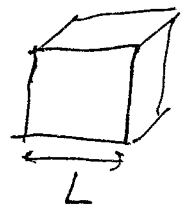
Reaction Rates

The Feynman rules provide a systematic procedure for computing probability amplitudes $\langle f|S|i\rangle$. But to compare such a calculation with experiment, we must work out the relation between S-matrix elements and the rates for physical processes. (We need a relativistic analog of "Fermi's Golden Rule")

In working this out, it is very convenient to put the world in a box with side L, and demand that all states obey periodic boundary conditions in the box. The reason that this is convenient is that, for a finite volume, the spectrum of the momentum operator is discrete. Hence, the momentum eigenstates become denumerable, and we can "count states."



As discussed back on page (1.18), the momenta in the box are restricted to the values

$$\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3) \quad \text{where } n_{1,2,3} \text{ are integers}$$

And the mode expansion of a scalar field has the form

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} a_{\vec{k}}^{\dagger} \right]$$

The reason for the $V^{-\frac{1}{2}}$ in front is that the creation and annihilation operators have have been chosen to have the discrete normalization,

$$[a_{\vec{k}}, a_{\vec{k}'}^{\dagger}] = \delta_{\vec{k},\vec{k}'}$$

And ϕ as above is normalized so that

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y}),$$

which follows from $\delta^3(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}$, $\quad V = L^3$

Here $\sum_{\vec{K}}$ means $\sum_{n_1,n_2,n_3}$. And since $\Delta\vec{K} = \frac{2\pi}{L}\Delta\vec{n}$,

we see that $\sum_{K_1} \longrightarrow \frac{L}{2\pi}\int dK_1$ in limit $L\to\infty$, where sums can be replaced by integrals, to good approximation

or $\sum_{\vec{K}} \longrightarrow \frac{V}{(2\pi)^3}\int d^3K$

(So we say that $V/(2\pi)^3$ is the "density of states" in momentum space.)

-- and $a_{\vec{K}} \to \left(\frac{(2\pi)^3}{V}\right)^{\frac{1}{2}} a(\vec{K})$, so that $\phi(x)$ becomes our continuum expression in the limit $V\to\infty$.

Now, we can rederive the Feynman rules for a world in a box, and initial and final states obeying the discrete normalization, e.g.

$$|\vec{K}_1,\vec{K}_2\rangle_V = a_{\vec{K}_1}^{+} a_{\vec{K}_2}^{+}|0\rangle$$

A contraction will be (for $x^0>y^0$)

$$\overbrace{\phi(x)\phi(y)} = [\phi^{(-)}(x), \phi^{(+)}(y)] = \frac{1}{V}\sum_{\vec{K}}\frac{1}{2\omega_K}e^{-iK\cdot(x-y)}$$

$$\longrightarrow \int\frac{d^3K}{(2\pi)^3 2\omega_K}e^{-iK\cdot(x-y)}$$

-- Becomes our usual expression for the contraction in the limit when sums are well approximated by integrals

But

$$\langle 0|\phi(x)|\vec{K}\rangle_V = \frac{1}{\sqrt{2\omega_K V}}$$

-- so the factors associated with initial (and final) particles are modified, in a way depending on V

With the modified Feynman rules, we find

$$\langle f|S-I|i\rangle_{VT} = i(2\pi)^4 A_{fi}^{(VT)}\,\delta^{(4)}_{(VT)}(K_{in}-K_{out})\,\prod_i(2E_iV)^{-\frac{1}{2}}$$

consider only S-I, because we are interested in prob. for something to happen

we impose cutoff on time also. These go smoothly to the corresponding continuum expressions for $V,T\to\infty$

factor for each initial and final particle

Now, we square this amplitude to find a transition probability. To square the δ function

$$[\delta^{(4)}(k)]^2 = \delta^4(k)\,\underbrace{\delta^4(0)}_{\to \int_{VT} d^4x\,/(2\pi)^4 = \frac{VT}{(2\pi)^4}}$$

Thus,

$$(\text{transition probability}) = |{}_V\langle f|S-1|i\rangle_V|^2$$

$$= |A_{fi}|^2 (2\pi)^4 \delta^4(k_{in}-k_{out})\, VT \prod_i (2E_iV)^{-1}$$

We expect factors of V to get cancelled, so we can take $V\to\infty$ limit. Factors of V associated with final state particles get cancelled by the density of final states. Factors of V associated with initial state particles will get cancelled when we divide by the flux to define a cross section. The factor of T indicates that the probability of a transition is finite per unit time.

The number of final one-particle states in infinitesimal momentum interval is

$$\frac{V}{(2\pi)^3}\, d^3k$$

(This measure can then be integrated over the finite momentum resolution of a detector.) We may write

$$\boxed{\begin{gathered}\left(\begin{array}{l}\text{Differential transition probability}\\ \text{per unit time}\end{array}\right) = \left[V \prod_{in}(2E_iV)^{-1}\right] |A_{fi}|^2\, D \\ \text{where} \\ \left(\begin{array}{l}\text{Relativistic density of final}\\ \text{states}\end{array}\right) \equiv D = \prod_{out} \frac{d^3k_i}{(2\pi)^3\, 2E_i}\,(2\pi)^4 \delta^{(4)}(k_{in}-k_{out})\end{gathered}}$$

Note that D (and $|A|^2$) are Lorentz-invariant quantities

Example: Two-body phase space

If there are two particles in the final state, then, in the zero momentum frame with $\vec{p}_{in} = \vec{p}_{out} = 0$, we have

$$D = \frac{d^3K_1}{(2\pi)^3 2E_1} \frac{d^3K_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^3(\vec{K}_1 + \vec{K}_2)\, \delta(E_1 + E_2 - \underbrace{E_T}_{\text{total energy of incoming particles}})$$

$$= \frac{d^3K_1}{(2\pi)^2 4E_1E_2} \delta(E_1 + E_2 - E_T)$$

write $d^3K_1 = \underbrace{K^2 dK}_{\text{magnitudes of both momenta are the same}} \, d\Omega$ solid angle

Using $\delta(f(p)) = \frac{1}{f'(p_0)} \delta(p - p_0)$ where p_0 is value with $f(p_0) = 0$

we have $\delta(E_1 + E_2 - E_T) = \frac{1}{\left|\partial E_1/\partial p + \partial E_2/\partial p\right|_{p=p_0}} \delta(p - p_0)$

and $E = \sqrt{p^2 + m^2} \Rightarrow \frac{\partial E}{\partial p} = p/E$

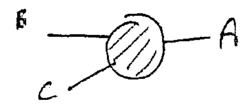
So the dK integral is trivial (momentum K is fixed by the total energy)

$$D = \frac{K^2 d\Omega}{(2\pi)^2 4E_1E_2} \frac{1}{(K/E_1 + K/E_2)} = \frac{K\, d\Omega}{16\pi^2(E_1 + E_2)}$$

or $\boxed{D = \frac{K\, d\Omega}{16\pi^2 E_T}}$ Two-body final state

where E_T is total energy, and K is momentum of either particle in the zero-momentum frame.

Example: Two-body Decay



Suppose the initial state is a single (unstable) particle. Then its (partial) decay rate in a particular two-body channel is

$$d\Gamma = \begin{pmatrix}\text{differential} \\ \text{decay rate}\end{pmatrix} = \frac{1}{2E_{initial}} |A_{fi}|^2 D$$

In the zero momentum frame, the particle decays from rest, and

$$\Gamma(A \to BC) = \frac{1}{2m_A} \int |A_{A\to BC}|^2 D$$

$$= \frac{1}{32\pi^2 m_A^{\,2}} k_B \int d\Omega \; |A|^2 = \frac{k_B}{8\pi m_A^2} |A|^2$$

(center of mass momentum of B and C) (for spinless particle)

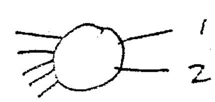
Note that, since $|A|^2$ and D are Lorentz invariant, in any other frame we have

$$\Gamma = \frac{m}{E} \Gamma_{0\text{-momentum}} = \frac{1}{\gamma} \Gamma_{0\text{-momentum}}$$

-- this is the usual time dilation factor in the lifetime of a state.

Cross Section for Two-Body Scattering

In two-body scattering, we define a cross section by dividing by the flux of initial state particles. According to our general formula, for a two body initial state,



$$(\text{differential transition rate}) = \frac{1}{4E_1E_2V} |A_{fi}|^2 D$$

$E_{1,2}$ are energies of initial particles. The explicit factor of $1/V$ occurs because our plane waves are normalized to represent a single particle in box of volume V.

Since our states are one particle states (rather than states of many identical particles), the density of particles in the box is $\frac{1}{V}$, and the incident flux is.

$$\text{flux} = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$$

(Notice that this is not a relativistic addition of velocities -- it is difference as observed in lab -- e.g., 2c for two back to back light beams)

Now, the differential cross section is

$$d\sigma = \frac{\text{differential transition rate}}{\text{incident flux}}, \text{ or}$$

$$\boxed{d\sigma = \frac{1}{4E_1E_2|\vec{v}_1 - \vec{v}_2|} |A_{fi}|^2 D}$$

Two-body scattering

In the zero-momentum frame,

$$|\vec{v}_1 - \vec{v}_2| = \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right| = |\vec{p}_1| \left(\frac{1}{E_1} + \frac{1}{E_2}\right)$$
$$= |\vec{p}_1| \frac{E_1 + E_2}{E_1 E_2}$$

So

$$\boxed{d\sigma = \frac{1}{4 E_T K_i} |A_{fi}|^2 D}$$

-- zero-momentum frame

E_T: total cm energy

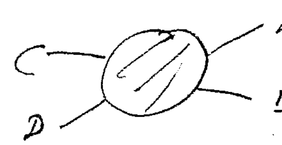
K_i: initial cm momentum

Under a Lorentz boost $\frac{1}{|E_2\vec{p}_1 - E_1\vec{p}_2|}$ is unchanged by a boost along the beam axis (increase in flux is compensated by time dilation).

$$\vec{p}_1 \longrightarrow \circledast \longleftarrow \vec{p}_2$$

Example: Two → Two Scattering

Putting together the above expression for $d\sigma$ with the formula for the two body density of final states, we have



$$\boxed{\frac{d\sigma}{d\Omega} = \frac{k_{final}}{k_{initial}} \frac{1}{64\pi^2 E_T^2} |A_{fi}|^2}$$ -- in zero-momentum frame

Detailed Balance (Time-Reversal Invariance)

A_{fi} and A_{if} are related in a theory with time-reversal symmetry. Heuristically, this comes about because reversing time interchanges i and f. More formally...

