3. Spin 1/2

3. SPIN 1/2

So far, we have restricted our attention to the relativistic quantum theory of scalar fields, those with the simplest possible transformation law under Poincare transformations. The corresponding particle excitations associated with such a field are spinless -- the particles are left invariant by a spatial rotation performed in their rest frame. This excludes many interesting cases of particles with intrinsic spin (electrons, photons, etc --)

We will next consider how to construct field theories that describe spin-$\frac{1}{2}$ particles. But first, we will formulate a general classification of fields and particles, according to their transformation properties under the Poincare group.

Representations of the Poincare Group

There are actually two (related) issues concerning the representation theory of the Poincare group that we must confront. They are: (1) How do the fields transform and (2) How do the particles transform. It is convenient to at first discuss these issues separately, and to see how they are related later.

How the Fields transform:

Scalar fields transform under Poincaré transformations in the simplest possible way:

$$U(a,\Lambda)\, \phi(x)\, U(a,\Lambda)^{-1} = \phi(\Lambda x + a)$$

-- where $U(a,\Lambda)$ is the unitary representation of the Poincaré group that acts in the Hilbert space (Here, by Poincaré transformations we mean the restricted transformations with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. We exclude from consideration, for now, the discrete spacetime symmetries.)

We wish to consider whether more general (linear) transformation laws are possible, of the form

$$U(a,\Lambda)\, \phi_\alpha(x)\, U(a,\Lambda)^{-1} = D_{\alpha\beta}(\Lambda^{-1})\, \phi_\beta(\Lambda x + a),$$

where ϕ has a finite number of components labeled by α. The form of the D's is restricted by the requirement that the U's form a representation (actually, a rep. up to a sign):

$$U(a_1,\Lambda_1)\, U(a_2,\Lambda_2) = U(\Lambda_1 a_2 + a_1, \Lambda_1\Lambda_2)$$

or, in particular $U(\Lambda_1)\, U(\Lambda_2) = U(\Lambda_1\Lambda_2)$, (where $U(\Lambda)$ denotes $U(a=0,\Lambda)$).
Thus,

$$U(\Lambda_1\Lambda_2)\, \phi_\alpha(x)\, U(\Lambda_1\Lambda_2)^{-1} = D_{\alpha\beta}(\Lambda_2^{-1}\Lambda_1^{-1})\, \phi_\beta(\Lambda_1\Lambda_2 x)$$

$$= U(\Lambda_1)\, U(\Lambda_2)\, \phi_\alpha(x)\, U(\Lambda_2)^{-1}\, U(\Lambda_1)^{-1}$$

$$= U(\Lambda_1)\, D_{\alpha\gamma}(\Lambda_2^{-1})\, \phi_\gamma(\Lambda_2 x)\, U(\Lambda_1)^{-1}$$

$$= D_{\alpha\gamma}(\Lambda_2^{-1})\, D_{\gamma\beta}(\Lambda_1^{-1})\, \phi_\beta(\Lambda_1\Lambda_2 x)$$

So we require

$$D(\Lambda_2^{-1}\Lambda_1^{-1}) = D(\Lambda_2^{-1})\, D(\Lambda_1^{-1})$$

-- the D's provide a representation of the (restricted) Lorentz group. (Actually, we'll also admit double-valued reps of $SO(3,1)$ -- analogous to the spinor representations of the rotation group $SO(3)$.)

Thus, to characterize all possible transformation laws for the fields, we need to classify (up to equivalence) all representations of the Lorentz group. An arbitrary representation can be expressed as a direct sum of <u>irreducible</u> representations, so it suffices to give a classification of the irreducible representations.

Note that the above argument does not require that D be a <u>unitary</u> representation. In fact, it is easy to see that the Lorentz group has no faithful finite-dimensional unitary representations. The reason is that $SO(3,1)$ is <u>noncompact</u>; the matrix elements $\Lambda^\mu{}_\nu$ are unbounded. But the group $U(n)$ of unitary $n \times n$ matrices is a compact group -- it has finite volume. If $SO(3,1)$ had an n-dimensional faithful unitary representation, then $SO(3,1)$ could be regarded as a subgroup of $U(n)$. But this is impossible -- a group of infinite volume cannot be contained in a group of finite volume. ("Faithful" means that only the identity is represented by the identity.)

We'll return later to the problem of constructing all irreducible representations of $SO(3,1)$. But first we'll examine the other context in which the representation theory of the Poincaré group arises.

<u>How the States Transform:</u>

In the scalar field theory, we found that the one particle states provided a unitary irreducible representation of the Poincaré group:

$$U(a,\Lambda)|k;m^2\rangle = e^{ia\cdot(\Lambda k)}|\Lambda k;m^2\rangle$$

In keeping with the above observation, we see that this representation is infinite dimensional; in fact, the states of the representation are labeled by the continuous parameter k, which takes all real values consistent with the constraints:

$$k^2 = k^\mu k_\mu = m^2 \qquad k^0 > 0$$

The representation is completely specified by the single nonnegative real number m^2.

Another way of seeing that the representations of the Poincare group must be infinite dimensional is to note that

$$U(\Lambda)^{-1} P\, U(\Lambda) = \Lambda P \qquad \text{("invariant abelian subalgebra")}$$

where P_μ is the translation generator. The P_μ's commute, and if they are Hermitian (as they must be if translations are unitarily represented), then they can be simultaneously diagonalized:

$$P_\mu|k\rangle = k_\mu|k\rangle$$

And

$$P_\mu U(\Lambda)|k\rangle = U(\Lambda)\, U(\Lambda)^{-1} P_\mu U(\Lambda)|k\rangle$$
$$= U(\Lambda)(\Lambda P)_\mu|k\rangle = (\Lambda k)_\mu U(\Lambda)|k\rangle$$

Thus, if k_μ is an eigenvalue of $P_\mu \neq$ zero, then all eigenvalues satisfying

$$k^2 = m^2 \quad k^0 > 0$$

<u>must</u> <u>occur</u>. So the representation must be infinite dimensional unless translations are represented <u>trivially</u> (i.e. by the identity).

The scalar particles have spin 0.
We may always (for $m^2 \neq 0$) choose a reference frame in which

$$K = n = (m, 0, 0, 0).$$

In this, the <u>rest</u> frame of the particle, the state $|n; m^2\rangle$ is left invariant by the subgroup of $SO(3,1)$ that leaves $K = n$ invariant. This subgroup is the rotation group $SO(3)$. So the particles are rotationally invariant (in their rest frame) -- they have no spin.

Thus, scalar fields cannot describe particles (electrons, photons, etc.) that have <u>intrinsic spin</u>. To construct a field theory for particles with spin we must include fields obeying the more general transformation law, as described above.

In general, particle states must transform as some (perhaps double valued) representation of the "little group" of rotations ($SO(3)$) that leave the four-momentum n fixed. The irreducible representations of $SO(3)$ satisfy

$$\vec{J}^2 = s(s+1)$$

and the s characterizing the representation of the little group is the spin of the particle. In fact, we will show that m^2 and s <u>completely characterize</u> an irreducible representation of the Poincare group (for $m^2 \neq 0$). In other words, given $m^2 \neq 0$ and s, we can <u>construct</u> the representation of the full Poincare group that satisfies $P_\mu P^\mu = m^2$ and reduces to the spin s representation of the little group. (The case of massless particles must be discussed separately.)

Massive Particles:

We want to show that $m^2 \neq 0$ and the spin S completely characterize the transformation properties of a state under the Poincare group.

First we must describe how to label the states of a representation of the Poincare group. Ordinarily, we label the states of a representation by their eigenvalues with respect to a maximal set of commuting generators. The Poincare group has the 10 generators

P_0, P_1, P_2, P_3 -- translations

J_{12}, J_{23}, J_{31} -- rotations

J_{01}, J_{02}, J_{03} -- boosts

(Notation of page 1.56)

The momenta P_μ are a maximal commuting set. (Boosts and rotations change the momenta) But we know that the momenta k_μ do not completely specify a state if the spin is $S \neq 0$.

We cannot completely distinguish the states with their eigenvalues under the commuting subalgebra, so we must label them another way. We'll denote the states (Actually, we take as the additional commuting generator $\vec{J} \cdot \vec{P} / |\vec{P}|$) -- Not in the algebra

$$|k, \alpha\rangle$$

where α is the index on which act the Lorentz transformations that leave k invariant (the "little group" of k).

For the states $|n, \alpha\rangle$, $n = (m, 0, 0, 0)$, the little group is the rotation group generated by J_{12}, J_{23}, J_{31}, and α may be chosen to be the eigenvalue of J_{12}, which takes the values $-S, -S+1, \ldots, S$. And for $R \in SO(3)$,

$$U(R)\, |n, \alpha\rangle = D_{\alpha\beta}(R)\, |n, \beta\rangle,$$

where D is the spin s representation of $SO(3)$.

Now, in order to deal with states of momentum K, we introduce a "standard" boost $L(k)$ for each value of K, which takes u to K

$$L(K): u \to K$$

($L(K)$ is ambiguous, since it can be replaced by $L(K)R(K)$ for $R \in SO(3)$, but this ambiguity turns out to be harmless.) The elements of the little group of K have the form

$$L(K)\, R\, L(K)^{-1}, \quad \text{for } R \in SO(3).$$

The states $|K,\alpha\rangle$ provide a representation of this little group, and, by continuity, there are $2s+1$ $|K,\alpha\rangle$'s, transforming as the spin s representation. Thus

$$U(L(K) R L^{-1}(K))|K,\alpha\rangle = D_{\alpha\beta}(S(K) R S^{-1}(K))|K,\beta\rangle.$$

And by a suitable redefinition of $L(k)$, we can remove $S(k)$. (Equivalently, we can rotate the $|K,\alpha\rangle$ basis to remove $S(K)$.) In other words, we may identify

$$|K,\alpha\rangle = U(L(K))|u,\alpha\rangle$$

But now the action of the representation on all states is completely determined. We may write

$$U(\Lambda)|K,\alpha\rangle = U(L(\Lambda K))\, U(L^{-1}(\Lambda K)\Lambda L(K))|u,\alpha\rangle$$

And $W(\Lambda,K) = L^{-1}(\Lambda K)\Lambda L(K)$ is a rotation (the "Wigner rotation" associated with Λ and K), which leaves u invariant.

$$U(\Lambda)|k,\alpha\rangle = U(L(\Lambda k))\, D_{\alpha\beta}(W(\Lambda,k))\,|n,\beta\rangle$$
$$= D_{\alpha\beta}(W(\Lambda,k))\,|\Lambda k,\beta\rangle$$

As claimed $U(\Lambda)|k,\alpha\rangle$ is determined by the spin s characterizing the representation D, and m^2.

