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4. Functional Integration

4. Functional Integration

Canonical quantization can be shown to be formally equivalent to another method of quantization -- path Integral quantization. We will reformulate the Feynman-Dyson perturbation theory for scalar and spinor field theories in the path integral language.

There are several motivations for doing this. The most important will emerge soon when we quantize spin-1 fields. For spin higher than $\frac{1}{2}$, the operator methods that we have used so far are quite clumsy, while the path integral methods are simple and elegant.

Canonical Quantization and the Path Integral:

To demonstrate the equivalence of canonical quantization and the path integral, we will consider the simplest quantum mechanical system -- one with one dynamical variable q and conjugate momentum p, and Hamiltonian

$$H(p,q) \qquad ([q,p] = i)$$

We'll use the Heisenberg picture:

$$q(t) = e^{iHt} q(0) e^{-iHt}$$
$$p(t) = e^{iHt} p(0) e^{-iHt}$$

At each fixed time t, both $q(t)$ and $p(t)$ have a complete set of eigenstates:

$$q(t)\,|q,t\rangle = q\,|q,t\rangle$$
$$p(t)\,|p,t\rangle = p\,|p,t\rangle$$

operator eigenvalue

where $\quad |q,t\rangle = e^{iHt}|q,0\rangle$

or $\quad |q,t_2\rangle = e^{iH(t_2-t_1)}|q,t_1\rangle$

The completeness of the states may be expressed as
$$\int dq\, |q,t\rangle\langle q,t| = \mathbb{1} = \int dp\, |p,t\rangle\langle p,t|$$
$$\langle q,t|q',t\rangle = \delta(q-q'),\quad \langle p,t|p',t\rangle = \delta(p-p').$$

We also know that
$$\langle q,t|p,t\rangle = \frac{1}{\sqrt{2\pi}}e^{ipq}$$
-- Because $p = \frac{1}{i}\frac{\partial}{\partial q}$ represents the canonical commutation relation $[q,p]=i$ (and the normalization is right
$$\int dq\, \langle p',t|q,t\rangle\langle q,t|p,t\rangle = \delta(p-p')\ ).$$

One more preliminary formula:

True for infinitesimal t_1-t_2, to linear order
$$\left\{\begin{aligned}\langle p_1,t_1|q_2,t_2\rangle &= \langle p_1,t_1|e^{iH(t_2-t_1)}|q_2,t_1\rangle \\ &= e^{-iH(p_1,q_1)(t_1-t_2)}\frac{1}{\sqrt{2\pi}}e^{-ip_1q_2}\end{aligned}\right.$$

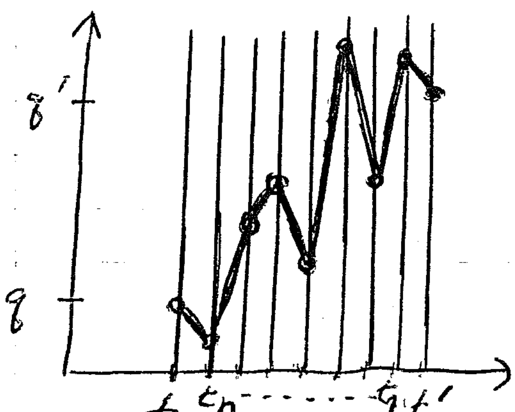
(If there is an <u>ordering</u> ambiguity, we must put all p's to the left of all q's in H, for this formula to be correct)

Now we wish to derive an expression for
$\langle q', t' | q, t \rangle$.