Suppose that antiunitary U_T commutes with full Hamiltonian H as well as free Hamiltonian H_0

$$[U_T, H] = [U_T, H_0] = 0$$

Then $$U_T\, U(t, t_0)\, U_T^{-1} = U_T\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)} U_T^{-1}$$

$$= U(t_0, t) = U(t, t_0)^+$$

And, since $S = \lim_{t\to\infty} \lim_{t_0 \to -\infty} U(t, t_0)$,

$$U_T\, S\, U_T^{-1} = S^+$$

Now, because U_T is antiunitary

$$(f, Si) = (U_T S i, U_T f) = (S^+ U_T i, U_T f)$$

But, at least for scalar particles $U_T i = i$, $U_T f = f$

so $(f, Si) = (i, Sf)$ $\Longrightarrow$ $\boxed{A_{fi} = A_{if}}$ Time-reversal invariant theory of scalar particles

Thus, the amplitude for $A+B \to C+D$
is same as for time reversed $C+D \to A+B$
(in a T-invariant theory)
But if the reaction is inelastic, the cross sections are different, since k_f/k_i gets turned upside down
the exothermic reaction goes faster, because there is more phase space for the final state particles

A slightly weaker form of detailed balance holds more generally, as a consequence of CPT invariance

$$\text{If} \quad U_{CPT} \, |i\rangle = |\bar{i}\rangle$$
$$U_{CPT} \, |f\rangle = |\bar{f}\rangle$$

Then above argument applied to the anti unitary operator U_{CPT} shows

$$\langle f | S | i \rangle = \langle \bar{i} | S | \bar{f} \rangle$$

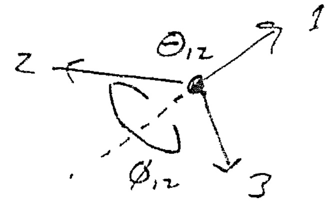
Example: Three-Body Phase Space

For a three body final state with specified 4-momentum, we have 9 variables (3 3-momenta) subject to 4 constraints, or 5 free variables

Consider the zero momentum frame, $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$.
$\vec{k}_3$ is evidently determined by $\vec{k}_1$ and $\vec{k}_2$, but $\vec{k}_1$ and $\vec{k}_2$ are not quite independent, because of the condition $E_T = E_1 + E_2 + E_3$:

$$D = \frac{(2\pi)^4}{(2\pi)^9 \, 8E_1E_2E_3} \, d^3k_1 \, d^3k_2 \, \delta(E_1 + E_2 + E_3 - E_T)$$

or $D = \frac{1}{(2\pi)^5 \, 8E_1E_2E_3} \; k_1^2 dk_1 \, d\Omega_1 \; k_2^2 dk_2 \, d\Omega_{12} \, \delta(E_1+E_2+E_3-E_T)$



Here Ω_{12} is solid angle of $\vec{k}_2$ relative to $\vec{k}_1$, parametrized by θ_{12} and ϕ_{12} as shown.

$$d\Omega_{12} = d\phi_{12} \, d\cos\theta_{12}$$

We would like to regard $\cos\theta_{12}$ as the constrained variable, and eliminate it. Thus $\vec{k}_1$, k_2, and ϕ_{12} are the 5 independent variables (ϕ_{12} determines the plane that the three momenta lie in.)

Now $$E_3 = \left[(\vec{p}_1+\vec{p}_2)^2 + m_3^2\right]^{\frac{1}{2}}$$
$$= \left[p_1^2 + p_2^2 + 2p_1p_2\cos\theta_{12} + m_3^2\right]^{\frac{1}{2}}$$

Thus $$\frac{\partial E_3}{\partial \cos\theta} = \frac{p_1p_2}{E_3},$$

and $$\int d\cos\theta \, \delta(E_1+E_2+E_3-E_T) = \frac{E_3}{p_1p_2}$$ (or zero if no allowed value of E_3 occurs in integration region.)

(with E_3 determined by energy conservation)

Thus
$$D = \frac{1}{8(2\pi)^5} \frac{1}{E_1E_2} \, p_1 dp_1 \, p_2 dp_2 \, d\Omega_1 \, d\phi_{12}$$

and $$E^2 = p^2 + m^2 \Rightarrow E\,dE = p\,dp$$

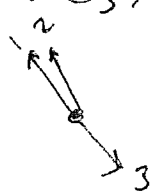
Finally
$$\boxed{D = \frac{1}{256\pi^5} \, dE_1 \, dE_2 \, d\Omega_1 \, d\phi_{12}}$$
(3 bodies, zero-momentum frame)

This is a remarkably simple expression. Ω_1 is to be integrated over a sphere and ϕ_{12} over a circle. But what makes 3 body phase space integrals complicated, is that the region of integration for E_1 and E_2 is complicated.

The integration region is determined by

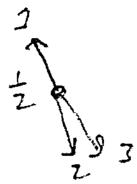
$$E_1 + E_2 + E_3 = E_T\,,$$

where E_3, as a function of $\cos\theta$, ranges between



$$E_3 = \left((p_1+p_2)^2 + m_3^2\right)^{\frac{1}{2}}$$

and $E_3 = \left[(p_1-p_2)^2 + m_3^2\right]^{\frac{1}{2}}$

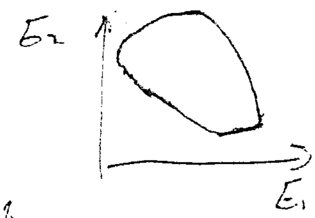


So region of integration is

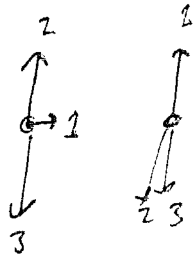
$$(p_1+p_2)^2 + m_3^2 \geq (E_T - E_1 - E_2)^2 \geq (p_1-p_2)^2 + m_3^2$$

$$p_1^2 + p_2^2 + m_3^2 + 2p_1p_2 \geq (E_T - E_1 - E_2)^2 \geq p_1^2 + p_2^2 + m_3^2 - 2p_1p_2$$

or $$\boxed{4(E_1^2 - m_1^2)(E_2^2 - m_2^2) \geq \left[(E_T - E_1 - E_2)^2 - E_1^2 - E_2^2 - m_3^2 + m_1^2 + m_2^2\right]^2}$$



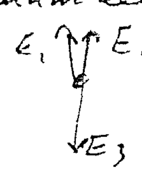
The allowed region is enclosed by some closed curve in the E_1-E_2 plane



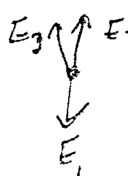
The case of 3 massless particles is relatively simple. The allowed values of E_1 obviously range from $E_1 = m_1 = 0$ to $E_1 = \frac{1}{2}E_T$

For fixed E_1  , the max allowed value of E_2 is $E_2 = \frac{1}{2}E_T$

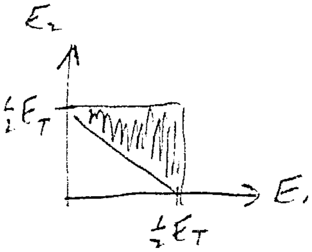
The minimum allowed value of E_2 is attained by either



$$\Rightarrow E_3 = E_1 + E_2 = E_T - E_1 - E_2 \Rightarrow E_2 = \tfrac{1}{2}E_T - E_1$$

or  $\Rightarrow E_2 = \frac{1}{2}E_T - E_3$ (whichever gives lower E_2)

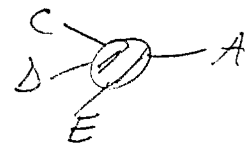
(But this requires $E_1 = \frac{1}{2}E_T$ -- so gives no condition)



Allowed region is triangle in the E_1 E_2 plane

Example: Three-Body Decay (of Spinless particle to Spinless Particles)

If decaying particle has no spin, then there is no preferred direction in space, and angular integrals are trivial. (Final particles must also have 0 spin)



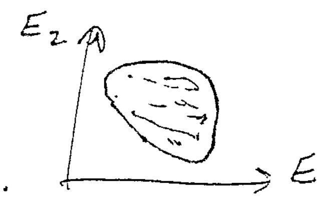
$$\Gamma(1\to 3) = \frac{1}{2m}\int |A|^2 D$$

$$= \frac{1}{2m}\int \frac{dE_1\, dE_2}{256\pi^5}\, d\Omega_1\, d\phi_{12}\, |A|^2 \qquad \text{But } \int d\Omega_1\, d\phi_{12} = 8\pi^2$$

$$= \frac{1}{64\pi^3 m}\int dE_1 dE_2\, |A|^2$$

From differential decay rate, we can determine energy dependence of $|A|$

E.g. if $|A|$ has no dependence on energy, events will uniformly fill the kinematically allowed region.



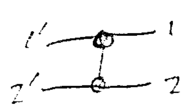
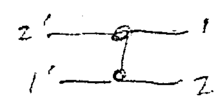
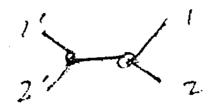
(Note that a factor of order $1/16\pi^2$ suppresses three-body decays, relative to two-body decays.)

Crossing Symmetry

Let us reconsider the calculation of $2 \to 2$ scattering in the scalar field theory with interaction

$$\mathcal{H}' = \frac{1}{3!}\lambda \phi^3$$

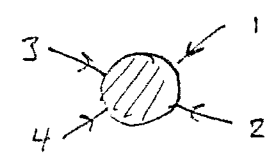
As we noted earlier, there are three Feynman diagrams that contribute, to order λ^2, to the amplitude

$iA =$  $+$  $+$ 

$$= (-i\lambda)^2 i \left[\frac{1}{(p_1-p_1')^2 - m^2 + i\epsilon} + \frac{1}{(p_1-p_2')^2 - m^2 + i\epsilon} + \frac{1}{(p_1+p_2)^2 - m^2 + i\epsilon} \right]$$

These diagrams are remarkably similar. Indeed, they all arise from the same diagram in the Wick expansion of S.

We can make the symmetry of the three diagrams even more manifest by slightly modifying our notational conventions.



Let's define all momenta to be coming in to the graph. So we introduce

$$p_3 = -p_1'$$
$$p_4 = -p_2'$$

as the momenta of the outgoing particles. In this convention, the sign of p^0 indicates whether the particle is incoming or outgoing. An external line with four-momentum p, $p^0 < 0$, represents an outgoing particle with energy $|p^0|$ and momentum $-\vec{p}$.

The amplitude A is a Lorentz invariant function of the momenta $p_{1,2,3,4}$ and the mass m. We can choose as our kinematic variables, on which A depends, the Lorentz invariants —

$$\left.\begin{aligned} s &= (p_1+p_2)^2 \\ t &= (p_1+p_3)^2 \\ u &= (p_1+p_4)^2 \end{aligned}\right\}\ \text{The "Mandelstam" variables}$$

Because of 4-momentum conservation, we may also write

$$\begin{aligned} s &= (p_3+p_4)^2 \\ t &= (p_2+p_4)^2 \\ u &= (p_2+p_3)^2 \end{aligned}$$

so permutations of the momenta interchange these variables.

In this notation, we may write the amplitude to order λ^2 as

$$iA = -i\lambda^2\left[\frac{1}{s-m^2}+\frac{1}{t-m^2}+\frac{1}{u-m^2}\right]$$

The three graphs are obtained from one another by permuting the invariants s, t, u. We speak of a s, t, or u "channel" diagram

Incidentally, although it is convenient to write A as a function of s, t, and u, to make its symmetry properties manifest, these are not really independent variables, because of momentum conservation

$$2s = (p_1+p_2)^2 + (p_3+p_4)^2 = 4m^2 + 2p_1\cdot p_2 + 2p_3\cdot p_4$$

Expressing t and u in the same way, we have

$$2(s+t+u) = 12m^2 + 2\sum_{i<j} p_i\cdot p_j$$

But momentum conservation $\Rightarrow$

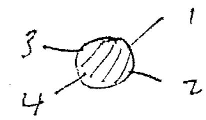
$$\left(\sum_{i=1}^{4} p_i\right)^2 = 4m^2 + 2\sum_{i<j} p_i\cdot p_j = 0$$

or $\quad s+t+u = 4m^2$

This symmetry of the amplitude under permutations of s, t, and u is called <u>crossing</u> <u>symmetry</u>.

It is clear that crossing symmetry will hold for a process involving four identical particles, not just in lowest order in λ^2, but to all orders of perturbation theory.

The reason is that, given any operator in the Wick expansion, we obtain the corresponding Feynman diagrams by matching external legs with external momenta in all possible ways.



The different matchings correspond to permutations of s, t, u, and summing over all matchings produces a permutation-invariant object.

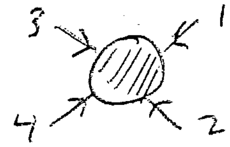
The idea of crossing (though not precisely speaking, crossing symmetry) can be generalized to processes in which the particles are not identical. In particular, if we consider a theory with complex scalar fields:

$$\varphi = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left[e^{-ik\cdot x}\, b(\vec{k}) + e^{ik\cdot x}\, c(k)^\dagger\right]$$

$$e^{-ik\cdot x} = \langle 0|\, \varphi(x)\, |k,+\rangle \quad \leftarrow \text{particle}$$

$$e^{ik\cdot x} = \langle k,-|\, \varphi(x)\, |0\rangle$$
$\uparrow$ antiparticle

So changing sign of the momentum corresponds to replacing an incoming particle by an outgoing antiparticle.



