Solution to Free Dirac Equation

Let us find the plane wave solutions to the free Dirac equation

$$(i\not{\partial} - m)\,\psi = 0$$

As in the case of the scalar field theory, we are motivated to find these solutions, not because we wish to interpret $\psi(x)$ as a single particle wave equation, but because we hope to construct a useful mode expansion of the field $\psi(x)$, which becomes an operator when we quantize the field theory.

ψ has 4 spinor components, and $\not{\partial} = \partial_\mu \gamma^\mu$ where the γ^μ's are 4×4 matrices. But we recall

$$(i\not{\partial} - m)\,\psi = 0 \Rightarrow (-i\not{\partial} - m)(i\not{\partial} - m)\,\psi = 0$$
$$= (\partial^2 + m^2)\,\psi$$

Therefore, the "plane-wave" solutions have the form

$$\psi = u_{\vec{p}}\, e^{-ip\cdot x}$$

where $p\cdot x = p^\mu x_\mu$, $u_{\vec{p}}$ is independent of x, and

$p^2 = m^2$ (the "mass shell" condition for a particle of mass m)

We will distinguish two types of solutions:

Positive frequency -- $\psi(x) = u_{\vec{p}}\, e^{-ip\cdot x}$
Negative frequency -- $\psi(x) = v_{\vec{p}}\, e^{ip\cdot x}$, $p^0 > 0$

When we carry out the mode expansion, the positive frequency part of $\psi(x)$ will destroy particles, and the negative frequency part will create particles

To write out explicit solutions, we must choose an explicit basis for the γ matrices. We'll choose Dirac's basis (the "standard" basis):

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

Positive frequency solution:

Plugging $\psi(x) = u_{\vec{p}} e^{-ip\cdot x}$ into $(i\not{\partial} - m)\psi(x) = 0$, we find

$$(\not{p} - m) u_{\vec{p}} = 0$$

The standard basis is particularly convenient because, in the zero-momentum ($\vec{p} = 0$) case, the "Dirac operator" $(\not{p} - m)$ is diagonal in this basis:

$$\vec{p} = 0 \Rightarrow (\not{p} - m) = p_0 \gamma^0 - m = m(\beta - 1)$$

So the Dirac equation becomes

$$\beta u_{\vec{0}} = u_{\vec{0}}$$

This equation evidently has two independent solutions, which we may denote

$$u_{\vec{0}}^{(1)} = \sqrt{2m}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_{\vec{0}}^{(2)} = \sqrt{2m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

By convention, the solutions are normalized so that

$$\bar{u}_{\vec{0}}^{(r)} u_{\vec{0}}^{(s)} = 2m\delta^{rs}$$

The solutions have been chosen to be eigenstates of

$$J_3 = \begin{pmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{pmatrix};$$

we have

$$J_3 u^{(1)}_{\vec{0}} = \tfrac{1}{2} u^{(1)}_{\vec{0}}, \quad J_3 u^{(2)}_{\vec{0}} = \tfrac{1}{2} u^{(2)}_{\vec{0}}.$$

It is also easy to see that

$$\left.\begin{aligned} \bar{u}^{(r)}_{\vec{0}} \gamma^0 u^{(s)}_{\vec{0}} &= 2m\delta^{r,s} \\ \bar{u}^{(r)}_{\vec{0}} \gamma^i u^{(s)}_{\vec{0}} &= 0 \end{aligned}\right\} \text{ or } \bar{u}^{(r)} \gamma^\mu u^{(s)} = 2p^\mu \delta^{r,s}$$

We can find the solution for arbitrary $\vec{p}$ by appealing to the covariance of the Dirac equation. Recalling that

$$\overline{D(\Lambda)}\, \gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu,$$

we see that

$$(\not{p} - m) u_{\vec{p}} = 0 \quad \Longrightarrow$$

$$\overline{D}(\Lambda^{-1})(\not{p} - m) D(\Lambda^{-1}) D(\Lambda) u_{\vec{p}} = 0$$

$$= [(\Lambda^{-1}\gamma)\cdot p - m] D(\Lambda) u_{\vec{p}} = 0$$

$$= [\gamma \cdot (\Lambda p) - m] D(\Lambda) u_{\vec{p}} = 0$$

That is, $D(\Lambda) u_{\vec{p}}$ is a plane wave solution with momentum Λp. Therefore, we may obtain a basis for $u_{\vec{p}}$ by applying $D(\Lambda)$ to $u^{(1,2)}_{\vec{0}}$, where Λ is a standard "boost" satisfying

$$\Lambda(m, \vec{0}) = p$$

There is, of course, an ambiguity in the choice of the standard boost -- an ambiguity corresponding to multiplication by an element of the little group. In other words, we may perform a rotation acting on $u_{\vec{0}}$ before we boost. But rotations, generated by ---

$$\vec{J} = \begin{pmatrix} \frac{1}{2}\vec{\sigma} & 0 \\ 0 & \frac{1}{2}\vec{\sigma} \end{pmatrix}$$

just mix up $u_{\vec{0}}^{(1)}$ and $u_{\vec{0}}^{(2)}$ with eachother. In other words, this ambiguity just means that we have the freedom to choose an arbitrary basis for the two linearly independent solutions

$$u_{\vec{p}}^{(1)} \text{ and } u_{\vec{p}}^{(2)}.$$

For example, suppose $\vec{p} = |\vec{p}|\hat{e}_3$ (momentum along the $+\hat{z}$ axis). We may choose

$$u_{\vec{p}}^{(1)} = \exp[-i\theta\hat{p}\cdot\vec{M}]\; u_{\vec{0}}^{(1)} = e^{\frac{\theta}{2}\alpha^3} u_{\vec{0}}^{(1)}$$

$$u_{\vec{p}}^{(2)} = e^{\theta/2\alpha^3} u_{\vec{0}}^{(1)}.$$

that is, we've chosen $u_{\vec{p}}^{(1,2)}$ to be J_3 eigenstates.

(Note the minus sign in the exponential; this is what is necessary in order that generators of boosts and rotations obey the commutation relations on page 3.17-18.)

The velocity parameter θ is chosen so that

$$\exp\left[\theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

gives the right momentum acting on $\begin{pmatrix} m \\ 0 \end{pmatrix}$; namely,

$$\cosh\theta = p^0/m \equiv E/m$$

Now, $\alpha^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$, and $(\alpha^3)^2 = \mathbb{1}$, so we have

$$e^{\frac{\theta}{2}\alpha^3} = \cosh\frac{\theta}{2} + \alpha^3\sinh\frac{\theta}{2} = \begin{bmatrix} \cosh\theta/2 & \sigma^3\sinh\theta/2 \\ \sigma^3\sinh\theta/2 & \cosh\theta/2 \end{bmatrix},$$

and therefore

$$u^{(1)}_{\vec{p}} = \sqrt{2m}\begin{pmatrix}\cosh\frac{\theta}{2}\\ 0\\ \sinh\frac{\theta}{2}\\ 0\end{pmatrix}, \quad u^{(2)}_{\vec{p}} = \sqrt{2m}\begin{pmatrix}0\\ \cosh\frac{\theta}{2}\\ 0\\ -\sinh\frac{\theta}{2}\end{pmatrix}.$$

use half-angle formulas:

$$\cosh\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(1+\cosh\theta)} = \sqrt{\frac{m+E}{2m}},$$

$$\sinh\frac{\theta}{2} = \sqrt{\tfrac{1}{2}(\cosh\theta - 1)} = \sqrt{\frac{E-m}{2m}},$$

to obtain

$$u^{(1)}_{\vec{p}} = \begin{pmatrix}\sqrt{E+m}\\ 0\\ \sqrt{E-m}\\ 0\end{pmatrix}; \quad u^{(2)}_{\vec{p}} = \begin{pmatrix}0\\ \sqrt{E+m}\\ 0\\ -\sqrt{E-m}\end{pmatrix}$$

(the motivation for our normalization convention is now clearer. With this convention, we may allow $m \to 0$ with E fixed, and obtain a smooth limit.) You may want to check explicitly that these satisfy $(\not{p} - m)u = 0$, in order to relieve minus sign anxiety.

We recall that, for $\vec{p} = 0$, the solutions satisfied

$$\bar{u}^{(r)}_{\vec{p}}\, u^{(s)}_{\vec{p}} = 2m\,\delta^{r,s}$$

$$\bar{u}^{(r)}_{\vec{p}}\, \gamma^\mu\, u^{(s)}_{\vec{p}} = 2p^\mu\,\delta^{r,s}$$

Now, covariance, or the identities

$$\bar{D}(\Lambda)\,D(\Lambda) = \mathbb{I} \qquad \bar{D}(\Lambda)\,\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu,$$

tell us that the solutions satisfy these conditions for all $\vec{p}$.

Negative frequency solution:

Construction is exactly analogous to positive frequency case. Equation is

$$(\not{p} + m)\, v_{\vec{p}} = 0$$

For $\vec{p} = 0$, this becomes $\gamma^0 v_{\vec{p}} = -v_{\vec{p}}$, and the two independent solutions may be chosen to be

$$v_{\vec{0}}^{(1)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad v_{\vec{0}}^{(2)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

These solutions have $J_3 = \mp \frac{1}{2}$

$$J_3 v_{\vec{0}}^{(1)} = -\tfrac{1}{2} v_{\vec{0}}^{(1)} \qquad J_3 v_{\vec{0}}^{(2)} = +\tfrac{1}{2} v_{\vec{0}}^{(2)}$$

(and are associated with the amplitude to create particles with $J_z = \pm \frac{1}{2}$.)

These solutions obey

$$\bar{v}_{\vec{0}}^{(r)} v_{\vec{0}}^{(s)} = -2m\delta^{r,s}, \quad (\text{since } \gamma^0 v = -v)$$

and $$\bar{v}_{\vec{0}}^{(r)} \gamma^\mu v_{\vec{0}}^{(s)} = 2p^\mu \delta^{r,s}.$$

We can obtain solutions for arbitrary $\vec{p}$ by applying a boost operator. These solutions satisfy

$$\bar{v}_{\vec{p}}^{(r)} v_{\vec{p}}^{(s)} = -2m\delta^{r,s},$$

$$\bar{v}_{\vec{p}}^{(r)} \gamma^\mu v_{\vec{p}}^{(s)} = 2p^\mu \delta^{r,s}.$$

Also $$\bar{v}_{\vec{0}}^{(r)} u_{\vec{0}}^{(s)} = \bar{u}_{\vec{0}}^{(r)} v_{\vec{0}}^{(s)} = 0$$

becomes $$\bar{v}_{\vec{p}}^{(r)} u_{\vec{p}}^{(s)} = \bar{u}_{\vec{p}}^{(r)} v_{\vec{p}}^{(s)} = 0,$$

after a boost

Projection operators:

Consider the 4×4 matrix:

$$M_{\vec{p}} = \sum_r u_{\vec{p}}^{(r)} \bar{u}_{\vec{p}}^{(r)}.$$

$M_{\vec{p}}$ may be determined from its action on any complete basis and the normalization conditions tell us that

$$M_{\vec{p}}\, u_{\vec{p}}^{(s)} = 2m\, u_{\vec{p}}^{(s)}$$

$$M_{\vec{p}}\, v_{\vec{p}}^{(s)} = 0$$

Therefore, we find that

$$M_{\vec{p}} = \sum_{r=1,2} u_{\vec{p}}^{(r)} \bar{u}_{\vec{p}}^{(r)} = \not{p} + m$$

(since $\not{p} u = mu$ and $\not{p} v = -mv$).

Similarly,

$$N_{\vec{p}} = \sum_{r=1,2} v_{\vec{p}}^{(r)} \bar{v}_{\vec{p}}^{(r)} = \not{p} - m.$$

Thus $\frac{1}{2m} M_{\vec{p}}$ is a projection onto positive frequency solutions, and $\frac{1}{2m} N_{\vec{p}}$ is a projection onto negative frequency solutions.

Canonical Formulation of Free Dirac Theory

Recall the Lagrange density is

$$\mathcal{L} = \pm \bar{\psi}(i\not{\partial} - m)\psi.$$

(We'll determine the sign soon.) Let's work out an expression for the Hamiltonian.

In doing so, we'll find it convenient to perform a change of variable similar to that invoked in the scalar field case. The general solution to the classical equations of motion (the free Dirac equation) is a superposition of plane wave solutions

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_{r=1,2} \left[b_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} + c_{\vec{p}}^{(r)\dagger} v_{\vec{p}}^{(r)} e^{ip\cdot x} \right]$$

Here $b_{\vec{p}}^{(r)}$ and $c_{\vec{p}}^{(r)\dagger}$ are arbitrary functions of $\vec{p}$, and independent of x. We would like to regard the b's and c's as new dynamical variables, and write H in terms of them.

The conjugate momenta are

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = \pm i (\bar{\psi}\gamma^0)_\alpha = \pm i \psi_\alpha^\dagger ,$$

and so

$$\mathcal{H} = \sum_\alpha \pi_\alpha \dot{\psi}_\alpha - \mathcal{L} = \pm i \bar{\psi}\gamma^0 \partial_0 \psi - \mathcal{L}$$

$$= \pm \left[-i\bar{\psi}\gamma^i \partial_i \psi + m\bar{\psi}\psi \right]$$

Now we'll evaluate $H = \int d^3x \mathcal{H}$ by plugging in the expression for ψ in terms of b and $c^\dagger$. But notice that this expression satisfies

$$i\gamma^0 \partial_0 \psi = -i\gamma^i \partial_i \psi + m\psi$$

if derivatives of b and $c^\dagger$ are set to zero. So we may write

$$H = \pm i \bar{\psi}\gamma^0 \partial_0 \psi ,$$

where it is understood that b and c are time-independent.

We have

$$i\gamma^0\partial_0\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{\frac{1}{2}\omega_p}} \sum_{r=1,2} \left[b_{\vec{p}}^{(r)} \gamma^0 u_{\vec{p}}^{(r)} e^{-ip\cdot x} - c_{\vec{p}}^{(r)\dagger} \gamma^0 v_{\vec{p}}^{(r)} e^{+ip\cdot x} \right]$$

and

$$\bar{\psi}(x) = \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega_{p'}}} \sum_{s=1,2} \left[b_{\vec{p}'}^{(s)\dagger} \bar{u}_{\vec{p}'}^{(s)} e^{ip\cdot x} + c_{\vec{p}'}^{(s)} \bar{v}_{\vec{p}'}^{(s)} e^{-ip\cdot x} \right]$$

Therefore,

$$H = \pm i \int d^3x \; \bar{\psi}(\vec{x},t)\gamma^0\partial_0\psi(\vec{x},t)$$

$$= \pm \int d^3p \, \frac{1}{2} \sum_{r,s} \Big[b_{\vec{p}}^{(s)\dagger} b_{\vec{p}}^{(r)} \bar{u}_{\vec{p}}^{(s)} \gamma^0 u_{\vec{p}}^{(r)} - c_{\vec{p}}^{(s)} c_{\vec{p}}^{(r)\dagger} \bar{v}_{\vec{p}}^{(s)} \gamma^0 v_{\vec{p}}^{(r)} - b_{-\vec{p}}^{(s)\dagger} c_{\vec{p}}^{(r)\dagger} \bar{u}_{-\vec{p}}^{(s)} \gamma^0 v_{\vec{p}}^{(r)} e^{2i\omega_p t} + c_{-\vec{p}}^{(s)} b_{\vec{p}}^{(r)} \bar{v}_{-\vec{p}}^{(s)} \gamma^0 u_{\vec{p}}^{(r)} e^{-2i\omega_p t} \Big]$$

(The x integration generates a δ-function that kills one of the p integrations.) Recall that

$$\bar{v}_{\vec{p}}^{(s)} \gamma^0 v_{\vec{p}}^{(r)} = \bar{u}_{\vec{p}}^{(s)} \gamma^0 u_{\vec{p}}^{(r)} = 2\omega_p \delta^{r,s}$$

Also $\gamma^0 \not{p} \gamma^0 = (p^0, -\vec{p})\cdot\gamma$. So

$$\gamma^0 [(p^0, -\vec{p})\cdot\gamma - m] \gamma^0 u_{\vec{p}} = 0$$

or $\gamma^0 u_{\vec{p}} = u_{-\vec{p}}$. Thus,

$$\bar{v}_{-\vec{p}}^{(s)} \gamma^0 u_{\vec{p}}^{(r)} = 0 = \bar{u}_{-p}^{(s)} \gamma^0 u_{\vec{p}}^{(r)}$$

We therefore have

$$H = \pm \int d^3p\ \omega_p \sum_{r=1,2} \left[b_{\vec{p}}^{(r)+} b_{\vec{p}}^{(r)} - c_{\vec{p}}^{(r)} c_{\vec{p}}^{(r)+} \right]$$

(Note that although the discussion so far has been "classical" we have been careful about "operator" ordering.)

