs $+$ 

(Recall remark on p 2.96 -- we can ignore radiative corrections to external legs.), this amplitude, with external legs on shell has the form


$$= \underbrace{\frac{(-i\lambda_R)^2 i}{s-m^2} + \frac{(-i\lambda_R)^2 i}{t-m^2} + \frac{(-i\lambda_R)^2 i}{u-m^2}} + \left(\begin{matrix}\text{non-pole}\\ \text{terms}\end{matrix}\right)$$

that is, each pole has this form as s, t, u approach pole

-- so $-i\lambda_R^2$ is the exact residue of each pole. Of course, the poles do not occur for physical values of the momenta, so their residues can be determined only by a "continuation" of the measured amplitude away from physical momenta.



(The pion-nucleon coupling is actually "measured" by such an analysis of NN scattering, and is found to be $\lambda_R \sim 14$; bad news for perturbation theory!)

Coupling renormalization proves convenient for much the same reason that mass renormalization does. The "fully quantum mechanical" coupling λ_R of three on-shell particles is not the same as the "classical" coupling λ_0. So it is sensible to compute in terms of the quantity λ_R that is accessible to experiment, rather than λ_0, which cannot be directly measured.

Nor should we be surprised that the exact coupling λ_R is not the same as the classical coupling λ_0. The vacuum of the quantum system is the ground state of a set of coupled oscillators, and the particles are able to interact "indirectly", by exciting an oscillator that then acts on another particle. There is no reason why the classical description of this process should be exact.

Infinite Renormalization

In our theory with interaction $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$, there are three types of connected diagrams that occur in order λ^2:

Tree

One Loop

Two Loop

So far, we have calculated only the tree graphs. I have been avoiding calculating the other graphs because they are *infinite*! Now we must finally confront these infinities, and decide how to deal with them.

Vacuum Energy

Actually, as we've discussed previously, the graph with no external lines is not a big problem. It just contributes a physically irrelevant overall phase to the S-matrix

$$S = \exp(\Sigma\,\text{connected}) = \exp(\ominus + --)$$

It can be interpreted as a (divergent) contribution to the vacuum energy density, since

$${}_{out}\langle 0|0\rangle_{in} = \langle 0|S|0\rangle = \exp\left(\Sigma \begin{smallmatrix}\text{connected, with}\\ \text{no external legs}\end{smallmatrix}\right)$$

$$\sim \exp(-iE_0T)$$

(If we turn the interaction on and off adiabatically, allowing it to persist for time T.) The vacuum energy

An S-matrix element with no external lines is proportional to $(2\pi)^4\delta^{(4)}(0)$ -- which we can interpret as the volume of spacetime. Or, returning to our *position space* Feynman rules --

$$= \frac{(-i\lambda)^2}{12}\int d^4x\,d^4y\,(\overbrace{\phi(x)\,\phi(y)})^3$$

(symmetry factor) But the contraction is translation invariant, so

$$= \frac{(-i\lambda)^2}{12}(VT)\int d^4x\,(\overbrace{\phi(x)\,\phi(0)})^3 = -i\mathcal{E}_0 VT$$

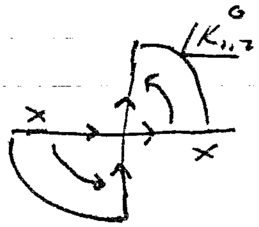
spacetime volume

vacuum energy density

We thus find that the vacuum energy density is

$$\mathcal{E}_0 = \frac{-i\lambda^2}{12}\int d^4x\,(\overbrace{\phi(x)\,\phi(0)})^3 \quad + \text{ higher order}$$

$$= \frac{-i\lambda^2}{12}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\,\frac{i}{(k_1^2-m^2+i\epsilon)}\,\frac{i}{(k_2^2-m^2+i\epsilon)}\,\frac{i}{((k_1+k_2)^2-m^2+i\epsilon)}$$



Just to check that the vacuum energy is real, note that the singularities of the integrand in the k_1^0 and k_2^0 planes are as shown (because of the $i\epsilon$) So the contour can be rotated to the imaginary axis as shown, without encountering any singularities (and with no contribution from the circle at infinity). So we can replace

$$k^0_{12} \to i k^0_{12},$$ (This is called a "Wick Rotation")

and forget about the $i\epsilon$.

$$\mathcal{E}_0 = \frac{\lambda^2}{12}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\,\frac{1}{k_1^2+m^2}\,\frac{1}{k_2^2+m^2}\,\frac{1}{(k_1+k_2)^2+m^2},$$
$$+ \text{ higher order}$$

where now k^2 has a Euclidean metric. We see that $\mathcal{E}_0 > 0$

But we also see that $\mathcal{E}_0$ is ∞. To make sense of the integrations, introduce an "ultraviolet cutoff" Λ. This means that we neglect the effects of quantum fluctuations with wave number larger than Λ. To impose such a cutoff in a way consistent with Lorentz invariance, we restrict our loop integrations to

$$K^2 < \Lambda^2$$

where K is "Euclideanized" 4-momentum.

It is obvious just from counting powers of K that the integrals are dominated by large values of K, and we have

$$\mathcal{E}_0 = C\lambda^2\Lambda^2 + \text{higher order},$$

where C is a (positive real) number that we could determine if we were really interested. Evidently, the vacuum energy is dominated by very short wavelength fluctuations. But this happened even in free field theory! It did not bother us before, and should not bother us now. The vacuum energy has no physical relevance, but we may, if we wish, introduce a counterterm

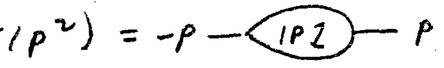
$$\otimes$$

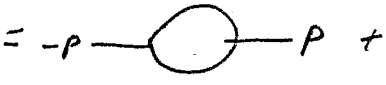
(an additive numerical constant in H) to subtract it away.

(This correction to the vacuum energy becomes interesting only when we consider coupling our theory to gravity -- see comment on page 1.14.)

Infinite Mass Renormalization

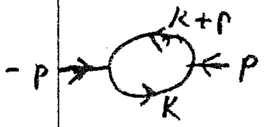
Let us now consider, in our model with $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$, the leading contribution to the quantity

$-i\Pi(p^2) =$ 

$=$ 

$+$ higher order,

where --



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k+p)^2-m^2+i\epsilon}$$

(symmetry factor)

We can immediately see that this integral is divergent. For large k, it behaves like

$$\int \frac{d^4k}{k^4}.$$

So it will proportional to the log of the ultraviolet cutoff:

 $\sim \ln\Lambda +$ finite part

-- we say that the ultraviolet divergence is "logarithmic"

We can also easily see, without doing an explicit calculation, that the "infinite" part of this diagram (the piece proportional to $\ln\Lambda$) is independent of the external momentum p. To see this differentiate the graph with respect to external momentum:

$\frac{\partial}{\partial p_\mu}$  $= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i2(p+k)_\mu}{((k+p)^2-m^2+i\epsilon)^2}$

Now counting powers of k shows that the integral converges. This shows that if the graph is expanded in powers of p^2 (about any value of p^2) in a Taylor series, only the constant term is divergent (proportional to $\ln \Lambda$).

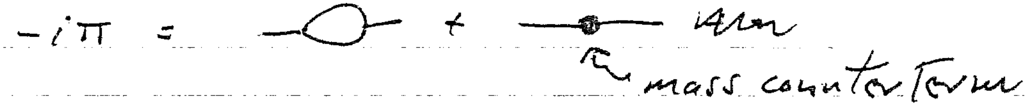
Therefore, without explictly calculating, we know that

$$\pi(p^2) = \lambda^2[C \ln \Lambda + \text{"finite part"}] + \text{higher order},$$

where C is a constant, independant of p^2

(The argument of the log should be dimensionless, so the finite part must include a term proportional to $\ln m$. Also, since the integral is dimensionless, C must be a numerical constant, independant of m)

Now suppose we perform mass renormalization, as described previously. That is, we include in $\pi(p^2)$ the contribution from the mass counterterm:



The mass counterterm is chosen so that $\pi(p^2)=0$ And thus we see that

$$\delta m^2 = -C\lambda^2 \ln(\Lambda/m) + \text{finite part} + \text{higher order}$$

The infinity (the $\ln\Lambda$ dependence) gets completely absorbed into the relation between the bare and physical masses,

$$m_0^2 = m^2 + \delta m^2 = m^2 - C\lambda^2 \ln(\Lambda/m) + --,$$

and the expression for π as a function of the physical mass m^2 is finite.

(Furthermore, since $\pi'(p^2)$ is finite, we see that the wave function renormalization is finite -- i.e. Z has no dependance on Λ.)

What is the meaning of this logarithmic divergence? It occurs because, for $k^2 >> p^2, m^2$, the quantum fluctuations have *no* *intrinsic scale*. If λ is small, the quantum corrections are in a sense small, but fluctuations on *all* *scales of length* contribute. For each factor of two, say, in distance scale, the contribution is small if λ is small, but because we get the *same* contribution from each factor of 2 in distance scale, the total contribution from fluctuations on *all* scales is infinite.

Evidently, then, even the low energy ($p^2 << \Lambda^2$) physics of our model is sensitive to the fluctuations at arbitrarily small wavelengths. This sounds like bad news. Because it may appear that, in order to predict low energy physics, we need to understand physics at arbitrarily short distances. But we can *never* expect *any* model field theory to provide an accurate description of physics at arbitrarily short distances (QED, for example, is not a valid description above 100 GeV, let alone 10^{19} GeV!) Since we can't really *know* what the physics is at very short distances, sensitivity to very short wavelengths appears to mean loss of predictive power.