Thus, a Feynman diagram with 4 (not necessarily identical) particles can be regarded as a contribution to any one of the three processes:

$$1 + 2 \to \bar{3} + \bar{4} \qquad = \text{“}s\text{-channel”}$$
$$1 + 3 \to \bar{2} + \bar{4} \qquad = \text{“}t\text{-channel”}$$
$$1 + 4 \to \bar{2} + \bar{3} \qquad = \text{“}u\text{-channel”}$$

depending on the values of the kinematic variables

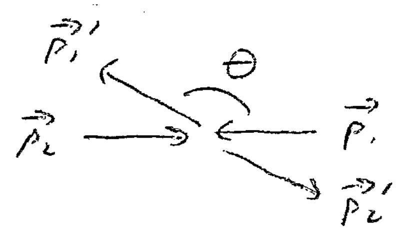
That is, the <u>same</u> <u>function</u> of the external momenta describes three distinct physical processes, when s, t, u take values in different regions. Now, interchanging s and t gives rise to a description of a <u>new</u> process. So we speak not of the crossing <u>symmetry</u> of a single amplitude, but of <u>crossing</u> <u>relations</u> between amplitudes.

In general, the four masses $m_{1,2,3,4}$ may be unequal, and s, t, u are constrained by

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

But let us, for simplicity, suppose that the masses of the four <u>distinct</u> particles are equal. What are the kinematically allowed values of s, t, u in each "channel".

In the zero-momentum frame we have, in the s-channel



$$s = (E+E)^2 = 4E^2 = E_T^2$$
$$t = (\vec{p}_1 - \vec{p}_1{}') = -2p^2(1-\cos\theta)$$
$$u = (\vec{p}_1 - \vec{p}_2{}') = -2p^2(1+\cos\theta),$$

where E, p are the energy, momentum of each particle in the center of mass, and θ is the center of mass scattering angle. Thus, allowed values in the s-channel are

$$s \geq 4m^2 \qquad t \leq 0,\ u \leq 0 \qquad \text{- s channel}$$

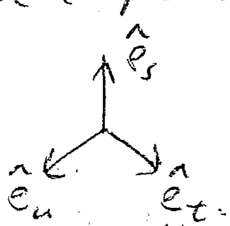
and, in the other channel we evidently have

$$t \geq 4m^2 \qquad s \leq 0,\ u \leq 0 \qquad \text{- t channel}$$

$$u \geq 4m^2 \qquad s \leq 0,\ t \leq 0 \qquad \text{- u channel}$$

The three distinct kinematic regions, corresponding to three distinct physical processes, are evidently nonoverlapping.

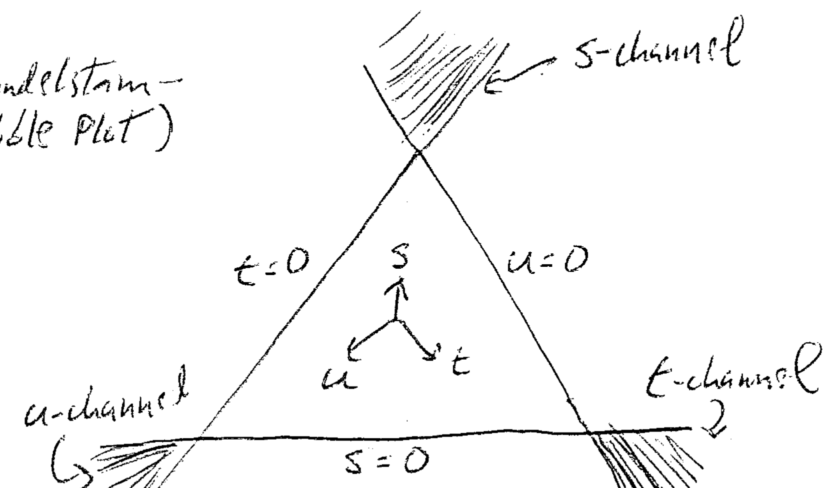
Since the three kinematic variables s, t, u are constrained by
$$s + t + u = 4m^2,$$
only two are independent, and they can be represented on a plane. To preserve the s, t, u symmetry introduce three unit vectors $\hat{e}_{s,t,u}$ arranged in an equilateral triangle: the constraint is taken into account if



$$s = \vec{x} \cdot \hat{e}_s + \tfrac{4}{3} m^2$$
$$t = \vec{x} \cdot \hat{e}_t + \tfrac{4}{3} m^2$$
$$u = \vec{x} \cdot \hat{e}_u + \tfrac{4}{3} m^2$$

where $\vec{x}$ is a point in the plane (since $\hat{e}_s + \hat{e}_t + \hat{e}_u = 0$)

(Mandelstam-Kibble Plot)



The kinematically allowed regions in the plane are sketched at left.

The point underlying the crossing relations is that there is a single function on this plane (provided by Feynman diagrams) that takes values in all three kinematically allowed regions. Putting it formally, we can go smoothly from one region to another by allowing the momenta to take unphysical (complex values). So if we know the amplitude in the s-channel region, it is determined by analytic continuation

[NOTE: This analytic continuation traverses a region in the plane with a size of order mc^2. Hence, the crossing relations have no nonrelativistic analog.

in the t-channel and u-channel regions.
The Feynman diagrams are analytic functions that don't care whether the momenta take physical or unphysical values.

(The diagrams do have singularities. We've seen that tree diagrams have poles, and loop diagrams have cuts. It is easy to continue around poles, but analytic continuation in a cut plane requires some care.)

Suppose we interchange incoming and outgoing particles

$1+2 \to \bar{3}+\bar{4}$ (change sign of all 4-momenta)

becomes $3+4 \to \bar{1}+\bar{2}$

This permutation, corresponding to analytic continuation:

$$p_{1,2,3,4} \to -p_{1,2,3,4}$$

does not change s, t, u at all, and hence does not change the (Lorentz-invariant) amplitude

So we have $$A_{fi} = A_{\bar{i}\bar{f}}$$

The amplitude is unchanged if we interchange initial and final and replace particles by antiparticles But this is just the CPT theorem (for scalar particles)! -- (CPT acting on a state does not change the 4-momentum)

As in our earlier (more general) discussion, the argument hinges on Lorentz invariance, and the idea of going from p to $-p$ by analytic continuation. This argument shows that CPT is a symmetry to all orders of perturbation theory, but we still need the more general argument to go beyond perturbation theory

Unitarity (Optical Theorem)

By construction, our S-matrix, which is the limit of the unitary operator $U(t, t_0)$, is itself a unitary operator in each order of perturbation theory. Unitarity, like crossing symmetry, places restrictions on the form of the amplitudes.

Let us write $S = 1 + iT$.

then $$1 = SS^\dagger = S^\dagger S = (1+iT)(1-iT^\dagger) = (1-iT^\dagger)(1+iT)$$

or $$T - T^\dagger = iTT^\dagger = iT^\dagger T$$

If we take matrix elements of both sides, and insert a sum over intermediate states on the RHS, we have

$$T_{fi} - T_{if}^* = i\sum_n T_{fn} T_{in}^* = i\sum_n T_{ni} T_{nf}^*$$

To find the unitarity condition on A, write

$$T_{fi} = A_{fi} \, (2\pi)^4 \delta^{(4)}(p_f - p_i),$$

where the initial and final states are eigenstates of four-momentum, and choose intermediate states to be momentum eigenstates, so

$$(2\pi)^4 \delta^4(p_i - p_f)\left(A_{fi} - A_{if}^*\right) = i\sum_n (2\pi)^4 \delta^{(4)}(p_i - p_n)(2\pi)^4 \delta^{(4)}(p_n - p_f) A_{fn} A_{in}^*$$

Factor out $\delta^4(p_i - p_f)$:

$$A_{fi} - A_{if}^* = i\sum_n (2\pi)^4 \delta^{(4)}(p_i - p_n) A_{fn} A_{in}^*$$

$$= i\sum_n (2\pi)^4 \delta^{(4)}(p_i - p_n) A_{ni} A_{nf}^*$$

If we choose $i=f$, we obtain the "Optical theorem"

$$\text{Im}\, A_{ii} = \frac{1}{2} \sum_n (2\pi)^4 \delta^{(4)}(p_i - p_n) A_{ni} A_{ni}^*$$

Now, recall that for a relativistically normalized one particle state

$$\mathbb{1} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} |k\rangle\langle k|$$

is unity acting on the one-particle subspace of Fock space. So, an insertion of the operator $\mathbb{1}$ is

$$\mathbb{1} = \sum_n \int \frac{d^3k_1}{(2\pi)^3 2E_1} \cdots \int \frac{d^3k_n}{(2\pi)^3 2E_n} |k_1 \cdots k_n\rangle\langle k_1 \cdots k_n|$$

Comparing with the density of final states (page 2.52)

$$D = \prod_{i=1}^{n} \int \frac{d^3k_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}(k_{in} - k_{out}),$$

We see that the optical theorem may be written

$$\text{Im}\, A_{ii} = \frac{1}{2} \sum_m \int |A_{mi}|^2 D_m$$

no of final state particles (m)

integral over final state phase space ($\int D_m$)

In the special case of a two-body initial state, we had (p 2.55)

$$\sigma_{total} = \frac{1}{4E_{tot} k_{initial}} \sum_m \int |A_{mi}|^2 D_m$$

in center of mass

The optical theorem is

$$\mathrm{Im}\, A_{ii} = 2KE\sigma$$

If we compare to the nonrelativistic formula

$$\mathrm{Im}\, f(k, \cos\theta = 1) = \frac{k}{4\pi}\sigma,$$

we see that the identification made earlier

$$A = 8\pi E_{total} f$$

was correct, even as to sign.

Unitarity and Poles

In our computation of the $2 \to 2$ amplitude to order λ^2, we found that the amplitude had poles at $s, t, u = m^2$. The computation shows only that such poles arise in perturbation theory in λ. But there is a much more general argument, based on unitarity that shows that the amplitude must have such poles. (Unitarity, of course, is a very general principle -- conservation of probability)

Consider the contribution to the unitarity equation

$$A_{fi} - A_{if}^* = \sum_n i(2\pi)^4 \delta^{(4)}(p_i - p_n) A_{fn} A_{in}^*$$

arising from one-particle intermediate states.