Quantization

If we were now to proceed as in the scalar case, we would impose canonical equal time commutation relations on ψ and $\pi = \pm i \psi^+$, and b and c would become quantized "oscillators!" But our proof of the connection of spin with statistics shows that this procedure must give rise to trouble, and we can see what the trouble is. However we choose the $\pm$ sign in H the excitations of the b or c oscillators will have negative energy. Hence, the energy is unbounded from below.

To obtain a theory respecting the right connection between spin and statistics, we show a deplorable lack of imagination and invoke canonical anticommutators. That is our fields are promoted to operators satisfying

$$[\psi_\alpha(\vec{x},t),\ \psi_\beta(\vec{y},t)]_+ = 0$$

$$[\psi_\alpha^+(\vec{x},t),\ \psi_\beta^+(\vec{y},t)]_+ = 0$$

$$[\psi_\alpha(\vec{x},t),\ \pm i \psi_\beta^+(y,t)]_+ = i \delta_{\alpha\beta} \delta^{(3)}(\vec{x}-\vec{y}).$$

This last may be rewritten:

$$[\psi_\alpha(\vec{x},t),\ \overline{\psi}_\beta(\vec{y},t)]_+ = \pm (\gamma^0)_{\alpha\beta} \delta^{(3)}(\vec{x}-\vec{y}),$$

which will sometimes be written in matrix notation

$$[\psi, \bar{\psi}]_{e.t.} = \pm \gamma^0 \delta^{(3)}(\vec{x}-\vec{y}),$$

with the spinor indices suppressed.

The anticommutators at equal time of ψ and $\bar{\psi}$ determine the anticommutators of the c's and b's. I claim that these are determined to be

$$[b_{\vec{p}}^{(r)}, b_{\vec{p}'}^{(s)\dagger}]_+ = \pm \delta^{r,s} \delta^3(\vec{p}-\vec{p}')$$

$$[c_{\vec{p}}^{(r)}, c_{\vec{p}'}^{(s)\dagger}]_+ = \pm \delta^{r,s} \delta^3(\vec{p}-\vec{p}')$$

$$[\text{else}]_+ = 0$$

To check this claim, we must verify that these give the correct equal time anticommutators for $\psi, \bar{\psi}$.

From the mode expansions

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_{r=1,2} \left[b_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} + c_{\vec{p}}^{(r)\dagger} v_{\vec{p}}^{(r)} e^{ip\cdot x} \right],$$

$$\bar{\psi}(y) = \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega_{p'}}} \sum_{s=1,2} \left[b_{\vec{p}'}^{(s)\dagger} \bar{u}_{\vec{p}'}^{(s)} e^{ip'\cdot y} + c_{\vec{p}'}^{(s)} \bar{v}_{\vec{p}'}^{(s)} e^{-ip'\cdot y} \right],$$

we see that

$$[\psi(x), \psi(y)]_+ = [\bar{\psi}(x), \bar{\psi}(y)]_+ = 0$$

are trivially satisfied, because $[b,b^\dagger]_+$, $[c,c^\dagger]_+$ do not occur. But

$$[\psi(\vec{x},t), \bar{\psi}(\vec{y},t)]_+$$

$$= \pm \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_r \left[u_{\vec{p}}^{(r)} \bar{u}_{\vec{p}}^{(r)} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v_{\vec{p}}^{(r)} \bar{v}_{\vec{p}}^{(r)} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

Now we use $\sum_r u^{(r)}_{\vec{p}} \bar{u}^{(r)}_{\vec{p}} = \not{p} + m$,

$\sum_r v^{(r)}_{\vec{p}} \bar{v}^{(r)}_{\vec{p}} = \not{p} - m$,

and perform the change of variable $\vec{p} \to -\vec{p}$ in the second term to obtain

$$[\psi(\vec{x},t), \bar{\psi}(\vec{y},t)]_\pm = \pm \gamma^0 \delta^{(3)}(\vec{x}-\vec{y}) \quad \text{-- as desired}$$

Suppose we choose the minus sign; i.e.

$$[b^{(r)}_{\vec{p}}, b^{(s)\dagger}_{\vec{p}'}]_+ = -\delta^{r,s}\delta^3(\vec{p}-\vec{p}'), \quad \text{etc...}$$

Or, for finite volume, using the discrete normalization as on p. (1.18) ff., we have

$$[b^{(r)}_{\vec{p}}, b^{(s)\dagger}_{\vec{p}'}]_+ = -\delta^{r,s}\delta_{\vec{p},\vec{p}'}$$

But the relation

$$b^{(1)}_{\vec{p}} b^{(1)\dagger}_{\vec{p}} + b^{(1)\dagger}_{\vec{p}} b^{(1)}_{\vec{p}} = -\underline{1}$$

does not make sense. The left-hand side is the sum of two positive terms and has only nonnegative eigenvalues, but the right-hand side is negative. So only the plus sign choice is consistent, and we have ---

$$H = \int d^3p \; \omega_p \sum_{r=1,2} [b^{(r)\dagger}_{\vec{p}} b^{(r)}_{\vec{p}} - c^{(r)}_{\vec{p}} c^{(r)\dagger}_{\vec{p}}]$$

$$[b^{(r)}_{\vec{p}}, b^{(s)\dagger}_{\vec{p}'}]_+ = \delta^{r,s}\delta^3(\vec{p}-\vec{p}')$$

$$[c^{(r)}_{\vec{p}}, c^{(s)\dagger}_{\vec{p}'}]_+ = \delta^{r,s}\delta^3(\vec{p}-\vec{p}')$$

$$[\text{ else }]_+ = 0$$

"Hole Theory"

Return to the discrete normalization:

$$H = \sum_{\vec{p},r} \omega_p \left(b_{\vec{p}}^{(r)\dagger} b_{\vec{p}}^{(r)} - c_{\vec{p}}^{(r)} c_{\vec{p}}^{(r)\dagger} \right)$$

$$= \sum_{\vec{p},r} \omega_p \left(b_{\vec{p}}^{(r)\dagger} b_{\vec{p}}^{(r)} + c_{\vec{p}}^{(r)\dagger} c_{\vec{p}}^{(r)} \right) - \sum_{\vec{p},r} \omega_p ,$$

using the anticommutation relation. We have an infinite set of uncoupled (fermionic) oscillators of the form

$$H = \omega b^\dagger b$$

where $[b^\dagger, b^\dagger]_+ = [b, b]_+ = 0, \quad [b, b^\dagger]_+ = 1$

For each such oscillator, we may define a ground state $|0\rangle$ such that

$$b|0\rangle = 0 \implies H|0\rangle = 0$$

An excited state is

$$|1\rangle = b^\dagger|0\rangle, \quad H|1\rangle = \omega b^\dagger b b^\dagger |0\rangle = \omega |1\rangle,$$

since $(b^\dagger)^2 = 0$ and $bb^\dagger = -b^\dagger b + 1$.

But $b^\dagger|1\rangle = 0$

$b|1\rangle = |0\rangle$

so the two states $|0\rangle$ and $|1\rangle$ are the complete Hilbert space of the fermionic oscillator. The "oscillator" is an energy level that is either occupied or unoccupied; it cannot be "doubly" occupied (Pauli exclusion principle).

For the full field theory, we may construct a Fock space state (as in the scalar field theory) by assigning to each oscillator an occupation number $n_{\vec{p},r}$. In the scalar case, the occupation number could be any nonnegative integer, but in the fermionic case it takes only the values 0 and 1.

Note that the vacuum has negative energy,

$$H|0\rangle = -\left(\sum_{\vec{p},r} \omega_p\right)|0\rangle$$

(The vacuum is the state with $n_{\vec{p},r} = 0$ for all $\vec{p}$ and r.) In the scalar case, we had

$$E_{\text{vacuum}} = \sum_{\vec{p}} \tfrac{1}{2}\omega_{\vec{p}},$$

which we interpretted as the sum of the zero-point energies of all oscillators. Of course, this vacuum energy is of no consequence; we can remove it by normal ordering in the fermionic case as we did in the scalar case. But, it seems curious that the "zero-point" energy of the fermionic oscillator is negative. Can we understand this heuristically?

Consider the pair of oscillators with

$$H = \omega(b^\dagger b - cc^\dagger)$$

Aside from the relative minus sign, b and $c^\dagger$ enter H symmetrically, and we may define a symmetrical "vacuum" state by

$$b|\text{vac}\rangle = 0 = c^\dagger|\text{vac}\rangle.$$

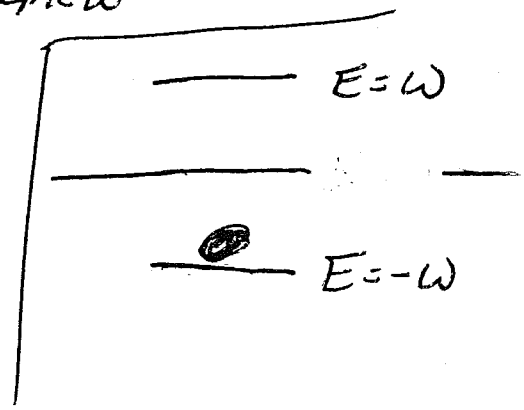
However, $|\text{vac}\rangle$ is not the actual ground state of this system. The Hilbert space consists of

The 4 states $b^+|vac\rangle$ $E = \omega$

$|vac\rangle$, $b^+c|vac\rangle$ $E = 0$

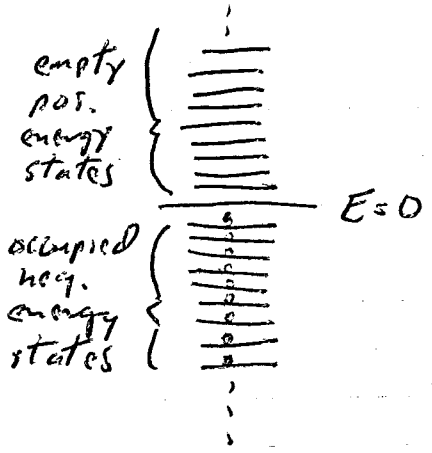
$c|vac\rangle$ $E = -\omega$

And we may interpret the spectrum by saying that there are 2 states, of energy $\pm\omega$, each of which may be either occupied or unoccupied. The true ground state is attained when the negative energy state is filled, and the positive energy state is empty. This is the state with $E = -\omega$ (the negative "zero-point energy"). Excitations are obtained by either occupying the positive energy state or evacuating the negative energy state (a hole with energy ω above ground state), or both.



The Hamiltonian of the full field theory is

$$H = \sum_{\vec{p},r} \omega_p \left(b^{(r)+}_{\vec{p}} b^{(r)}_{\vec{p}} - c^{(r)}_{\vec{p}} c^{(r)+}_{\vec{p}} \right)$$



For each value of $\vec{p}$ and r there is a pos. energy state and a neg. energy state. The true vacuum has all negative energy states filled (the "Dirac sea").

Note that we do not need to use this language. We can just as well write

$$:H: = \sum_{\vec{p},r} \omega_p \left(b^{(r)+}_{\vec{p}} b^{(r)}_{\vec{p}} + c^{(r)+}_{\vec{p}} c^{(r)}_{\vec{p}} \right)$$

normal ordered

-- so that we interpret all oscillators as having positive energy.

But it is sometimes convenient to think of $c^\dagger$ as an operator that creates a "hole" in the negative energy "sea". Note that this viewpoint is possible only because the oscillators are fermionic, and so can be only singly occupied.

It is obvious by covariance, and may be explicitly verified, that

$$P^\mu = \sum_{\vec{p},r} p^\mu \left(b^{(r)+}_{\vec{p}} b^{(r)}_{\vec{p}} - c^{(r)}_{\vec{p}} c^{(r)+}_{\vec{p}} \right)$$

Thus, the hole evacuated by $c^\dagger_{\vec{p}}$ has momentum $-\vec{p}$. Evacuating a hole with energy $-\omega_p$ and momentum $-\vec{p}$ is equivalent to creating an (anti-) particle with energy ω_p and momentum $\vec{p}$.

Charge:

The Lagrange density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under the symmetry operation

$\psi \to e^{i\alpha}\psi$, or $\delta\psi = i\psi$ in infinitesimal form.

The associated conserved Noether current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \delta\psi = \bar{\psi}\gamma^\mu\psi$$

Check: Equations of motion -- $i\partial\!\!\!/\psi = m\psi$, $i\partial_\mu\bar{\psi}\gamma^\mu = -m\psi$

$$\Rightarrow \quad \partial_\mu J^\mu = \bar{\psi}\,(-im + im)\psi$$

The conserved charge is $Q = \int d^3x\, \bar{\psi}\gamma^0\psi$.

Substitute

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_r \left(b_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} + c_{\vec{p}}^{(r)\dagger} v_{\vec{p}}^{(r)} e^{ip\cdot x} \right)$$

$$\Rightarrow Q(t=0) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_{r,s} \Big[b_{\vec{p}}^{(r)\dagger} b_{\vec{p}}^{(s)} \bar{u}_{\vec{p}}^{(r)} \gamma^0 u_{\vec{p}}^{(s)} + c_{\vec{p}}^{(r)} c_{\vec{p}}^{(s)\dagger} \bar{v}_{\vec{p}}^{(r)} \gamma^0 v_{\vec{p}}^{(s)}$$
$$+ b_{\vec{p}}^{(r)\dagger} c_{-\vec{p}}^{(s)\dagger} \bar{u}_{\vec{p}}^{(r)} \gamma^0 v_{-\vec{p}}^{(s)} + c_{\vec{p}}^{(r)} b_{-\vec{p}}^{(s)} \bar{v}_{\vec{p}}^{(r)} \gamma^0 u_{-\vec{p}}^{(s)} \Big]$$

Arguing as on p. (3.62), we have

$$Q = \int d^3p \sum_r \left(b_{\vec{p}}^{(r)\dagger} b_{\vec{p}}^{(r)} + c_{\vec{p}}^{(r)} c_{\vec{p}}^{(r)\dagger} \right)$$

$$= \int d^3p \sum_r \Big(\underbrace{b_{\vec{p}}^{(r)\dagger} b_{\vec{p}}^{(r)}}_{\text{charge due to particles}} + \underbrace{c_{\vec{p}}^{(r)\dagger} c_{\vec{p}}^{(r)}}_{\text{charge due to antiparticles}} + \underbrace{\delta(\vec{0})}_{\text{"vacuum charge"}} \Big)$$

The vacuum charge

$$\langle 0|Q|0\rangle = \sum_{\vec{p},r} (1)$$

is positive, and has an obvious "hole theory" interpretation. Both the negative and positive energy states have positive charge, and the vacuum is constructed by "filling" all negative energy states. Thus the holes (antiparticles) have negative charge.