But the procedure described above indicates that this need not be so! When we perform mass renormalization, we obtain expressions for amplitudes in terms of the *physical* mass, and these expressions are insensitive to the short-wavelength physics. If we wish to relate different measured "low energy" quantities, no dependence on Λ appears. All the sensitivity to short wavelengths can be isolated in the dependence of the physical parameters on the parameters in the "Hamiltonian of the World," e.g., the dependence of the physical mass on the bare mass

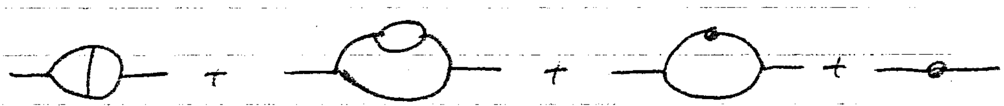
That all of our ignorance about short-wavelength physics can be absorbed into the relation between bare and renormalized parameters (and isolated from the predictions of relations among measurable quantities) is the crucial idea in the theory of renormalization. Indeed, it might even be regarded as the central concept of (relativistic) quantum field theory, because it is only when we grasp this concept that we recognize that we need not be so arrogant as to suppose that we understand physics at the Planck scale, in order to understand physics at 100 GeV.

Higher Orders

The "ultraviolet behavior" of our $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$ model is actually quite simple, because the sensitivity to short wavelengths becomes milder and milder in higher orders of perturbation theory in λ.

For example, consider the next order contribution to Π

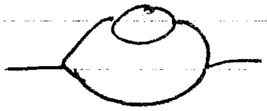
$$-i\Pi\Big|_{\text{order }\lambda^4} =$$



The behavior of the loop integrations in  for large loop momenta is evidently

$$\int d^4k_1\, d^4k_2\ \frac{1}{k_1^4}\ \frac{1}{k_2^4}\ \frac{1}{(k_1+k_2)^2},$$

and we can see by counting powers of K that this is convergent for large K. So the only cutoff dependence in π in order λ^4 comes from --



But we have already taken care of this with the order λ^2 mass counterterm. There is no new contribution to the infinite part of mass renormalization in order λ^4!

This situation persists in higher orders. Consider a general diagram with

L	loops
I	internal lines
E	external lines
V	vertices

The dimensionality of the loop integration (called the "superficial degree of divergence" of the diagram) is

$$D = 4L - 2I$$

But recall the topological identities:

$L = I - V + 1$ (the no. of integrations is the no. of propagators minus V for the momentum-conserving δ-function at each vertex, plus 1, since the overall momentum conserving δ-function can be factored out of the graph.)

$E + 2I = 3V$ (Both sides give the total no. of ends of lines that are absorbed by vertices.)

Therefore, $D = 4(I-V+1) - 2I = 2I - 4V + 4$
$= 3V - E - 4V + 4$

or $D = 4 - E - V$

The number of superficially divergent diagrams is small
For $D \geq 0$, we need

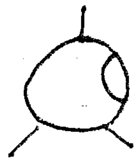
$E = 0, 1, 2, 3$
$V \leq 4, 3, 2, 1$

Vacuum diagrams are divergent up to order λ^4, but the only superficially divergent diagrams with external lines are

, , 

In particular,  is (superficially) finite

-- thus all the infinities in it come from graphs like



These infinities are removed by an insertion of the order λ^2 mass counterterm.

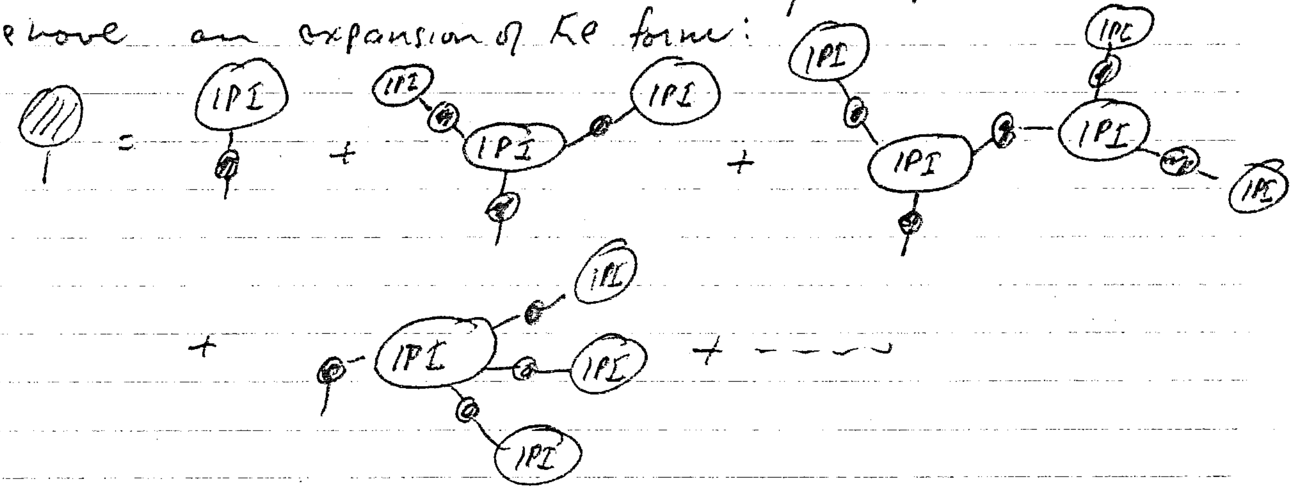
Thus, this theory has no infinite coupling (or field) renormalization. The only infinite renormalizations are vacuum energy renormalization in order λ^2 and λ^4, mass renormalization in order λ^2, and the renormalization of  in order λ and λ^3.

(More about this "tadpole" later.)

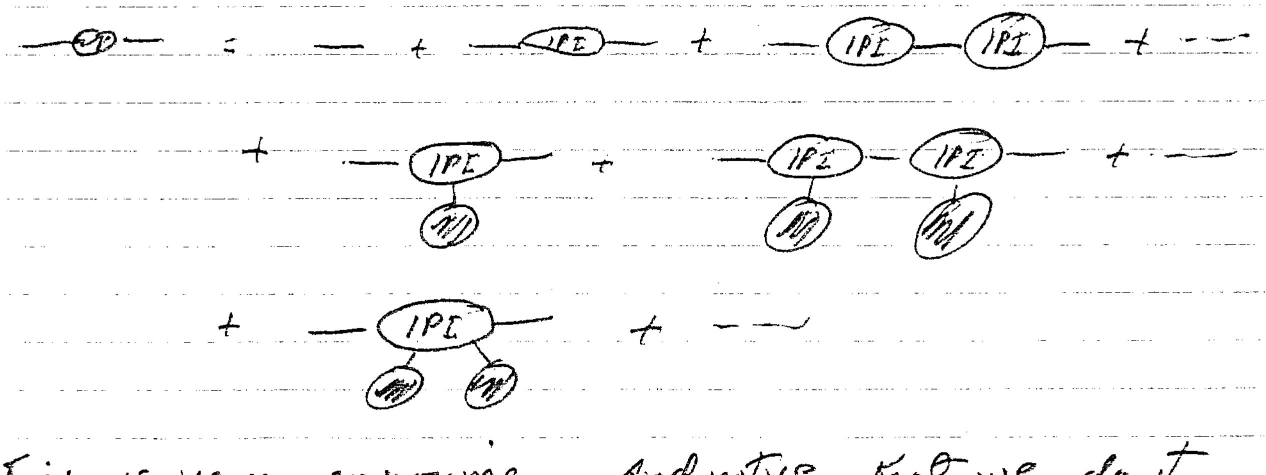
The "Tadpole"

Let  represent the sum of Feynman diagrams with one external line. (The external line necessarily carries zero four momentum, by momentum conservation.) Call it the tadpole.

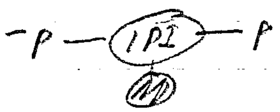
Tadpole graphs are not necessarily one-particle irreducible. We have an expansion of the form:



And if the tadpole is nonvanishing, our expression for the exact propagator ought to include two types of terms



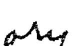
This is very annoying. And notice that we don't remove all these tadpole contributions by performing mass renormalization. The counterterm that cancels, for example



at $p^2=m^2$ does not cancel it at all p^2, since $\Gamma(p^2,p^2,0)\neq$ constan

It is therefore very convenient to introduce yet another (and our final) renormalization condition. Namely

 $= 0$

This determines a counterterm  $= -iC(\lambda, m^2)$ order by order in perturbation theory. Our bare Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_B)^2 - \frac{1}{2} m_0^2 \phi_B^2 + C\phi_B - \frac{1}{3!}\lambda_0 \phi_B^3$$

Notice that C is physically irrelevant in the classical theory, because it can be removed by "shifting" the field. If $\phi \to \phi + a$, then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_B)^2 - \frac{1}{2}(m_0^2 + \lambda_0 a)\phi_B^2 + \left(C - am_0^2 - \frac{\lambda_0}{2}a^2\right)\phi_B - \frac{1}{3!}\lambda_0 \phi_B^3 + \text{constant}.$$

We can choose a so that linear term now vanishes. We see that introducing C is just a convenient reparametrization of the bare theory, which allows us to dispense with tadpole insertions.

Notice also that it is enough to have  $= 0$ to ensure  $= 0$. So we find C order by order by demanding

 $= 0$,

and be assured that all tadpole insertions can be ignored. In practice, we never have to bother with computing C; it is really physically irrelevant.

This behavior of the superficial degree of divergence is easy to understand; it follows from simple dimensional analysis. Recall that for relativistically normalized states

$$\langle p | p' \rangle = (2\pi)^3 2p^0 \delta^3(\vec{p}-\vec{p}\,') \sim [M]^{-2}$$

So $|p\rangle$ has dimension -1 (that is $(\text{mass})^{-1}$) and

$|n \text{ particles}\rangle$ has dimension $-n$

Since the operator S is dimensionless

$\langle E-n \text{ particles} | S | n \text{ particles} \rangle$ has dimension $-E$.