This is the probability amplitude for a "particle" that is initially (at time t) in a "position" eigenstate at q to be measured at the final time (t') at the position q'.
This is just the Schrödinger wave function for a particle with initial wave function

$$\psi(x, t) = \delta(x - q).$$

Its value for all q' and t' is a complete specification of the quantum dynamics of the system



We begin by partitioning the time interval $[t, t']$ into $n+1$ equal segments, each of width

$$\epsilon = \frac{t' - t}{n+1}$$

(we will eventually consider the limit $\epsilon \to 0$)

Now we reexpress $\langle q', t' | q, t \rangle$ by summing over intermediate states many times:

$$\langle q', t' | q, t \rangle = \int dq_1 dp_1 \, dq_2 dp_2 \cdots dq_n dp_n \, dp'$$

$$\langle q', t' | p', t' \rangle \langle p', t' | q_1 t_1 \rangle \langle q_1 t_1 | p_1 t_1 \rangle \langle p_1 t_1 | q_2, t_2 \rangle \cdots$$

$$\cdots \langle q_n, t_n | p_n, t_n \rangle \langle p_n t_n | q, t \rangle$$

Now, invoke the above formulas

$$\langle q,t|p,t\rangle = \frac{1}{\sqrt{2\pi}}e^{ipq},$$

$$\langle p,t|q,t-\epsilon\rangle = \frac{1}{\sqrt{2\pi}}e^{-ipq}e^{-i\epsilon H(p,q)},$$

and obtain --

$$\langle q' t'|q,t\rangle = \int \frac{dp'}{2\pi}\frac{dq_1 dp_1}{2\pi}\cdots\frac{dq_n dp_n}{2\pi}$$

$$\exp i\left[p'q' - p'q_1 - \epsilon H(p',q_1) + p_1 q_1 - p_1 q_2 - \epsilon H(p_1,q_2) + \cdots + p_n q_n - p_n q - \epsilon H(p_n,q)\right]$$

As $\epsilon \to 0$, we may regard the q_j's and p_j's as an approximation to a differentiable trajectory in phase space. Then we write

$$q_{j+1} - q_j = \epsilon \dot{q}_j, \quad q_j - q_{j+1} = \epsilon \dot{q}_{j+1}$$

and we have

$$\langle q' t'|q t\rangle = \int \frac{dp'}{2\pi}\frac{dq_1 dp_1}{2\pi}\cdots$$

$$\exp i\epsilon\left[p'\dot{q}_1 - H(p',q_1) + p_1\dot{q}_2 - H(p_1,q_2) + \cdots + p_n\dot{q} - H(p_n,q)\right]$$

In the limit $\epsilon \to 0$, the argument of the exponential becomes a Riemann integral determined by the trajectory

$$q(t),\ p(t);$$

our formula becomes

$$\boxed{\langle q';t'|q,t\rangle = \int (dq)(dp)\, e^{iS_H[q,p]}}$$

where $S_H(q,p) = \int_t^{t'} dt''\,(p\dot{q} - H(p,q))$

is the action in Hamiltonian form, and the "functional measure" $(dq)(dp)$ is defined by the above limiting procedure; i.e. as the limit of

$$\int \frac{dp'}{2\pi}\,\frac{dq_1\,dp_1}{2\pi}\cdots$$

(Notice that the measure has the dimensions of dp', which is what is required for the dimensional consistency of our formula.) The trajectories included in the integral obey the boundary condition

$$q(t) = q\,, \quad q(t') = q'.$$

(There is no boundary condition on p.)

The boxed equation above is our main result. We have expressed the quantum mechanical amplitude as a "sum over histories" -- a coherent sum of all the phase space trajectories with given endpoints, weighted by the phase factor e^{iS_H}

If (as is typical) the Hamiltonian $H(p,q)$ is quadratic in momentum, then the (dp) integral is a gaussian, which we can do explicitly. Suppose

$$H = \frac{1}{2m}p^2 + V(q)$$

Then we can evaluate

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{i\epsilon(p\dot{q} - \frac{1}{2m}p^2)}.$$

We complete the square by writing

$$-\frac{1}{2m}p^2 + p\dot{q} = \frac{-1}{2m}(p - m\dot{q})^2 + \frac{1}{2}m\dot{q}^2,$$

and obtain

$$e^{i\epsilon\frac{1}{2}m\dot{q}^2}\int_{-\infty}^{\infty}\frac{dp}{2\pi} e^{-\frac{i\epsilon}{2m}p^2}$$

The integral is Gaussian, but the exponential has an imaginary argument. We can define the integral by giving ϵ a small negative imaginary part

$$\lim_{Im\epsilon\to 0^-}\int_{-\infty}^{\infty}\frac{dp}{2\pi} e^{-i\epsilon p^2/2m} = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}}$$

Some such limiting procedure is needed to make sense of the path integral anyway.