As with the Hamiltonian H, we may redefine (normal order) the charge by an additive constant, so that $\langle 0|Q|0\rangle = 0$. Ordinarily, by Q we will mean the normal ordered charge.

Causality

In order to be able to construct an algebra of causal observables (even in Fermi fields) we want our spinor fields to anticommute outside the light cone. This is trivially true for

$$[\psi(x), \psi(y)]_+ = [\bar{\psi}(x), \bar{\psi}(y)]_+ = 0,$$

but we should verify that

$$[\psi(x), \bar{\psi}(y)]_+ = 0 \quad \text{for} \quad (x-y)^2 < 0$$

the discussion is closely analogous to the corresponding one for scalar fields.

We may define

$$S_+(x-y)_{\alpha\beta} = [\psi^{(-)}_\alpha(x), \bar{\psi}^{(+)}_\beta(y)]_+ = \langle 0 | \psi^{(-)}_\alpha(x)\, \bar{\psi}^{(+)}_\beta(y) | 0 \rangle$$

($\psi^{(-)}$: annihilation part; $\bar{\psi}^{(+)}$: creation part)

Loosely speaking, this is an amplitude for a particle created at y to propagate to x. You may show (exercise)

$$S_+(x-y) = (i\partial\!\!\!/_x + m)\, \Delta_+(x-y)$$

where $\Delta_+(x-y) = [\phi^{(-)}(x), \phi^{(+)}(y)]$ -- scalar fields

This amplitude is "noncausal", it does not vanish for $(x-y)^2 < 0$. However,

$$i S(x-y)_{\alpha\beta} = [\psi_\alpha(x), \bar{\psi}_\beta(y)]_+ = (i\partial\!\!\!/_x + m)_{\alpha\beta}\, i\Delta(x-y)$$

where $i\Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x)$. This anticommutator is causal. As in the scalar case (compare p. 134 and p. 163) by destructive interference of the amplitude for a particle to go from y to x with that for an antiparticle to go from x to y. (for spacelike separation).

Statistics

The many particle states in the Fock space are of the form

$$\int d^3p_1 \cdots d^3p_n \sum_{r_1=1,2} \cdots \sum_{r_n=1,2} f(\vec{p}_1, r_1; \cdots \vec{p}_n, r_n)\, b^{(r_1)\dagger}_{\vec{p}_1} \cdots b^{(r_n)\dagger}_{\vec{p}_n} |0\rangle$$

(for a state with n particles and no antiparticles)
Since the $b^{(r)\dagger}_{\vec{p}}$ all anticommute, the wave function f must be antisymmetric under all interchanges of identical particles:

$$(\vec{p}_k, r_k) \longleftrightarrow (\vec{p}_\ell, r_\ell) \quad \Rightarrow \quad f \to -f$$

the many particle states respect Fermi statistics

"Classical Limit"

In canonical quantization, the canonical anticommutator is proportional to a factor of $\hbar$ that we have suppressed. Thus, formally, in the classical or $\hbar \to 0$ limit, all the fields anticommute. The "classical" Dirac theory is a theory of "anticommuting c-number" (or "Grassmann") fields

But physically, it does not make much sense to speak of the classical limit of the theory of a fermi (half-odd-integer spin) field. The classical limit is really the limit in which expectation values of fields are large compared to the quantum mechanical uncertainty in the value of the field. This is really a limit of large occupation number for the field oscillators. But the fermi field oscillators can have only occupation numbers 0 and 1 (not "large").

In view of this remark, our canonical quantization procedure for the Dirac theory seems rather dubious. Canonical quantization is designed to obtain a quantum theory that has a given classical theory as its classical limit. But the Dirac theory has no classical limit, so what is the point of applying canonical quantization to it?

In this connection, it is useful to recall the remarks on p 1.35 concerning the construction of the scalar field theory. We could have begun with a theory of free spin-$\frac{1}{2}$ particles, and introduced a field $\psi(x)$ that creates and destroys particles in the hope of constructing relativistically-invariant interactions using that field. We would have found it impossible to construct such a field that commutes at spacelike separations, but would have found it possible to construct an anticommuting causal field by taking ψ to be a particular linear combination of an operator that destroys a particle and one that creates an antiparticle. (This procedure is sometimes called "second quantization".)

So we see that the canonical quantization of the free Dirac theory is just a convenient and equivalent way of describing the construction of the second quantized theory. The canonical quantization language tends to be more useful, particularly when we consider interacting theories.

Discrete Symmetries: Parity

We have already seen that the operation

$$P: \quad \psi(\vec{x},t) \to \gamma^0 \psi(-\vec{x},t)$$

can be interpreted as parity. In the Weyl basis,

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^0\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

so multiplication by γ^0 has the effect of interchanging ψ_R and ψ_L.

We may define a unitary operator U_P that represents parity by specifying its action on the field:

$$U_P \, \psi(\vec{x},t) \, U_P^{-1} = \eta \, \gamma^0 \psi(-\vec{x},t).$$

Here η is a phase. In the free Dirac theory, multiplication by η is a charge rotation (a symmetry), so we have the freedom to rotate η to 1, by convention. But in a theory that doesn't have the U(1) conserved charge, we must fix the phase η so that the parity operation is a symmetry of the action (assuming that this is possible).

Since the action of parity on a 4-vector satisfies $P^2 = \mathbb{1}$, we require that

$$U_P^2 = \mathbb{1} \quad \text{or} \quad U_P^2 = -\mathbb{1}.$$

$U_P^2 = -\mathbb{1}$ is okay; it means that P^2 acting on a spinor is equivalent to a rotation by 2π. And P^2 leaves the vector $\bar{\psi}\gamma^\mu\psi$ invariant. These considerations restrict η to

$$\eta = 1, -1, i, -i.$$

Now we consider how the operator U_P acts on the states of the free Dirac theory. Since we have

$$\psi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_{r=1,2} \left(b_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} + c_{\vec{p}}^{(r)+} v_{\vec{p}}^{(r)} e^{ip\cdot x} \right)$$

and $$U_P \psi(\vec{x},t) U_P^{-1} = \gamma^0 \psi(-\vec{x},t)$$

($\eta = 1$ by convention), we infer that

$$U_P b_{\vec{p}}^{(r)} U_P^{-1} u_{\vec{p}}^{(r)} = b_{-\vec{p}}^{(r)} \gamma^0 u_{-\vec{p}}^{(r)}$$

(after the change of variable $\vec{p} \to -\vec{p}$).
Furthermore, as we saw on page (3.62)

$$\gamma^0 u_{-\vec{p}}^{(r)} = u_{\vec{p}}^{(r)}$$

if we choose the basis $u_{\vec{p}}^{(1)}, u_{\vec{p}}^{(2)}$ appropriately. With this choice of basis, $u_{\vec{p}}^{(r)}$ and $u_{-\vec{p}}^{(r)}$ transform the same way under rotations, because γ^0 commutes with the rotation generators (but anticommutes with the boost generators).

We see then, that parity acting on a state flips the 3-momentum. If

$$|\vec{p}, r, +\rangle = b_{\vec{p}}^{(r)+} |0\rangle$$

(the + identifies this state as a particle with $Q = +1$)

then $$U_P b_{\vec{p}}^{(r)+} U_P^{-1} = b_{-\vec{p}}^{(r)+}$$

implies

$$U_P |\vec{p}, r, +\rangle = |-\vec{p}, r, +\rangle.$$

Also, parity preserves the spin of a state.

From the Lorentz transformation property of ψ,

$$U(\Lambda)\,\psi(x)\,U(\Lambda^{-1}) = D(\Lambda^{-1})\,\psi(\Lambda x),$$

we derive that

$$U(R)\; b^{(r)}_{\vec{0}}\; u^{(r)}_{\vec{0}}\; U(R^{-1}) = b^{(r)}_{\vec{0}}\; D(R^{-1})\, u^{(r)}_{\vec{0}}$$

Here R is a rotation, an element of the little group of $p=(m,\vec{0})$, and r is summed over.

Equivalently

$$U(R)\, b^{(r)\dagger}_{\vec{0}}\; U(R^{-1})\; \bar{u}^{(r)}_{\vec{0}} = b^{(r)\dagger}_{\vec{0}}\; \bar{u}^{(r)}_{\vec{0}}\, D(R^{-1}),$$

and therefore

$$U(R)\,|\vec{0},r\rangle\,\bar{u}^{(r)}_{\vec{0}} = |\vec{0},r\rangle\,\bar{u}^{(r)}_{\vec{0}}\,D(R)$$

(using $\bar{D}(R^{-1}) = D(R)$). And since γ^0 commutes with $D(R)$, U_P commutes with $U(R)$. Parity preserves the transformation property of a state under the little group.

To summarize here, parity ...

flips the momentum $\vec{p}$.

preserves the spin $\vec{s}$.

This is the way we expect parity to act on the states -- $\vec{p}$ is a vector and $\vec{s}$ is an axial vector

But now consider the action of U_P on the antiparticle states. We recall that

$$\gamma^0 v_{\vec{0}}^{(r)} = - v_{\vec{0}}^{(r)}$$

and therefore $\gamma^0 v_{\vec{p}}^{(r)} = - v_{-\vec{p}}^{(r)}$

the above reasoning gives us

$$U_P \, C_{\vec{p}}^{(r)+} \, U_P^{-1} = - C_{-\vec{p}}^{(r)+}$$

or $$U_P \, | \vec{p}, r, - \rangle = - | -\vec{p}, r, - \rangle$$

Again, the momentum flips and the spin is preserved. But the *phase* of the state also changes sign.

As we noted, there is a conventional phase in the definition of parity, so

$$U_P \, | \vec{p}, r, + \rangle = \eta \quad | -\vec{p}, r, + \rangle$$

$$U_P \, | \vec{p}, r, - \rangle = - \eta \, | -\vec{p}, r, - \rangle$$

But the relative minus sign in the transformation properties of $|+\rangle$ and $|-\rangle$ is *not* conventional.

We say the the fermion and antifermion have *opposite* *intrinsic* *parity*. This behavior is different than what we found in the case of charged scalar particles. There, particle and antiparticle had the same parity.

Charge Conjugation

We recall that the theory of a (free) charged scalar field considered in Chapter 1 had a symmetry -- charge conjugation -- that interchanged particles and antiparticles. Similarly, particles and antiparticles enter the (free) Dirac theory in a symmetric fashion; the interchange

$$c_{\vec{p}}^{(r)} \longleftrightarrow b_{\vec{p}}^{(r)}$$

leaves the Hamiltonian invariant.

How does this charge conjugation operation act on the field $\psi(x)$? Guided by our experience with the charged scalar field, we might guess that charge conjugation acts on the field by complex conjugation, up to a linear transformation. That is,

$$C: \quad \psi(x) \longrightarrow A\,\psi^*(x) \equiv \psi^c(x)$$

where A is a matrix acting on spinor indices, and the last equality defines the charge conjugate field ψ^c.

If C is to be a symmetry, we require it to commute with Lorentz transformations; ψ and ψ^c must transform the same way under rotations and boosts. (C is an internal symmetry, not a spacetime symmetry. The spacetime argument x of $\psi(x)$ is unaffected by the transformation.) Under an infinitesimal Lorentz transformation parametrized by $\omega_{\mu\nu}$, ψ transforms as

$$\psi(x) \to \left(\mathbb{1} + \frac{i}{2} J^{\mu\nu}\omega_{\mu\nu}\right)\psi\left((\mathbb{1}-\omega)x\right),$$

where $\qquad J^{\mu\nu} = -\frac{i}{2}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

and ψ^c transforms as

$$\psi^c(x) = A\psi^*(x) \rightarrow A\left(\mathbb{1} - \tfrac{i}{2} J^{\mu\nu *}\omega_{\mu\nu}\right) A^{-1} \psi^c((\mathbb{1}-\omega)x)$$

So ψ and ψ^c transform the same way provided

$$A(\sigma^{\mu\nu})^* A^{-1} = -\sigma^{\mu\nu}.$$

Whatever basis we choose for the γ matrices, there is some matrix A that satisfies the above identity. But it is convenient, when discussing charge conjugation, to work in a particular basis in which A is especially simple. The convenient basis is the Majorana basis, in which the γ matrices are imaginary:

$$\gamma^{\mu *} = -\gamma^{\mu},$$

and hence $\sigma^{\mu\nu *} = -\sigma^{\mu\nu}$, of $A = \mathbb{1}$.
For example, we may choose

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = i\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^2 = i\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \gamma^3 = i\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}.$$

In this basis charge conjugation is the same as complex conjugation

(Actually, the matrix A is not unique; we can choose A to be any linear combination of $\mathbb{1}$ and γ_5 in the Majorana basis. This is because we are free to multiply the Weyl spinors ψ_R and ψ_L by different multiples of the identity without modifying their Lorentz transformation properties. But, by convention, we will fix the matrix A by specifying

$$A\gamma^{\mu *} A^{-1} = -\gamma^{\mu},$$

or $A = \mathbb{1}$ in the Majorana basis.)

Now we can determine how the charge conjugation operator acts on the states of the free Dirac theory. For this purpose we note that

$$u_{\vec{p}}^{(r)*} = v_{\vec{p}}^{(r)}, \text{ in the Majorana basis}$$

(if we choose appropriate phase conventions)

the point is that, if $u_{\vec{p}}^{(r)}$ satisfies

$$(\not{p} - m)\, u_{\vec{p}}^{(r)} = 0, \text{ then } -(\not{p} + m)\, u_{\vec{p}}^{(r)*} = 0,$$

if the γ^μ's are imaginary. So $u_{\vec{p}}^{*}$ is a negative frequency solution. Furthermore, if, e.g.

$$J^3 u_{\vec{0}}^{(1)} = \tfrac{1}{2} u_{\vec{0}}^{(1)} \qquad J^3 u_{\vec{0}}^{(1)*} = -\tfrac{1}{2} u_{\vec{0}}^{(1)}$$

$$J^3 u_{\vec{0}}^{(2)} = -\tfrac{1}{2} u_{\vec{0}}^{(2)} \quad \text{then} \quad J^3 u_{\vec{0}}^{(2)*} = \tfrac{1}{2} u_{\vec{0}}^{(2)}$$

because J^3 is imaginary in the Majorana basis. So we see that

$$u_{\vec{0}}^{(r)*} = v_{\vec{0}}^{(r)},$$

if we recall the peculiar conventions on page 3.59. Finally

$$u_{\vec{p}}^{(r)} = D(\Lambda(\vec{p}))\, u_{\vec{0}}^{(r)}$$

$$v_{\vec{p}}^{(r)} = D(\Lambda(\vec{p}))\, v_{\vec{0}}^{(r)}$$

where $D(\Lambda(\vec{p}))$ is a real matrix in the Majorana basis, because all of the Lorentz generators are imaginary in this basis. Hence,

$$u_{\vec{p}}^{(r)*} = v_{\vec{p}}^{(r)}, \text{ as claimed.}$$

(We should also check the normalization: If

$$\bar{u}u = u^\dagger \gamma^0 u = 2m, \text{ then}$$

$$u^{*\dagger} \gamma^0 u^* = \overline{u^*}\, u^* = -2m,$$

since γ^0 is imaginary. And this is the right norm. for v.)