And after we factor out $\delta^{(4)}(P_{total})$

$A_{(E \text{particles})}$ has dimension $4-E$

Now, what is the dimension of λ?

$$\text{Action} = \int d^4x \left[\tfrac{1}{2}\partial^\mu \phi\, \partial_\mu \phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{3!}\lambda\phi^3\right]$$

is dimensionless (in our units with $\hbar = c = 1$), so

ϕ has dimension 1 (or $\frac{d-2}{2}$, in d dimensions)

and, since $\lambda\phi^3$ must have dimension 4,

λ has dimension 1

Thus, dimensional analysis tells us that a contribution to A due to a Feynman diagram with V vertices is

$$A_{(E \text{particles})} \sim \lambda^V \left(\text{Integral of dimension } 4-E-V\right)$$

It is the <u>dimension</u> of the integral that determines D.

We see from this argument that the reason for the mild ultraviolet behavior of this theory is that the coupling constant λ has the dimensions of mass to a <u>positive</u> power. If we add a loop to a Feynman diagram, keeping the number of external lines fixed, we must add two vertices (since $V = 2L + E - 2$). Dimensional analysis thus shows that adding a loop decreases the dimension of the Feynman <u>integral</u>, and correspondingly reduces its <u>degree of</u> divergence.

In general, in a theory in which the coupling constant has <u>positive</u> dimension, there are a finite number of "primitive" divergences, and hence infinite renormalizations cease to be necessary after some finite order of perturbation theory. Such theories are said to be "<u>superrenormalizable</u>".

More interesting are the <u>renormalizable</u> theories, in which the coupling constant is dimensionless. In renormalizable theories, the primitive degree of divergence of a graph with a fixed number of external lines remains the same to all orders of perturbation theory, and thus the need for infinite renormalization persists to all orders.

Examples of renormalizable theories are scalar field theories with the interactions...

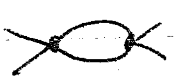
$\mathcal{H}' = \frac{1}{3!}\phi^3$ -- in 6 dimensions (dim $\phi = 2$)

$\mathcal{H}' = \frac{1}{4!}\phi^4$ .. in 4 dimensions (dim $\phi = 1$)

$\mathcal{H}' = \frac{1}{6!}\phi^6$ .. in 3 dimensions (dim $\phi = \frac{1}{2}$)

(Any polynomial interaction in ϕ is superrenormalizable in 2 dimensions, since ϕ is dimensionless.)

For example, in the four-dimensional theory with $\mathcal{H}' = \frac{\lambda}{4!}\phi^4$, the graph

 $\sim \lambda^2 \ln \Lambda + \cdots$

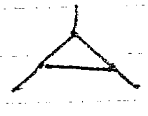
is logarithmically divergent, and hence generates (logarithmically) infinite coupling renormalization in order λ^2. The graph

 is quadratically divergent:

$$\sim \lambda^2 \left(\Lambda^2 + (p^2 - m^2) \ln \Lambda + \cdots \right)$$

It generates (quadratically) divergent mass renormalization, and (logarithmically) infinite field renormalization.

Dimensional analysis shows that coupling and field renormalization are *always* logarithmic, and mass renormalization is always quadratic in a renormalizable (scalar) field theory. E.g. consider the 6 dimensional theory with $\mathcal{H}' = \frac{\lambda}{3!}\phi^3$:

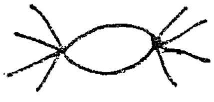
 is logarithmically divergent

 is quadratically divergent

A crucial feature of renormalizable and superrenormalizable theories is that primitive divergences occur only in graphs with some small number of external lines. This feature allows us to isolate the effects of short-wavelength quantum fluctuations in a few incalculable "renormalized parameters." This nice feature is not shared by theories in which the coupling constant has negative dimension, the so-called "*nonrenormalizable*" theories. In these theories, the ultraviolet convergence properties of Feynman diagrams get *worse and worse* in each order of perturbation theory.

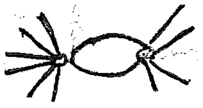
For example, consider a 4-dimensional theory with

$$H' = \frac{\lambda}{6!}\phi^6$$

We find that the diagram...  is logarithmically divergent. To remove this sensitivity to short-wavelengths, we need a ϕ^8 counterterm (since the graph has 8 external lines) Our interaction has become

$$H' = \frac{\lambda_6}{6!}\phi^6 + \frac{\lambda_8}{8!}\phi^8$$

— where now there are two free parameters λ_6 and λ_8. But we find that

 is now divergent.

To absorb *this* cutoff dependence into the relation between a bare and renormalized coupling, we must introduce a ϕ^{10} coupling. It is evident that we can absorb all the sensitivity to short-wavelength physics into our renormalized parameters only at the cost of introducing an infinite number of parameters. Thus, the theory has lost its predictive power. In nonrenormalizable theories, sensitivity to short wavelengths *does* destroy our ability to predict low energy physics.

So, in renormalizable and superrenormalizable theories, all the sensitivity to short-wavelength physics can be absorbed into a few parameters, which we can take to be the free parameters of the theory. And dimensional analysis determines what these parameters are. (In a renormalizable theory, they correspond to all the local terms in fields and derivatives of fields that have dimension less than or equal to D, in D spacetime dimensions.) In nonrenormalizable theories this procedure fails.

But, why are we so lucky that the world happens to be well-described by a renormalizable field theory (like QED)? Or did it have to be this way?

In fact, there is a very good general reason to believe that low energy physics can be described to excellent accuracy by a renormalizable field theory. To understand this, let us take the idea of a cutoff very seriously. That is, we will accept the idea that our scalar field theory is the mere "low energy phenomenology" of some more fundamental underlying field theory. The description of physics in terms of the Lagrangian

$$\mathcal{L}(\phi(x), \partial_\mu \phi(x), --)$$

is only approximate, and becomes inappropriate at wavelengths much smaller than Λ^{-1}, where Λ is the cutoff. This could happen for various reasons. Perhaps there are new elementary particles, with masses of order Λ. Or perhaps our scalar particles are not really elementary, but have a size of order Λ^{-1}, so that physics at shorter distances must be described in terms of their constituents.

When we adopt this point of view, there is no reason to expect the "phenomenological" Lagrangian to be particularly simple. It ought to be local, Poincaré-invariant, and should respect whatever other exact symmetries the theory has, but it need not be a polynomial in ϕ and derivatives, and could depend on derivatives higher than the first. Such a Lagrangian, in a four-dimensional theory of a real scalar field, when expanded in powers of ϕ, might have the form

$$\mathcal{L}_\Lambda = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M_0^2\phi^2 - \frac{1}{4!}\lambda_4\phi^4 - \frac{1}{6!}\tilde{\lambda}_6\phi^6$$

$$- \frac{1}{4}\tilde{h}(\partial_\mu\phi\partial^\mu\phi)\phi^2 + \cdots$$

(assuming a $\phi \to -\phi$ symmetry).
This is the "bare" Lagrangian of a theory with cutoff Λ; it is very complicated.

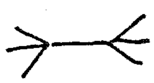
But now we come to a crucial point: the coefficients of the operators in $\mathcal{L}$ that are of dimension > 4 have dimensions of mass to a negative power. Since these coefficients are determined by physics at mass scale Λ and above, dimensional analysis would indicate that, for example,

$$\tilde{\lambda}_6 = \lambda_6 \Lambda^{-2},$$

where λ_6 is a dimensionless number expected to be of order one (it has no reason to be very large)

Now, imagine that we compute Feynman graphs using $\mathcal{L}_\Lambda$, with the understanding that all loop integrations are cut off at $k^2 = \Lambda^2$, and all external momenta p obey $p^2 \ll \Lambda^2$

We can see, just as a consequence of dimensional analysis, that the operators of dimension greater than 4 in $\mathcal{L}_\Lambda$ give a small contribution to the graphs, a contribution suppressed by powers of p^2/Λ^2. For example, consider graphs with 6 external lines:

 $\sim \lambda_4^2/p^2$

 $\sim \lambda_6/\Lambda^2$ -- suppressed by p^2/Λ^2

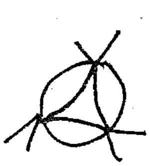
Similarly:

 $\sim \lambda_4^3 \int \frac{d^4k}{(k^2+p^2)^3} \sim \lambda_4^3/p^2$ (ignoring masses)

 $\sim \lambda_4 \frac{\lambda_6}{\Lambda^2} \int^\Lambda \frac{d^4k}{(k^2+p^2)^2} \sim \frac{\lambda_4\lambda_6}{\Lambda^2} \ln \Lambda^2/p^2$

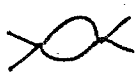
-- suppressed by p^2/Λ^2, up to a log

Adding more λ_6 interactions makes graphs more divergent, but the extra powers of Λ^2 from the loop integrations are compensated by extra powers of $1/\Lambda^2$ from the coupling. E.g.

 $\sim \left(\frac{\lambda_6}{\Lambda^2}\right)^3 \left(\int^\Lambda d^4k\right)^4 \frac{1}{(k^2)^6} \sim \frac{\lambda_6^3}{\Lambda^2}$

The dimension greater than four couplings are said to be "irrelevant" in the infrared. Their effects are suppressed by powers of p^2/Λ^2 relative to those of the renormalizable couplings.

There is an exception to this observation though. Namely, consider Feynman graphs with four or fewer external lines:

 $\sim \lambda_4^2 \int^{\Lambda} \frac{d^4K}{(K^2+p^2)^2} \sim \lambda_4^2 \ln\left(\Lambda^2/p^2\right)$

 $\sim \left(\frac{\lambda_6}{\Lambda^2}\right)^2 \left(\int^{\Lambda} d^4K\right)^3 \frac{1}{(K^2)^4} \sim \lambda_6^2$

--same order, up
to a logarithm

But this effect of the "irrelevant" couplings, which is not suppressed by powers of p^2/Λ^2, can simply be absorbed into the definition of the renormalized coupling $(\lambda_4)_R$.