We should check two things:
(i) that this representation is really unitary.
(ii) that replacing $L(k)\to L(k)R(k)$ does not change the representation (i.e. changes it by only a similarity transformation).

To check that $U(\Lambda)$ is really unitary, we write out the completeness relation:

$$\mathbb{1} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\alpha |k,\alpha\rangle\langle k,\alpha|$$

and thus

$$U(\Lambda)U(\Lambda)^\dagger = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\alpha D_{\alpha\beta}(W(\Lambda,k))\,|\Lambda k,\beta\rangle \langle \Lambda k,\gamma|\, D^\dagger_{\gamma\alpha}(W(\Lambda,k))$$

For each fixed k, D is a unitary rep: $D^\dagger D = \mathbb{1}$

so
$$U(\Lambda)U(\Lambda)^\dagger = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\alpha |\Lambda k,\alpha\rangle\langle \Lambda k,\alpha|$$
$$= \mathbb{1} \quad \text{(by invariance of the measure)}$$

To check that the replacement $L(k)\to L(k)R(k)$ does not give rise to a new representation, note that

$$W(\Lambda,k) \to R^{-1}(\Lambda k)\, W(\Lambda,k)\, R(k)$$

under this replacement. Therefore the

effect of replacing $L(k) \to L(k)R(k)$ on the representation of the Poincare group is to replace

$$U(\Lambda) \to V^{-1}U(\Lambda)V,$$

where $\quad V|k,\alpha\rangle = D_{\alpha\beta}(R(k))\,|k,\beta\rangle$

This means that the two representations are (unitarily) equivalent.

To summarize, we have constructed irreducible representations of the Poincare group from eigenstates of the momenta P_μ that transform as an irreducible representation of the "little group" that leaves the eigenvalues of the P_μ invariant. (This is Wigner's method of "induced representations".) The statement that the representation is irreducible means that any state can be obtained by allowing the representation to act on a given state; e.g. $|n,\alpha\rangle$

Massless Particles:

To find the irreducible unitary representations of the Poincare group with $m^2=0$, we will again use the method of "induced representations." But the implementation of the method is different, because the "little group" is not the rotation group.

Massless particles, of course, have no rest frame. But we can always choose a frame in which the four-momentum is

$$k = n = (1\,0\,0\,1);$$

that is, the particle moves in the $+\hat{z}$ direction with unit momentum (in suitable units).

Now we may ask, what is the "little group" that preserves n? It obviously contains the SO(2) group generated by J_{12} (rotations about $\hat{z}$). But there are, in fact, two other generators. E.g.,

$$\begin{bmatrix} 1 & -\epsilon & 0 & 0 \\ -\epsilon & 1 & 0 & \epsilon \\ 0 & 0 & 1 & 0 \\ 0 & -\epsilon & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

That is, n is preserved by an infinitesimal boost along $\hat{x}$ ($n \to (1\ {-\epsilon}\ 0\ 1)$) followed by an infinitesimal rotation in the x-z plane.
The third generator is obtained by rotating the one above; i.e. (boost along $\hat{y}$) $\times$ (rotation about $\hat{x}$)

In terms of the Lorentz generators, the three generators of the little group are

$$J = J_{12}$$
$$T_1 = J_{10} + J_{31}$$
$$T_2 = J_{20} + J_{23}$$

T_1 and T_2 commute, but they transform as a two-component vector under the rotations generated by J.
The <u>algebra</u> of the generators is

$$[T_1, T_2] = 0$$
$$[T_1, J] = i T_2$$
$$[T_2, J] = -i T_1$$

This is in fact the algebra of the generators of the "Euclidean group of the plane." We may think of T_1, T_2 as generating translations, and J as generating rotations in a two-dimensional space

The Euclidean group of the plane has a structure quite similar to that of the Poincare group itself. The commuting generators T_1 and T_2 form an invariant abelian subalgebra, just like the Poincare generators P_μ, and

$$T_1^2 + T_2^2$$

is an invariant, analogous to $P_\mu P^\mu$. Furthermore, just as for the Poincare group, we can see that, if translations are represented nontrivially, then the representations of the Euclidean group must be infinite dimensional:

If $$T_i |t\rangle = t_i |t\rangle,$$

then $$T_i\, U(R) |t\rangle = (Rt)_i\, U(R) |t\rangle$$

(where R is a rotation), so the eigenvalues of T_1, T_2 take <u>all</u> values on the circle

$$T_1^2 + T_2^2 = \text{constant},$$

just as the eigenvalues of P_μ take all values on the mass hyperboloid.

Once we choose an irreducible representation of the little group, the representation of the Poincare group is completely determined, just as in the massive case; we'll have

$$U(\Lambda) | k, \alpha\rangle = D_{\alpha\beta}\left(L^{-1}(\Lambda k)\Lambda L(k)\right) |\Lambda k, \beta\rangle,$$

where $L(k)$ is a boost that takes n to k, and D is the representation of the little group. But the peculiar new feature of the massless case is that the index α in general takes a continuous range of values, and the rep of the little group is infinite dimensional. The analog, for massless particles, of the spin states of a massive particle are <u>continuous</u> spin states.

But there is a class of particularly simple representations of the little group, those with

$$T_1 = T_2 = 0$$

these representations are finite-dimensional -- one-dimensional, in fact. A representation is specified by just the eigenvalue of J_{12}

$$J_{12} \equiv \lambda = \frac{n}{2}$$

which must be a half-integer if we are to have a single valued representation of (the covering group of) the Lorentz group. The quantity λ is called helicity. It is the component of angular momentum along the direction of motion $\vec{P}/|\vec{P}|$, a Lorentz-invariant quantity for massless particles.

In fact, the only massless representations of Poincaré that we will consider are those with $T_1 = T_2 = 0$, those for which "continuous spin" does not arise. It is believed that these are the only representations that are physically relevant. This belief is based in part on the observation that these are the only representations that seem to arise in relativistic quantum field theory. (Contin. spin → ∞ no. of massless species.)

Notice that in an irreducible representation of the Poincare group, the helicity λ takes only a single value. For example, states with $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$ do not transform into each other under rotations and boosts. This should be contrasted with the case of massive particles -- the component of angular momentum of a spin s massive particle takes $2s+1$ possible values. But rotations and boosts cannot change the component of angular momentum of a massless particle along its direction of motion.

Fields (again): Finite-dim. Reps of the Lorentz Group

Now that we have completely characterized how states can transform under the Poincare group, let us return to the question of how fields transform. As remarked earlier, the problem is to classify all finite-dimensional (nonunitary) irreducible representations of the (covering group of the) restricted Lorentz group. This turns out to be a fairly straightforward generalization of the representation theory of the rotation group.

As with the rotation group, we will find the finite-dimensional representations of (the covering group of the connected component of) the Lorentz group by determining the finite-dimensional representations of the <u>Lie algebra</u> of generators of infinitesimal rotations. This is sufficient, because finite rotation can always be recovered from infinitesimal ones.

The Lie algebra is a set of commutation relations of the form

$$[J_i, J_j] = i\, C_{ijk}\, J_k \qquad \text{(where } J\text{'s span a } \underline{\text{vector space}}\text{)}$$

(For the rotation group, J_1, J_2, J_3 are the components of angular momentum, and $C_{ijk} = \epsilon_{ijk}$, the totally antisymmetric 3-index tensor.) By a finite-dimensional representation of the Lie algebra we mean a set of finite-dimensional matrices that obey these commutation relations

The Lie algebra arises in the theory of continuous groups (or <u>Lie</u> groups) because an <u>infinitesimal</u> group element -- one arbitrarily close to the identity -- can be written in the form ---

$$R(\hat{e},\epsilon) = \mathbb{1} + i\epsilon\,\hat{e}\cdot\vec{J} + O(\epsilon^2)$$

The number of independent components of $\vec{J}$ is the dimension of the group, and $\hat{e}$ is a unit vector that picks out a direction in group space.

A finite transformation that is connected to the identity by a continuous path in the group can be obtained by composing many infinitesimal transformations:

$$R(\hat{e},\theta) = \lim_{N\to\infty}\left[R(\hat{e},\tfrac{\theta}{N})\right]^N$$

$$= \lim_{N\to\infty}\left[\mathbb{1} + i\frac{\theta}{N}\,\hat{e}\cdot\vec{J}\right]^N = \exp\left[i\theta\,\hat{e}\cdot\vec{J}\right]$$

(Compare a finite <u>rotation</u> about the $\hat{e}$ axis)
So a representation of the J_i's as finite-dim. matrices can be exponentiated to obtain a representation of the group.

Of course, the finite transformations obtained in this way must satisfy the group multiplication law, which has the schematic form:

$$\exp[i\vec{\alpha}\cdot\vec{J}]\,\exp[i\vec{\beta}\cdot\vec{J}] = \exp[i\vec{\gamma}(\vec{\alpha},\vec{\beta})\cdot\vec{J}]$$

(where $\vec{\alpha} = \theta\hat{e}$, in the notation used above). The $\vec{J}$'s must be chosen so that $\vec{\gamma}(\vec{\alpha},\vec{\beta})$ is the <u>right</u> functions. But it turns out that, for this <u>purpose</u>, it suffices to require that the $\vec{J}$'s represent the appropriate <u>Lie algebra</u>.