The sum over one particle states has the form

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} |k\rangle\langle k| = \int \frac{d^4k}{(2\pi)^3} \delta(k^2-m^2)\theta(k^0) |k\rangle\langle k|$$

Thus

$$\underbrace{A_{fi} - A_{if}^*}_{} = i\int \frac{d^4k}{(2\pi)^3} \delta(k^2-m^2)\theta(k^0)(2\pi)^4\delta^4(p_i-k) \langle f|A|k\rangle\langle k|A^+|i\rangle$$

same as $2i\,\mathrm{Im}\,A_{fi}$ in theory with T-invariance

$$+ \text{(more than one particle)}$$

$$= 2\pi i\, \delta(p_i^2-m^2)\,\underbrace{\theta(p_i^0)}_{\to 1}\, \underbrace{\langle f|A|k=p_i\rangle\langle k=p_i|A^+|i\rangle}_{\text{or } \langle f|A^+|k=p_i\rangle\langle k=p_i|A|i\rangle}$$

What form must A_{fi} then have? If we write

$$iA_{fi} = i\Delta(p_i) \langle f|iA|p_i\rangle\langle p_i|iA^+|i\rangle$$

then

$$iA_{if}^* = i\,\Delta(p_i)^* \langle f|iA|p_i\rangle\langle p_i|iA^+|i\rangle$$

since $p_i = p_f$

And we infer --- $\Delta(k) - \Delta(k)^* = -2\pi i\,\delta(k^2-m^2)$

Recalling the identity

$$\lim_{\epsilon\to 0^+}\left(\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right) = -2\pi i\,\delta(x)$$

we see that the general solution is

$$\Delta(k) = \frac{\lambda}{k^2-m^2+i\epsilon} - \frac{1-\lambda}{k^2-m^2-i\epsilon} + \text{(real and nonsingular)}$$

We can reject the solutions with $\lambda \neq 1, 0$ by demanding that only $A_{fi} - A_{if}^*$, and not A itself have a δ function singularity (two poles squeezed together). But both solutions are compatible with unitary -- $\Delta = \frac{1}{k^2 - m^2 + i\epsilon}$, or $\frac{-1}{k^2 - m^2 - i\epsilon}$

The choice of one or the other is rather arbitrary, as long as we make that choice consistently (corresponds to anti-time ordering, which reverses initial and final states, but is compatible with unitarity.) Making the conventional choice for the $i\epsilon$ prescription, we have, schematically

$iA_{fi} = \int$  i + (many-particle intermediate states — non pole terms)

$\frac{i}{p_i^2 - m^2 + i\epsilon}$ = "propagator"

-- The amplitude has a pole at $p_i^2 = m^2$

In a certain respect, this argument is not as general as desired. Unitarity requires the pole only when a one-particle state with $k = p_i$ is kinematically allowed. So the argument doesn't seem to apply to our ϕ^3 theory, where $s \geq 4m^2$. But amplitudes should be smooth functions of masses, as well as the kinematic variables. We can imagine varying the mass of the particle that appears in the sum over intermediate states so that it <u>is</u> kinematically allowed. We infer from that there is a pole at $s = m^2$ in general by analytic continuation in m^2. (Or we can imagine continuing to unphysical external momenta, with $s = m^2$.)

For the $2 \to 2$ scattering of identical particles, we found also poles at $t, u = m^2$. These are to be expected in general on the grounds of <u>crossing</u> symmetry. We discussed crossing before in the context of perturbation theory. But it too can be elevated to a general principle -- it is based only on our ability to analytically continue in the kinematic variables.

Two-Particle Unitarity

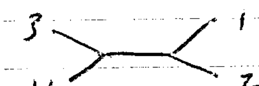
We have found already that unitarity has interesting implications for the behavior of the scattering amplitude as an analytic function. We saw in tree approximation that A has pole singularities at $s, t, u = m^2$. We now understand the pole at $s = m^2$ as a consequence of unitarity; the other poles are then a consequence of crossing symmetry.

Let's continue the study of A as an analytic function. First, recall that, to obtain an analytic function of kinematic variables from S, we must—

- Remove the forward piece:
$$S = 1 + iT$$
- Factor out the energy-momentum conserving δ-function
$$T_{fi} = (2\pi)^4 \delta^4(p_i - p_f) A_{fi}$$
- Remove disconnected part. We keep only the contribution to A coming from connected diagrams, so that there will not be additional δ-functions.

After these steps, the Feynman rules provide an integral representation of A, a smooth function of external momenta (except for isolated singularities).

In the theory with $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$, the pole in $2\to2$ scattering arises from the $2\to1$ coupling provided by the vertex.



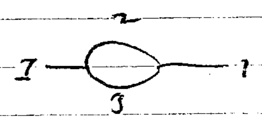
In order λ^2, a one particle intermediate state can occur in $2\to2$ scattering. The vertex provides also a $1\to2$ coupling, so, also in order λ^2, a two particle intermediate state occurs in the graph

What are the implications for the analytic structure of this graph, as a function of external momentum?

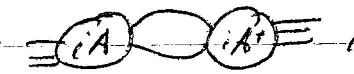
As in our discussion of the pole diagrams, we can consider the theory with 3 different scalars $\phi_{1,2,3}$, and coupling

$$\mathcal{L}' = \lambda \phi_1 \phi_2 \phi_3$$

The same diagram appears in this theory (except for a symmetry factor of $\frac{1}{2}$, but we can vary the masses $m_{1,2,3}$ so that the two particle intermediate state is kinematically allowed ($m_1 > m_2 + m_3$), and use unitarity to probe the singularity structure.



In fact, rather than analyze this particular graph, we can discuss in general the contribution to the unitarity eqn for A from two-particle intermediate states.



Unitarity $\Rightarrow$

$$A_{fi} - A^{+}_{fi} = \sum_n i(2\pi)^4 \delta^{(4)}(p_i - p_n) A_{fn} A^{+}_{ni}$$

The contribution to the RHS from two-particle states n is

$$\left(\tfrac{1}{2}\right) i \underbrace{\int \frac{d^3k_1}{(2\pi)^3\, 2E_1} \frac{d^3k_2}{(2\pi)^3\, 2E_2} (2\pi)^4 \delta^4(p_i - k_1 - k_2)}_{} \langle f|A|k_1k_2\rangle\langle k_1k_2|A^{+}|i\rangle$$

if 2-particles are identical

This is the two body phase space factor

$$\int d\Omega \frac{k}{16\pi^2 E_T}$$

where Ω is solid angle, k is momentum (of either particle) E_T is total energy, in cm frame

$$= \left(\tfrac{1}{2}\right) i \frac{k}{16\pi^2 E_T} \int d\Omega \langle f|A|k_1k_2\rangle\langle k_1k_2|A^{+}|i\rangle \;\theta\left(E_T^2 - (m_1+m_2)^2\right)$$

Now consider A_{fi} to be a function of the kinematic invariant $s = p_i^2 = E_T^2$ (with other invariants held fixed). For simplicity, suppose the two particles in the intermediate state have equal mass.

$$k^2+m^2 = \left(\frac{E_T}{2}\right)^2 \quad \text{or} \quad k = \sqrt{\tfrac{1}{4}s - m^2}$$

$$A_{fi} - A^+_{fi} = \frac{i}{64\pi^2}\sqrt{\frac{s-4m^2}{s}}\,\lambda\int d\Omega\, \langle f|A|k_1k_2\rangle\langle k_1k_2|A^+|i\rangle$$

$$\theta(s-4m^2) \qquad + \text{(other intermediate states)}$$

We have found that the two-particle unitarity eqn requires that $A_{fi} - A_{fi}^+$ be singular at $s=4m^2$ -- it has a square root branch point there.

In fact, below threshold, $s<4m^2$, no intermediate state is kinematically allowed and $A=A^+$, or

$$A_{fi}(s) = A_{if}(s)^* \quad , \; s \text{ real}, \; s<4m^2$$

If $A_{if}(s)$ is an analytic function of s in the complex s plane, then so is $A_{if}(s^*)^*$

(obeys Cauchy-Riemann conditions is $A_{if}(s)$ does)

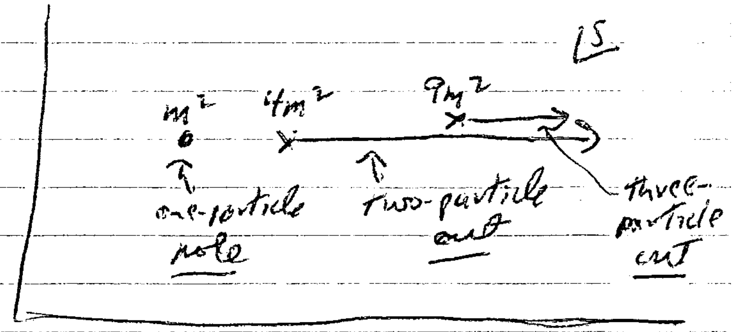
So

$$A_{fi}(s) = A_{if}(s^*)^*$$

is an equality of analytic functions on an open subset of the real line. By the uniqueness of analytic continuation, this equality holds throughout region of analyticity. Thus,

$$A(s) = A^+(s^*),$$

throughout the complex s-plane. The above unitarity equation is an equation for the discontinuity of $A(s)$ across the cut in the complex s plane along the positive real axis, beginning at $s=4m^2$.



That is,

$$A_{fi}(s+i\epsilon) - A^{\dagger}_{fi}(s+i\epsilon)$$
$$= A_{fi}(s+i\epsilon) - A_{fi}(s-i\epsilon)$$

Since there is a cut for $s > 4m^2$, we need to specify whether the physical value of A is above or below the cut. This is where the <u>$i\epsilon$-prescription</u> in the Feynman rules is essential. In all propagators, we replace

$$m^2 \to m^2 - i\epsilon$$

This moves the threshold branch point to

$$s = 4m^2 - i\epsilon$$

-- below the real axis. So real values of s sit just <u>above</u> the cut.

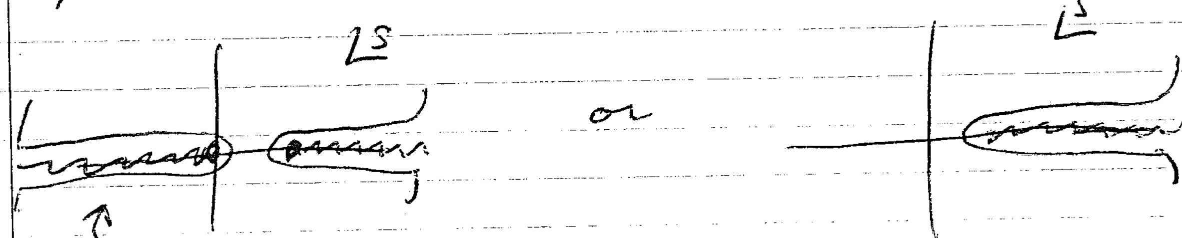
Other thresholds occur for higher values of s. E.g., the contribution due to <u>three-particle</u> intermediate states turns on at $s = 9m^2$, and there is a corresponding branch point. For $s > 9m^2$, there is a contribution to $A - A^{\dagger}$ due to the discontinuity across <u>this</u> cut.

In general, there are many cuts, and the topology of the Riemann surface becomes very complicated. But we can distinguish a "<u>physical sheet</u>" of the surface. Physical values of A are obtained by approaching the cuts from <u>above</u> on this sheet, and the physical values of $A^{\dagger}$ are obtained by approaching the cuts from <u>below</u>.

Dispersion Relations

We've seen that poles and cuts of $A(s)$ along the positive real axis of the complex s-plane arise as a consequence of unitarity. In fact, in Feynman graphs, singularities occur only along the real axis -- when momenta are complex, the singularities in propagators of Feynman integrals can be avoided.

on the physical sheet



crossed processes
conjugate cuts
for $s<0$

This means that it is possible to reconstruct the full amplitude from its discontinuity across the cuts on the real axis, by the method of dispersion relations.

(The S-matrix obeys a principle of maximal analyticity, in perturbation theory. The only singularities are necessary ones, required by unitarity.)

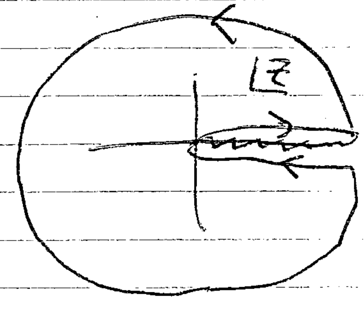
Particularly simple is the propagator, the two point amplitude

As a function of $s = p^2$ (the only invariant), its singularities are on the positive real axis only.

Now, for any analytic function $f(z)$, such that
- All singularities of $f(z)$ are on the positive real axis
- $|z|^\epsilon |f(z)| \to 0$ as $|z| \to \infty$,

we can easily derive ---

$$f(z) = \frac{1}{2\pi i}\int_0^\infty dx \, \frac{f(x+i\epsilon)-f(x-i\epsilon)}{x-z}$$

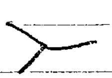


This is just an application of Cauchy's theorem to the contour shown. By assumption, the contour encloses no singularities, and the circle at ∞ does not contribute.