Thus, up to an accuracy of order p^2/Λ^2, our very complicated bare theory can be replaced by a renormalized theory, with just one renormalized coupling and a renormalized mass as free parameters. The relation between the bare Lagrangian and the renormalized Lagrangian is extremely complicated. But if we are interested in low energy physics at $p^2 << \Lambda^2$, we may compute to good accuracy using a simple renormalizable field theory (the most general renormalizable theory of one scalar field, with $\phi \to -\phi$ symmetry)

The shift in viewpoint that leads to the conclusion that nonrenormalizable theories are not without predictive power after all comes about when we take the idea of a physical short distance cutoff seriously, for we then recognize that the coupling constants should be functions of the cutoff, and scale roughly as indicated by dimensional analysis. The essential physics is that a "decoupling" of long-wavelength physics from complicated details of short-wavelength physics should occur no matter how complicated the short-wavelength physics is.

This new viewpoint is very useful in condensed matter physics as well as relativistic quantum field theory. It indicates that the long-wavelength behavior of any system should have a simple description, however complicated the microscopic physics. This idea is the key to the modern theory of second order phase transitions, for instance, as developed by Ken Wilson. (The idea is called "Universality".)

And it is, perhaps, THE MOST IMPORTANT IDEA IN PARTICLE PHYSICS. Because it enables us to understand why renormalizable quantum field theories work as a description of Nature, even though we may have no idea how things really are at extremely short distances. (And it explains, for example, why we can use QED at energies well below 100 GeV, even though to describe things accurately at energies of order 100 GeV, we need the Weinberg-Salam model.)

You might have noticed one implication of the above analysis. We ought to regard the mass term in the bare Lagrangian as just another coupling, and write

$$m_0^2 = \Lambda^2 \lambda_2$$

The same logic as above indicates that λ_2 is order one. And we therefore expect that the bare mass of our scalar is of order Λ^2

This is right, of course: —(1PI)— will be quadratically divergent, by dimensional analysis. We need a mass counterterm of order Λ^2.

But, in the phenomenological Lagrangian point of view, we are given a bare theory at scale Λ, determined by some underlying microscopic physics, and this bare theory determines the physical mass in turn. Why should it turn out that

$$(m^2)_{physical} << \Lambda^2 \quad ?$$

This seems to require an incredible conspiracy among all the bare couplings of the theory. Putting it differently, it requires that the bare mass be "fine tuned" to an accuracy of $m^2_{physical}/\Lambda^2$, since

$$m^2_{physical} = m_0^2 + O(\Lambda^2)$$

is small by virtue of a cancellation between two quantities of order Λ^2.

(This is a genuine problem. One says (beginning with Ken Wilson) that

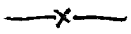
"Elementary scalars are <u>unnatural</u>."

A scalar particle wants to acquire a mass of order the cutoff. Things are different with fermions and vector mesons, because their masses can be forbidden by symmetries. There is, in fact, a symmetry called "supersymmetry" that can require that a scalar be massless, and our best reason for believing that supersymmetry has something to do with low energy ($\lesssim$ 100 TeV) physics is that an elementary ("Higgs") scalar seems to be required in the

Weinberg-Salam model. In any case, the need for an elementary scalar in the model encourages the hope that there is new physics --a new cutoff-- not far above 1 TeV. So-- maybe the SSC will not be a waste of money.)

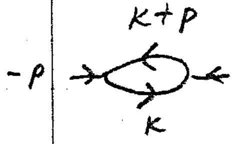
Computation of Loop Diagrams

It is about time we faced the computation of a divergent loop diagram. So consider, in our model with $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$, the leading contribution to Π

$-i\Pi(p^2) =$  $=$  $+$  + higher order

counterterm

Back on page 1.10, we had



$$= \underset{\uparrow\ \text{symmetry factor}}{\frac{1}{2}}(-i\lambda)^2\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{i}{(k+p)^2-m^2+i\epsilon}$$

To facilitate evaluation of the integral, we use the "Feynman trick." Note that

$$\int_0^1 dx\,\frac{1}{[ax+b(1-x)]^2} = \left(\frac{-1}{a-b}\right)\frac{1}{ax+b(1-x)}\Big|_0^1$$

$$= \frac{-1}{(a-b)}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}$$

Invoking this identity, we have

 $= \frac{1}{2}\lambda^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2+x\,2k\cdot p+xp^2-m^2+i\epsilon]^2}$

Completing the square in the denominator,

$$[\quad] = (k+xp)^2 + x(1-x)p^2 - m^2 + i\epsilon .$$

So we may shift the variable of integration to obtain

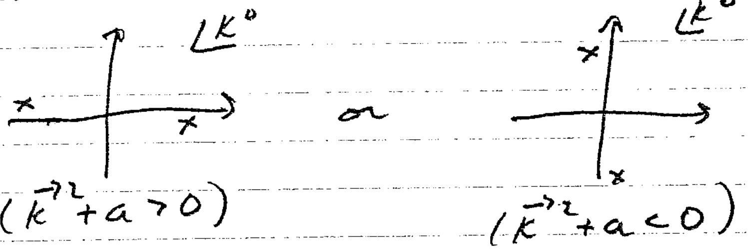
$$-\!\bigcirc\!- = \tfrac{1}{2}\lambda^2 \int_0^1 dx\ I(a)$$

$$\text{where}\quad I(a) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - a + i\epsilon]^2}, \quad a = m^2 - x(1-x)p^2$$

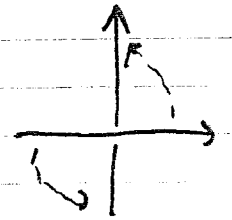
Now we must evaluate $I(a)$. The next step is to rotate the k^0 integration contour, replacing k^0 by ik^0. This step is felicitous, because it converts the Minkowski metric to a Euclidean metric. Furthermore, singularities of the integrand occur at

$$(k^0)^2 = \vec{k}^2 + a - i\epsilon ;$$

they are located at



$(\vec{k}^2 + a > 0)$ $(\vec{k}^2 + a < 0)$



In either case we may rotate the contour of the k^0 integral as shown, without encountering any singularities. And the integrand falls off rapidly enough for $|k^0| \to \infty$ so that the contribution from the quarter circles at infinity can be safely ignored. Performing this "Wick rotation" we obtain

$$I(a) = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[-k_E^2 - a + i\epsilon]^2}$$

Have $K_E^2 = (K^0)^2 + \vec{K}^2$. I'll drop the subscript E on the "Euclidean" momentum from now on.

To proceed with the evaluation of $I(a)$, we notice that the integrand is invariant under four-dimensional Euclidean rotations of K_μ (this is a remnant of Lorentz invariance in Minkowski space). Therefore, we may replace

$$d^4K = \Omega_3 K^3 dK$$

volume of the "three-sphere"

To evaluate Ω_3 (or just as easily Ω_{D-1}), we note that

$$\int d^D x \, e^{-x^2} = \pi^{D/2}$$

-- It factorizes into D Gaussian integrals

or using rotational invariance

$$= \Omega_{D-1} \int_0^\infty x^{D-1} dx \, e^{-x^2}$$

$$= \Omega_{D-1} \tfrac{1}{2} \int_0^\infty dx^2 \, (x^2)^{D/2 - 1} e^{-x^2} = \tfrac{1}{2} \Omega_{D-1} \Gamma(\tfrac{D}{2}),$$

where $\Gamma(z)$ is the "Gamma Function"

Thus

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

since an integration by parts shows that

$$\Gamma(z+1) = z\,\Gamma(z),$$

and

$$\Gamma(1) = 1$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

are elementary integrals,

we can easily find Ω_{D-1} for any D. Since $\Gamma(2) = 1$, we have, in particular,

$$\Omega_3 = 2\pi^2$$

Returning to the evaluation of the diagram,

$$I(a) = i\frac{2\pi^2}{16\pi^4}\int_0^\infty dk\, k^3 \frac{1}{[k^2+a-i\epsilon]^2}$$

$$= \frac{i}{8\pi^2}\frac{1}{2}\int_0^\infty dk^2\, k^2 \frac{1}{[k^2+a-i\epsilon]^2}$$

The remaining integration is elementary:

$$\int_0^{\Lambda^2} dz \frac{z}{(z+b)^2} = \int_0^{\Lambda^2} dz\left[\frac{1}{z+b} - \frac{b}{(z+b)^2}\right]$$

$$= \left[\ln(z+b) + \frac{b}{(z+b)}\right]\Bigg|_0^{\Lambda^2}$$

$$= \ln\left(\frac{\Lambda^2+b}{b}\right) + \frac{b}{\Lambda^2+b} - 1$$

$$= \ln \Lambda^2/b - 1 + O(b/\Lambda^2)$$

So if we introduce a "Lorentz-invariant ultraviolet cutoff" $k^2 \leq \Lambda^2$ (as on p. (2.109)), and discard terms that vanish for $\Lambda^2/b \to \infty$, we have...

$$\boxed{I(a) \equiv \int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-a+i\epsilon]^2} = \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{a-i\epsilon}\right) - 1\right]}$$

(we've retained the $i\epsilon$ so that we'll know which branch of the log to take when $a<0$.)