We now have ---

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi} e^{i\epsilon(p\dot{q} - \frac{1}{2m}p^2 - V(q))} = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}} e^{i\epsilon[\frac{1}{2}m\dot{q}^2 - V(q)]}$$

$$= \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}} e^{i\epsilon L(q,\dot{q})}$$, where L is the Lagrangian

And therefore

$$\langle q', t' | q, t \rangle = \frac{1}{\sqrt{2\pi i\epsilon/m}} \int \prod_i \frac{dq_i}{\sqrt{2\pi i\epsilon/m}} e^{i\epsilon[L(q_i, \dot{q}_i)]}$$

(from dp' integral)

or

$$\boxed{\langle q', t' | q, t \rangle = \int (dq)\, e^{iS_L[q]}}$$

in the $\epsilon \to 0$ limit, where the integration measure is again defined by the limiting procedure, and

$$S_L = \int_t^{t'} dt'' L(q, \dot{q})$$

is the action in Lagrangian form, where q obeys the B.C.

$$q(t) = q \quad , \quad q(t') = q'.$$

The boxed equation is Feynman's famous "sum over histories" formula for quantum mechanics.

Why is this formula important?

- First of all, it provides us with deep insights into the nature of the classical limit. We have set $\hbar = 1$, but it is easy to restore it in Feynman's formula by dimensional analysis, since S has the dimensions of $\hbar$:

$$\langle q', t' | q, t \rangle = \int (dq)\, e^{iS[q]/\hbar}$$

(understood to be Lagrangian action)

(There are also $\hbar$'s in the measure, but they are not relevant in the present discussion.) The quantum mechanical particle is very clever. It "tries out" all trajectories that go from q to q' in time $t'-t$, and adds together the phases $e^{iS/\hbar}$ associated with all of them to decide where it wants to go. It likes to go where these phases tend to add constructively, and does not like to go where they add destructively. Now, the "classical limit" is the limit $\hbar \to 0$, or, in other words, the limit in which the action is typically very large, in units of $\hbar$. In this limit, the phase factor $e^{iS/\hbar}$ changes very rapidly when a small change is made in the trajectory. Thus, nearby trajectories tend to cancel each other out; their phases add destructively. An exception is a trajectory that is a stationary point of $S[q]$. Then neighboring paths have essentially the same action, and their phases add constructively. In the $\hbar \to 0$ limit, the functional integral is dominated by trajectories near the stationary point of S, and we have

$$\langle q',t'|q,t\rangle \sim e^{iS_0/\hbar} \quad (\hbar\to 0)$$

action at the stationary point

("stationary phase approximation.") But these trajectories, according to the least action principle, are just the classical trajectories. Now we understand how the classical particle is able to know which trajectory extremizes S. It really tries them all, and the phases $e^{iS/\hbar}$ happen to interfere constructively along the classical trajectory.

But when S_0 is not large (comparable to $\hbar$) the "quantum fluctuations" about the classical trajectory are not suppressed, and classical physics fails to provide an adequate description of the physics.

- Feynman's formula reduces quantum mechanics to "quadrature"! In classical mechanics, we are usually satisfied if we have reduced the solution to the equations of motion to the evaluation of integrals ("quadrature"), which can in principle be done by a computer. Now any quantum mechanics problem can be solved by a mere evaluation of a functional integral. This sounds like a joke, but it is not. The most intractible quantum mechanics problems are being solved these days (in particle physics, condensed matter physics, etc.) by numerical evaluation of the path integral on high speed computers.