But this means, if $A(s)$ falls off fast enough at large s, that we can recover the full amplitude $A(s)$ on the physical sheet from just its discontinuity across the cuts, which is determined by unitarity. This means that we can generate the amplitude recursively.

Begin the recursive procedure with a "tree approximation" to the amplitude, which has poles, and no cuts. Include in the (perhaps) tree approximation a (small) parameter α, that allows us to count "orders of perturbation theory." Now, unitarity and the tree approx determine the discontinuity of $A(s)$ to order α^2. From the dispersion relation, we then find all of $A(s)$ to order α^2. But now we can use unitarity to find discontinuity of $A(s)$ to order α^3. And so on.

In fact, our Feynman rules are precisely a realization of this strategy. In ϕ^3 theory we begin with the three point amplitude

 $= -i\lambda$

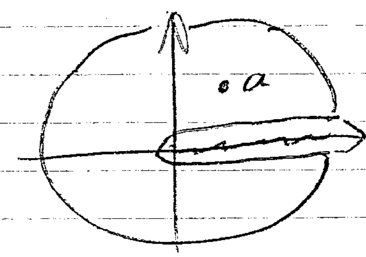
then, computing with Feynman rules, we find the form of A to all orders in λ required by unitarity.

Are there ambiguities in this recursive procedure? There are, if $s^{\epsilon}A(s)$ does not $\to 0$ as $s \to \infty$. Then a "subtraction" is required in the dispersion relation.

Suppose that $|z|^{\epsilon-1}|f(z)| \to 0$ as $|z| \to \infty$ then derive a dispersion relation for

$$\frac{f(z)}{z-a}$$

where a is a constant



$$\frac{f(z)}{z-a} = \frac{f(a)}{z-a} + \frac{1}{2\pi i}\int_0^\infty dx\, \frac{\mathrm{Disc}\, f(x)}{(x-z)(x-a)}$$

residue of pole at $z'=a$

or $$f(z) = f(a) + \frac{1}{2\pi i}(z-a)\int_0^\infty dx\, \frac{\mathrm{Disc}\, f(x)}{(x-z)(x-a)}$$

Here $f(a)$ is an arbitrary additive constant in $f(z)$, not determined by the discontinuity (a constant has no discontinuity)

Correspondingly, when we reconstruct A order by order in α, we are free to add to A in each order a polynomial in external momentum, that does not contribute to the discontinuity of A

(This ambiguity tells us how much freedom we have to modify the Feynman rules order by order in λ, without spoiling unitarity. This is an important idea in renormalization theory, as we will see.)

Unstable Particles

Consider again the (exact) propagator: It is the sum of connected Feynman diagrams with two external lines, and also the "continuation" of the mass shell $p^2 = m^2$ of

$$-p - \text{(blob)} - p$$

$$A(p^2) = \langle p | A | p \rangle$$

Our earlier analysis shows that the contribution to the imaginary part of $A(p^2)$ due to two particle intermediate states is

$$2i \operatorname{Im} A(p^2) = i \int |\langle k_1 k_2 | A | p \rangle|^2 D(k_1, k_2) + \cdots$$

$$\text{or} \quad \operatorname{Im} A(p^2) = \tfrac{1}{2}(2m)\, \Gamma + \cdots$$

where Γ is the total decay rate for a particle at rest with mass $m^2 = p^2$ to decay to the two-body state

Now, suppose that such a decay really is kinematically allowed. That is, there are particles with mass m, m_1, m_2 and $m_1 + m_2 < m$. So particle of mass m can decay to two bodies. It is unstable. Hence, it can no longer appear in the asymptotic states of scattering theory. But the unstable particle can still make its presence felt as a <u>resonance</u> in amplitudes for scattering of the other (stable) particles. Let's investigate how this arises.

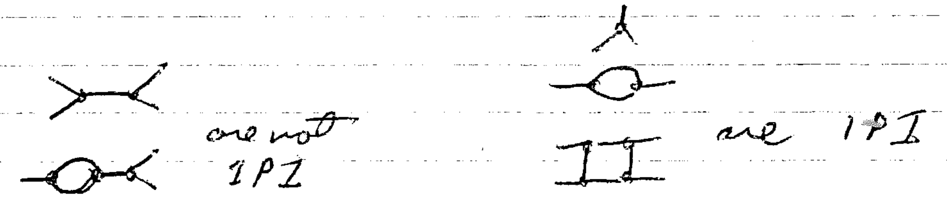
The decay of the particle actually moves the pole in $A(p^2)$ at $p^2 = m^2$ off the real axis. To see this ---

It is convenient to define a subclass of connected Feynman graphs:

<u>Definition:</u>

A connected Feynman graph is said to be <u>one-particle irreducible</u> (or 1PI) if it cannot be disconnected by cutting a single internal line.

Thus



We will denote by --- —(1PI)— , (1PI) , etc

The sum of all 1PI diagrams with the indicated number of external lines.

Therefore, a resummation of the perturbation expansion for the propagator can be written

$$iA = \underbrace{-\!\!\!-\!\!\!\bullet\!\!\!-\!\!\!-}_{\text{sum of all diagrams}} = \text{——} + \text{—(1PI)—} + \text{—(1PI)—(1PI)—} + \cdots$$

This is an infinite series that we can sum. (Converges for large p^2 -- and determined for all p^2 by analytic continuation)

We define

$$\text{—(1PI)—} = -i\Pi(p^2).$$

Then

$$\text{—}\bullet\text{—} = \frac{i}{p^2-m^2+i\epsilon} + \frac{i}{p^2-m^2+i\epsilon}\,(-i\Pi(p^2))\,\frac{i}{p^2-m^2+i\epsilon} + \cdots$$

$$= \frac{i}{p^2-m^2+i\epsilon}\left[\frac{1}{1-\frac{\Pi}{p^2-m^2+i\epsilon}}\right] = \frac{i}{p^2-m^2-\Pi(p^2)+i\epsilon}$$

Now, how does $\Pi(p^2)$ modify the pole in the propagator?

- It may have a constant piece that moves the position of the pole. Thus, the <u>physical mass</u> of the particle need not be the same as the mass m that appears in the Feynman rules. (More about this when we discuss <u>mass renormalization</u>.)

- It may have a piece proportional to p^2, and thus change the residue of the pole. (More about this when we discuss <u>field renormalization</u>.)

If neither of above happen (as we can arrange by renormalization), then still

- $\Pi(p^2)$ has an imaginary part if particle is unstable, which shifts pole off the real axis.

-- this we cannot remove, since it is required by <u>unitarity</u>

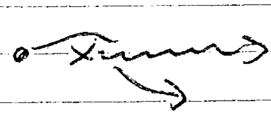
$$\mathrm{Im}\,\Pi(p^2=m^2) = -\,\mathrm{Im}\,A(p^2=m^2)\Big|_{\text{2 or more particles}}$$

$$= -m\Gamma$$

And the propagator becomes

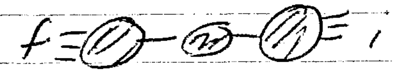
$$-\!\!\bigcirc\!\!- = \frac{i}{p^2-m^2+im\Gamma} \quad (\text{for } p^2\sim m^2)$$

the pole drops below the real axis onto the "second sheet" (and is so consistent with our claims that all singularities occur for s real on <u>physical</u> sheet)



As m increases and reaches threshold for decay, it ducks around branch point and slides under the cut.

If the lifetime Γ^{-1} is long ($\Gamma \ll m$), then the pole is close to real axis, and causes rapid variation of amplitudes for $s \approx m^2$. The sum of graphs of the form --



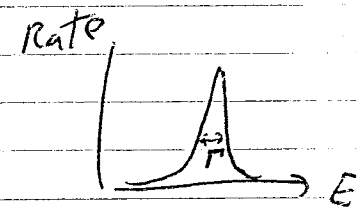
contribute

$$A_{fi}(s) \propto \frac{1}{s-m^2+im\Gamma}$$

For $s = E_T^2 \approx m^2$, we have $s-m^2 \sim (E_T-m)(E_T+m) \sim 2m(E_T-m)$

$$\text{so } A \propto \frac{1}{E_T-m+i\Gamma/2} \text{ or}$$

$$\text{Rate} \propto |A|^2 \propto \frac{1}{(E_T-m)^2+\Gamma^2/4}$$



A narrow peak ("Breit-Wigner Resonance") with Γ = full width at half maximum

Partial Wave Amplitudes (and Multichannel Resonance Theory)

For the purpose of discussing more fully the theory of resonances in multichannel scattering, it is convenient to introduce partial wave amplitudes

E.g. suppose we consider $2 \to 2$ scattering, but with various possible 2-body final states (="inelastic channels") Since angular momentum commutes with S, we can expand asymptotic states in eigenstates of $\vec{J}^2$, instead of $\vec{P}$, and scattering preserves $\vec{J}^2$.

To relate this to the partial wave expansion of potential scattering, we "factor out" center of mass motion:

In zero-momentum frame:

$$\langle \vec{p}\,', \vec{p}\,'' | S | \vec{p}, -\vec{p} \rangle = \delta^{(3)}(\vec{p}\,' + \vec{p}\,'') \langle \vec{p}\,' | S | \vec{p} \rangle_{cm}$$

planewave normalization
(not relativistic)

In nonrelativistic scattering theory, $\langle S \rangle_{cm}$ is conventionally expressed in terms of amplitude f

$$\langle \vec{p}\,' | S | \vec{p} \rangle_{cm} = \delta^{(3)}(\vec{p} - \vec{p}\,') + \frac{i}{2\pi} \delta(E - E') f(\vec{p}, \vec{p}\,')$$

Then $\quad \dfrac{d\sigma}{d\Omega} = |f(\vec{p}, \vec{p}\,')|^2$

In basis of angular momentum eigenstates

$$\langle E', l', m' | S | E, l, m \rangle = \delta(E' - E) \delta_{l'l} \delta_{m'm} S_l$$

By expanding plane wave in angular momentum eigenstates, find --

also called $e^{i 2\delta_l}$ ← phase shift

$$f(p, \theta) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \frac{(S_l - 1)}{2ip}$$

angle between $\vec{p}, \vec{p}\,'$

$$\sigma_{Total} = 2\pi \int_{-1}^{1} d\cos\theta \, |f|^2 = \frac{\pi}{p^2} \sum_l (2l+1) |S_l - 1|^2$$

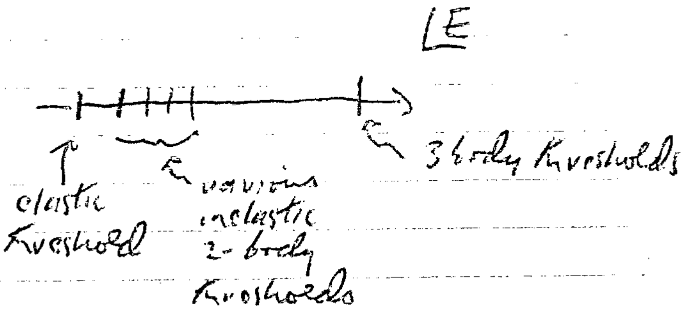
When there are inelastic channels, S_l becomes a matrix -- An $N \times N$ matrix if there are N accessible two body channels (including elastic channel). Unitarity requires ---

$$S_l(E)^\dagger S_l(E) = \mathbb{1} \leftarrow N \times N \text{ unit matrix}$$

Now, let's consider the analytic properties of the function $S_l(E)$ -- (We may consider it to be a function of E rather than s; just introduces a cut at $s = 0$)

As we saw before, when we continue to complex E

$$A(s) = A(s^*)^\dagger$$



In the angular momentum basis, S splits into $N\times N$ block for each l, each block satisfying

$$S_l(E) = S_l(E^*)^\dagger \qquad \text{-- "Reflection Principle"}$$

Thus, above threshold,

$$S_l(E+i\epsilon) = S_l^\dagger(E-i\epsilon) \qquad \text{("Physical sheet")}$$

-- physical value is just above cut in the E plane, and discontinuity across cut on "physical sheet" is $S_l - S_l^\dagger$

Now, to discuss resonances, we want to analytically continue S_l below the cut onto the "2nd sheet". First note that unitarity of the physical S-matrix

$$S_l^\dagger(E+i\epsilon)\, S_l(E+i\epsilon) = \mathbb{1},$$

together with reflection principle, imply

$$S_l(E-i\epsilon)\, S_l^\dagger(E-i\epsilon) = \mathbb{1}$$

-- so S_l is unitary just below cut, as well as above, on the physical sheet.