Note that we can also evaluate the convergent integral

$$I_n(a) \equiv \int \frac{d^4K}{(2\pi)^4}\frac{1}{[K^2-a+i\epsilon]^n}, \quad n>2 \text{ (and integer)}$$

by simply differentiating with respect to a

$$I_n(a) = \frac{1}{(n-1)!}\left(\frac{d}{da}\right)^{n-2} I(a) = \frac{1}{(n-1)!}(n-3)!(-1)^n(a)^{2-n}\frac{i}{16\pi^2}$$

or

$$\boxed{I_n(a) = \frac{i}{(n-1)(n-2)16\pi^2}(-a)^{2-n}, \quad n>2}$$

Mass Renormalization:

We have now evaluated:

$$-\bigcirc- = \int_0^1 dx\,\frac{i\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{m^2-x(1-x)p^2-i\epsilon}\right) - 1\right]$$

Let's perform mass renormalization. The renormalization condition is

$$\left[-\bigcirc- \;+\; -i(Z\delta m^2)\right]\Big|_{p^2=m^2} = 0$$

So we must choose the counterterm to be

$$Z\delta m^2\Big|_{\text{order }\lambda^2} = -i\,-\bigcirc-\Big|_{p^2=m^2}$$

$$= \frac{\lambda^2}{32\pi^2}\left[\int_0^1 dx\,\ln\left(\frac{\Lambda^2}{m^2(1-x(1-x))}\right) - 1\right]$$

$$= \frac{\lambda^2}{32\pi^2}\left[\ln\frac{\Lambda^2}{m^2} + (\text{Known finite constant})\right]$$

Now we have

 $+ \underset{\nwarrow \text{(mass counterterm)}}{-\times-} = \frac{-i\lambda^2}{32\pi^2}\int_0^1 dx \ln\left[\frac{m^2 - x(1-x)p^2 - i\epsilon}{m^2(1-x(1-x))}\right]$

Field Renormalization:

Next we impose our other renormalization condition:

$\frac{d}{dp^2}\Big[$  $+ i(Z-1)(p^2-m^2)\Big]\Big|_{p^2=m^2} = 0$

As noted earlier, the field renormalization is finite

$(Z-1) = \frac{id}{dp^2}$  $\Big|_{p^2=m^2}$

$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dx \frac{-x(1-x)}{m^2(1-x(1-x))} = \frac{\lambda^2}{32\pi^2 m^2} \times (\text{known constant})$$

And including both counterterms, we have (to this order)

$-i\Pi(p^2) =$  $+ \underset{\text{(mass)}}{-\times-} + \underset{\text{(field)}}{-\times-} + \cdots$

$$= \frac{-i\lambda^2}{32\pi^2}\int_0^1 dx \left\{ \ln\left[\frac{m^2 - x(1-x)p^2 - i\epsilon}{m^2(1-x(1-x))}\right] + \frac{(p^2-m^2)\,x(1-x)}{m^2(1-x(1-x)}\right\}$$

$+$ higher order

This is our final expression for $\Pi(p^2)$, to this order, expressed in terms of the renormalized mass (and coupling) with our renormalization conditions correctly imposed.

In our discussion of two-particle unitarity on page (2.73) ff, we anticipated that π would have a cut along the positive real axis in the complex p^2 plane, with the discontinuity across the cut given by

$$\underset{\substack{\text{two}\\\text{particles}\\\text{(identical)}}}{\text{Disc} \langle p|A|p\rangle\Big|} = \frac{i}{64\pi^2}\sqrt{\frac{p^2-4m^2}{p^2}}\,\Theta(p^2-4m^2)\int d\Omega\, \langle p|A^\dagger|k_1 k_2\rangle \langle k_1 k_2|A|p\rangle$$

Does our computed $\pi(p^2)$ obey this unitarity relation to the relevant order (order λ^2)? In order λ, we have

$$\langle k_1 k_2|A|p\rangle = (-i\lambda)$$

and $\int d\Omega = 4\pi$ (the particles are identical), so to order λ^2 we need to verify:

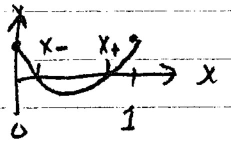
$$\text{Disc}\ \pi(p^2) = \frac{-i\lambda^2}{16\ \ \pi}\sqrt{\frac{p^2-4m^2}{p^2}}\,\Theta(p^2-4m^2)$$

where $$\pi(p^2) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \underbrace{\ln\left[m^2 - x(1-x)p^2 - i\epsilon\right]}_{\text{this is the only nonanalytic piece}} + \cdots$$

The discontinuity of π comes from the discontinuity of the logarithm.

Now $\ln(x^2 - x + m^2/p^2 - i\epsilon)$ has discontinuity $-2\pi i$ whenever its argument is negative (Note that we need the $i\epsilon$ to get the sign right) Solving a quadratic eqn, this occurs for

$$x_- < x < x_+, \text{ where } \quad x_\pm = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{m^2}{p^2}\right)^{\frac{1}{2}}$$



the argument of the log goes negative in the region of the x integration only for

$$p^2 > 4m^2$$

And we have.—

$$\mathrm{Disc}\ \Pi(p^2) = \frac{\lambda^2}{32\pi}(-2\pi i)\ \theta(p^2-4m^2)\int_{x_-}^{x_+} dx$$

$$= \frac{-i\lambda^2}{16\pi}\, 2\left(\frac{1}{4}-\frac{m^2}{p^2}\right)^{\frac{1}{2}} = \frac{-i\lambda^2}{16\pi}\left(\frac{p^2-4m^2}{p^2}\right)^{\frac{1}{2}}$$

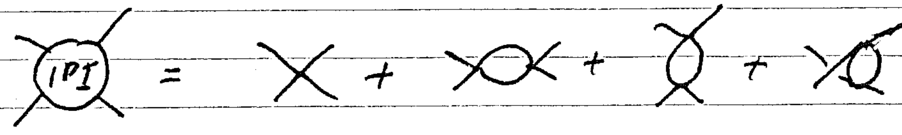
we have successfully checked the two-particle unitarity relation, to order λ^2 in perturbation theory

A Renormalizable Theory

In the model $H' = \frac{\lambda}{3!}\phi^3$, coupling renormalization is finite, as discussed previously. It is much more interesting to consider coupling renormalization in a renormalizable (rather than superrenormalizable) theory. An example, in four dimensions, is

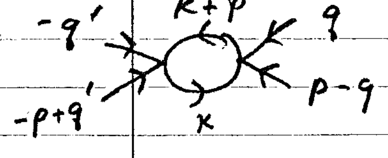
$$H' = \frac{1}{4!}\lambda\phi^4$$

In this model, the first nontrivial coupling renormalization arises in order λ^2:



+ counterterms + higher order

For example



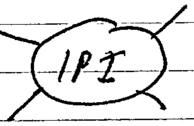
$$= \underbrace{\frac{1}{2}}_{\text{(sym. factor)}}(-i\lambda)^2\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{i}{(k+p)^2-m^2+i\epsilon}$$

But this is exactly the same expression that we just evaluated! And so

$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{m^2 - x(1-x)p^2 - i\epsilon}\right) - 1\right]$$

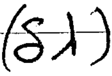
(Here $p^2 = s$, and the other two crossed graphs are obtained by replacing s by t or u)

The "symmetric point" renormalization condition for this theory, analogous to the one chosen in ϕ^3 theory, would be



$$\Big|_{s=t=u=\frac{4}{3}m^2} = -i\lambda$$

Thus, the counterterm to order λ^2 is chosen to be



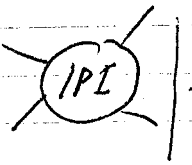
$$-i(\delta\lambda) = -3\left(\quad\right)\Big|_{p^2=\frac{4}{3}m^2}$$

or $$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{m^2\left(1-\frac{4}{3}x(1-x)\right)}\right) - 1\right] + \cdots$$

$$= \frac{3\lambda^2}{32\pi^2}\left(\ln\left(\Lambda^2/m^2\right) + \text{finite constant}\right) + \text{higher order}$$

(The coupling renormalization depends logarithmically on the cutoff, as in any renormalizable theory)

And, with coupling renormalization performed, we have



$$\Big|_{\text{order }\lambda^2} = \frac{-i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\frac{m^2-x(1-x)s-i\epsilon}{m^2(1-\frac{4}{3}x(1-x))} + (s\to t)+(s\to u)\right]$$

A Classification of Diagrams

We can classify all Feynman diagrams into three groups, depending on the nature of their sensitivity to short-wavelength quantum fluctuations

- **Convergent**

These are the diagrams with no sensitivity to Λ, for $\Lambda^2 \gg m^2, p^2$. The loop integrations are dominated by $k^2 \sim m^2, p^2$. Thus, these graphs incorporate only the effects of the 'long-wavelength' quantum fluctuations

- **Power Divergent**

These are the graphs with a positive superficial degree of divergence. They behave like Λ^p, $p>0$. In other words, the integrals are completely dominated by large k, and these graphs represent effects of the short-wavelength fluctuations. As a result, they have a trivial dependence on m^2 and p^2, and get completely absorbed into renormalized parameters

- **Log Divergent**

These are the graphs with vanishing degree of divergence. The logarithmic divergence indicates that quantum fluctuations on all scales of length from p^2, m^2 to Λ^2 are contributing; these graphs describe the fluctuations without any intrinsic scale. Hence the diagrams in this class are the most interesting ones. They generate nontrivial dependence

$$\sim \ln \Lambda^2/p^2 , \ln \Lambda^2/m^2$$

on external momenta (and physical masses) arising from this scale-free fluctuations

Breakdown of Perturbation theory

Since we encounter corrections in perturbation theory of the form

$$\lambda_0^2 \ln(\Lambda^2/p^2) ,$$ (where λ_0 is bare coupling)

we might worry about perturbation theory being a very poor approximation. Even if λ_0 is very small, $\ln \Lambda^2/p^2$ becomes as large as we please for $\Lambda \to \infty$. We get more powers of $\ln(\Lambda)$ with each power of λ_0 (in e.g. $\lambda \phi^4$ theory), so our perturbative expansion appears to make little sense.

We can cover up this problem by reexpressing the perturbative expansion in terms of a renormalized coupling. Then the large $\ln(\Lambda^2/p^2)$ factors do not appear. But this is not entirely satisfactory. Because if we adopt the "phenomenological Lagrangian" viewpoint, these large logarithms are going into the relation between bare and renormalized quantities, so they appear to make $\lambda_{renorm.}$ large, even if λ_0 is small.