First, we wish to see that the commutator of two generators is itself a generator: For this, use the identity —

$$e^A e^B e^{-A} e^{-B} = \exp\left[[A,B] + \text{higher order in } A, B\right]$$

The left-hand side is a product of group elements and is hence a group element. If

$$A = i\epsilon J_i$$
$$B = i\epsilon J_j$$

Then $\quad RHS = \exp\left[-\epsilon^2 [J_i, J_j] + O(\epsilon^4)\right]$

But this is itself an infinitesimal element of the group, which can be expanded in the J's:

$$= \exp\left[-\epsilon^2 i\, C_{ijk} J_k + \cdots\right]$$

Thus, $\quad [J_i, J_j] = i\, C_{ijk} J_k \quad$ (C_{ijk}'s are called "structure constants" of the algebra.)

The generators obey a <u>Lie algebra</u> that is determined by the group multiplication law.

Next, we wish to see that the group multiplication law for finite group elements can also be recovered from the Lie algebra. This follows from an identity of the form:

$$e^A e^B = \exp\left[A + B + \tfrac{1}{2}[A,B] + \cdots\right]$$

(Baker-Campbell-Hausdorff)

(Each term expressible in terms of <u>commutators</u>.)

The identity uniquely determines any product of finite group elements, in terms of an infinite series (that always converges if A, B are finite dimensional matrices). And to compute this series, we need only be able to perform <u>addition</u> and <u>commutation</u> of generators.

So we see that the problem of finding finite-dim. representations of a Lie group reduces to the problem of finding finite-dimensional representations of the associated Lie algebra. In fact, the exponentiation of a Lie algebra always produces a simply connected space (closed loops can be contracted to a point) so we obtain representations of the covering group of the connected component of the Lie group.

For example, each representation of the angular momentum algebra corresponds to a representation of the group $SU(2)$, the covering group of the rotation group $SO(3)$

("Projected" to the group, a rep of the covering group becomes a rep "up to a phase" or "projective representation." Such reps are of interest in quantum mechanics, because states in Hilbert space are defined only up to an arbitrary phase anyway.)

Our program is now clear: We must find the Lie algebra of the Lorentz group $SO(3,1)$, and then find its irreducible representations.

The group has 6 generators

$$\left.\begin{aligned} J_1 &\equiv J_{23} \\ J_2 &\equiv J_{31} \\ J_3 &\equiv J_{12} \end{aligned}\right\} \text{rotation generators} \qquad \left.\begin{aligned} M_1 &\equiv J_{01} \\ M_2 &\equiv J_{02} \\ M_3 &\equiv J_{03} \end{aligned}\right\} \text{boost generators}$$

If we choose to define $SO(3,1)$ in terms of its action on a 4-vector

$$x \to \Lambda x,$$

then we know how to represent these generators as 4×4 matrices.

write $\Lambda = \mathbb{I} + \omega$ (for ω infinitesimal)

then

$$\Lambda^T \eta \Lambda = \eta = \eta + \omega^T \eta + \eta \omega + \cdots$$

or

$$\eta \omega \eta = -\omega^T$$

The most general 4×4 matrix ω satisfying this condition is

$$\omega = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & \omega_{12} & -\omega_{31} \\ \omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ \omega_{03} & \omega_{31} & -\omega_{23} & 0 \end{pmatrix}$$

$$= i\,\omega_{0k} M_k + i\,\tfrac{1}{2}\epsilon_{k\ell m}\,\omega_{\ell m} J_k$$

E.g. $M_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, etc. (antihermitian)

$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$, etc. (hermitian)

Now, we compute the commutators:

$[J_i, J_j] = i\,\epsilon_{ijk} J_k$ -- the familiar angular momentum algebra

$[J_i, M_j] = i\,\epsilon_{ijk} M_k$ -- this is just the statement that the M_i's transform as a vector under rotations.

$(\,U(R)\,\vec{M}\,U(R)^{-1} = R\vec{M}\,)$

$$[M_i, M_j] = -i\,\epsilon_{ijk}\, J_k$$

-- this is less obvious.

(x boost)(y boost) − (y boost)(x boost) = (Rotation in x-y plane)

And it is quite important to get the <u>sign</u> of the rotation right:

$$[M_1, M_2] = (-1)\left[\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}\right] = \begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$= -i J_3$$

We can reduce the problem of finding the irreducible representations of this algebra to a problem we already know how to solve by a neat trick. Define two sets of <u>hermitian</u> generators

$$\vec{J}^{(+)} = \tfrac{1}{2}(\vec{J} + i\vec{M}) \qquad \vec{J}^{(-)} = \tfrac{1}{2}(\vec{J} - i\vec{M})$$

Then

$$[J_i^{(+)}, J_j^{(+)}] = \tfrac{1}{4} i\epsilon_{ijk}(J_K + J_K + i M_K + i M_K)$$
$$= i\epsilon_{ijk} J_K^{(+)}$$

Similarly,

$$[J_i^{(-)}, J_j^{(-)}] = i\epsilon_{ijk} J_K$$

But $[J_i^{(+)}, J_j^{(-)}] = \tfrac{1}{4} i\epsilon_{ijk}(J_K - J_K + M_K - M_K) = 0$

We have two commuting SU(2) algebras of hermitian generators. (This trick works only in 3+1 dimensions.)

We already know everything about the irreducible representations of the SU(2) algebra.

Some of their properties:

- Labeled by a half integer $s = \frac{n}{2}$, $n = 0, 1, 2, 3 \ldots$
- $(2s+1)$ - dimensional
- States can be chosen eigenstates of J_3, with eigenvalues $m = -s, -s+1, \ldots s-1, s$
- Matrix elements of $J_{1,2}$ between such states are readily inferred from the algebra.
- They exponentiate to (in general double-valued) unitary reps $D^{(s)}(R)$ of the rotation group.
- Reality: $D^{(s)}(R)^*$ is unitarily equivalent to $D^{(s)}(R)$ (This is clear because the $(2s+1)$-dim. rep is unique, up to equivalence)
- Direct Product:

$$D^{(s_1)} \otimes D^{(s_2)} = D^{(s_1+s_2)} \oplus D^{(s_1+s_2-1)} \oplus \cdots \oplus D^{|s_1-s_2|}$$

(addition of angular momenta)

- Symmetry:

In the product $D^{(s)} \otimes D^{(s)}$ of two identical representations,

$D^{(2s)}, D^{(2s-2)}, \ldots$ are symmetric

$D^{(2s-1)}, D^{(2s-3)}, \ldots$ are antisymmetric

under interchange of the two factors

Properties of the Irreducible Representations of SO(3, 1)

From the above properties of the irreducible representations of the rotation group, and the observation that the SO(3,1) algebra decomposes into two commuting SU(2) algebras, we may infer the corresponding properties of the SO(3,1) representations

- The representations may be denoted $D^{(S_+, S_-)}(\Lambda)$. They are labeled by two half integers S_+, S_- -- the representations of the $\vec{J}^{(+)}$ and $\vec{J}^{(-)}$ algebras. We have

$$\vec{J} = \vec{J}^{(+)} + \vec{J}^{(-)}$$
$$\vec{M} = (-i)(\vec{J}^{(+)} - \vec{J}^{(-)})$$

Finite rotations are represented as

$$D[R(\hat{e}, \theta)] = \exp[i\theta(\vec{J}^{(+)} + \vec{J}^{(-)})\cdot\hat{e}] \quad \text{-- \underline{unitary}}$$

Finite boosts are represented as

$$D[B(\hat{e}, \theta)] = \exp[\theta(\vec{J}^{(+)} - \vec{J}^{(-)})\cdot\hat{e}] \quad \text{-- \underline{non-unitary}}$$

- Dimensionality:

The representations have dimension

$$(2S_+ + 1)(2S_- + 1).$$

A basis for the representation is the states

$$|m_+, m_-\rangle$$

satisfying

$$J_3^{(+)}|m_+, m_-\rangle = m_+|m_+, m_-\rangle$$
$$J_3^{(-)}|m_+, m_-\rangle = m_-|m_+, m_-\rangle$$

(Since $J^{(+)}$ and $J^{(-)}$ commute, the representations $D^{(s_+)}$ and $D^{(s_-)}$ of $SU(2)$ act independently on these states, without interfering with each other)

- Complex Conjugation:
  we know that $D^{(s)} \sim D^{(s)*}$ (unitarily equivalent) for $SU(2)$ representations. Hence
$$(\vec{J}^{(\pm)})^* \sim -\vec{J}^{(\pm)}$$
Since
$$\vec{J} = \vec{J}^{(+)} + \vec{J}^{(-)}$$
$$\vec{M} = (-i)(\vec{J}^{(+)} - \vec{J}^{(-)}) ,$$
we see that
$$\vec{J}^* \sim -\vec{J}$$
$$\vec{M}^* \sim \vec{M}$$
In other words, the effect of complex conjugation on the boost
$$D[B(\hat{e},\theta)] = \exp[\theta \hat{e}\cdot(\vec{J}^{(+)} - \vec{J}^{(-)})]$$
is to interchange $\vec{J}^{(+)}$ and $\vec{J}^{(-)}$. Thus
$$[D^{(s_+,s_-)}(\Lambda)]^* \sim D^{(s_-,s_+)}(\Lambda)$$
The complex conjugate of a state transforming as the representation (s_+,s_-) transforms as (s_-,s_+)

- Parity:
  The $SO(3,1)$ Lie algebra is left invariant by a "parity" operation:
$$\vec{J} \to \vec{J}$$
$$M \to -\vec{M}$$

this operation therefore takes one representation of $SO(3,1)$ to another. Since

$$\vec{J}^{(+)} \longleftrightarrow \vec{J}^{(-)}$$

under this parity operation. We see that parity (like complex conjugation) takes

$$D^{(s_+,s_-)} \longrightarrow D^{(s_-,s_+)}$$

- Direct Product:

The two "angular momenta" generated by $\vec{J}^{(+)}$ and $\vec{J}^{(-)}$ add independently:

$$D^{(s_+,s_-)} \otimes D^{(s_+',s_-')} = \bigoplus_{s_+''=|s_+-s_+'|}^{s_++s_+'} \ \bigoplus_{s_-''=|s_--s_-'|}^{s_-+s_-'} D^{(s_+'',s_-'')}$$

- Symmetry

In the product $D^{(s_+,s_-)} \otimes D^{(s_+,s_-)}$

the reps

$(2s_+, 2s_-), (2s_+-2, 2s_-), (2s_+-1, 2s_--1)$, etc are symmetric

$(2s_+, 2s_--1), (2s_+-1, 2s_-)$, etc are antisymmetric

under interchange of the two factors

- Rotations

Under the rotation subgroup of $SO(3,1)$ generated by $\vec{J} = \vec{J}^{(+)} + \vec{J}^{(-)}$

the irreducible representation $D^{(s_+, s_-)}$ transforms reducibly.

$$D^{(s_+, s_-)}(R) \sim \bigoplus_{s=|s_+-s_-|}^{s_++s_-} D^{(s)}(R)$$

-- addition of angular momentum.