- The Feynman formula is readily generalized to more complicated systems with more degrees of freedom. It obviously applies if there are any finite number of p's and q's, and, formally at least, it applies to a system with an infinite number of degrees of freedom, like a quantum field theory. To make this more precise, we may discretize space as well as time, and consider an approximation to the functional integral that -- in a finite volume at least -- involves a finite number of ordinary integrals. Then we may attempt to define the quantum field theory as a limit in which the discrete spacetime lattice becomes arbitrarily fine. Taking this limit may be subtle; e.g., renormalizations may be required for it to make sense.

Ignoring such subtleties, we have for a scalar field theory coupled to a source ρ

$$\langle \phi(\vec{x}', t') \mid \phi(\vec{x}, t) \rangle_\rho = \int (d\phi)\, e^{i(S[\phi] + \int \rho\phi)}$$

If we prepare an eigenstate of the field, this expression gives the amplitudes for various outcomes of a field measurement at a later time, in the presence of the source ρ

Time ordering and Green Functions:

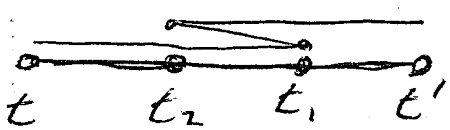
We can derive a path integral expression for the expectation value of a string of operators. Consider for example

$$\langle q',t' | q(t_1)\, q(t_2) | q, t\rangle$$

$$= \int dq_1 dq_2 \langle q',t' | q(t_1) | q_1, t_1\rangle \langle q_1, t_1 | q(t_2) | q_2, t_2\rangle \langle q_2, t_2 | q, t\rangle$$

-- inserting a sum over intermediate states at times t_1, t_2

$$= \int dq_1 dq_2\; q_1 q_2 \langle q',t' | q_1, t_1\rangle \langle q_1, t_1 | q_2, t_2\rangle \langle q_2, t_2 | q, t\rangle$$



Now, the three matrix elements above can all be replaced by functional integrals. Therefore, *if* $t_1 > t_2$, we have

$$\langle q', t' | q(t_1)\, q(t_2) | q, t\rangle = \int_{\substack{q(t)=q \\ q(t')=q'}} (dq)\, e^{iS}\, q(t_1)\, q(t_2)$$

But note that if $t_2 > t_1$ it does not work. We must sum over trajectories that take 3 values between times t_1 and t_2. The path integral gives an expectation value of a time-ordered product.

These comments can evidently be generalized to give

$$\int_{(B.C.)} (dq)\, e^{iS} q(t_1) \cdots q(t_n) = \langle q' t' | T[q(t_1) \cdots q(t_n)] | q, t \rangle$$

(if all times $t_1 - t_n$ are in $[t, t']$).

Field Theory

Applying this formula to field theory, we have

$$\int_{B.C.} (d\phi)\, e^{iS} \phi(x_1) \cdots \phi(x_n)$$

$$= \langle \phi(\vec{x}', t') | T[\phi_H(x_1) \cdots \phi_H(x_n)] | \phi(\vec{x}, t) \rangle$$

Heisenberg fields

therefore, if we expand $\langle \phi(\vec{x}', t') | \phi(\vec{x}, t) \rangle_\rho$ in powers of ρ

$$\langle \phi(\vec{x}'; t') | \phi(\vec{x}, t) \rangle_\rho = \int d\phi\, e^{iS} e^{iS\rho\phi}$$

$$= \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int d^4x_1 \cdots d^4x_n\, \rho(x_1) \cdots \rho(x_n)$$
$$\langle \phi(\vec{x}'; t') | T[\phi_H(x_1) \cdots \phi_H(x_n)] | \phi(\vec{x}, t) \rangle$$