We want to find an analytic function in the lower half plane that matches the physical S_l on the real axis -- this function is the analytic continuation of S_l to second sheet. Call this continuation $\hat{S}_l$. It must satisfy

$$\lim_{\epsilon \to 0} \tilde{S}_\ell(E - i\epsilon) = S_\ell(E + i\epsilon)$$

continuation to second sheet — physical sheet

But reflection principle $\Rightarrow$ $S_\ell(E+i\epsilon) = S_\ell^{+}(E-i\epsilon)$
and unitarity $\Rightarrow$ $S_\ell^{+}(E-i\epsilon) = S_\ell^{-1}(E-i\epsilon)$ (as $\epsilon \to 0$)

Thus $S_\ell^{-1}(E-i\epsilon)$ is an analytic function (the inverse of a matrix of analytic functions) that matches $S_\ell(E+i\epsilon)$ as $\epsilon \to 0$. That is

$$\tilde{S}_\ell(E) = S_\ell^{-1}(E)$$

is the desired analytic continuation of S_ℓ onto the second sheet.

Thus, if S_ℓ has zeros (isolated points where $\det S_\ell = 0$) on first sheet in lower half plane, poles on the second sheet will arise. These will be <u>resonance poles</u>.

Let's consider the consequences of such a pole close to the real axis. In the vicinity of the pole,

$$S = S^{(B)} + \frac{M}{E - E_R + i\Gamma/2} \qquad \text{(Near second sheet pole)}$$

where $S^{(B)}$ is a "background value" of S, that can be regarded as roughly constant in the vicinity of the pole. Here $S^{(B)}$ and M are $N \times N$ matrices, and the subscript ℓ on S_ℓ has been dropped.

How does unitarity constrain $S^{(B)}$ and M?

If the resonance is very narrow, Γ small, then S is dominated by the second sheet pole even for E real, where S is <u>unitary</u>. It is convenient to write

$$S = S^{(B)}\left(\mathbb{1} - \frac{i\Gamma \underline{P}}{E - E_R + i\Gamma/2}\right)$$

$$SS^{+} = \mathbb{1} \Rightarrow \left(\mathbb{1} - \frac{i\Gamma \underline{P}}{E - E_R + i\frac{\Gamma}{2}}\right)\left(\mathbb{1} + \frac{i\Gamma \underline{P}^{+}}{E - E_R - i\Gamma/2}\right) = S^{(B)+}S^{(B)}$$

Since the RHS has no poles, residues of poles on LHS must vanish

$$\text{or} \quad \underline{P}(\mathbb{1} - \underline{P}^{+}) = 0 = (\mathbb{1} - \underline{P})\underline{P}^{+}$$

$$\Rightarrow \quad \underline{P} = \underline{P}^{+} \text{ and } \underline{P}^{2} = \underline{P}$$

(Then LHS is $\mathbb{1}$, so $S^{(B)}$ must be <u>unitary</u>.)

We conclude that $\underline{P}$ is an orthogonal projection in the $N \times N$ space of channels. Its general form is

$$\underline{P} = \sum_{r=1}^{R} e^{(r)} e^{(r)+} \qquad \text{where } (e^{(r)}, e^{(s)}) = \delta_{rs} \text{ and } R \le N$$

For $R > 1$, there are several (R) resonances with <u>same mass and width</u>; can arise as a consequence of internal symmetry. Barring internal symmetry, we expect $R = 1$. And

$$S = S^{(B)}\left(\mathbb{1} - \frac{i\Gamma e e^{+}}{E - E_R + i\Gamma/2}\right)$$

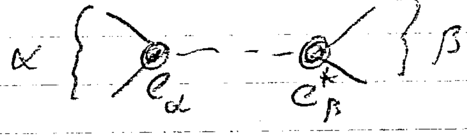
<u>Examples:</u>

① First, suppose background scattering is negligible:

$$(S-\mathbb{1})_{\alpha\beta} = \frac{-i\Gamma e_\alpha e_\beta^*}{E-E_R+i\Gamma/2}$$ ← Residue "Factorizes"

Here $e_\alpha = (\alpha, e)$ -- component of e along channel α. We have a Breit-Wigner resonance that couples to channel α with "strength" $\sim \sqrt{\Gamma}\, e_\alpha$

The formula can be interpreted as saying that production and decay of the resonance are "independent". The narrow resonant state forgets how it was formed and decays into available final states α with rates proportional to $|e_\alpha|^2$, regardless of the entrance channel. This is the key physical feature of multichannel resonance theory; we have found it to be a consequence of unitarity.



For resonance in lth partial wave, we have (if BG scattering is negligible)

$$\frac{\sigma_l(\beta\to\alpha)}{\sigma_l(\beta\to \text{all})} = \frac{|e_\alpha|^2}{\sum_\gamma |e_\gamma|^2} = |e_\alpha|^2$$

The quantities $\gamma_\alpha = \Gamma|e_\alpha|^2$ are called partial widths of the resonance. They sum up to Γ, and can be interpreted as decay rate for the resonance into the α-channel.

The total cross section is (in lth partial wave)

$$\sigma_l(\alpha\to\text{all}) = \frac{\pi}{p^2}(2l+1)\sum_\beta |S_l^{\alpha\beta}-1|^2$$

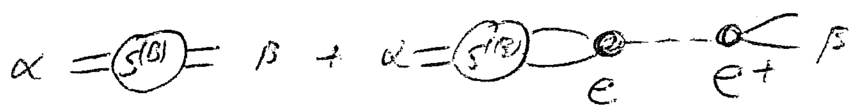
$$= \frac{\pi}{p^2}(2l+1)\frac{\Gamma^2}{(E-E_R)^2+\Gamma^2/4}\,|e_\alpha|^2$$

Therefore, by measuring the total cross section in the vicinity of the resonance, we can determine not only total width Γ of the resonance (from the observed width of the peak) but also, from the height of the peak, we determine $|e_\alpha|^2$, and hence the partial width of the resonance, for decay into the entrance channel.

If there is background scattering, then

$$(S-\mathbb{1})_{\alpha\beta} = (S^{(B)}-\mathbb{1})_{\alpha\beta} - i\Gamma \frac{(S^{(B)}e)_\alpha e_\beta^*}{E-E_R+i\Gamma/2}$$

-- or schematically



The heuristic interpretation is -- the resonance is produced, then decays, and then the decay products undergo "background scattering". We say that $S^{(B)}$ causes a "final state interaction". Actually, it is just a convention to put background scattering at the end instead of the beginning. We could just as easily speak of "initial state interactions". The important point is that we must not include both.

(This observation is sometimes dignified by the name "Final State Interaction Theorem")

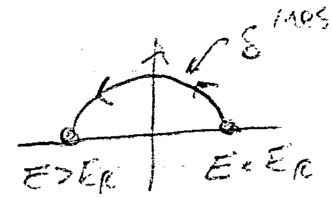
Suppose there is just one channel (elastic scattering) then $ee^+=1$, and we have

$$S = S^{(B)}\left(\frac{E-E_R-i\Gamma/2}{E-E_R+i\Gamma/2}\right)$$

Both $S=e^{2i\delta}$ and $S^{(B)}=e^{2i\delta^{(B)}}$ are pure phases

Then $S = e^{2i(\delta^{(B)} + \delta^{(res)})}$

where $\delta^{(res)} = \text{Arctan} \frac{-\Gamma}{2(E-E_R)}$

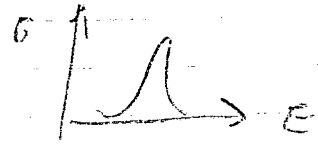


$\delta^{(res)}$ rapidly advances by π in vicinity of the resonance.

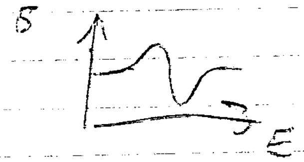
In general $\delta^{(B)}$ and $\delta^{(res)}$ interfere:

$\sigma \propto |S-1|^2 = 4\sin^2(\delta^{(B)} + \delta^{(res)})$

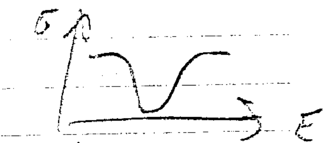
We can have --



for $\delta^{(B)} \sim 0$,



for $\delta^{(B)} \sim \pi/4$, or even



for $\delta^{(B)} \sim \pi/2$

(see page (1.126))

~~Computation of Loop Diagrams~~

~~In our model with $\mathcal{L}' = \frac{\lambda}{3!}\phi^3$, we encountered in order λ^2 the diagram~~

~~this is the leading contribution to $-i\Pi(p^2)$. We have avoided calculating it until now.~~

~~$-i\Pi(p^2) = $ + higher order~~

~~$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k+p)^2-m^2+i\epsilon}$~~

~~(sym. factor)~~

~~To do the integral, we use a trick:~~

Unitarity and Statistical Mechanics

Back on page (2.56) was a brief discussion of "detailed balance"; it was noted that time-reversal invariance relates the rates of the process $i \to f$ to the rate for the process $f \to i$. In discussions of thermal equilibrium in books on statistical physics, the impression is sometimes given that detailed balance (hence T-invariance) is essential for guaranteeing that "thermal equilibrium" is preserved by microscopic collisions. I remarked in the lecture that statistical mechanics can actually be founded on the very general principle of unitarity, rather than T-invariance, which is known to be violated in Nature. I did not have time to explain this, so I am including this supplement to the notes, which gives a brief explanation.

The basic observation is:

If $S = 1 + iT$, then

$$SS^{+} = S^{+}S = 1 \Rightarrow \sum_{k} |T_{ik}|^2 = \sum_{k} |T_{ki}|^2 \quad (TT^{+} = T^{+}T)$$

And there, the rates satisfy

$$\sum_{k} \Gamma(i \to k) = \sum_{k} \Gamma(k \to i)$$

(Recall that phase space factors in Γ enter symmetrically with initial and final states -- essentially a factor of $(1/2E)$ for each initial or final particle.)

This observation allows us to write down a rate equation for the number of particles $n(p_1)$ that "occupy" the mode with momentum p_1.

To be completely general, we must include in our amplitudes occupation number factors, as discussed, e.g., on page 1.19 of the notes. Since

$$\langle n(p)+1 |\, a^{\dagger}(p)\, | n(p) \rangle = \sqrt{1+n(p)},$$

$$\langle n(p)-1 |\, a(p)\, | n(p) \rangle = \sqrt{n(p)},$$

this rate equation takes the form

$$\frac{dn(p_1)}{dt} = \sum_{k,l} \int dp_2 \cdots dp_k \, dp_1' \cdots dp_l'$$

$$\Big\{ \Gamma(p_1', \cdots, p_l' \to p_1, \cdots, p_k)\, n(p_1') \cdots n(p_l') [1+n(p_1)] \cdots [1+n(p_k)]$$

$$- \Gamma(p_1, \cdots, p_k \to p_1', \cdots, p_l')\, n(p_1) \cdots n(p_k) [1+n(p_1')] \cdots [1+n(p_l')] \Big\}$$

Here the Γ's are the rates for the processes in vacuum; that is, with occupation numbers for initial particles equal to one, and occupation numbers for final particles equal to zero.