Examples

- The simplest nontrivial representations are
$$D^{(\frac{1}{2},0)}, \quad D^{(0,\frac{1}{2})}.$$
They are two-dimensional, and transform as spin-$\frac{1}{2}$ under the rotation group. They are parity conjugates and complex conjugates of one another.

- The representation $D^{(\frac{1}{2},\frac{1}{2})} = D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})}$ is the four-vector. To see that, note that there are three different 4-dimensional reps
$$D^{(\frac{1}{2},\frac{1}{2})}, \quad D^{(3/2,0)}, \quad D^{(0,3/2)}$$
But the latter two are not invariant under parity conjugation and complex conjugation. Furthermore, under SO(3) rotations,
$$D^{(\frac{1}{2},\frac{1}{2})} \longrightarrow 1 + 0,$$
a vector and a scalar. This is the way a 4-vector is supposed to transform under rotations.

- Two-index tensor

A direct product of two four-vectors, $V_\mu W_\nu$, transforms as

$$D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})} = D^{(0,0)} + D^{(1,0)} + D^{(0,1)} + D^{(1,1)}$$

dimension:	16 =	1 +	3 +	3 +	9
symmetry:		S	A	A	S

What are these irreducible representations in tensor language?

The symmetric part of the direct product may be decomposed into a trace and a traceless tensor

$$\tfrac{1}{2}(V_\mu W_\nu + V_\nu W_\mu)$$

$$= \underbrace{\tfrac{1}{4}\eta_{\mu\nu} V\cdot W}_{1} + \underbrace{\tfrac{1}{2}(V_\mu W_\nu + V_\nu W_\mu - \tfrac{1}{2}\eta_{\mu\nu} V\cdot W)}_{9}$$

Both are preserved by Lorentz transformations. These are the 1- and 9-dimensional representations.

The antisymmetric part

$$A_{\mu\nu} = \tfrac{1}{2}(V_\mu W_\nu - V_\nu W_\mu)$$

has 6 components, and can also be decomposed into two irreducible pieces. This decomposition is accomplished with the dual operation. The dual of an antisymmetric tensor is defined by

$$*A_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma} A^{\lambda\sigma}$$

Note that, e.g.

$$*A_{01} = A_{23} \qquad *A_{23} = -A_{01}$$

so

$$**A_{\mu\nu} = -A_{\mu\nu}$$

Thus, the dual operation $*$ has eigenvalues $\pm i$

The eigenstates are $A_{\mu\nu} \mp i \,{}^*\!A_{\mu\nu}$,

since $*(A_{\mu\nu} \mp i\,{}^*\!A_{\mu\nu}) = \pm i(A_{\mu\nu} \mp i\,{}^*\!A_{\mu\nu})$

These eigenvalues are preserved by Lorentz transformations, since

$$*(\Lambda A \Lambda^T) = \Lambda({}^*\!A)\Lambda^T,$$

which follows from $\det\Lambda = 1$.

The Covering Group $SL(2,\mathbb{C})$

We have remarked several times that some of the representations of $SO(3,1)$ are "double-valued." This is a familiar phenomenon. The rotation group $SO(3)$ also has double-valued representations (those with s = half-odd-integer), which are single-valued representations of the covering group $SU(2)$. We would like to find the corresponding covering group for $SO(3,1)$. In the process, we will find the explicit connection between the 4-vector representation $D^{(\frac{1}{2},\frac{1}{2})}$ and the spinor representation $D^{(\frac{1}{2},0)}$.

If we recall the trick for constructing the 2 to 1 mapping $SU(2) \to SO(3)$, it is easy to see how to generalize it to the rotation group. We associate with each real 4-vector V^μ a matrix $\underset{\sim}{V}$

$$V^\mu \longrightarrow \underset{\sim}{V} = \begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix} = V^\mu \sigma_\mu$$

where $\sigma_\mu = (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)$

The matrix $\underset{\sim}{V}$ is hermitian if V^μ is real, and

$$\det \underset{\sim}{V} = V_0^2 - \vec{V}^2 = V^\mu V_\mu$$

-- the invariant inner product.

$SO(3,1)$ transformations are linear transformations that preserve the reality of V^μ and the length $V^\mu V_\mu$. In this representation, they must preserve the hermiticity of $\underset{\sim}{V}$ and $\det \underset{\sim}{V}$. Transformations with this property are

$$\underset{\sim}{V} \longrightarrow \underset{\sim}{A}\,\underset{\sim}{V}\,\underset{\sim}{A}^\dagger, \quad \text{where } |\det \underset{\sim}{A}| = 1$$

An overall phase in $\underset{\sim}{A}$ has no effect on $\underset{\sim}{V}$, so we may as well choose

$$\det \underset{\sim}{A} = 1$$

We have thus constructed a homomorphism

$$SL(2,\mathbb{C}) \longrightarrow SO(3,1;\mathbb{R})$$

where $SL(2,\mathbb{C})$ is the multiplication group of 2×2 matrices of complex numbers with determinant 1. This homomorphism has a nontrivial kernel, since both $\mathbb{1}$ and $-\mathbb{1}$ in $SL(2,\mathbb{C})$ correspond to the identity in $SO(3,1)$. It is a 2-to-1 mapping, analogous to the mapping $SU(2) \rightarrow SO(3)$. $SL(2,\mathbb{C})$ is the covering group of the Lorentz group.

The fundamental two-dimensional representation of $SL(2,\mathbb{C})$ must be the $D^{(\frac{1}{2},0)}$ representation considered earlier (or $D^{(0,\frac{1}{2})}$ -- a choice made by convention). The generators of

$SL(2,\mathbb{C})$ are traceless 2×2 matrices. A convenient basis is --

$$\left.\begin{aligned}\vec{J} &= \tfrac{1}{2}\vec{\sigma}\\ \vec{M} &= \tfrac{1}{2}i\vec{\sigma}\end{aligned}\right] \quad \text{the representation } D^{(\frac{1}{2},0)} \qquad \begin{aligned}U(R(\hat{e},\theta)) &= e^{i\vec{\sigma}\cdot\hat{e}\,\theta/2}\\ U(B(\hat{e},\theta)) &= e^{-\vec{\sigma}\cdot\hat{e}\,\theta/2}\end{aligned}$$

or

$$\left.\begin{aligned}\vec{J} &= \tfrac{1}{2}\vec{\sigma}\\ \vec{M} &= -\tfrac{1}{2}i\vec{\sigma}\end{aligned}\right] \quad \text{the representation } D^{(0,\frac{1}{2})} \qquad \begin{aligned}U(R(\hat{e},\theta)) &= e^{i\vec{\sigma}\cdot\hat{e}\,\theta/2}\\ U(B(\hat{e},\theta)) &= e^{\vec{\sigma}\cdot\hat{e}\,\theta/2}\end{aligned}$$

The transformation property of the matrix $\underset{\sim}{V}$, writing out indices, is

$$V_{\alpha\dot{\alpha}} \longrightarrow A_{\alpha\beta}\, A^{*}_{\dot{\alpha}\dot{\beta}}\, V_{\beta\dot{\beta}}$$

Here we have introduced a notational convention, according to which indices acted on by $\underset{\sim}{A} \in SL(2,\mathbb{C})$ are "undotted" and indices acted on by $\underset{\sim}{A}^*$ are "dotted." This notation makes it evident that $V_{\alpha\dot{\alpha}}$ transforms as the $D^{(\frac{1}{2},\frac{1}{2})}$ representation -- the direct product of the defining representation of $SL(2,\mathbb{C})$ and the conjugate representation.

An arbitrary irreducible representation of $SL(2,\mathbb{C})$ is easily expressed in this tensor notation. We merely note that

$$D^{(s_+,s_-)} = D^{(s_+,0)} \otimes D^{(0,s_-)}$$

or...

$$D^{(S_+, S_-)} = \underbrace{D^{(\frac{1}{2},0)} \otimes \cdots \otimes D^{(\frac{1}{2},0)}}_{2S_+ \text{ times, symmetrized}} \otimes \underbrace{D^{(0,\frac{1}{2})} \otimes \cdots \otimes D^{(0,\frac{1}{2})}}_{2S_- \text{ times, symmetrized}}$$

Thus

$$D^{(\frac{n}{2}, \frac{m}{2})} \sim \psi_{\alpha_1 \cdots \alpha_n, \dot\beta_1, \cdots \dot\beta_m}$$

The tensor ψ in the $(\frac{n}{2}, \frac{m}{2})$ rep is completely symmetric in its n undotted indices, and in its m dotted indices.