If charge conjugation is implemented by an operator U_c satisfying

$$U_c\, \psi(x)\, U_c^{-1} = \psi(x)^* \quad \text{(in the Majorana basis)},$$

then

$$\int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_r \left[U_c\, b_{\vec{p}}^{(r)} U_c^{-1}\, u_{\vec{p}}^{(r)} e^{-ip\cdot x} + U_c\, c_{\vec{p}}^{(r)+} U_c^{-1}\, v_{\vec{p}}^{(r)} e^{ip\cdot x} \right]$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_r \left[b_{\vec{p}}^{(r)+} u_{\vec{p}}^{(r)*} e^{ip\cdot x} + c_{\vec{p}}^{(r)} v_{\vec{p}}^{(r)*} e^{-ip\cdot x} \right].$$

Or, since $u_{\vec{p}}^{(r)*} = v_{\vec{p}}^{(r)}$, $v_{\vec{p}}^{(r)*} = u_{\vec{p}}^{(r)}$, we have

$$U_c\, b_{\vec{p}}^{(r)}\, U_c^{-1} = c_{\vec{p}}^{(r)}$$

$$U_c\, c_{\vec{p}}^{(r)}\, U_c^{-1} = b_{\vec{p}}^{(r)}$$

As expected, charge conjugation interchanges particle and antiparticle.

Also, C preserves the spin. To see this recall that

$$U(R)\, b_{\vec{0}}^{(1)}\, u_{\vec{0}}^{(1)}\, U(R^{-1}) = b_0^{(1)}\, D(R^{-1})\, u_{\vec{0}}^{(1)}$$

$$U(R)\, c_{\vec{0}}^{(1)+}\, v_{\vec{0}}^{(1)}\, U(R^{-1}) = c_0^{(1)+}\, D(R^{-1})\, v_{\vec{0}}^{(1)}$$

where R is a rotation (an element of the "little group" of $\vec{p} = 0$). If R is a rotation by θ about the $\hat{e}_3$ axis, then

$$\langle 0 |\, b_{\vec{0}}^{(1)}\, u_{\vec{0}}^{(1)}\, e^{-i\theta J_3} | \vec{0}, 1, + \rangle$$

$$= \langle 0 | b_{\vec{0}}^{(1)} | 0, 1 \rangle\, e^{i\frac{\theta}{2}\sigma_{12}}\, u_{\vec{0}}^{(1)}$$

$$\langle \vec{0}, 1, -1 |\, e^{i\theta J_3}\, c_{\vec{0}}^{(1)+}\, v_{\vec{0}}^{(1)} | 0 \rangle$$

$$= \langle \vec{0}, 1, -1 |\, c_{\vec{0}}^{(1)+} | 0 \rangle\, e^{i\theta/2\,\sigma_{12}}\, v_{\vec{0}}^{(1)}$$

And therefore

$$J^3 |\vec{0}, 1, +\rangle = \tfrac{1}{2} |\vec{0}, 1, +\rangle ,$$
$$J^3 |\vec{0}, 1, -\rangle = \tfrac{1}{2} |\vec{0}, 1, -\rangle ;$$

our conventions are such that $|\vec{p}, r, +\rangle$ and $|\vec{p}, r, -\rangle$ transform the same way under the little group. Charge conjugation, acting as

$$U_C |\vec{p}, r, +\rangle = |\vec{p}, r, -\rangle$$
$$U_C |\vec{p}, r, -\rangle = |\vec{p}, r, +\rangle ,$$

preserves the spin of a particle. (This is clear, since C was constructed to commute with Lorentz transformations.)

Charge conjugation can also be interpreted in terms of hole theory. The Hamiltonian

$$H = \int d^3p \sum_r \left[b_{\vec{p}}^{(r)+} b_{\vec{p}}^{(r)} - c_{\vec{p}}^{(r)} c_{\vec{p}}^{(r)+} \right]$$

becomes after a charge conjugation transformation

$$\rightarrow \int d^3p \sum_r \left[c_{\vec{p}}^{(r)+} c_{\vec{p}}^{(r)} - b_{\vec{p}}^{(r)} b_{\vec{p}}^{(r)+} \right]$$

Charge conjugation turns the spectrum of H upside down, the positive and negative energy modes trade places, and creation and annihilation operator are interchanged. Hence, a positive energy excitation is mapped by C to a hole in the negative energy sea. This is why $V_{\vec{0}}^{(r)}$ is chosen to have opposite J^3 of $U_{\vec{0}}^{(r)}$ -- so that the negative energy mode associated with $c_{\vec{p}}^{(r)+}$ has opposite spin of the positive energy mode associated with $b_{\vec{p}}^{(r)}$. For then the particle and hole have the <u>same</u> spin.

Transformation Properties of Bilinears under C:

We will now determine how various fermion bilinears transform under

$$C: \psi \to A\psi^*, \quad \text{where} \quad A\gamma^{\mu *}A^{-1} = -\gamma^\mu.$$

Since these bilinears are basis independent (they have no uncontracted spinor indices) we may determine their transformation properties in any convenient basis. We will continue to use the Majorana basis in which $A = \mathbb{1}$.

Consider a bilinear of the form

$$\bar{\psi}^{(1)} M \psi^{(2)} = \psi^{(1)\dagger}\gamma^0 M \psi^{(2)}$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are, in general, two <u>different</u> spinors.

Under charge conjugation,

$$C: \bar{\psi}^{(1)} M \psi^{(2)} \to \psi^{(1)}_\alpha (\gamma^0 M)_{\alpha\beta} \psi^{(2)*}_\beta$$

$$= (-1)\, \psi^{(2)*}_\beta (M^T(\gamma^0)^T)_{\beta\alpha} \psi^{(1)}_\alpha$$

$$+ (\text{c-no. anticommutator term})$$

where we have anticommuted the two spinors. If we consider all bilinears to be normal ordered, then there is no anticommutator term. Furthermore, in the Majorana basis

$$(\gamma^0)^T = (\gamma^{0\dagger})^* = -\gamma^0.$$

So the transformation property of a <u>normal-ordered</u> bilinear is

$$:\bar{\psi}^{(1)} M \psi^{(2)}: \to :\bar{\psi}^{(2)} \underline{M}^* \psi^{(1)}:$$

where $\underline{M}^* = \gamma^0 M^T \gamma^0$, <u>in</u> the Majorana basis.

Examples

Using $C: \bar{\psi}^{(1)} M \psi^{(2)} \to \bar{\psi}^{(2)} \bar{M}^* \psi^{(1)}$

(with normal-ordering understood) in the Majorana basis, we can determine the transformation properties, which are independent of basis, of the various bilinears.

- $\bar{\psi}\psi \to \bar{\psi}\psi$
- $\bar{\psi}\gamma_5\psi \to \bar{\psi}\gamma_5\psi$, because $\bar{\gamma}_5 = -\gamma_5$, and γ_5 is imaginary in the Majorana basis ($\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$).

Evidently then,

$$\bar{A}\tfrac{1}{2}(1+\gamma_5)B \to \bar{B}\tfrac{1}{2}(1+\gamma_5)A$$

or $$\bar{A}_L B_R \to \bar{B}_L A_R$$

This behavior is unsurprising. We already knew that complex conjugation interchanges the $(\frac{1}{2},0)$ and the $(0,\frac{1}{2})$ representations of the Lorentz group, or interchanges the Weyl spinors ψ_L and ψ_R.

- $\bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^\mu\psi$, because $\bar{\gamma}^\mu = \gamma^\mu$ and $\gamma^{\mu *} = -\gamma^\mu$ in the Majorana basis.

The sign of the current $J^\mu = \bar{\psi}\gamma^\mu\psi$ flips under C, as does the sign of the charge $Q = \int d^3x\, J^0$. Of course, this is how charge conjugation gets its name. It changes the sign of Q by interchanging particle and antiparticle.

- $\bar{\psi}\gamma^\mu\gamma_5\psi \to \bar{\psi}\gamma^\mu\gamma_5\psi$, because $(\overline{\gamma^\mu\gamma_5})^* = -(\gamma_5\gamma^\mu)^*$
  $= \gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$

Thus $$\bar{A}\gamma^\mu \tfrac{1}{2}(1+\gamma_5)B \to -\bar{B}\gamma^\mu \tfrac{1}{2}(1-\gamma_5)A$$

or $$\bar{A}_R\gamma^\mu B_R \to -\bar{B}_L\gamma^\mu A_L$$

Charge conjugation, as we expect, interchanges right-handed and left-handed currents.

- $\bar{\psi}\sigma^{\mu\nu}\psi \to -\bar{\psi}\sigma^{\mu\nu}\psi$, because $\bar{\sigma}^{\mu\nu} = \sigma^{\mu\nu}$ and $\sigma^{\mu\nu *} = -\sigma^{\mu\nu}$ in Majorana basis

Majorana Fermion

Because C commutes with proper Lorentz transformation, it is consistent to impose a constraint on the Dirac spinor:

$$\psi = \psi^c.$$

This is called the Majorana condition, and it reduces by half the number of independent components of ψ. In the Majorana basis, this condition becomes

$$\psi = \psi^*$$

A majorana fermion is real in the Majorana basis.

In the Weyl basis, the matrix A is

$$A = i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$$

(The relation

$$A\gamma^{\mu *}A^{-1} = -\gamma^{\mu}$$

is satisfied because in this basis $\gamma^0, \gamma^1, \gamma^3$ are real and γ^2 is imaginary.) And the Majorana condition

$$\psi = A\psi^*$$

is satisfied by

$$\psi = \begin{pmatrix} \psi_R \\ -i\sigma_2 \psi_R \end{pmatrix}$$

So we see that imposing the Majorana condition means that ψ_R and ψ_L are not independent Weyl spinors, but rather are charge conjugates of one another. A Majorana spinor is merely a way of describing in our 4-component Dirac notation a 2-component Weyl spinor.

Other Discrete Symmetries

CP:

C and P acting on spinors do not commute; they anticommute. This is because of (or, if you prefer, it is the explanation of) the fact that fermion and antifermion have opposite intrinsic parity.

In terms of the action of C and P on the fields, we note that in the Majorana basis

$$(\psi^P)^C = (\gamma^0\psi)^C = \gamma^{0*}\psi^* = -\gamma^0\psi^*$$
$$= (\psi^C)^P = (\psi^*)^P = \gamma^0\psi^* = -(\psi^P)^C$$

(because $\gamma^{0*} = -\gamma^0$ in this basis).

In terms of the action of C and P on states, we note that

$$C: \quad b_{\vec{p}}^{(r)} \to c_{\vec{p}}^{(r)}, \quad c_{\vec{p}}^{(r)} \to b_{\vec{p}}^{(r)}$$

$$P: \quad b_{\vec{p}}^{(r)} \to b_{-\vec{p}}^{(r)}, \quad c_{\vec{p}}^{(r)} \to -c_{-\vec{p}}^{(r)},$$

and hence

$$CP: \quad b_{\vec{p}}^{(r)} \to c_{-\vec{p}}^{(r)}, \quad c_{\vec{p}}^{(r)} \to -b_{-\vec{p}}^{(r)}$$

$$PC: \quad b_{\vec{p}}^{(r)} \to -c_{-\vec{p}}^{(r)}, \quad c_{\vec{p}}^{(r)} \to b_{-\vec{p}}^{(r)}$$

or

$$U_C U_P |\vec{p}, r, +\rangle = |-\vec{p}, r, -\rangle$$

$$U_C U_P |\vec{p}, r, -\rangle = -|-\vec{p}, r, +\rangle$$

$$U_P U_C |\vec{p}, r, +\rangle = -|-\vec{p}, r, -\rangle$$

$$U_P U_C |\vec{p}, r, -\rangle = |-\vec{p}, r, +\rangle$$

And since U_C and U_P anticommute,

$$(U_{CP})^2 = (U_C U_P)^2 = -U_C^2 U_P^2 = -\mathbb{1}.$$

So $(CP)^2$ is equivalent to a rotation by 2π; it is the identity acting on bosons but $-\mathbb{1}$ acting on fermions

CPT and PT

In a homework exercise, we saw that in any theory involving spin $\frac{1}{2}$ fermions, there is always a symmetry of the form:

$$\psi^{(\frac{1}{2},0)} \to (i)\,\psi^{(\frac{1}{2},0)*}$$
$$\psi^{(0,\frac{1}{2})} \to (-i)\,\psi^{(0,\frac{1}{2})*}$$

implemented by an antiunitary operator U_{CPT} (This is actually a special case of what was shown in the homework.) In the 4-component Dirac notation, this may be written

$$CPT: \quad \psi(x) \to i\gamma_5 \psi^*(-x),$$

since γ_5 is $+1$ acting on the $(\frac{1}{2},0)$ part, and -1 acting on the $(0,\frac{1}{2})$ piece of a Dirac spinor ψ. (The sign in this transformation is purely conventional, but the phase (i) is important.)

In the Majorana basis, then, the operation PT acts as

$$PT: \quad \psi(x) \to i\gamma_5 \psi(-x),$$

(since $i\gamma_5$ is real) and is also implemented by an antiunitary operator.

Let us work out how PT and CPT act on the <u>states</u>.

We have

$$PT: \quad \psi(x) \to U_{PT}\,\psi(x)\,U_{PT}^{-1}$$

$$= \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_r \left[U_{PT}\, b_{\vec{p}}^{(r)}\, U_{PT}^{-1}\, u_{\vec{p}}^{(r)*} e^{ip\cdot x} + U_{PT}\, c_{\vec{p}}^{(r)+}\, U_{PT}^{-1}\, v_{\vec{p}}^{(r)*} e^{ip\cdot x} \right]$$

$$= i\gamma_5 \psi(-x) = \int \sum_r \left[b_{\vec{p}}^{(r)}\, i\gamma_5 u_{\vec{p}}^{(r)} e^{ip\cdot x} + c_{\vec{p}}^{(r)+}\, i\gamma_5 v_{\vec{p}}^{(r)} e^{-ip\cdot x} \right]$$

where the antilinearity of U_{PT} has been used.

We need to express $\gamma_5 u_{\vec{p}}^{(r)}$ in terms of $v_{\vec{p}}^{(s)}$'s. It is easy to see that, up to a phase $\gamma_5 u_{\vec{p}}^{(1)}$ is the same as $v_{\vec{p}}^{(2)}$. This is because γ_5 anticommutes with γ^μ and commutes with the Lorentz generators. Therefore, $\gamma_5 u$ is a negative frequency solution,

$$(\not{p} + m)\, \gamma_5 u_{\vec{p}}^{(r)} = 0$$

and, e.g., $J^3 \gamma_5 u_{\vec{0}}^{(1)} = \frac{1}{2} \gamma_5 u_{\vec{0}}^{(1)} \Rightarrow \gamma_5 u_{\vec{0}}^{(1)} = \eta\, v_{\vec{0}}^{(2)}$ (where η is a phase).
And also

$$\gamma_5 u_{\vec{p}}^{(1)} = D(\Lambda(\vec{p}))\, \gamma_5 u_{\vec{0}}^{(1)} = \eta\, v_{\vec{p}}^{(2)}$$

To determine the phase η, we must be careful about our phase conventions. Let us establish phase conventions for the u's and v's such that

$$u_{\vec{p}}^{(r)\,*} = v_{\vec{p}}^{(r)}$$

in the Majorana basis, as in our discussion of charge conjugation. In this basis

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

and the eigenstates of γ^0 are (up to normalization)

$$u_{\vec{0}}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix} \qquad u_{\vec{0}}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$$

$$v_{\vec{0}}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix} \qquad v_{\vec{0}}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$$

(where phases have been chosen such that $u^* = v$).