This is a real problem of principle with the perturbation theory. Strictly speaking, for perturbation theory to work well, we need not just λ_0 small, but

$$\lambda_0 \ln(\Lambda^2/m^2) \ll 1, \quad \lambda_0 \ln \Lambda^2/p^2 \ll 1$$

For any fixed Λ, there is always a λ_0 small enough so that perturbation theory is a good approximation.

As for the power divergences, they seem to require an even *stronger* condition on λ_0. But these are the "uninteresting" contributions with no dependence on external momentum. So the problem is just that we can't easily arrange for renormalized quantities with dimensions of $(\text{mass})^p$ to be small compared to Λ^p $(p > 0)$. This is just the "naturalness" problem again.

Green Functions

As we noted previously (p. 2.92) our formulation of scattering theory is flawed. We had assumed that the full interacting Hamiltonian H can be replaced by the free Hamiltonian H_0 at asymptotically large times $t \to \pm\infty$. But, in fact, the self-interactions of the particles cannot be turned off even when particles are widely separated. And those self interactions give rise to (potentially infinite) mass shifts and field rescalings!

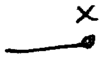
We "solved" this problem by dividing up the action into a "free" part and an interaction that included "counterterms" to cancel the mass shift and field rescaling, but the logical foundation of what we did may still be unclear. In particular, we originally derived the Feynman rules as a procedure for calculating S-matrix elements -- all external momenta were firmly fixed on the "mass shell" $p^2 = m_0^2$, where m_0^2 is the mass in H_0. But then we found that, in order to satisfy unitarity, we had to consider "continuing" external momenta off the mass shell, and then identify the physical momenta as those satisfying $p^2 = m^2$, where m^2 is the location of the pole in the exact propagator. The justification that we gave for this procedure was at best rather heuristic.

Now we would like to give a new formulation of scattering theory that is more appropriate for relativistic quantum field theory. The first step will be to give a physical interpretation to off-shell Feynman diagrams. Then we will relate the Feynman diagrams to S-matrix elements.

In order to give an interpretation to off-shell diagrams, imagine coupling our interacting field theory to an external source

$$\mathcal{H} \to \mathcal{H} + \mathcal{H}', \quad \mathcal{H}' = \rho(x)\,\phi(x)$$

(Compare p. 2.17) In this modified theory, there is a new (position space) Feynman rule



$= -i\rho(x)$

Now consider the vacuum Feynman diagrams in the presence of the source. (Imagine that $\rho \to 0$ for $t \to \pm\infty$, so that the source does not change the asymptotic vacuum state) Define the functional

$$Z[\rho] = \frac{\langle 0|S|0\rangle_\rho}{\langle 0|S|0\rangle_{\rho=0}}$$

-- the vacuum-to-vacuum transition amplitude in the presence of the source ρ.

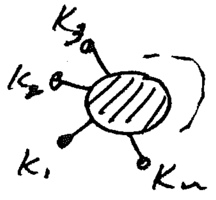
(divides out the vacuum bubbles of the source free theory)

Expanding in a power series in the source, we have

$$Z[\rho] = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \,--\, d^4x_n\, \rho(x_1) -- \rho(x_n)\, G^{(n)}(x_1, -- x_n)$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} -- \int \frac{d^4k_n}{(2\pi)^4}\, \tilde\rho(-k_1) -- \tilde\rho(-k_n)\, \tilde G^{(n)}(k_1, -- k_n)$$

where $\tilde G^{(n)}(k_1, -- k_n) =$



is the sum of all Feynman diagrams with n external lines (vacuum bubbles divided out).

The momenta $k_1 -- k_n$ can take any value (constrained by the $\delta^4(k_1 + -- + k_n)$ in $\tilde G^n$) -- they are not constrained by the mass shell condition.

The functions $G^{(n)}$ are called Green functions; they describe the (nth order) nonlinear response of the system to the external source e. The functional $Z[e]$ is the "generating functional" of the Green functions.

(Recall that Σgraphs = exp(Σconnected graphs) therefore $\ln Z[e]$ is the generating functional for connected Green functions. When $\ln Z[e]$ is expanded in powers of e, the nth order term is the sum of all connected Feynman diagrams with n external lines.)

The physical interpretation of $Z[e]$ is still somewhat unclear, because the "asymptotic vacuum" $|0\rangle$ is not the real vacuum of the theory -- it is the ground state of the "free" Hamiltonian H_0. But we will now show that $Z[e]$ can actually be interpreted as the "persistence amplitude" in the presence of the source e of the physical vacuum -- the ground state of the full Hamiltonian including interactions. (A remarkable observation, first made by Gell-Mann and Low in 1951.)

That is, we claim that

$$Z[e] = \lim_{\substack{t\to\infty \\ t'\to-\infty}} {}_P\langle 0| U_S(t,t';e)|0\rangle_P$$

where U_S is the Schrodinger picture time-evolution operator in the presence of the source e, and $|0\rangle_P$ is the physical vacuum, the ground state of the exact theory with the source turned off:

$$H|0\rangle_P = \underset{\uparrow}{E_0}\,|0\rangle_P$$

zero, if we remove vac. bubbles with c.t.

First, recall our derivation of the formula for the interaction picture time evolution operator (p 2.4 ff.) We write the Hamiltonian as

$$H = \underset{\uparrow\,\text{"free"}}{H_0} + H' \leftarrow \text{"perturbation"}$$

and define

$$U_I(t,t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$$

(where t_0 is the time at which interaction picture and Heisenberg picture operators are equated). Then we derive

$$U_I(t,t') = T \exp\left[-i\int_{t'}^{t} dt''\, H_I(t'')\right]$$

where $H_I(t) = e^{iH_0(t-t_0)} H' e^{-iH_0(t-t_0)}$

Now, apply this derivation to

$$H_\rho = H + \int d^3x\, \rho(x)\, \phi_s(x) \quad \leftarrow \text{Schrodinger picture operator}$$

And we have:

$$e^{iH(t-t_0)}\, U_s(t,t';\rho)\, e^{-iH(t'-t_0)}$$

$$= T\exp\left[-i\int_{t'}^{t} dt''\, d^3x\, \rho(x)\, \phi_H(x)\right]$$

where $\phi_H(x) = e^{iH(t-t_0)}\phi_s(x)\, e^{-iH(t-t_0)}$

Thus,

$${}_{\underline{\rho}}\langle 0|\, U_s(t,t';\rho)\,|0\rangle_{\underline{\rho}} = {}_{\underline{\rho}}\langle 0|\, T\exp\left[-i\int_{t'}^{t} d^4x\, \rho(x)\,\phi_H(x)\right]|0\rangle_{\underline{\rho}},$$

if $H|0\rangle_{\underline{\rho}} = 0$

Expanding in powers of ρ, we see that our claim is

$$G^{(n)}(x_1, -- x_n) = {}_{\rho}\langle 0|T(\phi_H(x_1) -- \phi_H(x_n))|0\rangle_{\rho}$$

On the other hand, the Feynman rules give

$$Z[\rho] = \frac{\langle 0|\, U_I(\infty,-\infty;\rho)|0\rangle}{\langle 0|\, U_I(\infty,-\infty;\rho=0)|0\rangle} \quad (\text{= "asymptotic vacuum"})$$

$$= \frac{\langle 0|\, T\exp[-i\int d^4x(\mathcal{H}_I + \rho\,\phi_I)]\,|0\rangle}{\langle 0|T\exp[-i\int d^4x\, \mathcal{H}_I]\,|0\rangle}$$

And if we expand in powers of ρ:

$$G^{(n)}(x_1,--x_n) = (\langle 0|U_I(\infty,t_1)\,\phi_I(x_1)\,U_I(t_1,t_2)\,\phi_I(x_2) -- \phi_I(x_n)\,U_I(t_n,-\infty)\,|0\rangle)/\langle 0|U_I(\infty,-\infty)|$$

-- for the particular time-ordering $t_1 \geq t_2 \geq -- \geq t_n$

So we want to show that this expression can be equated with

$${}_{\rho}\langle 0|\ \phi_H(x_1) - \phi_H(x_n)|0\rangle_{\rho}$$

-- for this particular time ordering

But recall that $\phi_I(x_j) = U_I(t_j,t_0)\,\phi_H(x_j)\,U_I(t_0,t_j)$,

so we see that the Feynman rules give

$$G^{(n)}(x_1 - x_n) = \frac{\langle 0|\, U_I(\infty,t_0)\,\phi_H(x_1) -- \phi_H(x_n)\,U_I(t_0,-\infty)|0\rangle}{\langle 0|\, U_I(\infty,-\infty)\,|0\rangle}$$

Now we need to consider how to take a limit of the form

$$\lim_{t'\to-\infty} \langle\psi|U_I(t_0,t')|0\rangle.$$

Since $U_I(t_0,t') = e^{-iH(t_0-t')}e^{-iH_0(t'-t_0)}$,

and $H_0|0\rangle = 0$, this is

$$\lim_{t'\to-\infty} \langle\psi|e^{-iH(t_0-t')}|0\rangle$$

$$= \langle\psi|0\rangle_{I}\,{}_{I}\langle 0|0\rangle + \lim_{t'\to-\infty} \sum_n \langle\psi|n\rangle\langle n|0\rangle e^{-iE_n(t_0-t')},$$

where we have inserted a complete set of eigenstates of H.

But in the limit $t'\to-\infty$, only the contribution from the intermediate state $|0\rangle_I$ survives. The reason is that $|0\rangle_I$ is the only normalizable eigenstate ~~of~~ of H; it exhausts the discrete spectrum of H. The energies of particle states are a continuum, and the rapid oscillations of the phase in the limit $t'\to-\infty$ kills their contribution. This is just an application of the "Riemann-Lebesgue lemma", which says that the Fourier transform $f(t)$ of a smooth function $\tilde{f}(\omega)$ approaches zero for $t\to\pm\infty$. (The function $\tilde{F}(\omega)$ must be as singular as a Dirac δ-function to prevent $f(t)$ from decaying asymptotically.)