With rates expressed in this form, our unitarity relation becomes

$$0 = \sum_{l} \int dp_1' \cdots dp_l' \left[\Gamma(p_1', \cdots, p_l' \to \text{state}) - \Gamma(\text{state} \to p_1' \cdots p_l') \right]$$

$$[1+n(p_1')] \cdots [1+n(p_l')]$$

Now we may write

$$\frac{dn(p_1)}{dt} = \sum_{k,l} \int dp_2 \cdots dp_k \, dp_1' \cdots dp_l' \, [1+n(p_1)] \cdots [1+n(p_k)]$$

$$\left\{ \Gamma(p_1' \cdots p_l' \to p_1 \cdots p_k) \frac{n(p_1')}{1+n(p_1')} \cdots \frac{n(p_l')}{1+n(p_l')} - \Gamma(p_1 \cdots p_k \to p_1' \cdots p_l') \frac{n(p_1)}{1+n(p_1)} \cdots \right\}$$

$$\times [1+n(p_1')] \cdots [1+n(p_l')]$$

And so we see that unitarity ensures that $\frac{dn(p_1)}{dt} = 0$, provided

$$\frac{n(p_1')}{1+n(p_1')} \cdots \frac{n(p_\ell')}{1+n(p_\ell')} = \frac{n(p_1)}{1+n(p_1)} \cdots \frac{n(p_k)}{1+n(p_k)}$$

$= \text{constant}$ (independent of $\ell, p_1' \cdots p_\ell'$, for fixed $k, p_1 \cdots p_k$)

But the microscopic collisions conserve energy, so this is satisfied provided

$$\frac{n(p)}{1+n(p)} = e^{-\beta E_p}$$

since the both sides above will be $e^{-\beta(E_{total})}$, which is independent of ℓ and $p_1' \cdots p_\ell'$.
A more general solution is

$$\frac{n(p)}{1+n(p)} = e^{-\beta E_p + \beta\mu Q_p}$$

where Q_p is some other quantity conserved by the collisions. This is just the Bose-Einstein distribution. It is a sufficient (and necessary too) condition for the occupation numbers to be preserved by collisions. We have shown that unitarity alone implies that a Bose-Einstein distribution is in equilibrium!

Unitarity and CPT

Unitarity has some interesting further consequences when combined with CPT. Back on page (2.57) we saw

$$T_{ik} = T_{\bar{k}\bar{i}} \quad \text{where } \bar{k} \text{ denotes CPT conjugate of } k$$

From the unitarity relation on page (2.90A), we therefore have

$$\sum_k |T_{ik}|^2 = \sum_k |T_{i\bar{k}}|^2 = \sum_k |T_{k\bar{i}}|^2 = \sum_k |T_{\bar{i}k}|^2$$

sum over all k same as sum over all $\bar{k}$ — CPT — unitarity

Thus $\Gamma(i \to \text{all}) = \Gamma(\bar{i} \to \text{all})$;
particle and antiparticle have the same total width but partial widths to channels j and $\bar{j}$ can be different

i.e. $\Gamma(i \to j) \neq \Gamma(\bar{i} \to \bar{j})$, in general
(if there is CP violation).

Note also that unitarity implies

$$T_{ij} - T_{ji}^* = i \sum_k T_{ik} T_{jk}^*$$

$$= T_{ij} - T_{\bar{i}\bar{j}}^* \quad \text{-- by CPT}$$

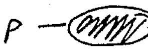
Therefore, in the lowest order of perturbation theory (Born, or "tree" approximation), we always have

$$|T_{ij}|^2 = |T_{\bar{i}\bar{j}}|^2, \text{ and hence } \Gamma(i \to j) = \Gamma(\bar{i} \to \bar{j})$$

CP violation in rates only occurs in higher orders; it is necessarily a loop effect.

Mass Renormalization

Recall the expression derived on p.(2.81) for the exact propagator

$-p$  p $= \dfrac{i}{p^2 - m^2 - \Pi(p^2) + i\epsilon}$,

where $-i\Pi(p^2) =$ 

We've considered the implications if $\Pi(p^2=m^2)$ has an imaginary part. (Pole on the second sheet, etc.)

But it is also possible for $\Pi(p^2=m^2)$ to have a real part, even if the particle is stable and its imaginary part vanishes. Indeed, in our model with

$$\mathcal{H}' = \frac{1}{3!}\lambda\phi^3,$$

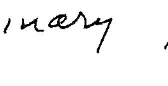
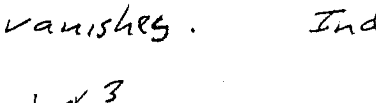
the leading contribution to Π is

$-i\Pi =$  $+$ (higher order),

a one-loop graph contributes in order λ^2. We will soon see by an explicit calculation of this diagram that $\Pi(p^2=m^2) \neq 0$ in order λ^2.

What does this mean? Consider, for example, the sum of all Feynman Diagrams that contribute to the $2\to 2$ scattering amplitude:

Ignoring corrections to external legs -- see later

 $=$  $+$ 2 crossed graphs $+$ $+$

This amplitude has an s-channel pole which is the pole in the exact propagator. Furthermore, the unitarity argument of p.(2.70) shows that the position of this pole is the mass of a physical particle.

But if $\Pi(p^2=m^2)\neq 0$ in order λ^2, then this pole is not located at $p^2=m^2$. It is shifted by an amount of order λ^2. The physical mass, the position of the pole, is determined by solving

$$p^2 - m^2 - \Pi(p^2) = 0,$$

which for $\Pi(m^2) = O(\lambda^2)$, is solved by

$$m^2_{physical} = m^2 + O(\lambda^2)$$

The physical mass of the particle need not be (and is not) the same as the mass the appears in the Feynman rules.

This in itself is not such a big surprise. The quantity m^2 is the "classical" mass that appears in the action of the theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}\lambda\phi^3,$$

and is also the particle mass in the limit of free field theory $\lambda \to 0$. But in the interacting theory, why shouldn't the fully quantum mechanical mass be different from the classical mass? The vacuum of the quantum theory is the ground state of (actually coupled) oscillators; the one particle state moving through the vacuum interacts with these oscillators, and we should expect it to have a different inertia than the classical particle

We see then, that our formulation of scattering theory on page (2.9) ff was not really adequate. We argued that, because particles separate, we can replace the full Hamiltonian H by the free Hamiltonian H_0 for $t \to \pm\infty$. But this is misleading, because the self-interactions of a particle (or its "interactions" with the vacuum) do not "turn off", even when we consider an isolated one particle state. The true asymptotic states are states of noninteracting particles, but because of the self-interactions, their masses are shifted.

However, it is possible to modify the Feynman rules so that the mass m appearing in the propagator really is the same as the physical mass. (And this is a very convenient thing to do.) Recall from our discussion of unitarity that we have the freedom, in each order of perturbation theory in the coupling λ, to add to an amplitude a polynomial in external momentum. Adding such a term will not spoil the unitarity of the S-matrix, because it does not contribute to the discontinuities across the cuts.

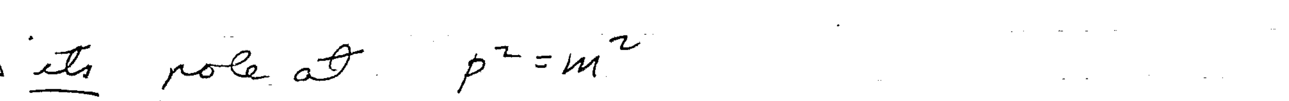
Let us prescribe then, as we may without interfering with unitarity, that in each order in λ, a constant is added to the $\Pi(p^2)$ computed from the Feynman rules. We define

$$\tilde{\Pi}^{(n)}(p^2) = \underbrace{\Pi^{(n)}(p^2)}_{\text{computed from Feynman rules}} - \underbrace{\Pi^{(n)}(m^2)}_{\text{an additive constant, not depending on } p}$$

(n) denotes nth order in λ

this quantity $\tilde{\Pi}^{(n)}(p^2)$ obeys $\tilde{\Pi}^{(n)}(p^2)\Big|_{p^2=m^2} = 0$,

and the propagator obtained from



has its pole at $p^2 = m^2$

Here is a convenient way of describing this modification of the Feynman rules: Replace the interaction Hamiltonian density by

$$\mathcal{H}' \to \mathcal{H}' + \underbrace{\tfrac{1}{2}(\delta m^2)\phi^2}_{\text{= "mass counterterm"}}$$

and so include in the Feynman rules a new vertex

 $= -i(\delta m^2)$

Now, δm^2 is a function of m^2 and λ, and the expression for $\tilde{\Pi}$ in order λ^n becomes

$$\tilde{\Pi}^{(n)}(p^2) = \Pi^{(n)}(p^2) + (\delta m^2)^{(n)}$$

← order λ^n piece of δm^2

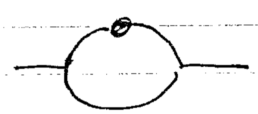
We determine the function

$\delta m^2(\lambda, m^2)$

order by order in λ by demanding that

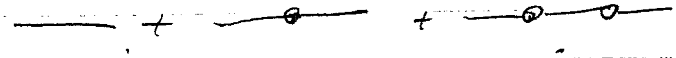
$$\tilde{\Pi}^{(n)}(p^2)\Big|_{p^2=m^2} = 0$$

This formulation of the procedure makes it clearer than in the preceding one that it must be carried out recursively. That is, in computing the graphs contributing to $\Pi^{(n)}(p^2)$, we must include insertions of the counterterms that have been generated in lower order. E.g., in order λ^4



contributes, where  is δm^2 to order λ^2. (This is actually necessary for unitarity.)

Of course, inserting counterterms in all possible ways is the same as using the propagator



$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}(i\delta m^2)\frac{i}{p^2-m^2} = \frac{i}{p^2-m^2-\delta m^2+i\epsilon}$$

-- The mass term in the Lagrangian is

$$\mathcal{L}_{mass} = -\tfrac{1}{2} m_0^2 \phi^2, \qquad m_0^2 = m^2 + \delta m^2,$$

and the amplitudes must not be affected by our (arbitrary) choice of a way of splitting m_0^2 into a "free" part and an interaction.

We could just use as a propagator $\frac{i}{p^2 - m_0^2}$.

Then all amplitudes $\quad A(\lambda, m_0^2, \text{momenta})$

and the physical mass $\quad m^2(\lambda, m_0^2)$

are calculable functions of λ and m_0^2 (in perturbation theory)

But it is far more convenient to use as a propagator $\frac{i}{p^2 - m^2}$, and find a counterterm $\delta m^2(\lambda, m^2)$ in each order of λ, such that the pole in the propagator remains fixed at $p^2 = m^2$ to all orders.
Then m^2 is the physical mass, and we compute directly

$$A(\lambda, m^2, \text{momenta})$$

as a function of the physical mass, which is a measurable quantity. (To do this using the other procedure, we'd have to invert the function $m^2(\lambda, m_0^2)$ order by order to find $m_0^2(\lambda, m^2)$, and plug into $A(\lambda, m_0^2)$. Adding the counterterms order by order is much more direct.)

A better way of describing what we do when we add counterterms is -- we are not really modifying the theory at all. We are merely reparametrizing it in a useful way, so that our calculations will be expressed directly in terms of the directly measurable mass m^2 where the propagator has its pole, rather than the experimentally inaccessible mass m_0^2 that appears in the classical Lagrangian.

(We'll have an even stronger motivation to do this when we find that m_0^2 is <u>infinite</u>, but it is important to appreciate that we would prefer to express our calculations in terms of m^2 even if m_0^2 were finite.)