We can see explicitly that

$$\gamma_5 u_{\vec{0}}^{(1)} = v_{\vec{0}}^{(2)} \qquad \gamma_5 u_{\vec{0}}^{(2)} = -v_{\vec{0}}^{(1)}$$

When we insert this in the mode expansion of $U_{PT}\,\psi\,U_{PT}^{-1}$, we see that

$$U_{PT}\, b_{\vec{p}}^{(1)}\, U_{PT}^{-1} = -i\, b_{\vec{p}}^{(2)}$$

$$U_{PT}\, b_{\vec{p}}^{(2)}\, U_{PT}^{-1} = i\, b_{\vec{p}}^{(1)}$$

$$U_{PT}\, c_{\vec{p}}^{(1)}\, U_{PT}^{-1} = -i\, c_{\vec{p}}^{(2)}$$

$$U_{PT}\, c_{\vec{p}}^{(2)}\, U_{PT}^{-1} = i\, c_{\vec{p}}^{(1)}$$

And that, for example

$$U_{PT}\,|\vec{p}, 1, +\rangle = -i\,|\vec{p}, 2, +\rangle$$

The PT operator preserves the momentum of a state but flips its spin. This is how we expect PT to act.

We also note that $U_{PT}^2 = -\mathbb{1}$ acting on fermion states. (Recall that U_{PT} is antiunitary.) We could have anticipated this, knowing the transformation property of the field.

$$U_{PT}\,\psi(x)\,U_{PT}^{-1} = i\gamma_5\,\psi(-x) \quad \text{(Majorana)}$$

$$U_{PT}^2\,\psi(x)\,U_{PT}^{-2} = (i\gamma_5)^2\,\psi(x) = -\psi(x),$$

since $i\gamma_5$ is real in the Majorana basis. (This property is actually independent of the phase in the definition of PT, because U_{PT} is antiunitary.)

We already know how C acts on the states, and we find that under CPT:

$$\text{CPT}: \quad b_{\vec{p}}^{(1)} \to -i\, c_{\vec{p}}^{(2)}$$

$$b_{\vec{p}}^{(2)} \to i\, c_{\vec{p}}^{(1)}$$

$$c_{\vec{p}}^{(1)} \to -i\, b_{\vec{p}}^{(2)}$$

$$c_{\vec{p}}^{(2)} \to i\, b_{\vec{p}}^{(1)}$$

U_{CPT} acting on a state takes a particle to an antiparticle with the same momentum, but with spin flipped (and hence opposite helicity)

In a CPT invariant theory, then, if there is a state with helicity λ then a state with helicity $-\lambda$ must also occur. This is why, although a massless particle with helicity λ is a representation of the Poincare group, such states always appear in pairs with opposite helicity in local field theory.

Interacting Fermi Fields

So far we have considered only the free theory of a spinor field. Now we wish to consider interacting spinor fields. Of particular interest are interactions of "renormalizable type" -- those described by a term in the Lagrange density that is a local operator of dimension 4 or less. As we have seen, these are the interactions that are "relevant" in the low-energy limit.

The field ψ is normalized so that $\bar{\psi} i\partial\!\!\!/ \psi$ has dimension 4, and therefore

ψ has dimension $3/2$.

The lowest dimension Lorentz-invariant operator that can be constructed from ψ is, e.g.,

$(\bar{\psi}\psi)^2$ -- dimension 6.

(other than operators that are quadratic). So there is no possible renormalizable self-coupling for ψ.

(This is the correct conclusion in 4 spacetime dimensions. But in 2 spacetime dimensions, ψ has dimension $1/2$ and $(\bar{\psi}\psi)^2$ has dimension 2, so it is a renormalizable interaction.)

Since there is no renormalizable theory of a self coupled spinor, let us consider instead a theory of a spinor coupled to a scalar. A scalar field ϕ has dimension 1, so operators of dimension 4 can be constructed:

$$\bar{\psi}\psi\phi, \quad \bar{\psi} i\gamma_5 \psi\phi$$

these are both Lorentz-invariant and hermitian (if ϕ is a real scalar field). the most general renormalizable theory of a single Dirac spinor and a single real scalar (with a conserved U(1) fermion number) is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2)$$
$$- \lambda_S \bar{\psi}\psi\phi - \lambda_P \bar{\psi} i\gamma_5 \psi\phi$$
$$- \frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4$$

where m, μ are masses, and $\lambda_{S,P,3,4}$ are real coupling constants.
Of course, we could introduce several fermions and several scalars; then there would be many more allowed couplings.

As we have seen,

$\bar{\psi}\psi$ is C even, P even
$\bar{\psi} i\gamma_5 \psi$ is C even, P odd

therefore, this most general theory of ψ and ϕ violates P and CP. $\bar{\psi}\psi\phi$ is parity invariant only if ϕ is a scalar field (under P),

$$P: \phi(\vec{x}, t) \to \phi(-\vec{x}, t).$$

And $\bar{\psi} i\gamma_5 \psi\phi$ is parity invariant only if ϕ is a pseudoscalar,

$$P: \phi(\vec{x}, t) \to -\phi(-\vec{x}, t).$$

So if λ_S and λ_P are both nonzero, then there is no possible choice of a parity transformation property for ϕ such that the action is invariant under parity. But C is a good symmetry if ϕ is chosen to be C even. (Hence CP is typically violated).

The theory becomes parity (and CP) invariant if either $\lambda_P = 0$ (ϕ is a scalar), or $\lambda_S = \lambda_3 = 0$ (ϕ is a pseudoscalar). The theory of a pseudoscalar coupled (in a P-invariant way) to a spinor has **2** dimensionless coupling constants, λ_P and λ_4.

(Incidently, couplings $\bar{\psi}\psi\phi$, $\bar{\psi}_i\gamma_5\psi$ or a spinor to a scalar are sometimes called "Yukawa couplings".)

We found that a renormalizable theory of a single Dirac fermion and a single real scalar is automatically C invariant (if there is a conserved fermion number). This property does not persist in more complicated theories with several fermions and scalars.

Consider, for example, a theory of two Dirac fermions coupled to a real scalar ϕ:

$$\mathcal{L} = \bar{\psi} i\partial \!\!\!/ \; \bar{A}(i\partial \!\!\!/ - m_A)A + \bar{B}(i\partial \!\!\!/ - m_B)B + \tfrac{1}{2}((\partial\phi)^2 - \mu^2\phi^2) - \lambda_3/3!\,\phi^3 - \lambda_4/4!\,\phi^4 + \mathcal{L}_{Yuk}$$

where

$$\mathcal{L}_{Yuk} = -\left(\lambda_{S,A}\,\bar{A}A + \lambda_{P,A}\,\bar{A}i\gamma_5 A + \lambda_{S,B}\,\bar{B}B + \lambda_{P,B}\,\bar{B}i\gamma_5 B + \lambda_S\,\bar{A}B + \lambda_S^*\,\bar{B}A + \lambda_P\,\bar{A}i\gamma_5 B + \lambda_P^*\,\bar{B}i\gamma_5 A \right)\phi$$

Here $\lambda_{S,A}$ $\lambda_{P,A}$, $\lambda_{S,B}$, $\lambda_{P,B}$ are real and $\lambda_{S,P}$ are complex. So this most general Yukawa coupling contains 8 real parameters.

There is a convenient way to reparametrize these Yukawa couplings. Let ψ_i be a spinor carrying a "flavor" index i,

$$\psi_1 = A \qquad \text{or} \qquad \psi = \begin{pmatrix} A \\ B \end{pmatrix}$$
$$\psi_2 = B$$

Then the Yukawa couplings may be written

$$\mathcal{L}_{Yuk} = -(\bar{\psi}_{Li}\,\lambda_{ij}\,\psi_{Rj} + \bar{\psi}_{Rj}(\lambda^\dagger)_{ji}\,\psi_{Li})\,\phi$$
$$= -(\bar{\psi}_L \underset{\sim}{\lambda}\,\psi_R + \bar{\psi}_R \underset{\sim}{\lambda}^\dagger \psi_L)\,\phi,$$

where $\underset{\sim}{\lambda}$ is a complex 2×2 matrix (8 real parameters).

Now recall the transformation properties of bilinears under C, P, and CP:

$$P: \bar{\psi}_{Li}\,\psi_{Rj} \to \bar{\psi}_{Ri}\,\psi_{Lj}$$
$$C: \bar{\psi}_{Li}\,\psi_{Rj} \to \bar{\psi}_{Lj}\,\psi_{Ri}$$
$$CP: \bar{\psi}_{Li}\,\psi_{Rj} \to \bar{\psi}_{Rj}\,\psi_{Li}$$

Suppose that $\lambda_3 \neq 0$, so that ϕ must be P and CP even. Then it is easy to see that $\mathcal{L}_{Yuk}$ is invariant under P, C, CP under the conditions.

P: $\underline{\lambda} = \underline{\lambda}^+$ ($\underline{\lambda}$ hermitian)

C: $\underline{\lambda} = \underline{\lambda}^T$ ($\underline{\lambda}$ symmetric)

CP: $\underline{\lambda} = \underline{\lambda}^*$ ($\underline{\lambda}$ real)

Imposing any of the above three discrete symmetries reduces the number of independent real Yukawa couplings. If all three are imposed, $\underline{\lambda}$ must be a real symmetric 2×2 matrix (three real parameters). We note that C invariance is not automatic and that CP invariance is possible even if P and C are not good symmetries. ($\underline{\lambda}$ is real but not symmetric.)

The case of one Dirac spinor is special because a 1×1 matrix is always symmetric.
(Note: this analysis applies to any number of spinors coupled to ϕ.)

Perturbation Theory in a Spinor Field Theory

We will next consider how to formulate the covariant perturbation expansion in a theory of an interacting Dirac spinor field. This is mainly a straightforward generalization of what we did in Chapter 2, but there are a few new features. We will need to keep track of the spinor indices carried by the fields, and there are also some important minus signs that arise, associated with Fermi statistics.

We will first consider interaction picture perturbation theory, as formulated at the beginning of chapter 2. We will discuss later in this chapter how the results can be reinterpretted in terms of Green functions and the LSZ reduction formula.

We need to perform a Wick expansion, an expansion of Dyson's interaction picture time evolution operator

$$T \exp\left[-i \int d^4x \; \mathcal{H}_I [\psi_I, \bar{\psi}_I, \phi_I]\right]$$

in terms of normal ordered products of fields. So we need to extend Wick's theorem, which tells us how to expand a time-ordered product of free fields in terms of normal-ordered products, to the case of free Fermi fields.

We will define time-ordering for spinor fields differently than for scalar fields:

$$T[\psi_\alpha(x) \bar{\psi}_\beta(y)] = \theta(x^0 - y^0) \psi_\alpha(x) \bar{\psi}_\beta(y) - \theta(y^0 - x^0) \bar{\psi}_\beta(y) \psi_\alpha(x)$$

Notice the minus sign. This sign is necessary if we want $T[\psi\bar{\psi}]$ to be a Lorentz covariant object. The point is that time-ordering is frame-dependent when $(x-y)$ is spacelike. But with time ordering defined as above, the sign of $x^0 - y^0$ is actually irrelevant when $(x-y)^2 < 0$, because $\psi(x)$ and $\bar{\psi}(y)$ then anticommute. Note also that the T product defined this way changes sign when the arguments are permuted:

$$T[A(x)B(y)] = -T[B(y)A(x)]$$

Similarly, the T product of a string of fields is defined so that a (-1) appears if an odd permutation is required to put the fields in time-ordered sequence. E.g.,

$$T[A(x)B(y)C(z)] = \begin{array}{ll} A\,B\,C & x^0 > y^0 > z^0 \\ -B\,A\,C & y^0 > x^0 > z^0 \\ C\,A\,B & z^0 > x^0 > y^0 \\ -A\,C\,B & x^0 > z^0 > y^0 \\ B\,C\,A & y^0 > z > x^0 \\ -C\,B\,A & z^0 > y^0 > x^0 \end{array}$$

where A, B, C are spinor fields.

We are entitled to interpret the T in Dysons formula as this covariant T product, because the interaction H_I is necessarily an observable, bilinear in Fermi fields. Thus, an even number of permutations of Fermi fields always occur when we reorder the factors in

$$T[H_I(x_1) \cdots H_I(x_n)].$$

Normal ordering of a product of free fields should also be defined with such an oscillating minus sign. For example

$$:\psi^{(-)}(x)\,\bar{\psi}^{(+)}(x): = -\bar{\psi}^{(+)}(y)\,\psi^{(-)}(x)$$

(where $\psi^{(+)}$ and $\psi^{(-)}$ denote the creation and annihilation parts, respectively, of the free field ψ). The point is that we

should think of the normal ordered product as what we would obtain by anticommuting the fields as necessary and dropping all c-no. anticommutators. Evidently, then

$$:A(x)B(y): = -:B(y)A(x):;$$

the normal-ordered product, too, changes sign when the spinor fields A and B are permuted. Similarly, in the normal ordered product of a string of fields, a (-1) is included whenever an odd number of permutations of Fermi fields has been performed. For example

$$:ABC: = :(A^+ + A^-)(B^+ + B^-)(C^+ + C^-)$$

$$= A^+B^+C^+ + A^+B^+C^- - A^+C^+B^- + A^+B^-C^-$$

$$+ B^+C^+A^- + B^+C^-A^- - C^+B^-A^- + B^-C^-A^-.$$

Now we can define a <u>contraction</u> of two spinor fields as

$$\overbrace{A(x)B(y)} = :A(x)B(y): + \overbrace{A(x)B(y)}$$

or

$$\overbrace{A(x)B(y)} = \theta(x^0-y^0)[A^{(-)}(x), B^{(+)}(y)]$$

$$- \theta(y^0-x^0)[B^-(x), A^+(y)]_+$$

We see that

$$\overbrace{\psi(x)\psi(y)} = \overbrace{\bar\psi(x)\bar\psi(y)} = 0,$$

because $\psi^{(+)}$ and $\psi^{(-)}$ anticommute, and that

$$\psi_\alpha(x)\overbrace{\bar\psi_\beta(y)} = -\overbrace{\bar\psi_\beta(y)\psi_\alpha(x)},$$

Now we are finally ready to state and prove Wick's theorem. The statement and proof are almost the same as for scalar fields, except there is now a (-1) that keeps track of the parity (odd or even) of the permutation of spinor fields that gives rise to a given term.

Wick's theorem:

$$T(A_1 A_2 \cdots A_n) = :A_1 A_2 \cdots A_n:$$
$$+ \overbrace{A_1 A_2}\, :A_3 \cdots A_n: - \overbrace{A_1 A_3}\, :A_2 A_4 \cdots A_n: + \overbrace{A_1 A_4}\, :A_2 A_3 A_5 \cdots A_n:$$
$$+ \text{all other terms with one contraction}$$
$$+ \overbrace{A_1 A_2}\, \overbrace{A_3 A_4}\, :A_5 \cdots A_n: - \overbrace{A_1 A_3}\, \overbrace{A_2 A_4}\, :A_5 \cdots A_n:$$
$$+ \text{all other terms with two contractions}$$
$$+ \text{all other contractions}$$

(The (-1) precedes a term whenever an odd number of permutations of spinor fields have been performed to put the fields in the order shown.)

The proof, by induction on n, is almost identical to that for scalar fields on page (2.16):

Since both sides are antisymmetric under all permutations of the spinor fields A_k, it suffices to verify the theorem for a particular time ordering

$$x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0, \quad \text{so } T(A_1 \cdots A_n) = A_1 \cdots A_n$$

The theorem is trivially true for a product of $n=1$ or $n=2$ fields. We must show that if it is true for $n-1$ and $n-2$ fields, then it is true for n fields

The RHS can be written as

RHS = terms with A_1 not contracted
+ terms with A_1 contracted

$$= A_1^{(+)}(A_2 \cdots A_n) - (-1)^n (A_2 \cdots A_n) A_1^{(-)}$$
$$+ \overbrace{A_1 A_2}\,(A_3 \cdots A_n) - A_2 \overbrace{A_1 A_3}\, A_4 \cdots A_n + A_2 A_3 \overbrace{A_1 A_4}\, A_5 \cdots A_n$$
$$+ \cdots \qquad + (-1)^n A_2 \cdots A_{n-1} \overbrace{A_1 A_n}$$

(by the induction hypothesis). And since x_1^0 is the latest time

$$\overbrace{A_1 A_K} = [A_1^{(-)}, A_K]_+$$

So,

terms with A_1 contracted

$$= A_1^{(-)} A_2 A_3 \cdots A_n + A_2 A_1^{(-)} A_3 \cdots A_n - A_2 A_1^{(-)} A_3 \cdots A_n - A_2 A_3 A_1^{(-)} A_4 \cdots A_n$$
$$+ \cdots \quad + (-1)^n A_2 \cdots A_n A_1^{(-)},$$

and

$$RHS = (A_1^{(+)} + A_1^{(-)}) A_2 \cdots A_n = A_1 \cdots A_n = LHS,$$

which verifies Wick's theorem.