More about Spin-$\frac{1}{2}$

We will denote a two-component spinor in the $D^{(\frac{1}{2},0)}$ rep. by ψ_R, and one in the $D^{(0,\frac{1}{2})}$ rep by ψ_L; we will refer to them as right-handed and left-handed <u>Weyl spinors</u>. The notation anticipates the helicity of the states annihilated by fields $\psi_R(x)$ and $\psi_L(x)$, which we will check later.

The Weyl spinor $\psi_{R\alpha}$, $\alpha = 1,2$ has an undotted index. We may construct from it a 4-vector:

$$\psi_{R\alpha} \psi^*_{R\dot\alpha} = V^\mu \sigma_\mu$$

To extract the components of V^μ, we note that

$$\mathrm{tr}\, \sigma_\mu \sigma_\nu = 2\delta_{\mu\nu}.$$

Hence,

$$2V^\mu = \psi_{R\alpha} \psi^*_{R\dot\alpha} (\sigma_\mu)_{\dot\alpha\alpha} = \psi_R^\dagger \sigma_\mu \psi_R$$

We see that the bilinear $(\psi_R^\dagger \psi_R, \psi_R^\dagger \vec{\sigma} \psi_R)$ transforms as a 4-vector under $SO(3,1)$

We can construct such a 4-vector from ψ_L also. However, it is not correct to write $\psi_{L\dot\alpha}$, and assume that ψ_L transforms under $SL(2,C)$ exactly as ψ_R^* does. It transforms the same way only up to a unitary change of basis. Comparing the $D^{(\frac{1}{2},0)}$ and $D^{(0,\frac{1}{2})}$ generators on page 3.27, we see that we need to find unitary V such that

$$\left.\begin{aligned} V\vec{J}^*V^\dagger &= -\vec{J} \\ V\vec{M}^*V^\dagger &= \vec{M} \end{aligned}\right\} \text{ or } \quad V\vec{\sigma}^*V^\dagger = -\vec{\sigma}$$

An apropriate choice is $V = -i\sigma_2$ (the $-i$ is merely conventional). So the right way to obtain a R-handed Weyl spinor from a L-handed one is

$$\psi_{R\alpha} = [-i(\sigma_2) \quad \psi_L^*]_\alpha$$

To construct a 4 vector from ψ_L, we use

$$\psi_R^\dagger \sigma_\mu \psi_R = \psi_L^T \sigma_2 \sigma_\mu \sigma_2 \psi_L^* = \psi_L^\dagger \sigma_2 \sigma_\mu^T \sigma_2 \psi_L$$

$$= \psi_L^\dagger \tilde{\sigma}_\mu \psi_L \quad \text{where } \tilde{\sigma}_\mu = (\mathbb{1}, -\vec{\sigma})$$

It is also easy to construct Lorentz-singlets. The singlet is the antisymmetric product of two $D^{(\frac{1}{2},0)}$'s. That is,

$$\epsilon^{\alpha\beta}\psi_{R\alpha}\psi_{R\beta} = \psi_R^T i\sigma_2 \psi_R \quad (\epsilon^{01} = 1 = -\epsilon^{10})$$

is a singlet. So are

$$\psi_L^T i\sigma_2 \psi_L, \qquad \psi_R^T (i\sigma_2)(-i\sigma_2)\psi_L^* = \psi_L^\dagger \psi_R$$

and $\psi_R^\dagger \psi_L$.

Spin and Statistics

In our discussion of scalar field theories, a useful notion in the proof of the CPT theorem and the connection between spin and statistics is the idea of complexification of the Lorentz group -- the extension of $L(\mathbb{R})$ to $L(\mathbb{C})$, the group of complex-valued matrices satisfying $\Lambda^T \eta = \Lambda$. This notion is useful because when we continue the Wightman functions

$$W(x_1, \ldots x_n) = \langle 0 | \phi(x_1) \ldots \phi(x_n) | 0 \rangle$$

to complex values of the difference variables $x_1 - x_2$, $x_2 - x_3$, etc., these functions are invariant under complex Lorentz transformations throughout their domain of analyticity

To extend the CPT theorem and spin-statistics connection to fields of arbitrary spin, we will need to consider how fields in arbitrary $SL(2,\mathbb{C})$ representations transform under complexified Lorentz transformations.

This is simple. We want to generalize the transformation law

$$\underset{\sim}{V} \to \underset{\sim}{A}\,\underset{\sim}{V}\,\underset{\sim}{A}^{+}, \quad A \in SL(2,\mathbb{C})$$

so that $\det \underset{\sim}{V}$ is still preserved, but the hermiticity of $\underset{\sim}{V}$ is not preserved.

So we consider

$$\underset{\sim}{V} \to \underset{\sim}{A}\,\underset{\sim}{V}\,\underset{\sim}{B} \qquad A, B \in SL(2,\mathbb{C})$$

The complexified Lorentz group has the covering group

$$SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$$

the dotted and undotted indices are acted on by <u>independent</u> $SL(2,\mathbb{C})$ transformations.

Of special interest are the transformation properties of fields under $-\mathbb{1} \in L(\mathbb{C})$. Like all elements of the Lorentz group, this element is covered by 2 elements of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ that is,

$$\left.\begin{array}{c} (\mathbb{1}, -\mathbb{1}) \\ (-\mathbb{1}, \mathbb{1}) \end{array}\right\} \longrightarrow -\mathbb{1}$$

$$\begin{array}{cc} \cap & \cap \\ SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) & L(\mathbb{C}) \end{array}$$

A general field $\phi^{(S_+, S_-)}$ may be regarded as a tensor with $2S_+$ undotted indices and $2S_-$ dotted indices. Hence

$$\phi^{(S_+,S_-)} \longrightarrow \left\{\begin{array}{c} (-1)^{2S_+} \\ (-1)^{2S_-} \end{array}\right\} \phi^{(S_+,S_-)}$$

under $-\mathbb{1} \in L(\mathbb{C})$. There are, for half-odd-integer spin representations ($2S_+ + 2S_- =$ odd), two choices, depending on which representative of $-\mathbb{1}$ in $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ is chosen.

Now we are ready to state and prove the spin-statistics connection for fields in arbitrary representations of $SL(2,\mathbb{C})$, generalizing our earlier proof for scalar fields. (Problem 4)

We assume the Wightman axioms of page 1.73ff, except that ① the relativistic transformation law is generalized to

$$U(a,\Lambda)\,\phi(x)\,U(a,\Lambda)^{-1} = D^{(S_+,S_-)}(\Lambda^{-1})\,\phi(\Lambda x+a)$$

-- each field transforms according to some irreducible representation (S_+,S_-) of $SL(2,\mathbb{C})$, and ④ causality is stated as --

For any two fields ϕ_1, ϕ_2, either

$$[\phi_1(x), \phi_2(y)]_- = 0 \quad \text{for } (x-y)^2<0$$

or

$$[\phi_1(x), \phi_2(y)]_+ = 0 \quad \text{for } (x-y)^2<0$$

Either commutators (Bose statistics) or Anticommutators (Fermi statistics) vanish outside the light cone. (In the latter case, we may construct an algebra of commuting observables from expressions which are even in Fermi fields.)

The point of the connection between spin and statistics is that consistency requires the statistics of fields to be correlated with their representation content under $SL(2,\mathbb{C})$. Specifically,

$2(S_++S_-)$ even (integer angular momentum) $\Longrightarrow$ Bose statistics

$2(S_++S_-)$ odd (half-odd-integer angular momentum) $\Longrightarrow$ Fermi statistics

We'll sketch the proof of this, in several steps.

Theorem:

Suppose the commutator of ϕ with ϕ^+ is chosen to be "abnormal": i.e.

$$\left.\begin{array}{ll} [\phi(x), \phi^+(y)]_- = 0, & 2(s_+ + s_-) = \text{odd} \\ \text{or} \quad [\phi(x), \phi^+(y)]_+ = 0, & 2(s_+ + s_-) = \text{even} \end{array}\right\} \text{ for } (x-y)^2 < 0$$

Then $\qquad \phi(x)|0\rangle = 0$

Proof: Consider the functions

$$W(z) = \langle 0| \phi(z) \phi^+(0) |0\rangle$$
$$\hat{W}(z) = \langle 0| \phi^+(z) \phi(0) |0\rangle$$

From the commutation relations, we have

$$0 = \langle 0| \phi(z) \phi^+(0) |0\rangle \mp \langle 0| \phi^+(0) \phi(z) |0\rangle$$
$$= W(z) \mp \hat{W}(-z) \quad \left(\begin{smallmatrix}\text{Bose}\\ \text{Fermi}\end{smallmatrix}\right) \quad \text{for } z \text{ real}, z^2 < 0$$

(using translation invariance of the vacuum.)

This identity is satisfied throughout the domain of analyticity of the function $W(z) \mp \hat{W}(-z)$. Within this domain $\hat{W}(-z)$ is "covariant" under complex Lorentz transformations.
In particular, under the transformation $-\mathbb{1} \in L(\mathbb{C})$, we have

$$\phi(z) \to (-1)^{2s_+} \phi(-z)$$
$$\phi^+(z) \to (-1)^{2s_-} \phi^+(-z)$$

(ϕ^+ transforms as the representation (s_-, s_+) if ϕ transforms as (s_+, s_-)) Thus,

$$\hat{W}(-z) = (-1)^{2(s_+ + s_-)} \hat{W}(z)$$

Note that the sign here suffers from no ambiguity, it is independent of how we choose to represent $-\mathbb{1}$ in $SL(2,\mathbb{C})$. Now

$$0 = \mathcal{W}(z) \mp (-1)^{2(s_+ + s_-)} \hat{\mathcal{W}}(z) \qquad \left(\begin{smallmatrix}\text{Bose}\\ \text{Fermi}\end{smallmatrix}\right)$$

holds throughout the domain of analyticity, and in particular for real z, which lies on the boundary of this domain. If the statistics is "abnormal", then $\mathcal{W}(z) + \hat{\mathcal{W}}(z) = 0$, or

$$0 = \langle 0|\phi(x)\phi^\dagger(x+y)|0\rangle + \langle 0|\phi^\dagger(x)\phi(x+y)|0\rangle$$

If we choose $y=0$, then the RHS is a sum of two nonnegative terms, and we have

$$\| \phi(x)|0\rangle \|^2 = \langle 0|\phi^\dagger(x)\phi(x)|0\rangle = 0,$$

or

$$\phi(x)|0\rangle = 0$$

If ϕ were the only field in the theory, this result and the "completeness" axiom would imply that $\mathcal{H} = \{0\}$ -- the Hilbert space is trivial (as in the homework problem) But we may want to consider the case that ϕ is one of many fields. In that case we appeal to ---

Theorem:

If $A(x)$ is a local field (commutes or anticommutes with all other fields at spacelike separation) then

$$A(x)|0\rangle = 0 \quad \text{implies} \quad A(x) = 0$$

To prove this, first we'll need —

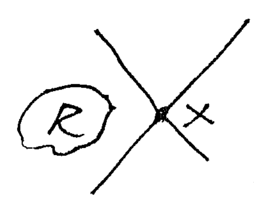
Lemma: If R is any open set in spacetime, then (polynomials in) fields smeared in R acting on the vacuum $|0\rangle$ are dense in the Hilbert space $\mathcal{H}$.