So, by inserting intermediate states, we obtain...

$$G^{(n)}_{(x_1,\dots,x_n)} = \frac{\langle 0|\, U_I(\infty,t_0)\, \phi_H(x_1) - \phi_H(x_n)\, U_I(t_0,-\infty)\,|0\rangle}{\langle 0|\, U_I(\infty,t_0)\, U_I(t_0,-\infty)\,|0\rangle}$$

$$= \frac{\langle 0|0\rangle_{\underline{P}}\; {}_{\underline{P}}\langle 0|\, \phi_H(x_1) -- \phi_H(x_n)\,|0\rangle_{\underline{P}}\; {}_{\underline{P}}\langle 0|0\rangle}{\langle 0|0\rangle_{\underline{P}}\; {}_{\underline{P}}\langle 0|0\rangle}$$

$$= {}_{\underline{P}}\langle 0|\, \phi_H(x_1) - \phi_H(x_n)\,|0\rangle_{\underline{P}}$$

for the particular time ordering $t_1 \geq t_2 \geq \dots \geq t_n$, or, in general

$$\boxed{G^{(n)}_{(x_1,\dots x_n)} = {}_{\underline{P}}\langle 0|\, T[\phi_H(x_1) -- \phi_H(x_n)]\,|0\rangle_{\underline{P}}}$$

which is what we wanted to show

We have found that the sum of all (off-shell) Feynman diagrams with n external lines is the expectation value in the <u>physical</u> vacuum of the time-ordered product of n Heisenberg fields.

(if vacuum bubbles are excluded from diagrams.)

And the generating functional $Z[\rho]$ is

$$Z[\rho] = {}_{\underline{P}}\langle 0|\, U_S(\infty,-\infty;\rho)\,|0\rangle_{\underline{P}},$$

the probability amplitude for the physical vacuum to persist in the presence of the source ρ.

From now on, I will drop the subscripts on $|0\rangle_P$ and ϕ_H. It will be understood that $|0\rangle$ denotes the physical vacuum, and ϕ_H the Heisenberg picture field, unless explicitly stated otherwise.

Let's consider some of the consequences of this connection between off-shell diagrams and vacuum expectation values.

Tadpole

Recall  = (Σ connected Feynman diagrams with one external line)

We now know how to interpret this tadpole --

 $= \quad \langle 0|\phi(x)|0\rangle$;

it is the expectation value in the (physical) vacuum of the (Heisenberg) field.

This relation helps to clarify our tadpole renormalization condition (p. 2.115B). By demanding that  $= 0$, we are "shifting" the field:

$$\phi(x) = \langle 0|\phi(x)|0\rangle + \phi'(x),$$

obtaining as our new quantum field $\phi'(x)$ an operator with vanishing vacuum expectation value. The field ϕ' is more convenient than ϕ for studying the quantum fluctuations about the <u>physical</u> vacuum.

Propagator

We also have

= ($\sum$ connected Feynman diagrams with two external lines)

$$= \langle 0| T\, \phi(x)\, \phi(y) |0\rangle$$

(if ϕ is chosen so that the tadpole vanishes). We can derive some interesting properties of the propagator by inserting a sum over a complete set of intermediate states.

Consider the time ordering $x^0 > y^0$. Then

$$\langle 0|\phi(x)\,\phi(y)|0\rangle =$$

$$\underbrace{\int \frac{d^3k}{(2\pi)^3 2\omega_k}}_{\text{sum over (rel. norm.) one-particle states}} \langle 0|\phi(x)|k\rangle\langle k|\phi(y)|0\rangle + \underset{\text{two or more particles}}{\sum_n{}'} \langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle .$$

The (physical) vacuum is translation invariant, so

$$\langle 0|\phi(x)|k\rangle = e^{-ik\cdot x}\langle 0|\phi(0)|k\rangle$$

Furthermore, the vacuum is Lorentz-invariant, so

$$\langle 0|\phi(0)|k\rangle = \langle 0|\phi(0)|\Lambda k\rangle$$

must be independent of k. Denote it

$$\langle 0|\phi(0)|k\rangle = \sqrt{Z}$$

Now we have...

$$\langle 0|\phi(x)\phi(y)|0\rangle = Z\int\frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik\cdot(x-y)} + {\sum_n}' e^{-iP_n\cdot(x-y)}|\langle 0|\phi(0)|n\rangle|^2$$

$$= Z\Delta_+(x-y;\underset{\substack{\nearrow \\ \text{physical} \\ \text{mass}}}{m^2}) + \text{Remainder},$$

where

$$\text{Remainder} = \int d^4k\, e^{-ik\cdot(x-y)}{\sum_n}'\delta^4(k-P_n)|\langle 0|\phi(0)|n\rangle|^2$$

$$= \int\frac{d^4k}{(2\pi)^3}e^{-ik\cdot(x-y)}\sigma(k^2)\theta(k^0)$$

and $\sigma(k^2)\theta(k^0) = (2\pi)^3{\sum_n}'\delta^4(k-P_n)\,|\langle 0|\phi(0)|n\rangle|^2$

Or we may write

$$\text{Remainder} = \int d\mu^2\sigma(\mu^2)\int\frac{d^4k}{(2\pi)^3}e^{-ik\cdot(x-y)}\theta(k^0)\delta(k^2-\mu^2)$$

$$= \int d\mu^2\sigma(\mu^2)\,\Delta_+(x-y;\mu^2),$$

and thus

$$\boxed{\langle 0|\phi(x)\phi(y)|0\rangle = Z\Delta_+(x-y;m^2) + \int d\mu^2\sigma(\mu^2)\Delta_+(x-y;\mu^2)}$$

We've expressed $\langle 0|\phi(x)\phi(y)|0\rangle$ as a superposition of free field values, summed over the mass. (This is called a <u>spectral</u> <u>representation</u>, or Lehmann-Källen representation.) The n particle contribution to $\sigma(\mu^2)$ is a smooth function beginning at the threshold value $\mu^2 = (nm)^2$

Summing over the two time orderings, we have

$$\langle 0|T[\phi(x)\phi(y)]|0\rangle$$
$$= Z\, i\Delta_F(x-y; m^2) + \int d\mu^2\, \sigma(\mu^2)\, i\Delta_F(x-y;\mu^2),$$

where
$$i\Delta_F(x-y) = \theta(x^0-y^0)\Delta_+(x-y) + \theta(y^0-x^0)\Delta_+(y-x),$$

or, if we Fourier transform,

$$\boxed{\begin{aligned} &= \int d^4x\, e^{ip\cdot x}\langle 0|T[\phi(x)\phi(0)]|0\rangle \\ &= \frac{iZ}{p^2-m^2+i\epsilon} + \int d\mu^2\, \sigma(\mu^2)\frac{i}{p^2-\mu^2+i\epsilon} \end{aligned}}$$

-- the spectral representation for the exact propagator.

This general argument has verified what we had inferred earlier from unitarity -- that the physical mass is the position of the pole in the propagator.

Furthermore, we have learned that if we perform field renormalization so that the residue of the pole is $Z=1$, then ϕ_H is properly normalized to create a one particle state when acting on the <u>physical</u> vacuum, $\langle 0|\phi(0)|k\rangle = 1$.

There is evidently also a spectral representation for the field <u>commutator</u>:

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = Z\, i\Delta(x-y;m^2) + \int d\mu^2\, \sigma(\mu^2)\, i\Delta(x-y;\mu^2),$$

where $i\Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x)$.

Suppose that ϕ is a canonically normalized Heisenberg field; that is, it obeys the conventional canonical commutation relation. Then, by differentiating with respect to y^0 and taking $x^0 = y^0$, we find

$$1 = Z + \int d\mu^2\, \sigma(\mu^2)$$

$$= Z\left(1 + \int d\mu^2\, \tilde{\sigma}(\mu^2)\right)$$

(where I have rescaled σ by replacing ϕ by the renormalized field in the expression for σ)

We see that

$$0 \le Z \le 1$$

and that $Z < 1$ unless $\tilde{\sigma} = 0$ (because $\tilde{\sigma}$ is nonnegative)

(Note that this shows that the interaction picture cannot "exist" unless $\tilde{\sigma} = 0$. Interaction picture fields are normalized so that $\langle 0|\phi_I|k\rangle = 1$, and have conventional equal time commutators. But the renormalized Heisenberg field has an unconventional commutator unless $Z=1$, and so cannot be unitarily equivalent to ϕ_I. But $Z=1$ requires $\tilde{\sigma} = 0$, or that the field ϕ_H couples *only* to one-particle states. And that means that there is no scattering -- so ϕ_H must be a free field. We see that the interaction picture exists only in the trivial case of free fields.)

Asymptotic Condition and the S-Matrix

We have now found an interpretation of Feynman diagrams in terms of vacuum expectation values of time-ordered products, but to complete a formulation of scattering theory, we must relate the diagrams to S-matrix elements.

As stressed on e.g. p. 2.98, the key feature that makes a scattering theory possible is not that interactions "turn off" at $t \to \pm\infty$, but rather that wave-packets separate, so that particle states become noninteracting. This means that it is legitimate to replace our fields by free fields for $t \to \pm\infty$; they are just not the fields of the free theory obtained by taking the coupling constant to zero.