Feynman Rules for Fermions

By carrying out the Wick expansion as prescribed above, we can easily evaluate matrix elements

$$\langle \beta \mid T \exp\left(-i \int d^4x \, \mathcal{H}_I\right) \mid \alpha \rangle$$

of the interaction picture time evolution operator between interaction picture (asymptotic) states.

As in the scalar field theory, we will keep track of the different contractions in the Wick expansion by drawing Feynman diagrams that represent contributions to the scattering amplitude. But because the contraction

$$\overbrace{\psi(x)\,\bar{\psi}(y)}$$

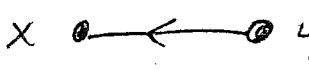
must be distinguished from $\overbrace{\bar{\psi}(x)\,\psi(y)}$ we will draw an arrow on the internal line that signifies a contraction. In position space,

$$= \overbrace{\psi_\alpha(x)\,\bar{\psi}_\beta(y)}\,.$$

The arrow points to the vertex at which ψ is contracted, and away from the vertex at which $\bar{\psi}$ is contracted. (Recall that there are no contractions of ψ with ψ or $\bar{\psi}$ with $\bar{\psi}$.)

The arrow does not represent a direction of momentum flow, but rather a direction of flow of fermionic charge.

That is, we may think of



as signifying the propagation of either a (virtual) particle from y to x or an antiparticle from x to y (depending on the time ordering).

Indeed, the contraction, or <u>propagator</u>, can be expressed as

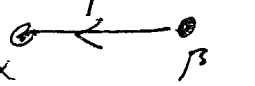
$$\langle 0 | T[\psi(x)\,\bar{\psi}(y)] | 0 \rangle .$$

So the above interpretation is supported by the observation that $\bar{\psi}$ can create a particle or destroy an antiparticle, while ψ can create an antiparticle or destroy a particle.

In a homework exercise, you showed that

$$\overbrace{\psi_\alpha(x)\,\bar{\psi}_\beta(y)} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} .$$

Thus, the momentum space fermion propagator is



$$= \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}$$

which we will also write as

$$= \left(\frac{i}{\not{p}-m+i\epsilon}\right)_{\alpha\beta}$$

That is, $(\not{p}+m)/(p^2-m^2)$ is the inverse of the matrix $(\not{p}-m)$.

Thus, the propagator is the inverse of the differential operator

$$i\not{\partial}-m$$

that appears in the quadratic part of the action (with an $i\epsilon$ prescription to make the inverse well defined). Compare the scalar field theory, in which the scalar propagator is the inverse of

$$-\partial^\mu\partial_\mu - m^2,$$

the differential operator in the action of the free scalar field. When we learn functional methods in chapter 4, we will better appreciate that this is no coincidence, but a special case of a general connection.

We note that the fermion propagator has a pole at $p^2=m^2$. This is the same as the scalar propagator, and we have understood this pole as a consequence of unitarity. The residue of this pole is a projection operator:

$$\left.\begin{aligned}\not{p}+m &= \text{pos. frequency projector } (p^0>0)\\ &= \text{neg. frequency projector } (p^0<0)\end{aligned}\right\} \text{at } p^2=m^2$$

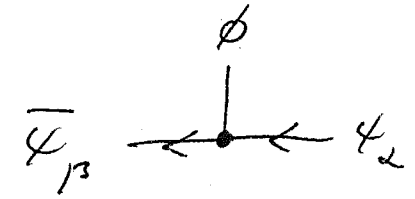
This observation confirms again that the propagator represents a flow of fermionic charge in the direction of the arrow. The intermediate states associated with the pole are particles for $p^0>0$ and antiparticles for $p^0<0$.

Feynman Rules

It should now be clear how we construct the Feynman diagram expansion for the amplitudes in a theory of a coupled spinor and scalar. If the interaction is

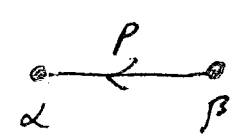
$$\mathcal{L}_{int} = -\lambda \bar{\psi} \Gamma \psi$$

(where Lorentz invariance requires Γ to be a linear combination of $\mathbb{I}$ and γ_5), then the factor associated with a vertex is



$$= -i\lambda \Gamma_{\beta\alpha}$$

while the propagator factors are



$$= \left[\frac{i}{\not{p} - m + i\epsilon}\right]_{\alpha\beta}$$

$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

These rules enable us to write out the Wick expansion of Dyson's T product in terms of normal ordered products.

To complete the rules for computing amplitudes, we need the factors associated with particles in the initial and final states. We recall the mode expansion of the interaction picture (free) field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_r \left[b_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} + c_{\vec{p}}^{(r)\dagger} v_{\vec{p}}^{(r)} e^{ip\cdot x} \right],$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}} \sum_r \left[b_{\vec{p}}^{(r)\dagger} \bar{u}_{\vec{p}}^{(r)} e^{ip\cdot x} + c_{\vec{p}}^{(r)} \bar{v}_{\vec{p}}^{(r)} e^{-ip\cdot x} \right].$$

Also, recall that for states with the relativistic normalization

$$\langle 0 | \frac{b_{\vec{p}}^{(r)}}{(2\pi)^{3/2}\sqrt{2\omega_p}} | p; s \rangle = \delta^{r,s} \delta^3(\vec{p} - \vec{p}'),$$

or

$$\langle 0 | \psi(x) | p, r, + \rangle = u_{\vec{p}}^{(r)} e^{-ip\cdot x},$$

$$\langle 0 | \bar{\psi}(x) | p, r, - \rangle = \bar{v}_{\vec{p}}^{(r)} e^{-ip\cdot x},$$

$$\langle p, r, + | \bar{\psi}(x) | 0 \rangle = \bar{u}_{\vec{p}}^{(r)} e^{ip\cdot x},$$

$$\langle p, r, - | \psi(x) | 0 \rangle = v_{\vec{p}}^{(r)} e^{ip\cdot x}.$$

Thus, in position space, the Feynman rules for incoming and outgoing particles are

Incoming fermion: $\alpha \longleftarrow p, r = \left[u_{\vec{p}}^{(r)} \right]_\alpha e^{-ip\cdot x}$

Incoming antifermion: $\alpha \longrightarrow p, r = \left[\bar{v}_{\vec{p}}^{(r)} \right]_\alpha e^{-ip\cdot x}$

Outgoing fermion: $p, r \longleftarrow \alpha = \left[\bar{u}_{\vec{p}}^{(r)} \right]_\alpha e^{ip\cdot x}$

Outgoing ant.fermion: $p, r \longrightarrow \alpha = \left[v_{\vec{p}}^{(r)} \right]_\alpha e^{ip\cdot x}$

As for the momentum space rules, the $e^{\pm ip\cdot x}$ factors merely enforce momentum conservation at the vertices (as for scalar fields).

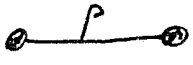
So we may summarize the rules in momentum space:

Feynman Rules

- Assign to each internal line the most general momentum consistent with conservation of momentum at all vertices.
- For each vertex:

$$= -i\Gamma_{\alpha\beta}$$

- For each scalar internal line:

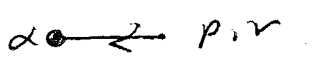


$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

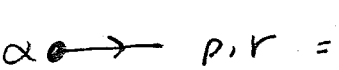
- For each fermion internal line:

$$= \left[\frac{i}{\not{p} - m + i\epsilon}\right]_{\alpha\beta}$$

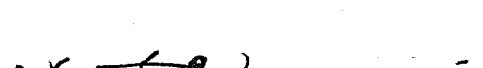
- For each external line:



$$= [u_{\vec{p}}^{(r)}]_\alpha$$



$$= [\bar{v}_{\vec{p}}^{(r)}]_\alpha$$



$$= [\bar{u}_{\vec{p}}^{(r)}]_\alpha$$

$$= [v_{\vec{p}}^{(r)}]_\alpha$$

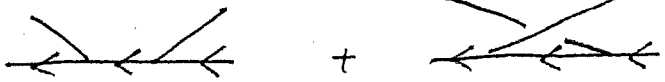
- Integrate $\int \frac{d^4k}{(2\pi)^4}$ for each undetermined (loop) momentum.
- Rules for Fermi (-1)'s : see below

-- these rules associate each diagram with a contribution to iA (with the $(2\pi)^4 \delta^4(p_{in} - p_{out})$ divided out).

(Of course, we might also want to include the ϕ^4 and ϕ^3 couplings, in which case there are additional vertices associated with these couplings.)

For example in order λ^2 we have the following diagrams.

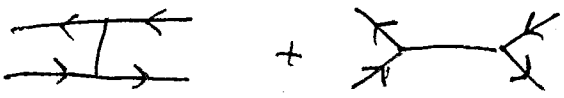
Tree Graphs (scattering) --

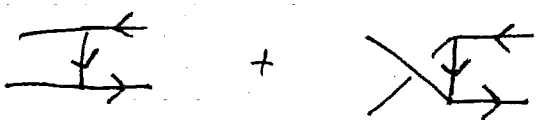
$f\phi \to f\phi$: 

($\bar{f}\phi \to \bar{f}\phi$: reverse the arrows)

$ff \to ff$; 

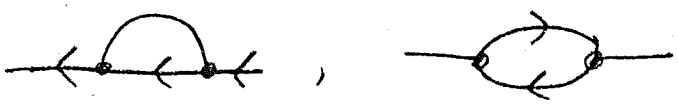
($\bar{f}\bar{f} \to \bar{f}\bar{f}$: reverse the arrows)

$f\bar{f} \to f\bar{f}$: 

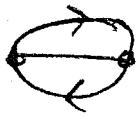
$f\bar{f} \to \phi\phi$: 

3.99.10

One Loop (self-Energy) --



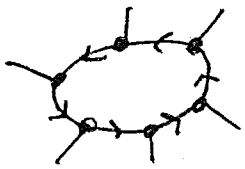
Two Loop (Vacuum) --



Minus Signs From Fermi Statistics

The Feynman rules specified above are not complete, because we have not been careful about (-1)'s that come from anticommuting the spinor fields. These factors of (-1) are important, because they are necessary to ensure that our amplitudes are fully consistent with Fermi statistics.

Minus signs can arise in two ways -- they are associated with either a closed loop of fermion propagators

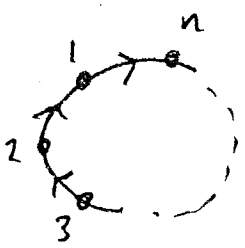


inside a diagram, or with an initial or final state that contains two or more fermions.

Closed Loops:

The contractions associated with a closed loop inside a diagram have the generic form:

3.99.11



$$= \overline{\psi_1}\psi_1\ \overline{\psi_2}\psi_2\ \overline{\psi_3}\psi_3 \cdots \overline{\psi_n}\psi_n$$

(Spinor indices are suppressed here; the closed loop corresponds to the trace of a product of matrices in the spinor indices, which depends only on the cyclic order of the propagators.) But now, recall that

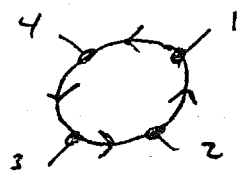
$$\overline{\psi_\alpha \psi_\beta} = -\overline{\psi_\beta \psi_\alpha}$$

The above contractions are all $\psi\bar{\psi}$ contractions except for one $\bar{\psi}\psi$ contraction, so there is one minus sign. This is true of all contractions that form a closed loop. (Since the $\bar{\psi}\psi$'s all commute with each other inside the T product, we can always order them so that the contractions take the form given above.) So we have the Feynman rule:

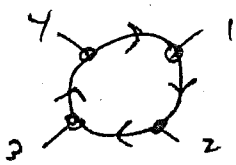
- For each closed fermion loop:

$$\Rightarrow (-1)$$

When we consider diagrams with closed fermion loops, we must be careful to count correctly all the independent contractions. For example:



and



(arrows running in opposite directions) are distinct contractions. They represent different cyclic orderings of the vertices. But

are identical contractions. (Indeed, one diagram can be obtained from the other by lifting it out of the plane of the paper, flipping it over, and rotating it.)

For a loop with, e.g., 4 vertices, there are 6 distinct contractions, corresponding to the distinct cyclic orderings

1 2 3 4
1 2 4 3
1 3 4 2
1 3 2 4
1 4 2 3
1 4 3 2

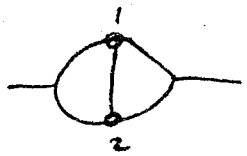
All 6 give rise to the same Feynman integral, but it is important to include the combinatoric factor 6 that counts the distinct contractions.

The combinatorics described here is related to the combinatorics involved in the computation of symmetry factors in the self-coupled scalar field theory. Symmetry factors arise when there are permutations of the vertices or lines that do not give rise to new contractions. Such symmetry factors do not arise in the Yukawa theory, except in certain vacuum diagrams.

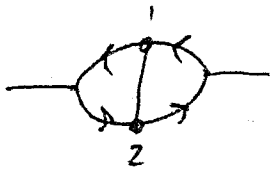
For example,

in ϕ^3 theory has symmetry factor $\frac{1}{2}$, because the two internal lines are indistinguishable, but

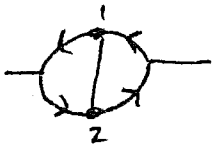
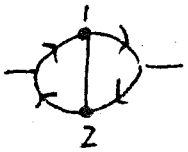
in $\bar{\psi}\psi\phi$ theory has no symmetry factor because the internal lines are distinguishable.



in ϕ^3 theory has factor $\frac{1}{2}$ because vertices 1 and 2 are indistinguishable.



in $\bar{\psi}\psi\phi$ theory has no factor because vertices 1 and 2 are distinguishable

(or, in other words  and  are distinct contractions).

But —

has factor $\frac{1}{2}$, because vertices are indistinguishable.

Initial and Final states:

For matrix elements between initial and final states that contain two or more fermions, we need to establish sign conventions for the states in order to assign a sign to the matrix element. For example. Does

$|p_1, r_1, +; p_2, r_2, +\rangle$ denote $b_1^\dagger b_2^\dagger |0\rangle$

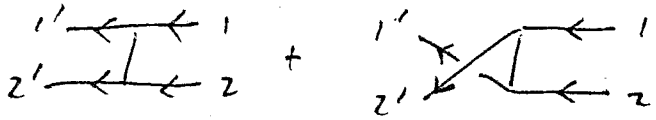
or $b_2^\dagger b_1^\dagger |0\rangle = -b_1^\dagger b_2^\dagger |0\rangle$?

Of course, such sign conventions carry no physical significance; they fix only an overall sign of an amplitude. On the other hand, the relative sign of different diagrams that contribute to the same amplitude are significant. We have to be careful to choose our sign conventions for the states consistently in order to get these relative signs right.