Proof: The function $\langle\psi|\phi_1(x_1)\cdots\phi_n(x_n)|0\rangle = W_\psi$ has the same analyticity properties as a vacuum expectation value. Hence, if it vanishes for $x_1,\dots,x_n \in R$, some open set, it vanishes throughout its domain of analyticity, and therefore

$$W_\psi = 0 \quad \text{for all real } x_1,\dots,x_n$$

If fields smeared in R are not dense, then we can find some state $|\psi\rangle$ such that $W_\psi^{(n)}=0$ for all n, for all fields $\phi_1\cdots\phi_n$ and for all $x_1,\dots,x_n\in R$. But the above argument shows that, in that case, the state ψ is not in the span of fields acting on the vacuum with x unrestricted, which contradicts completeness of the fields.

Proof of Theorem:



If $A(x)|0\rangle=0$, let R be some open region of spacetime spacelike separated from x. If P is any polynomial in fields smeared in R, then $[P, A(x)]_\mp = 0$, since A is a local field. Thus, for any state $|\psi\rangle$

$$0 = \langle\psi|\, P\, A(x)\,|0\rangle = \mp\langle\psi|\, A(x)\, P\,|0\rangle$$

But polynomials in fields smeared in R span $\mathcal{H}$, so $A(x)|\psi\rangle = 0$, or $A(x)=0$.

We have now succeeded in showing that, if ϕ obeys an abnormal commutation relation with its adjoint, then $\phi = 0$. Therefore, the field ϕ has no physical contents and can be dropped from the theory.

Next, we can show that commutator of ϕ with itself must be normal.

Theorem:

If $[\phi(x), \phi^\dagger(y)]_\pm$ is "normal", then so is $[\phi(x), \phi(y)]_\pm$

This will follow from --

Theorem (Cluster Decomposition):

$$\lim_{x \to \infty} \langle 0 | \phi_1(x) \phi_2(0) | 0 \rangle = \langle 0 | \phi_1(0) | 0 \rangle \langle 0 | \phi_2(0) | 0 \rangle$$

"Proof": Insert a sum over intermediate states

$$\langle 0 | \phi_1(x) \phi_2(0) | 0 \rangle = \sum_n \langle 0 | \phi_1(0) | n \rangle \langle n | \phi_2(0) | 0 \rangle e^{-iP_n \cdot x}$$

The result follows if we let $x \to \infty$ and use the Riemann-Lebesgue argument of page (2.143). (It does not matter whether limit is taken for x timelike or spacelike.)

Now, to prove the theorem above, note that, if ϕ has an abnormal commutator with itself, and normal with $\phi^\dagger$, then

$$0 \le \| \phi(y)\phi(x) \|^2 = \langle 0| \phi^+(x)\phi^+(y)\phi(y)\phi(x)|0\rangle$$
$$= -\langle 0|\phi^+(y)\phi(y)\phi^+(x)\phi(x)|0\rangle,$$

by the assumed commutation relations, for $(x-y)^2 < 0$.
Now, use the cluster property in the limit $(x-y)\to\infty$ with $x-y$ spacelike:

$$0 \le -\langle 0|\phi^+(y)\phi(y)|0\rangle\langle 0|\phi^+(x)\phi(x)|0\rangle$$
$$= -\| \phi(y)|0\rangle \|^4$$

A nonnegative quantity is ≤ 0; therefore it $=0$

$$\phi(y)|0\rangle = 0 \Rightarrow \phi(y) = 0$$

We've seen that the commutator of ϕ with itself and with its adjoint must be "normal", but what about the commutator of two different fields?

The standard choice is

$$\left.\begin{array}{l} [\text{Boson}, \text{Boson}]_- = 0 \\ [\text{Boson}, \text{Fermion}]_- = 0 \\ [\text{Fermion}, \text{Fermion}]_- = 0 \end{array}\right\} \text{ for spacelike separation}$$

This choice, in fact, applies in general, but the correct statement is not that we must make this choice, but rather that we always can make this choice.

I will not describe the most general version of this theorem, but will describe an example that illustrates how it works. The general theorem is discussed in detail in --

R. Streater and A. Wightman, "PCT, Spin and Statistics, and all that."

Example:

Suppose that $\phi(x)$ is a Bose field and $\psi(x)$ is a Fermi field. The commutator of ϕ with itself and the (anti)commutator of ψ with itself are normal. But ψ and ϕ obey the abnormal relation

$$[\psi(x), \phi(y)]_+ = 0 \quad , \quad (x-y)^2 < 0.$$

Polynomials in ψ and ϕ acting on the vacuum are dense in the Hilbert space $\mathcal{H}$, so we may divide $\mathcal{H}$ into two sectors

$\mathcal{H}_+$ -- spanned by polynomials even in ψ acting on $|0\rangle$
$\mathcal{H}_-$ -- spanned by polynomials odd in ψ acting on $|0\rangle$

Obviously, both sectors are preserved by the operator ϕ,

$$\phi(x): \ \mathcal{H}_\pm \to \mathcal{H}_\pm ,$$

while ψ interchanges them

$$\psi(x): \ \mathcal{H}_\pm \to \mathcal{H}_\mp$$

Now perform a change of variable. Define a new field ϕ' by

$$\phi'(x) = \phi(x) \qquad \text{acting on } \mathcal{H}_+$$
$$\phi'(x) = -\phi(x) \qquad \text{acting on } \mathcal{H}_-$$

or
$$\phi'(x) = (-1)^F \phi(x)$$

where F is the fermion (or ψ) number.
Since ψ interchanges $\mathcal{H}_+$ and $\mathcal{H}_-$, we see that

$$(-1)^F \psi = -\psi(-1)^F,$$

and
$$\psi(x)\phi'(y) = \psi(x)(-1)^F\phi(y) = \phi(y)(-1)^F\psi(x), \quad (x-y)^2 < 0$$

or
$$[\psi(x)\phi'(y)]_- = 0.$$

We have managed to redefine the field ϕ so that its commutator with ψ is normal. We have not disturbed the normal commutator of ϕ with itself, or the Poincare transformation property of ϕ, because the unitary operator $(-1)^F$ is in fact a symmetry; it commutes with Poincare transformations. (Actually, this symmetry operation is just a rotation by 2π.)

A change of variable like the above that changes a commutator to an anticommutator (or vice-versa) is called a Klein transformation. The general connection between spin and statistics may be stated this way: A theory with abnormal commutators always respects symmetries, such that a Klein transformation can be performed that makes the commutators normal. (The Klein transformation is just a change of variable and does not modify the physics of the theory.)

Free Field Theory for Spin $\frac{1}{2}$

Now we would like to construct a classical free field theory for the right-handed Weyl spinor $\psi_{R\alpha}(x)$, a field transforming as the representation $(\frac{1}{2},0)$ of $SL(2,\mathbb{C})$. As when we constructed the free scalar field theory, we demand that the action respect a few general principals. It should --

- Depend only on fields and their <u>first</u> derivatives.
- Be local.
- Be Poincare invariant

Thus we write $S = \int d^4x\, \mathcal{L}(\partial_\mu \psi_R(x), \psi_R(x))$, where $\mathcal{L}$ is a Lorentz singlet. We also demand that

- $\mathcal{L}$ is hermitian,

so that the action is real.

The defining property of free field theory is

- $\mathcal{L}$ is quadratic (at most) in fields,

so that the classical eqn of motion is a linear equation.

Let us catalog the terms we can construct from ψ_R that are Lorentz-invariant and quadratic. On page 3.28 ff, we noted that

$\psi_R^+ \tilde{\sigma}^\mu \psi_R = (\psi_R^+ \psi_R, \ \psi_R^+ \vec{\sigma}\, \psi_R)$ transforms as a 4-vector

$\psi_R^T \sigma_2 \psi_R$ is a singlet

We can construct a singlet linear in derivatives

$$\psi_R^{\dagger} \tilde{\sigma}^{\mu} \partial_\mu \psi_R = \psi_R^{\dagger} \partial_0 \psi_R + \psi_R^{\dagger} \vec{\sigma} \cdot \vec{\nabla} \psi_R,$$

which is anti hermitian, or one quadratic in derivatives

$$\partial^\mu \psi_R^T \sigma_2 \partial_\mu \psi_R$$

This completes the list of invariants. The most general $\mathcal{L}_{free}$ would be a linear combination of all three (multiplied by i or plus hermitian conjugate, as necessary). But we will make one further assumption:

- There is a conserved fermion number. $\mathcal{L}$ is invariant under the symmetry operation

$$\psi_R(x) \rightarrow e^{i\alpha} \psi_R(x)$$

This eliminates from consideration $\psi_R^T \sigma_2 \psi_R$ and $\partial \psi_R^T \sigma_2 \partial \psi_R$. We are left with just one term. The most general form for $\mathcal{L}$, after a field rescaling, is

$$\mathcal{L} = \pm i \; \psi_R^{\dagger} \tilde{\sigma}^{\mu} \partial_\mu \psi_R \qquad \tilde{\sigma}^{\mu} = (\mathbb{1}, \vec{\sigma})$$

The i makes $\mathcal{L}$ hermitian. We won't be able to decide which choice of sign is appropriate until we quantize the theory. If our two component spinor had been ψ_L, we would have written

$$\mathcal{L} = \pm i \; \psi_L^{\dagger} \sigma^{\mu} \partial_\mu \psi_L \qquad \sigma^{\mu} = (\mathbb{1}, -\vec{\sigma})$$

Note that a theory constructed just involving ψ_R (or just involving ψ_L) cannot be parity invariant, since ψ_L and ψ_R are parity conjugate.