This procedure of trading in the unphysical m_0^2 for the physical m^2 is called <u>mass renormalization</u>.

m_0^2 is called the "bare mass"

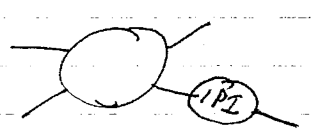
m^2 is called the "renormalized mass"

As described above, order by order in perturbation theory, we can compute m_0^2 in terms of m^2

$$m_0^2 = m_0^2(m^2, \lambda)$$

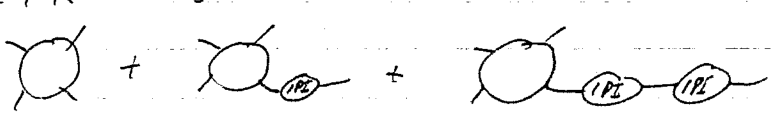
say this after explaining wave function renormalization.

There is another noteworthy advantage to performing mass renormalization as described above. When we compute radiative corrections to scattering, among the diagrams that arise are diagrams of the form --



But this diagram is actually <u>infinite</u> -- when the external particle is "on the mass-shell", the propagator blows up.

We must sum the series:



So the factor associated with the external line is

$$1 + \frac{\Pi(p^2)}{p^2-m^2} + \left(\frac{\Pi(p^2)}{p^2-m^2}\right)^2 + \cdots = \frac{1}{1-\frac{\Pi}{p^2-m^2}} = \frac{p^2-m^2+i\epsilon}{p^2-m^2-\Pi(p^2)+i\epsilon}$$

But if we impose the renormalization conditions

$$\Pi(m^2) = 0 \quad \text{"mass renormalization"}$$
$$\Pi'(m^2) = 0 \quad \text{"field renormalization"} \text{ -- see below}$$

then this factor is trivial (i.e. 1) when $p^2 = m^2$ ("on shell") and the radiative corrections to the external lines may simply be <u>ignored</u>.

Wave Function Renormalization (Field Renormalization)

Even when we perform mass renormalization, there is another problem with our usual way of calculating. The exact propagator

 $= \dfrac{i}{p^2 - m^2 - \Pi(p^2) + i\epsilon}$

has its pole at $p^2 = m^2$ if $\Pi(m^2) = 0$ ("mass renormalization condition"). But what about the <u>residue</u> of the pole?

If $\quad \Pi'(m^2) \equiv \dfrac{d}{dp^2}\Pi(p^2)\Big/_{p^2=m^2} \neq 0$,

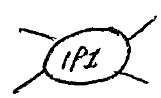
then residue is modified

 $= \dfrac{i}{(p^2-m^2)(1-\Pi'(m^2)) + O(p^2-m^2)^2}$

in vicinity of the pole, or ---

 $\sim \dfrac{iZ}{p^2-m^2+i\epsilon}, \quad Z^{-1} = 1 - \Pi'(m^2)$

Reconsidering "one-particle unitarity" again, we see that the discontinuity in

 =  + crossed + 

due to the pole will be --

$$(A-A^\dagger)_{fi} = Z\, 2\pi i\, \delta(p^2-m^2)\, \langle f|A|p\rangle\langle p|A^\dagger|i\rangle$$

This factor of Z can arise only because our one particle state $|p\rangle$ is not a properly normalized one-particle state, i.e. it obeys

$$\langle p'|p\rangle = \frac{1}{Z}\, 2p^0 (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,'),$$

so that it is

$$\sqrt{Z}\,|p\rangle$$

that has the proper relativistic normalization

The state $|p\rangle$ is the state annihilated by the interaction picture field ϕ_I. We had supposed that ϕ_I was a free field (which is true) with the standard normalization. But we now see that the normalization of ϕ_I is actually nonstandard, if $Z \neq 1$, because of the self-interaction of the field, which does not turn off asymptotically. Because of the self-interactions, the correct "asymptotic behavior" for the field $\phi(\vec{x},t)$ is

$$\underset{\text{"Heisenberg field"}}{\phi(\vec{x},t)} \xrightarrow[t\to\mp\infty]{} \sqrt{Z}\, \underbrace{\phi_{\substack{in\\out}}(\vec{x},t)}_{\text{"properly normalized free field"}}$$

Compare page (2.13)

That is, our interaction picture field ϕ_I, which coincides with ϕ as $t\to-\infty$, obeys

$$\langle 0|\,\phi_I(x)\,|p\rangle = e^{-ip\cdot x}$$

the defined $|p\rangle$. But for a properly normalized state

$$|p\rangle_{norm} = \sqrt{Z}\,|p\rangle$$

$$\langle 0|\,\phi_I(x)\,|p_{norm}\rangle = \sqrt{Z}\, e^{-ip\cdot x}$$

So it is $Z^{-\frac{1}{2}}\phi_I$ that is a properly normalized field.

If we want our amplitudes to be properly normalized, we need to rescale ϕ. Before we had

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - \frac{1}{2} m_0^2 \phi_B - \frac{1}{3!}\lambda \phi_B^3$$

$\uparrow$ mass renormalization performed

The field ϕ_B has the canonical normalization;
Under canonical quantization, its commutator is

$$[\phi_B(\vec{x},t), \dot{\phi}_B(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})$$

But ϕ_B is not properly normalized to destroy a one particle state

$$\langle 0|\phi_B(x)|p\rangle_{norm} = \sqrt{Z}\,e^{-ip\cdot x}$$

It is called the "bare field", hence the subscript B.

If we rescale, then

$$\mathcal{L} = \frac{1}{2} Z\,\partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} m_0^2 Z \phi_R^2 - \frac{1}{3!}\lambda Z^{3/2}\phi_R^3$$

where $\phi_R = Z^{-\frac{1}{2}}\phi_B$. Its commutator is not canonical, but it is properly normalized to annihilate a one-particle state.
ϕ_R is called the "renormalized field"

The field rescaling is called "Field Renormalization" (or Wave Func. Renormalization)

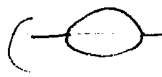
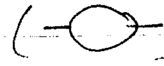
To see explicitly that this rescaling really fixes the problem with the propagator, consider how it changes the Feynman graph. Evidently

$$\overbrace{\phi_R(x)\,\phi_R(y)} = \frac{1}{Z}\,\overbrace{\phi_B(x)\,\phi_B(y)},$$

so in the Feynman rules appropriate for renormalized fields, the propagator is ---

$$\frac{i}{Z(p^2 - m_0^2)} \leftarrow$$ assuming we've summed up insertions of the mass counterterm

In determining how diagrams are rescaled, we must recall that rescaling fields changes both the propagator and the vertices. E.g.

$$\left(\right.$$  $$\left.\right)_{new} = \underbrace{\left(\frac{1}{Z}\right)^2}_{\text{for propagators}} \underbrace{\left(Z^{3/2}\right)^2}_{\text{for vertices}} \left(\right.$$  $$\left.\right)_{old}$$

In general, the effect of rescaling the field on the Feynman rules is such that a graph is rescaled by

$$Z^{-I + \frac{3}{2}V}$$, if it has I internal lines and V vertices

But for *any* diagram $2I + E = 3V$
where E is the number of external lines. (the number of "ends" of lines is no. of lines "consumed" by the vertices)
So factor is

$$Z^{\frac{1}{2}(3V-2I)} = Z^{\frac{1}{2}E}$$

or — just Z for $-i\Pi =$ 

So, with the new Feynman rules, the exact propagator is

$$\left(\right.$$  $$\left.\right)_{new} =$$

$$\frac{i}{Z(p^2 - m_0^2)} + \frac{i}{Z(p^2-m_0^2)} \, {-iZ\Pi(p^2)} \, \frac{i}{Z(p^2-m_0^2)} + \cdots$$

$$= \frac{i}{Z\left(p^2 - m_0^2 - \underbrace{\Pi(p^2)}_{\text{old } \Pi}\right)}$$

If the "old" residue of the pole was Z, the new residue is 1. This is just what we expect, if the "new" Feynman rules give properly normalized amplitudes.

Of course, we do not <u>have</u> to rescale the fields this way, it is merely a convenience. If we don't do the field rescaling, then we need to remember to rescale our n-point <u>amplitudes</u> by the factor

$$Z^{-n/2},$$

at the end of the calculation.

The rescaling of the fields can be carried out <u>iteratively</u>, order by order in perturbation theory, much as we performed the mass renormalization. In terms of renormalized fields, write the Lagrange density as

$$\mathcal{L} = \tfrac{1}{2} Z \partial_\mu \phi_R \partial^\mu \phi_R - \tfrac{1}{2}(m^2+\delta m^2) Z \phi_R^2 - \tfrac{1}{3!} Z^{3/2} \lambda \phi_R^3$$

$$= \tfrac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \tfrac{1}{2} m^2 \phi_R^2 - \tfrac{1}{3!} \lambda' \phi_R^3$$

λ' — a <u>rescaled</u> coupling

$$+ \underbrace{\tfrac{1}{2}(Z-1)\left(\partial_\mu \phi_R \partial^\mu \phi_R - m^2 \phi_R^2\right) - \tfrac{1}{2} Z \delta m^2 \phi_R^2}_{\text{"counterterms"}}$$

Now the propagator in the Feynman rules is

$$\frac{i}{p^2 - m^2}$$

-- which has its pole at the physical mass, and has the standard residue. We may determine $(Z-1)$ and $Z\delta m^2$ order by order in perturbation theory by imposing the two "renormalization conditions":

$\Pi(m^2) = 0$ -- mass renormalization condition

$\frac{d}{dp^2}\Pi(m^2) = 0$ -- field renormalization condition that is, in the nth order of perturbation theory in λ', we have

$$\Pi^{(n)}(p^2) = \underbrace{f^{(n)}(p^2)}_{\substack{\text{sum of diagrams,}\\ \text{including insertion}\\ \text{of lower order}\\ \text{counterterms}}} + (Z\delta m^2)^{(n)} - (Z-1)^{(n)}(p^2-m^2)$$

(Here (n) denotes nth order in λ')

To satisfy the renormalization conditions, we choose

$$(Z\delta m^2)^{(n)} = -f^{(n)}(p^2)\Big|_{p^2=m^2}$$

$$Z^{(n)} = \frac{d}{dp^2} f^{(n)}(p^2)\Big|_{p^2=m^2}$$

Thus, we can compute Z and δm^2 order by order, as functions of λ' and m^2.

It was implicit in the above discussion that the correct Feynman rule for the vertex

$$\mathcal{L}' = \tfrac{1}{2}(Z-1)\left(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2\right)$$

is

$$-p \text{ ---}\otimes\text{--- } p = i(p^2-m^2)(Z-1)$$

This requires a word of explanation, since we have derived Feynman rules only in the case of nonderivative interactions. That is, the perturbation

$$\mathcal{H}' = \tfrac{1}{2}(Z-1)\left(\pi^2 + \vec{\nabla}\phi^2 + m^2\phi^2\right),$$

and in our derivation of Wick's theorem, we considered strings of fields ϕ, without any conjugate momenta. But in this case, it is clear that the rule is the right one, by a self-consistency argument.

That is, we are entitled to demand that the amplitudes be <u>independent</u> of an arbitrary choice of how we divide up the Lagrangian into a "free" part and an interacting part. E.g.

$$\mathcal{L} = Z\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + \cdots$$

$$= \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 + (Z-1)\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \cdots$$

In the first case, the propagator is $\dfrac{i}{Zp^2-m^2}$

In the 2nd case, if we sum up <u>insertions</u> of the interaction, we have

$$\text{——} + \text{——o——} + \text{——o——o——} = \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\, i(Z-1)p^2\, \frac{i}{p^2-m^2} + \cdots$$

$$= \frac{i}{p^2-m^2+(Z-1)p^2} = \frac{i}{Zp^2-m^2}$$

-- which agrees with the other expression, and checks the Feynman rule.

<u>Coupling Renormalization</u>

At one point in the above discussion, we replaced the coupling λ by

$$\lambda' = Z^{3/2}\lambda,$$

without making special comment. Obviously we are free to do this. λ is a free parameter, and we can always replace it by some function times λ, and express our amplitudes as functions of λ' rather than λ. <u>Physics</u> (e.g. relations among measurable quantities) cannot be affected when we trade in λ for λ' (just as it is unaffected when we