Because of the factor of $\sqrt{Z}$ in the coupling of the canonically normalized field ϕ to one-particle states, the proper "asymptotic condition" satisfied by the field must be

$$\phi(\vec{x},t) \xrightarrow[t\to\mp\infty]{} \sqrt{Z}\,\phi_{\substack{in\\out}}(\vec{x},t)$$

Heisenberg field

properly normalized free field

Note that this limit must be interpreted in the "weak sense." The asymptotic condition means

$$\langle\alpha|\phi|\beta\rangle \longrightarrow \langle\alpha|\phi_{\substack{in\\out}}|\beta\rangle$$

for any states $|\alpha\rangle$, $|\beta\rangle$. Interactions become unimportant asymptotically in the evaluation of the matrix element, because wave packets separate and spread. But we cannot expect

$$\| (\phi - \phi_{\substack{in \\ out}}) |\beta\rangle \| \to 0 \quad \text{(strong limit)}$$

It is only the overlap of $(\phi - \phi_{\substack{in \\ out}})|\beta\rangle$ with each fixed state that approaches zero.
If we could replace

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle \text{ by } \langle 0 | \phi_{in}(x) \phi_{in}(y) | 0 \rangle$$

asymptotically, then the commutators $[\phi, \phi]$ and $[\phi_{in}, \phi_{in}]$ would have to have the <u>same</u> normalization.

Before (e.g. back on p 2.96) we argued that the proper way to compute S-matrix elements was to perform mass and field renormalization and then compute Feynman graphs <u>ignoring</u> all radiative corrections to the <u>external</u> legs. Now we wish to <u>derive</u> this result from the asymptotic condition and the Green function interpretation of diagrams. In other words, we conjecture that n-particle S-matrix elements can be obtained as residues of the nth order poles in the n-particle Green function $G^{(n)}$.
For example,

$$\langle k_3, k_4 | S - I | k_1, k_2 \rangle$$
$$= \prod_r \left[(-i)(k_r^2 - m_r^2) Z^{-\frac{1}{2}} \right] \tilde{G}^{(4)}(k_1, k_2, -k_3, -k_4).$$

The factor of $1/\sqrt{Z}$ for each field ϕ implements field renormalization, and the factors $(-i)(k_r^2 - m_r^2)$ remove the propagator on the external line (which blows up for physical momenta $k_r^2 = m_r^2$, where m_r^2 is a <u>physical</u> mass) Incoming and outgoing momenta are distinguished by the sign of k^0. Let us now verify the conjecture.

Our starting point is the asymptotic condition:

$$\frac{1}{\sqrt{Z}}\,\phi(x) \xrightarrow[t\to\mp\infty]{} \int \frac{d^3k}{(2\pi)^3(2\omega_k)}\left[e^{-ik\cdot x}\,\alpha_{\substack{in\\ out}}(k) + e^{ik\cdot x}\,\alpha^{\dagger}_{\substack{in\\ out}}(k)\right].$$

$\nwarrow$ (relativistic normalization)

Here,

$$\langle 0|\,\alpha_{\substack{in\\ out}}(k)\,|k'\rangle_{\substack{in\\ out}} = (2\pi)^3\, 2\omega_k\, \delta^3(\vec{k}-\vec{k}').$$

It is convenient to express $\alpha, \alpha^{\dagger}$ in terms of ϕ. In a compact notation, the appropriate expression is

$$\frac{i}{\sqrt{Z}}\int d^3x \begin{bmatrix} e^{ik\cdot x} \\ e^{-ik\cdot x}\end{bmatrix} \overleftrightarrow{\partial_0}\, \phi(x) \xrightarrow[t\to\mp\infty]{} \begin{bmatrix} \alpha(k) \\ -\alpha(k)^{\dagger}\end{bmatrix}_{\substack{in\\ out}}$$

(where $A\overleftrightarrow{\partial}B \equiv A\,\partial B - (\partial A)\,B$).

Therefore we have

$$\alpha_{out}(k) - \alpha_{in}(k) = \left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right)\frac{i}{\sqrt{Z}}\int d^3x\, e^{ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \frac{i}{\sqrt{Z}}\int d^4x\, \partial_0\, e^{ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \frac{i}{\sqrt{Z}}\int d^4x\left[e^{ik\cdot x}\partial_0^{2}\phi(x) - (\partial_0^{2}e^{ik\cdot x})\,\phi(x)\right]$$

But $\partial_0^2 e^{ik\cdot x} = (\vec{\nabla}^2 - m^2)e^{ik\cdot x}$, and for matrix elements between wave-packet states <u>spatial</u> integration by parts is justified, so we may write ---

$$\alpha_{out}(k) - \alpha_{in}(k) = \frac{i}{\sqrt{Z}}\int d^4x\, e^{ik\cdot x}(\partial^\mu\partial_\mu + m^2)\,\phi(x)$$

Although integration by parts in time generates a surface term (and it is just that surface term that we are interested in), we may consider $\phi(x)$ to be inserted inside a Green function that is evaluated graphically using the Feynman rules. We may then differentiate ϕ after contractions are performed, and obtain:

$$\boxed{\alpha_{out}(k) - \alpha_{in}(k) = \frac{-i}{\sqrt{Z}}(k^2-m^2)\int d^4x\, e^{ik\cdot x}\phi(x)}$$

By identical reasoning, we obtain

$$\boxed{\alpha^+_{in}(k) - \alpha^+_{out}(k) = \frac{-i}{\sqrt{Z}}(k^2-m^2)\int d^4x\, e^{-ik\cdot x}\phi(x)}$$

These are the fundamental identities, from which the rest of scattering theory will follow. (It is understood that $k^0 = \sqrt{\vec{k}^2+m^2} > 0$ in both formulas.)

Now consider

$$\int d^4x\, e^{-ik\cdot x}\, {}_{out}\langle\alpha|T[\phi(x)\,\mathcal{O}]|\beta\rangle_{in}\,,$$

where $\mathcal{O}$ represents any string of fields. We apply the identity, noting that because of the time-ordering, α^+_{out} acts on the left, and α^+_{in} on the right. Therefore,

$$\int d^4x\, e^{-ik\cdot x}\, {}_{out}\langle\alpha|T[\phi(x)\,\mathcal{O}]|\beta\rangle_{in}$$

$$= \frac{i\sqrt{Z}}{k^2-m^2}\Big[{}_{out}\langle\alpha|\,\mathcal{O}\,\alpha^+_{in}(k)|\beta\rangle_{in} - {}_{out}\langle\alpha|\alpha^+_{out}(k)\,\mathcal{O}\,|\beta\rangle_{in}\Big]$$

We can appreciate better the significance of the time-ordering in Green functions. Time-ordered products are the natural operator products that arise in scattering theory, because the time-ordering allows α_{in}^+, α_{out}^+ to act on the appropriate asymptotic state.

Similarly, for a negative frequency field appearing in a Green function:

$$\int d^4x\, e^{ik\cdot x} {}_{out}\langle\alpha|'T'\,\phi(x)\,\mathcal{O}|\beta\rangle_{in}$$

$$= \frac{i\sqrt{Z}}{k^2-m^2}\left[{}_{out}\langle\alpha|\alpha_{out}(k)\,\mathcal{O}|\beta\rangle_{in} - {}_{out}\langle\alpha|\mathcal{O}\,\alpha_{in}|\beta\rangle_{in}\right]$$

So positive frequency fields create incoming states or destroy outgoing states, and negative frequency fields create outgoing states or destroy incoming states.

Now, let's apply this formalism to two-body scattering. We'll denote

$$[\tilde{\phi}(k)]_{amp} \equiv \frac{-i}{\sqrt{Z}}(k^2-m^2)\int d^4x\, e^{-ik\cdot x}\phi(x)$$
$$= \alpha_{in}^+(k) - \alpha_{out}^+(k)$$

$$[\tilde{\phi}(-k)]_{amp} = \frac{-i}{\sqrt{Z}}(k^2-m^2)\int d^4x\, e^{ik\cdot x}\phi(x)$$
$$= \alpha_{out}(k) - \alpha_{in}(k)$$

-- so "amp" means that the propagator $i\sqrt{Z}/k^2-m^2$ has been amputated.

Consider:

$$\langle 0 | T\left[\tilde{\phi}(k_1) \tilde{\phi}(k_2) \tilde{\phi}(-k_3) \tilde{\phi}(-k_4) \right]_{\text{amp}} | 0 \rangle$$

$$= \langle 0 | T\left[\tilde{\phi}(-k_3) \tilde{\phi}(-k_4) \right]_{\text{amp}} | k_1, k_2 \rangle_{\text{in}}$$

$$= {}_{\text{out}}\langle k_3 | \; \tilde{\phi}(-k_4)_{\text{amp}} | k_1, k_2 \rangle_{\text{in}} - \langle 0 | \tilde{\phi}(-k_4)_{\text{amp}} \, \alpha_{\text{in}}(k_3) | k_1, k_2 \rangle$$

$$= {}_{\text{out}}\langle k_3, k_4 | k_1, k_2 \rangle_{\text{in}} - {}_{\text{out}}\langle k_3 | \alpha_{\text{in}}(k_4) | k_1, k_2 \rangle$$
$$- {}_{\text{out}}\langle k_4 | \alpha_{\text{in}}(k_3) | k_1, k_2 \rangle + {}_{\text{in}}\langle k_3, k_4 | k_1, k_2 \rangle_{\text{in}}$$

But single particle in and out states are identical, so

$$= {}_{\text{out}}\langle k_3, k_4 | k_1, k_2 \rangle_{\text{in}} - {}_{\text{in}}\langle k_3, k_4 | k_1, k_2 \rangle_{\text{in}}$$

The second term can be expressed in terms of δ-functions, and is just the disconnected contribution to the scattering. We have

$$= \langle k_3, k_4 | (S - I) | k_1, k_2 \rangle$$

As we conjectured, the "amputated" Green function gives the matrix element of $S - I$.

The relation between Green functions and S-matrix elements that we have derived is called the "reduction formula."

A very important feature of this derivation is that we never had to demand that the fields $\phi(x)$ be elementary, only that it have a nonvanishing coupling to one-particle states. This means that our scattering formalism applies just as well to bound states that are created when a composite operator acts on the vacuum.