Equation of Motion

To derive a classical equation of motion from this action, one could treat the real and imaginary parts of each component of ψ_R as independent fields. But this is actually equivalent to the algebraically simpler procedure of treating ψ_R and $\psi_R^\dagger$ as independent fields.

This procedure works because the action is real. In general, if ψ is a complex field, and S is real, then the variation of S has the form

$$\delta S = \int d^4x \, (X \delta\psi + X^* \delta\psi^*)$$

If we write $\psi = \psi_R + i\psi_I$, $X = X_R + i X_I$, in terms of real and imaginary parts, then

$$\delta S = \int d^4x \, 2 (X_R \delta\psi_R + X_I \delta\psi_I)$$

Treating ψ_R and ψ_I as independent, we have

$$\delta S = 0 \implies X_R = X_I = 0$$

But this is the same as what we would obtain from treating ψ and ψ^* as independent

$$\delta S = 0 \implies X = X^* = 0$$

Thus, from

$$S = \pm i \int d^4x \, \psi_R^\dagger \tilde{\sigma}^\mu \partial_\mu \psi_R$$

we may derive the classical eqn of motion

$$\frac{\partial \mathcal{L}}{\partial \psi^{+}_{R\alpha}} = 0 \quad \Rightarrow \quad \tilde{\sigma}^{\mu} \partial_{\mu} \psi_R = 0$$

or, equivalently,

$$\partial_{\mu} \frac{\partial \mathcal{L}}{\partial \partial_{\mu} \psi_{R\alpha}} = 0 \quad \Rightarrow \quad \partial_{\mu} \psi_R^{+} \tilde{\sigma}^{\mu} = 0$$

Plane-wave solutions:

The field equation

$$(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\, \psi_R(x) = 0$$

has solutions of the form

$$\psi_R(x) = \psi_{RK}\, e^{-iK\cdot x}$$

where $(-iK^0 + i\vec{\sigma}\cdot\vec{K})\,\psi_{RK} = 0$

or $(\vec{\sigma}\cdot\vec{K})\,\psi_{RK} = K^0 \psi_{RK}$

Since $(\vec{\sigma}\cdot\vec{K})^2 = \vec{K}^2$, the eigenvalues of $\vec{\sigma}\cdot\vec{K}$ are $\pm|\vec{K}|$ and $|K^0| = \pm|\vec{K}|$ -- Dispersion relation of a massless free particle

For a particle moving along the $+\hat{z}$ axis
$\vec{K} = |\vec{K}|\,\hat{e}_3$
the solution with $K^0 = +|\vec{K}|$ satisfies

$$\sigma_3 \psi_{RK} = \psi_{RK} \quad \text{or} \quad \psi_{RK} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When we quantize the theory, the amplitude to destroy a one-particle state satisfies the field equation

$$\langle 0|\, \psi_R(x)\, |K,\lambda\rangle \propto e^{-iK\cdot x}\, \psi_{RK}$$

From the form of the solution, we can infer the helicity of the particle annihilated by the field. The helicity λ is defined by

$$e^{iJ_3\theta}|k,\lambda\rangle = e^{i\lambda\theta}$$

From the transformation property

$$U(\Lambda)\,\psi_R(x)\,U(\Lambda)^{-1} = D(\Lambda^{-1})\,\psi_R(\Lambda x),$$

we have

$$e^{iJ_3\theta}\,\psi_R\, e^{-iJ_3\theta} = e^{-i\frac{\theta}{2}\sigma_3}\,\psi_R .$$

Therefore

$$e^{-i\lambda\theta}\langle 0|\,\psi_R(0)\,|k,\lambda\rangle = e^{-i\frac{\theta}{2}\sigma_3}\,\langle 0|\,\psi_R(0)\,|k,\lambda\rangle,$$

or

$$\underline{\lambda = \tfrac{1}{2}}$$

Particles destroyed by ψ_R have positive helicity.

Similarly,

$$\langle k,\lambda|\,\psi_R(x)\,|0\rangle \propto e^{ik\cdot x}\,\psi_{Rk}$$

And

$$e^{+i\lambda\theta}\langle k,\lambda|\,\psi_R(0)\,|0\rangle = e^{-i\frac{\theta}{2}\sigma_3}\,\langle k,\lambda|\,\psi_R(0)\,|0\rangle,$$

so

$$\underline{\lambda = -\tfrac{1}{2}}.$$

Particles created by ψ_R have negative helicity.

On the other hand, the equation of motion for ψ_L is

$$(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\psi_L(x) = 0,$$

(Since $\psi_L^+ \sigma^\mu \psi_L$ is a 4-vector, $\sigma^\mu = (1, -\vec{\sigma})$)

so ψ_L annihilates a particle of helicity $\lambda = -\frac{1}{2}$, and creates a particle of helicity $\lambda = \frac{1}{2}$. This is the reason for the notation L and R.

The Free Dirac Theory

Now we will devise a free field theory that is invariant under the parity operation that interchanges ψ_L and ψ_R. We continue to demand a conserved fermion number; that is $\mathcal{L}$ must be invariant under a $U(1)$ symmetry

$$\psi_R \to e^{i\alpha}\psi_R$$
$$\psi_L \to e^{i\alpha}\psi_L$$

The new Lorentz-invariant operator, quadratic in fields and hermitian, that we can construct is

$$\psi_R^+\psi_L + \psi_L^+\psi_R,$$

and the most general Lagrangian, after a field rescaling, is

$$\mathcal{L} = \pm\left[i\psi_R^+\tilde{\sigma}^\mu\partial_\mu\psi_R + i\psi_L^+\sigma^\mu\partial_\mu\psi_L - m\psi_R^+\psi_L - m\psi_L^+\psi_R\right]$$

Actually, we could include $\partial\psi_L^+\partial\psi_R$ + h.c., but exclude it because it is dimension 5, and hence unimportant at low momentum.

(we can make m real by redefining phase of ψ_R). Parity, or invariance under $\psi_L \leftrightarrow \psi_R$, did not need to be imposed. It is an automatic property of an action that satisfies all our other requirements, once we decide that the fields will be ψ_L and ψ_R.

Equation of motion

Varying with respect to $\psi_R^\dagger$, $\psi_L^\dagger$ we obtain the classical equations

$$i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\,\psi_R - m\,\psi_L = 0$$
$$i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\psi_L - m\,\psi_R = 0$$

We may introduce 4×4 matrices

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$
$$\beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

so that these two equations may be combined into

$$\left[i(\partial_0 + \vec{\alpha}\cdot\vec{\nabla}) - m\beta\right]\psi = 0\,, \quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

-- The free <u>Dirac Equation</u>

The 4 matrices $\beta, \vec{\alpha}$ all anticommute (with each other) and square to $\mathbb{1}$. We have represented these matrices in the particular basis in which the Lorentz generators

$$\vec{J} = \begin{pmatrix} \frac{1}{2}\vec{\sigma} & 0 \\ 0 & \frac{1}{2}\vec{\sigma} \end{pmatrix} \qquad \vec{K} = \begin{pmatrix} \frac{1}{2}i\vec{\sigma} & 0 \\ 0 & -\frac{1}{2}i\vec{\sigma} \end{pmatrix} = \frac{1}{2}i\vec{\alpha}$$

(Weyl basis)

are block diagonal. Another basis that is sometimes convenient is the basis in which β is diagonal

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix}$$

(standard basis)

For many purposes, it will be unnecessary to specify which basis we are working with.

γ Matrices

If we multiply by β, the Dirac equation becomes

$$[i(\beta\partial_0 + \beta\vec{\alpha}\cdot\vec{\nabla}) - m]\psi = 0$$

To emphasize the covariance of this equation, we may write

$$\gamma^0 = \beta \qquad \vec{\gamma} = \beta\vec{\alpha}$$

Then $\psi^+\beta\gamma^\mu\psi$ is a 4-vector, and the Dirac equation may be written

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

The γ's satisfy the anticommutation relations

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu} \qquad (\text{-- "covariant"})$$

(since $\beta\alpha_i\beta\alpha_i = -\beta^2\alpha_i^2 = -\mathbb{1}$) -- Note that the γ^i's are antihermitian, $\vec{\gamma}^+ = -\vec{\gamma}$

We may define a "Dirac adjoint" by

$$\bar{\psi} = \psi^+\gamma^0.$$

Then the 4-vector bilinear may be written

$$\bar{\psi}\gamma^\mu\psi,$$

and the scalar is

$$\psi^+\beta\psi = \bar{\psi}\psi.$$

The Dirac adjoint of a matrix is defined by

$$\bar{M} = \gamma^0 M^+ \gamma^0.$$

Hence $\quad \bar{\gamma}^\mu = \gamma^\mu \qquad$ (γ's are "self-bar")

We will sometimes denote a 4-vector contracted with the γ matrices by

$$\not{a} = a_\mu \gamma^\mu \qquad \text{(Feynman's "slash")}$$

In our new notation, the Lagrange density is

$$\mathcal{L} = \pm \bar{\psi}(i\not{\partial} - m)\psi$$

The γ matrices tend to be handier for calculation than $\vec{\alpha}, \beta$, because their anticommutator

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}$$

is consistent with the spacetime metric. Thus we have identities such as

$$\not{a}\not{a} = a^\mu a^\nu \tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = a^2$$

If ψ solves the Dirac eqn,

$$(i\not{\partial} - m)\psi = 0,$$

then

$$(-i\not{\partial} - m)(i\not{\partial} - m)\psi = 0$$

$$= (\not{\partial}\not{\partial} + m^2)\psi = (\partial^\mu\partial_\mu + m^2)\psi.$$

It also solves the Klein-Gordon equation. The Dirac operator $i\not{\partial} - m$ is sort of a "square root" of the Klein-Gordon operator $\partial^\mu\partial_\mu + m^2$.