Now, by a familiar argument, we may extract the vacuum expectation value by taking limits $t \to -\infty$, $t' \to \infty$ (as on page 2.143)

$$| \phi(\vec{x}, t) \rangle = e^{iHt} | \phi(\vec{x}, 0) \rangle$$

and

$$\lim_{t \to -\infty} \langle \psi | e^{iHt} | \phi(\vec{x}, 0) \rangle = e^{iE_0 t} \langle \psi | 0 \rangle_E \; {}_E\langle 0 | \phi(\vec{x}, 0) \rangle$$

vacuum energy

physical vacuum

And therefore

$$\lim_{\substack{t'\to\infty \\ t\to-\infty}} \langle \phi(\vec{x}';t') | T[\phi_H(x_1) \cdots \phi_H(x_n)] | \phi(\vec{x};t) \rangle$$

$$= e^{-iE_0(t'-t)} \langle \phi(\vec{x}';0) | 0 \rangle_E \, {}_E\langle 0 | \phi(\vec{x};0) \rangle$$

$${}_E\langle 0 | T \, \phi_H(x_1) \cdots \phi_H(x_n) | 0 \rangle_E$$

$$= C \; G^{(n)}(x_1 \cdots x_n)$$

C a constant that depends on the boundary conditions satisfied by the fields, but is independent of $n, x_1 \cdots x_n$.

We have found

$$Z[e] = N \int (d\phi) \, e^{i(S + \int e\phi)}$$

where $Z[e]$ is the generating functional for the Green functions, and N is a normalization condition that depends on boundary conditions but is independent of e. Since $Z[e] = 1$, we see that

$$N^{-1} = \int (d\phi) \, e^{iS}$$

or

$$\boxed{Z[e] = \frac{\int (d\phi) \, e^{i(S + \int e\phi)}}{\int (d\phi) \, e^{iS}}}$$

The expression $Z[e]$ has all information about the physics of the theory in it; in particular, the full S-matrix. The dependence on the boundary values of the fields has dropped out of this expression, so we may choose the boundary conditions as we please; e.g., $\phi = 0$ at $t \to -\infty$, $t' \to \infty$. Note also that the expression does not depend on how we normalize the functional measure $(d\phi)$. So, for the purpose of doing physics, we need not consider the issue of what normalization is "proper."

Example: The Free Scalar Field

To check our functional integral expression, let us apply it to a case where we know the right answer, and can do the integral exactly. Consider:

$$S = \int d^4x \ \tfrac{1}{2} \phi(x) (-\Box - m^2) \phi(x)$$

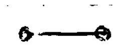
(integration by parts)

$$= \int \frac{d^4k}{(2\pi)^4} \ \frac{1}{2} \tilde{\phi}(k) (k^2 - m^2) \tilde{\phi}(-k)$$

We know that

$$Z[e] = \Sigma(\text{vacuum diagrams in presence of source})$$

$$= \exp(\Sigma \text{ connected diagrams})$$

$= \exp($  $)$

$$= \exp\left[\tfrac{1}{2}(i)^2 \int \frac{d^4k}{(2\pi)^4} \ \tilde{e}(k) \left(\frac{i}{k^2 - m^2 + i\epsilon}\right) \tilde{e}(-k) \right]$$

(symmetry factor)

We want to see whether we can get this same expression for $Z[e]$ by doing the functional integral.

$$Z[e] = N \int d\phi \, e^{i(S + \int e\phi)}$$

$\nwarrow$ normalization

It is possible to evaluate the functional integral in free field theory because it is just a Gaussian. Compare

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2}ax^2} = (2\pi/a)^{\frac{1}{2}} \quad (\text{for } \mathrm{Re}\, a > 0).$$

This is readily generalized to an N-dimensional integral

$$\int d^N x \, e^{-\frac{1}{2}(\vec{x}, A\vec{x})}$$

where A is a symmetric $N \times N$ matrix with an orthonormal basis of eigenvectors.

Suppose: $A\hat{e}_n = \lambda_n \hat{e}_n \quad (\hat{e}_n, \hat{e}_m) = \delta_{n,m}$.