Consider for example the process (in tree approximation)

$$ff \to ff.$$

Applying the Feynman rules:

$iA =$ 

$$= (-i\lambda)^2 \Big[\bar{u}_1' \Gamma u_1 \, \bar{u}_2' \Gamma u_2 \, \frac{i}{(p_1 - p_1')^2 - \mu^2 + i\epsilon} + (-1)\, \bar{u}_1' \Gamma u_2 \, \bar{u}_2' \Gamma u_1 \, \frac{i}{(p_1 - p_2')^2 - \mu^2 + i\epsilon} \Big]$$

Why the relative minus sign in the sum of the diagrams?

The point is that

$$T\left[(\bar{\psi}\psi\phi)_A \, (\bar{\psi}\psi\phi)_B\right] = :\overbrace{(\bar{\psi}\psi\phi)_A \, (\bar{\psi}\psi\phi)_B}: + \text{other terms}$$

and

$$:(\bar{\psi}\psi)_A (\bar{\psi}\psi)_B: \; = - b_A^+ b_B^+ b_A b_B$$

But now when we evaluate

3.99.15

$$\langle 1', 2' | b_A^\dagger b_B^\dagger b_A b_B | 1, 2 \rangle,$$

there are two types of terms. We can have

$$\begin{pmatrix} b_A \text{ annihilates } 1 \\ b_B \text{ annihilates } 2 \end{pmatrix} \text{ and either } \begin{pmatrix} b_A^\dagger \text{ creates } 1' \\ b_B^\dagger \text{ creates } 2' \end{pmatrix}$$

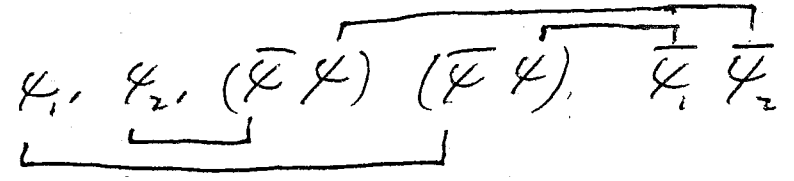
$$\text{or } \begin{pmatrix} b_A^\dagger \text{ creates } 2' \\ b_B^\dagger \text{ creates } 1' \end{pmatrix}$$

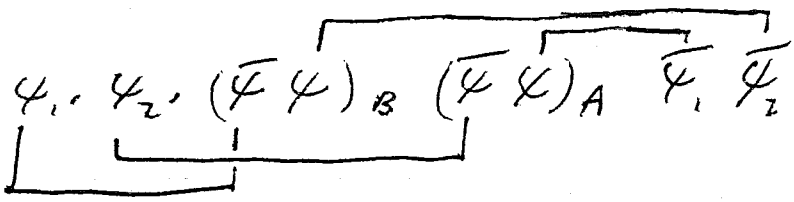
The two possibilities correspond to the two Feynman diagrams. And it is clear that, however we choose our phase conventions for the states $|1,2\rangle$ and $|1',2'\rangle$ the two contributions will differ in sign, because $b_A^\dagger$ and $b_B^\dagger$ anticommute.

Of course, we can also see how the relative (-1) arises if we interpret the Feynman diagrams as contributions to Green functions rather than matrix elements. In the computation of

$$\langle 0 | T[\psi_1, \psi_2, \bar{\psi}_1 \bar{\psi}_2] | 0 \rangle$$

in order λ^2, the contractions





occur. There is a relative minus sign in

the sum of these contractions, arising from anticommuting ψ_1 and ψ_2'.

The relative sign in the sum of these diagrams is necessary in order for the amplitude to change sign when the identical fermions in the initial or final state are interchanged, as required by Fermi statistics. The requirement that such signs be included completes our specification of the Feynman rules.

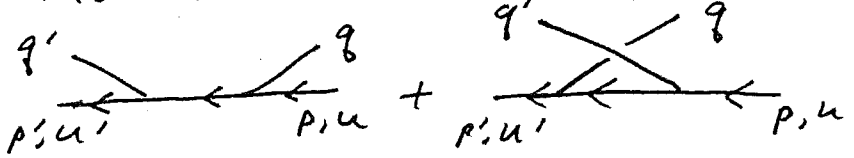
In higher order diagrams, getting these signs right can be tricky. The fool-proof method is to write out a string of $\bar{\psi}\psi$'s and do the contractions by hand.

A Sample Computation

To further illustrate the use of the Feynman rules, let's compute the tree approximation to the process

$$f\phi \to f\phi$$

The diagrams are --

$iA =$ 

$$= (-i\lambda)^2 \left[\bar{u}' \Gamma \frac{i}{\not p + \not q - m + i\epsilon} \Gamma u + \bar{u}' \Gamma \frac{i}{\not p' - \not q - m + i\epsilon} \Gamma u \right]$$

To be concrete, suppose $\Gamma = i\gamma_5$.

Then

$$iA = i\lambda^2\left[\frac{\bar{u}'\gamma_5(\not{p}+\not{q}+m)\gamma_5 u}{(p+q)^2-m^2} + \frac{\bar{u}'\gamma_5(\not{p}'-\not{q}+m)\gamma_5 u}{(p'-q)^2-m^2}\right]$$

Now we recall that u and $\bar{u}$ obey

$$(\not{p}-m)u = 0$$
$$\bar{u}'(\not{p}'-m) = 0,$$

so we can anticommute $\not{p}$ and $\not{p}'$ through γ_5 to obtain

$$A = \lambda^2\,\bar{u}'\not{q}u\left[\frac{1}{(p'-q)^2-m^2} - \frac{1}{(p+q)^2-m^2}\right]$$

Spin Averaging

The above amplitude has dependence on the spin of the initial and final fermion contained in the factor $\bar{u}'\not{q}u$. But often we are interested in the cross section for a process averaged over initial spins and summed over final spins. We may want to compare our calculation to an experiment in which the beam and target are unpolarized (and the final spins are not measured). After spin averaging, the answer can be expressed in terms of Lorentz invariant functions of the momenta.

So consider an amplitude like that above which has the form:

$$A_{s,r} = \bar{u}^{(s)}\mathcal{M}u^{(r)}\,f(\text{invariants})$$

(where M is a 4×4 matrix acting on spinor indices). Then

$$|A|^2_{\text{average}} = \frac{1}{2}\sum_{r,s} |f|^2\, \bar{u}^{(s)} M u^{(r)}\, \bar{u}^{(r)} \bar{M} u^{(s)}$$

But the factor $\sum_{r,s} \bar{u}'^{(s)} M u^{(r)} \bar{u}^{(r)} \bar{M} u'^{(s)}$

can be rewritten as

$$\mathrm{tr}\left[M \left(\sum_r u^{(r)} \bar{u}^{(r)}\right) \bar{M} \left(\sum_s u'^{(s)} \bar{u}'^{(s)}\right)\right],$$

and we recall that

$$\sum_r u^{(r)} \bar{u}^{(r)} = \not{p} + m$$
$$\sum_s u'^{(s)} \bar{u}'^{(s)} = \not{p}' + m.$$

Thus we obtain

$$|A|^2_{\text{ave}} = |f|^2 \frac{1}{2} \mathrm{tr}\left[M (\not{p}+m) \bar{M} (\not{p}'+m)\right].$$

For the process considered above, the relevant trace is

$$\frac{1}{2}\mathrm{tr}\left[\not{q} (\not{p}+m) \not{q} (\not{p}'+m)\right].$$

In general, to calculate spin averaged cross sections we must evaluate traces of products of γ matrices. This is facilitated by a few basic identities.

3.99.19

Trace Identities

The fundamental relation satisfied by the γ matrices

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu},$$

can be expressed as $\not{a}_1 \not{a}_2 + \not{a}_2 \not{a}_1 = 2 a_1 \cdot a_2$. Using this relation repeatedly, we can evaluate the trace of a product of any number of γ matrices. First we note that:

- $\mathrm{Tr}(\not{a}_1 \not{a}_2 \cdots \not{a}_n) = 0$, n odd

This is so because $\gamma_5^2 = \mathbb{I}$ and $\gamma_5 \not{a} = -\not{a} \gamma_5$. Hence,

$$\mathrm{Tr}(\not{a}_1 \cdots \not{a}_n) = \mathrm{Tr}(\gamma_5^2 \not{a}_1 \cdots \not{a}_n) =$$
$$= -\mathrm{Tr}\, \gamma_5 (\not{a}_1 \cdots \not{a}_n) \gamma_5 \quad \text{(for } n \text{ odd)}$$
$$= -\mathrm{Tr}\, \gamma_5^2 (\not{a}_1 \cdots \not{a}_n) = -\mathrm{Tr}(\not{a}_1 \cdots \not{a}_n)$$
(by cyclicity of the trace).

For n even, we can evaluate the trace by recursively applying

- $$\mathrm{Tr}(\not{a}_1 \cdots \not{a}_n) = (a_1 \cdot a_2)\mathrm{Tr}(\not{a}_3 \cdots \not{a}_n) - (a_1 \cdot a_3)\mathrm{Tr}(\not{a}_2 \not{a}_4 \cdots \not{a}_n) + (a_1 \cdot a_4)\mathrm{Tr}(\not{a}_2 \not{a}_3 \not{a}_5 \cdots \not{a}_n)$$
$$+ \cdots + a_1 \cdot a_n \mathrm{Tr}(\not{a}_2 \cdots \not{a}_{n-1}), \quad n \text{ even}$$

This identity reduces a trace of a product of n γ matrices to traces of products of $(n-2)$ γ matrices.

3.99.20

To prove it, use $\not{a}_1\not{a}_2 = -\not{a}_2\not{a}_1 + 2a_1\cdot a_2$.

$$\mathrm{tr}\,\not{a}_1\not{a}_2\cdots\not{a}_n = 2a_1\cdot a_2\,\mathrm{tr}(\not{a}_3\cdots\not{a}_n) - \mathrm{tr}(\not{a}_2\not{a}_1\not{a}_3\cdots\not{a}_n)$$

$$-\mathrm{tr}(\not{a}_2\not{a}_1\not{a}_3\cdots\not{a}_n) = -2a_1\cdot a_3\,\mathrm{tr}(\not{a}_2\not{a}_4\cdots\not{a}_n) + \mathrm{tr}(\not{a}_2\not{a}_3\not{a}_1\not{a}_4\cdots\not{a}_n)$$

and so on. Therefore,

$$\begin{aligned}\mathrm{tr}(\not{a}_1\not{a}_2\cdots\not{a}_n) = {} & 2(a_1\cdot a_2)\,\mathrm{tr}(\not{a}_3\cdots\not{a}_n) \\ & - 2(a_1\cdot a_3)\,\mathrm{tr}(\not{a}_2\not{a}_4\cdots\not{a}_n) \\ & + 2(a_1\cdot a_4)\,\mathrm{tr}(\not{a}_2\not{a}_3\not{a}_5\cdots\not{a}_n) \\ & + \cdots \\ & + 2(a_1\cdot a_n)\,\mathrm{tr}(\not{a}_2\not{a}_3\cdots\not{a}_{n-1}) \\ & - \mathrm{tr}(\not{a}_2\not{a}_3\cdots\not{a}_n\not{a}_1),\end{aligned}$$

from which the desired result follows, using cyclicity of the trace.

Of course for two matrices we have

$$\mathrm{tr}\,\not{a}_1\not{a}_2 = \tfrac{1}{2}\,\mathrm{tr}(\not{a}_1\not{a}_2 + \not{a}_2\not{a}_3) = (a_1\cdot a_2)\,\mathrm{tr}(\mathbb{1}) = 4a_1\cdot a_2.$$

And for four matrices,

$$\begin{aligned}\mathrm{tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = {} & 4\big[(a_1\cdot a_2)(a_3\cdot a_4) - (a_1\cdot a_3)(a_2\cdot a_4) \\ & + (a_1\cdot a_4)(a_2\cdot a_3)\big].\end{aligned}$$

3.99.21

Also of interest are traces of products of γ^μ's and one γ_5:

- $\mathrm{tr}\,\gamma_5 = 0$ (we already know this.)
- $\mathrm{tr}(\gamma_5 \not{a}_1 \not{a}_2) = 0$

  (Because $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow$ e.g.,
  $\mathrm{tr}(\gamma_5\gamma^2\gamma^3) = -i\,\mathrm{tr}(\gamma^0\gamma^1) = \frac{-i}{2}\mathrm{tr}(\gamma^0\gamma^1+\gamma^1\gamma^0) = 0.$)
- $\mathrm{tr}(\gamma_5 \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4) = 4i\,\epsilon_{\mu\nu\lambda\sigma}\, a_1^\mu a_2^\nu a_3^\lambda a_4^\sigma$

  (where $\epsilon_{0123} = +1$)

To prove this one, use $\not{a}_1\not{a}_2 = -\not{a}_2\not{a}_1 + (a_1\cdot a_2)$ and $0 = \mathrm{tr}(\gamma_5 \not{a}_1 \not{a}_2)$ to show that $\mathrm{tr}(\gamma_5 \not{a}_1 \not{a}_2 \not{a}_3 \not{a}_4)$ is totally antisymmetric under interchange of the a_k's. Then calculate

$$\mathrm{tr}(\gamma_5\gamma_0\gamma_1\gamma_2\gamma_3) = i\,\mathrm{tr}\,\gamma_5^2 = 4i$$

to find the normalization.

Finally, to complete the calculation of the spin-averaged $|A|^2$ for ϕ scattering, we evaluate

$$\tfrac{1}{2}\mathrm{tr}\left[\not{q}(\not{p}+m)\not{q}(\not{p}'+m)\right] = \tfrac{1}{2}m^2\,\mathrm{tr}\,\not{q}^2 + \tfrac{1}{2}\mathrm{tr}(\not{q}\not{p}\not{q}\not{p}')$$

$$= 2\left(m^2 q^2 + (q\cdot p)(q\cdot p') - q^2(p\cdot p') + (q\cdot p')(q\cdot p)\right)$$

$$= 2m^2(m^2 - p\cdot p') + 4(q\cdot p)(q\cdot p').$$

Renormalization of Spinor Field Theory

In the Yukawa theory with interaction

$$\mathcal{L}_{int} = -g\,\bar{\psi}\Gamma\psi\phi$$

let us repeat the dimensional analysis of p. 1.116, but this time in the Green function language. Consider

$$G^{(n,n,m)} = \langle 0|T(\underbrace{\bar{\psi}\cdots\bar{\psi}}_{n}\ \underbrace{\psi\cdots\psi}_{n}\ \underbrace{\phi\cdots\phi}_{m})|0\rangle$$

(There must be as many ψ's as $\bar{\psi}$'s for G to be nonvanishing, because of charge conservation.) Since ψ has dimension $3/2$, ϕ has dimension 1

$G^{(n,n,m)}$ has dimension $3n+m$

Fourier transform:

$\tilde{G}^{(n,n,m)}$ has dimension $-5n-3m$

Amputate propagators on external legs:

$$\tilde{G}^{(n,n,m)}\Big|_{amp} \text{ has dimension } -3n-m$$

Remove energy-momentum conserving δ-function:

Connected Diagram with $F=2n$ Fermion external lines and $B=m$ Bose external lines has dimension $4-B-\frac{3}{2}F$

Since coupling is dimensionless $D = 4-B-\frac{3}{2}F$ is expression for superficial degree of divergence of a diagram

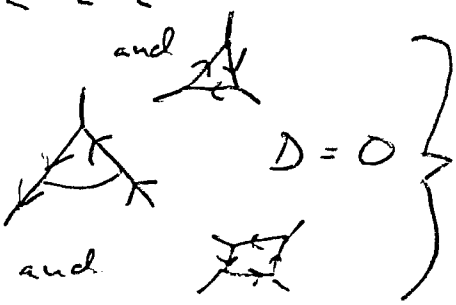
Examples (Divergent one loop diagrams):

$D = 4$ Vacuum Energy

$D = 3$ Tadpole
(absent in pseudoscalar theory, $\Gamma = i\gamma_5$, because of parity symmetry: $\langle 0 | \phi(x) | 0 \rangle = 0$)

$D = 2$ Mass and Field renormalization for ϕ

$D = 1$ Mass and Field renormalization for ψ



$D = 0$ Coupling renormalization

The discussion of the renormalization of the vacuum energy, tadpole, scalar mass and field are no different from in scalar field theories considered earlier. But let us examine more closely the renormalization of the fermion mass and field, and the coupling. These have some new features, because of the spinor indices floating around.