Lorentz transformation properties of γ matrices:

The representation $D(\Lambda)$ of the Lorentz group according to which the Dirac spinor ψ transforms is not unitary, $D^{-1} \neq D^{+}$, but D^{-1} is equal to its Dirac adjoint! We know that $\bar\psi\psi$ is a scalar, and

$$\Lambda: \bar\psi\psi \to \psi^{+} D(\Lambda)^{+}\gamma^0 D(\Lambda)\psi = \bar\psi \bar D(\Lambda) D(\Lambda)\psi = \bar\psi\psi$$

This is true for all $\psi \Rightarrow \quad \bar D(\Lambda) D(\Lambda) = \mathbb{1}$.

Since $\bar\psi\gamma^\mu\psi$ transforms as a 4-vector:

$$\Lambda: \bar\psi\gamma^\mu\psi \to \bar\psi \bar D(\Lambda)\gamma^\mu D(\Lambda)\psi = \Lambda^\mu{}_\nu \bar\psi\gamma^\nu\psi$$

for any ψ. Thus

$$\bar D(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu .$$

This is the sense in which the four γ matrices constitute a 4-vector

Lorentz generators in terms of γ matrices:

In the Weyl basis,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$J^i = \begin{pmatrix} \frac{1}{2}\sigma^i & 0 \\ 0 & \frac{1}{2}\sigma^i \end{pmatrix}, \quad M^i = \begin{pmatrix} \frac{1}{2}i\sigma^i & 0 \\ 0 & -\frac{1}{2}i\sigma^i \end{pmatrix}$$

Thus $M^i = \frac{1}{2} i \gamma^0\gamma^i$.

To express J^i in terms of γ's, note that

$$[\gamma^i, \gamma^j] = -\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix}$$

$$= -2i\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

We have $J^i = \frac{i}{4}\frac{1}{2}\epsilon^{ijk}[\gamma^j, \gamma^k] \qquad M^i = \frac{i}{4}[\gamma^0, \gamma^i]$

Recall (see e.g. page 3.16) that

$$\mathcal{M}^i = J^{0i} \qquad J^k = \tfrac{1}{2}\epsilon^{kij}J^{ij}$$

we see that the Lorentz generators can be expressed as

$$J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Following convention, we define $\sigma^{\mu\nu} = \frac{1}{2i}[\gamma^\mu, \gamma^\nu]$ and write

$$J^{\mu\nu} = -\tfrac{1}{2}\sigma^{\mu\nu}$$

The statement that the $J^{\mu\nu}$'s are Lorentz generators means that they satisfy the commutation relations of the $SO(3,1)$ Lie algebra. Notice that the commutators of the $J^{\mu\nu}$'s are completely determined by the anticommutation relations of the γ's, and are independent of the basis in which we express the γ's. So, while we checked that $J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ only in the Weyl basis, it is clearly true in any basis.

The matrix γ_5:

we may define

$$\gamma_5 \equiv \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Then, from $[\gamma^\mu, \gamma^\nu]_- = 2\eta^{\mu\nu}$, we see that

$$[\gamma^\mu, \gamma_5]_+ = 0$$

$$\gamma_5^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = \mathbb{1}$$

$$\overline{\gamma}_5 = -i\gamma^3\gamma^2\gamma^1\gamma^0 = -\gamma_5$$

$$\gamma_5^\dagger = \gamma_5$$

γ_5 has eigenvalues ± 1, and commutes with the Lorentz generators $J^{\mu\nu}$. So Lorentz transformations preserve its eigenvalues. In fact,

$$\gamma_5 = \frac{i}{4}[\gamma_0, \gamma_1][\gamma_2, \gamma_3] = -4i\, M^1 J^1,$$

and in the Weyl basis $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So γ_5 is $+1$ acting on ψ_R and -1 acting on ψ_L. Thus (in any basis) we may construct from γ_5 projection operators that project out the components of ψ transforming as the $(\frac{1}{2}, 0)$ rep and the $(0, \frac{1}{2})$ rep

$$P_R = \tfrac{1}{2}(1 + \gamma_5) \quad P_R^2 = P_R \qquad P_R \psi = \psi_R$$
$$P_L = \tfrac{1}{2}(1 - \gamma_5) \quad P_L^2 = P_L \qquad P_L \psi = \psi_L$$

Parity:

Parity interchanges ψ_L and ψ_R. In the Weyl basis $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$

So $\quad P: \psi(\vec{x}, t) \to \gamma^0 \psi(-\vec{x}, t)$

defines the action of the parity operator on the Dirac spinor ψ. Note that the bilinears we constructed transform as expected under P:

Scalar - $\bar{\psi}\psi \to \bar{\psi}\gamma^0\gamma^0\psi = \bar{\psi}\psi$ is invariant

Vector - $\bar{\psi}\gamma^\mu\psi \to \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi = (\bar{\psi}\gamma^0\psi, -\bar{\psi}\vec{\gamma}\psi)$

We may also construct:

Pseudoscalar: $\bar{\psi}\gamma_5\psi \to \bar{\psi}\gamma^0\gamma_5\gamma^0\psi = -\bar{\psi}\gamma_5\psi$

Axial Vector: $\bar{\psi}\gamma^\mu\gamma_5\psi \to \bar{\psi}\gamma^0\gamma^\mu\gamma_5\gamma^0\psi = (-\bar{\psi}\gamma^0\psi, \bar{\psi}\vec{\gamma}\psi)$

(Since γ_5 commutes with Lorentz generators, $\bar\psi\gamma_5\psi$ and $\bar\psi\gamma^\mu\gamma_5\psi$ transform as a scalar, vector respectively under Lorentz transformations.)

These bilinears are related to those we constructed from the Weyl spinors by

$$\bar\psi\tfrac{1}{2}(I+\gamma_5)\psi = \psi_L^\dagger\psi_R, \quad \bar\psi\tfrac{1}{2}(I-\gamma_5)\psi = \psi_R^\dagger\psi_L$$

$$\bar\psi\gamma^\mu\tfrac{1}{2}(1+\gamma_5)\psi = \psi_R^\dagger\bar\sigma^\mu\psi_R, \quad \bar\psi\gamma^\mu\tfrac{1}{2}(1-\gamma_5)\psi = \psi_L^\dagger\sigma^\mu\psi_L$$

(parity interchanges these)

Catalog of Bilinears:

We have now constructed 5 types of bilinears in ψ.

S	$\bar\psi\psi$	V	$\bar\psi\gamma^\mu\psi$	T	$\bar\psi\sigma_{\mu\nu}\psi$
P	$\bar\psi\gamma_5\psi$	A	$\bar\psi\gamma^\mu\gamma_5\psi$		

This is actually a complete list of the independent representations of the Lorentz group that can be constructed from ψ and $\bar\psi$. To see this, recall—

ψ transforms as $(\tfrac{1}{2},0)+(0,\tfrac{1}{2})$, and

$$[(\tfrac{1}{2},0)+(0,\tfrac{1}{2})]\otimes[(\tfrac{1}{2},0)+(0,\tfrac{1}{2})]$$

$$= \underbrace{(1,0)+(0,1)}_{T} + \underset{S}{(0,0)} + \underset{P}{(0,0)} + \underset{V}{(\tfrac{1}{2},\tfrac{1}{2})} + \underset{A}{(\tfrac{1}{2},\tfrac{1}{2})}$$

One might have suspected that $\bar\psi\sigma^{\mu\nu}\gamma_5\psi$ is an independent object. But in fact

$$\sigma^{01}\gamma^5 = (-i\gamma^0\gamma^1)(i\gamma^0\gamma^1\gamma^2\gamma^3) = \gamma^2\gamma^3 = i\sigma^{23}, \text{ etc}$$

or-- $\quad \sigma^{\mu\nu}\gamma_5 = i\,{}^*\sigma^{\mu\nu}$

So $\quad \bar\psi\sigma^{\mu\nu}\tfrac{1}{2}(I\pm\gamma_5)\psi$

are the "self dual" and "anti self dual" pieces of an antisymmetric tensor, and each transforms irreducibly under Lorentz. (These are the (1,0) and (0,1) representations.)

Hermiticity Properties:

Taking the adjoint of a bilinear is the same as taking its "bar". So

$\bar\psi\psi$ is hermitian

$\bar\psi\gamma_5\psi$ is antihermitian ($\bar\gamma_5 = -\gamma_5$)

$\bar\psi\gamma^\mu\psi$ is hermitian

$(\bar\psi\gamma^\mu\gamma_5\psi)^\dagger = \bar\psi(-\gamma_5)\gamma^\mu\psi = \bar\psi\gamma^\mu\gamma_5\psi$ -- hermitian

$(\bar\psi\sigma^{\mu\nu}\psi)^\dagger = \left(\frac{-1}{2i}\right)\bar\psi[\gamma^\nu,\gamma^\mu]\psi = \bar\psi\sigma^{\mu\nu}\psi$ -- hermitian

$(\bar\psi\sigma^{\mu\nu}\gamma_5\psi)^\dagger = \quad \bar\psi(-\gamma_5)\sigma^{\mu\nu}\psi = -\bar\psi\sigma^{\mu\nu}\gamma_5\psi$ -- antihermitian