Fermion Propagator

what restrictions are placed on the form of the exact Fermion propagator by Lorentz invariance?

Recall that

$$U(\Lambda)^{-1}\psi(x)U(\Lambda) = D(\Lambda)\psi(\Lambda^{-1}x)$$
$$U(\Lambda)\bar{\psi}(x)U(\Lambda) = \bar{\psi}(\Lambda^{-1}x)\bar{D}(\Lambda)$$

and that $U(\Lambda)|0\rangle = |0\rangle$

therefore

$$G(x) = \langle 0|T(\psi(x)\bar{\psi}(0))|0\rangle$$
$$= D(\Lambda)G(\Lambda^{-1}x)\bar{D}(\Lambda)$$

If we fourier transform, $\tilde{G}(p) = \int d^4x\, e^{ip\cdot x}G(x)$, then

$$\tilde{G}(p) = D(\Lambda)\tilde{G}(\Lambda^{-1}p)\bar{D}(\Lambda)$$

Now $\tilde{G}(p)$ is a 4×4 matrix, with, in principle, 16 independent components. We may choose as a basis for the 4×4 matrices

$$\mathbb{1},\ \gamma_5,\ \gamma_\mu,\ \gamma_\mu\gamma_5,\ \sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu,\gamma_\nu]$$

all of which have definite transformation properties under Lorentz transformations (and parity):

$$D(\Lambda)\gamma^\mu\bar{D}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu$$

So the most general allowed form for $\tilde{G}$ is

$$\tilde{G}(p) = a(p^2) + \gamma_5 b(p^2) + \not{p}\, c(p^2) + \not{p}\gamma_5 d(p^2)$$
$$+ \sigma_{\mu\nu}p^\mu p^\nu e(p^2)$$
$$+ \sigma_{\mu\nu}p_\lambda p_\sigma \tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma} f(p^2)$$

It depends, in general, on 4 independent scalar functions of p^2.

vanishes because of antisym of $\sigma_{\mu\nu}$

The number of independent functions can be further reduced if we assume that the theory has a parity symmetry ($\Gamma = \mathbb{1}$ or $\Gamma = \gamma_5$). Recall

$$U_P \psi(\vec{x},t) U_P^{-1} = \gamma^0 \psi(-\vec{x},t)$$
$$U_P \bar{\psi}(\vec{x},t) U_P^{-1} = \bar{\psi}(-\vec{x},t) \gamma^0$$
$$U_P |0\rangle = |0\rangle \quad \text{(if parity is a symmetry)}$$

so
$$G(\vec{x},t) = \gamma^0 G(-\vec{x},t) \gamma^0$$
$$\tilde{G}(p^0,\vec{p}) = \gamma^0 G(p^0,-\vec{p}) \gamma^0$$

We see that the matrices

$$\mathbb{1}, \not{p} \quad \text{(parity even)}$$

are consistent with this condition, while

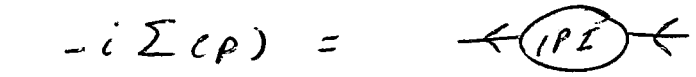
$$\gamma_5, \not{p}\gamma_5 \quad \text{(parity odd)}$$

are not. Thus, in a parity conserving theory,

$$\tilde{G}(p) = a(p^2) + \not{p}\, c(p^2)$$

— only <u>two</u> independent functions.

As in the scalar theory (p 2.81 ff) we may define

$$-i\Sigma(p) =$$ 

then the full propagator is seen (by summing a geometric series) to be

$$= \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon}$$

Let's consider now the parity conserving case, which is somewhat simpler than the general case. Since the inverse propagator $\tilde{G}(p)^{-1}$ obeys the same invariance condition as $\tilde{G}$, we see that

$$\Sigma(p) = A(p^2) + \not{p} B(p^2) = \Sigma(\not{p})$$

The loop corrections to the propagator have the effect, in general, of shifting the position of the pole in the propagator, and rescaling the residue of the pole. We can compensate for these effects by introducing counterterms that ensure that m is really the physical mass (position of the pole) and ψ is a properly normalized field. The appropriate renormalization conditions are

$$\Sigma(\not{p}) = \frac{d\Sigma(\not{p})}{d\not{p}} = 0 \quad \text{at} \quad \not{p} = m$$

These fix the mass and field counterterms order-by-order. We have

$$\mathcal{L} = \bar{\psi}_B (i\not{\partial} - m_0)\psi_B + \cdots$$

$$= \bar{\psi}_R (i\not{\partial} - m)\psi_R + (Z-1)\bar{\psi}_R (i\not{\partial} - m)\psi_R - Z\delta m\, \bar{\psi}_R \psi_R$$

where $\psi_R = Z^{-\frac{1}{2}} \psi_B$, $\delta m = m_0 - m$

(and ψ_R denotes renormalized field, not right-handed component)

Counterterms are

$i(Z-1)(\not{p} - m)$ -- Field renorm

$-iZ\delta m$ -- Mass renorm

Computing Σ to a given order in perturbation theory we have

$$-i\Sigma = -i\Big[C_0 + C_1(\not p - m) + \mathcal{O}(\not p - m)^2 + Z\delta m - (Z-1)(\not p - m)\Big]$$

($Z\delta m - (Z-1)(\not p - m)$: counterterms)

δm and Z are chosen order by order to cancel C_0 and C_1. This imposes the renormalization condition.

Power counting also shows that these counterterms suffice to make Σ finite. Diagrams contributing to Σ have $D = 1$ So

$$C_0 = O(\Lambda) \qquad \text{(where } \Lambda \text{ is an ultraviolet cutoff)}$$
$$C_1 = O(\ln \Lambda)$$

Remainder = Finite

Computation actually shows that there is no $O(\Lambda)$ divergence. Actually

$$C_0 = m\, O(\ln \Lambda).$$

Mass renormalization, like field renormalization, is logarithmic. Mathematically, the linear divergence is absent because it comes from a Feynman integral of the form

$$\int d^4p \, \frac{p_\mu}{(p^2)^2}$$

which is required to vanish by Lorentz invariance.

As in the scalar case, we can relate the wave function renormalization condition to a condition on the amplitude for the field ψ to annihilate a particle. Consider

$$\langle 0 | \psi(x) | \vec{p}, r \rangle = e^{-ip\cdot x} \langle 0 | \psi(0) | \vec{p}, r \rangle$$

But $|\vec{p}, r\rangle$ can be expressed as a Lorentz transformation acting on a standard one particle state

$$|\vec{p}, r\rangle = U(\Lambda(\vec{p}, r)) \; | \; \vec{0}, r=1 \rangle$$

(i.e. $J_3 = +\frac{1}{2}$)

and $U(\Lambda)^{-1} \psi(x) U(\Lambda) = D(\Lambda) \psi(\Lambda^{-1} x)$

$\langle 0 | U(\Lambda)^{-1} = \langle 0 |$ implies

$$\langle 0 | \psi(0) | \vec{p}, r \rangle = D(\Lambda(\vec{p}, r)) \langle 0 | \psi(0) | \vec{0}, 1 \rangle$$

Now, rotational invariance tells us that only the $J_z = \frac{1}{2}$ components of $\psi(0)$ can annihilate the $J_3 = \frac{1}{2}$ state

$$U(\theta) | r=1 \rangle = e^{-\frac{i}{2}\theta} | r=1 \rangle$$

$$U(\theta)^{-1} \psi(0) U(\theta) = e^{-iJ_3\theta} \psi(0)$$

So $e^{-i\frac{\theta}{2}} \langle 0 | \psi(0) | \vec{0}, 1 \rangle = e^{-iJ_3\theta} \langle 0 | \psi(0) | \vec{0}, 1 \rangle$

or

$$\langle 0 | \psi(0) | \vec{0}, 1 \rangle = \sqrt{2m} \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

-- Depends on two independent numbers, in general

But in the parity conserving case, this is reduced to one number. Recall that U_P acting on state does not flip spin. So, if $U_P |0\rangle = |0\rangle$ (parity is a symmetry) then

$$\langle 0 | \psi(0) | \vec{0}, 1 \rangle = \langle 0 | U_P \psi(0) U_P | \vec{0}, 1 \rangle = \delta^0 \langle 0 | \psi(0) | \vec{0}, 1 \rangle$$

Thus $b = 0$, and $\langle 0|\psi(x)|\vec{p},r\rangle$ depends on just one normalization constant and we have

$$\langle 0|\psi(x)|\vec{p},r\rangle = e^{-ip\cdot x} u_{\vec{p}}^{(r)} Z_\psi^{1/2}$$

↑ relativistically normalized

We may define renormalized fields by

$$\psi_R = Z_\psi^{-\frac{1}{2}} \psi, \text{ so that } \langle 0|\psi_R(x)|\vec{p},r\rangle = e^{-ip\cdot x} u_{\vec{p}}^{(r)}$$

We can derive a spectral representation to show that the exact propagator for the renormalized field will have a conventionally normalized residue of the pole at $p^2 = m^2$.

Insert a complete sum over intermediate states to find

$$\langle 0|\psi(x)\bar{\psi}(0)|0\rangle = \int \frac{d^3p \sum_r}{(2\pi)^3 2\omega_p} e^{-ip\cdot x} \langle 0|\psi(0)|p,r\rangle\langle p,r|\bar{\psi}(0)|0\rangle$$

$$+ \sum_n{}' e^{-ip_n\cdot x} \langle 0|\psi(0)|n\rangle\langle n|\bar{\psi}(0)|0\rangle$$

↑ (sum over two or more particles)

$$= S_+(x;m) + \text{Remainder}$$

↑ (for renormalized fields)

Remainder

$$= \int d^4p\, e^{-ip\cdot x} \sum_n{}' \delta^4(p-p_n) \langle 0|\psi(0)|n\rangle\langle n|\bar{\psi}(0)|0\rangle$$

But Lorentz invariance and parity invariance tells that

$$(2\pi)^3 \sum_n' \delta^4(p - P_n) \langle 0|\psi(0)|n\rangle\langle n|\bar{\psi}(0)|0\rangle$$

$$= [A(p^2) + \not{p} B(p^2)]\,\theta(p^0)$$

$$= \theta(p^0)\left[\rho_+(p^2)(\not{p} + \sqrt{p^2}) + \rho_-(p^2)(\not{p} - \sqrt{p^2})\right]$$

So -- Remainder

$$= \int da^2 \left[\rho_+(a^2) \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot x}\,\theta(p^0)(\not{p}+a)\,\delta(p^2-a^2)\right.$$

$$\left. + \rho_-(a^2) \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot x}\theta(p^0)(\not{p}-a)\,\delta(p^2-a^2)\right]$$

$$= \int da^2 \left[\rho_+(a^2)\,(i\not{\partial}+a)\,\Delta_+(x;a)\right.$$

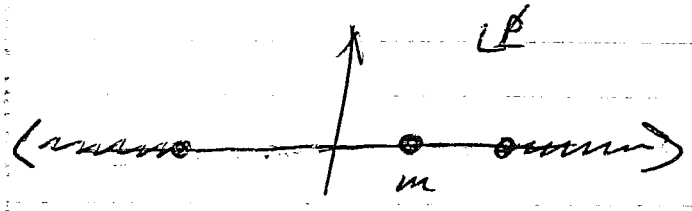
$$\left. + \rho_-(a^2)\,(i\not{\partial}-a)\,\Delta_+(x;a)\right]$$

If we sum over time orderings and Fourier transform, we have the spectral rep for the propagator (for renormalized fields)

$$t = \frac{i}{\not{p} - m + i\epsilon} + \int da^2 \rho_+(a^2) \frac{i}{\not{p} - a + i\epsilon}$$

$$+ \int da^2\, \rho_-(a^2) \frac{i}{\not{p} + a + i\epsilon}$$

As an analytic function of $\not{p}$, the exact propagator has a pole at $\not{p} = m$, and cuts running along both positive and negative real axes, beginning at the two-particle threshold.



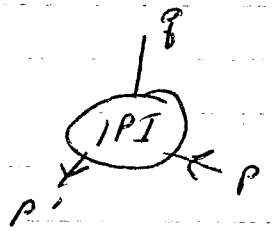
The field ψ couples to both parity + and parity − states. And σ_+, σ_- can be interpreted as the contributions from + parity and − parity intermediate states. (Sum over parity plus produces $\sum u\bar{u} = \not{p} + m$; sum over negative parity produces $\sum v\bar{v} = \not{p} - m$)

We can now derive a reduction formula that relates Green functions to S-matrix elements. This goes just as in scalar case, and need not be repeated.

Coupling Renormalization

Consider:

$$-i\Gamma(p, p'; q) =$$



Like the propagator, it has a form restricted by Lorentz invariance

$$\Gamma = a + \gamma_5 b + \quad \cdot \quad O(p, p'; q)$$

(where a, b are scalar functions of e.g. p^2, p'^2, q^2)

But power counting tells us that terms linear and higher order in external momentum are (superficially) finite. The only divergence is in

$$a, b = O(\ln \Lambda)$$

To make Λ finite, we need only the counterterms of the form

$$\bar{\psi}\psi\phi, \quad \bar{\psi} i\gamma_5 \psi\phi$$

In fact, if theory has a parity symmetry, we need only one type of counterterm or the other, but not both. Consider the two cases..

$\mathcal{L}_{int} = -g\,\bar{\psi}\psi\phi$ $\quad\phi$ is a scalar $\quad P: \phi(\vec{x},t) \to \phi(-\vec{x},t)$

$\mathcal{L}_{int} = -g\,\bar{\psi} i\gamma_5 \psi\phi$ $\quad\phi$ is a pseudoscalar $\quad P: \phi(\vec{x},t) \to -\phi(-\vec{x},t)$

How does parity invariance constrain

$$G^{(1,1,1)} = \langle 0 | T(\phi\,\bar{\psi}\,\psi) | 0 \rangle \; ?$$

$$\tilde{G}^{(1,1,1)}(p^0,\vec{p};\, p'^0,\vec{p}';\, q^0,\vec{q}\,)$$

$$= \pm\, \gamma^0 \tilde{G}^{(1,1,1)}(p^0,-\vec{p};\, p'^0,-\vec{p}';\, q^0,-\vec{q}\,)\gamma^0$$

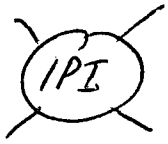
$+$ if ϕ is scalar
$-$ if ϕ is pseudoscalar

Thus $(a + \gamma_5 b) = \pm\, \gamma^0(a+\gamma_5 b)\gamma^0$

So ϕ is scalar $\Rightarrow b = 0$, ϕ is pseudoscalar $\Rightarrow a = 0$.

The parity conserving theories have just a single renormalized Yukawa coupling

Of course, the scalar self-couplings are also renormalized, and there are contributions to the renormalization from fermion loops. E.g.



is log divergent by power counting, for example, in one loop,

is log divergent, a divergence that must be cancelled with a ϕ^4 counterterm (i.e., is absorbed into the renormalized $\frac{1}{4!}\phi^4$ coupling)

In the parity violating theories, or the scalar parity conserving theory, there is also in general a ϕ^3 coupling. The graph

renormalizes this coupling in one-loop order.

But  $= 0$ in the parity conserving pseudoscalar theory.