We expand $\vec{x}$ in this basis

$$\vec{x} = \sum_{n=1}^{N} x_n \hat{e}_n$$

Then $(\vec{x}, A\vec{x}) = \sum_{n,m} x_n x_m \lambda_m (\hat{e}_n, \hat{e}_m) = \sum_{n=1}^{N} \lambda_n x_n^2$

Thus

$$\int d^N x \, e^{-\frac{1}{2}(\vec{x}, A\vec{x})} = \prod_{n=1}^{N} \int dx_n \, e^{-\frac{1}{2}\lambda_n x_n^2}$$

$$= \prod_{n=1}^{N} (2\pi/\lambda_n)^{\frac{1}{2}} = (2\pi)^{N/2} (\det A)^{-\frac{1}{2}}$$

(if $\mathrm{Re}\, \lambda_n > 0$)

Defining a normalized measure

$$\int (dx) = (2\pi)^{-N/2} \int d^N x \,,$$

we have

$$\boxed{\int (dx)\, e^{-\frac{1}{2}(\vec{x}, A\vec{x})} = (\det A)^{-1/2}}$$

(if all eigenvalues of A have a positive real part)
But we can apply this to an <u>infinite</u>-dimensional matrix also, assuming the determinant exists.

We can also evaluate a Gaussian integral with a more general quadratic form that includes a linear term, by completing the square. Consider

$$\int (dx) \exp\left[-\tfrac{1}{2}(x, Ax) + (x, b)\right]$$

where b (and x) are real (A is symmetric)

$$= \int (dx) \exp -\tfrac{1}{2}\left(x - A^{-1}b, A(x - A^{-1}b)\right) \exp\left[\tfrac{1}{2}(b, A^{-1}b)\right]$$

$$= e^{\frac{1}{2}(b, A^{-1}b)} \int (dx)\, e^{-\frac{1}{2}(x, Ax)}$$

(shifting variable of integration)

$$= (\det A)^{-\frac{1}{2}}\, e^{\frac{1}{2}(b, A^{-1}b)}$$

(A is invertible if all of its eigenvalues are nonzero.)
This formula can also be applied if A is an operator in an infinite dimensional space, provided the determinant exists.

Now we can apply this formula to the integral

$$\int (d\phi) \exp i \int d^4x \left[\tfrac{1}{2} \phi(x) (-\Box - m^2) \phi(x) + \phi(x) \rho(x) \right]$$

$$= \left[\det i(\Box + m^2) \right]^{-\frac{1}{2}} \exp \left[-\tfrac{1}{2} \int d^4x\, d^4y\, \rho(x) \left[i(\Box + m^2) \right]^{-1}_{(x,y)} \rho(y) \right]$$

$$= \left[\det i(\Box + m^2) \right]^{-\frac{1}{2}} \exp \left[-\tfrac{1}{2} \int \frac{d^4K}{(2\pi)^4} \left(\frac{i}{K^2 - m^2} \right) \rho(K) \rho(-K) \right]$$

To normalize properly, we divide by this expression when $\rho = 0$. This divides out the determinant, and we have

$$Z[\rho] = N \int d\phi\, e^{i(S + \int \rho\phi)}$$
$$= \exp \left[-\tfrac{1}{2} \int \frac{d^4K}{(2\pi)^4} \rho(K) \frac{i}{K^2 - m^2} \rho(-K) \right]$$

-- which agrees perfectly with our other expression. But there are a few subtleties that deserve comment.

- The determinant does not really exist:

$$\det A = \prod_n \lambda_n = \exp\left(\sum_n \ln \lambda_n \right)$$

So we can write (formally) $\det i(\Box + m^2)$ as an exponential of the sum of the logs of its eigenvalues:

$$\det \left[i(\Box + m^2) \right] = \exp \left[\int \frac{d^4K}{(2\pi)^4} \ln \left[(-i)(K^2 - m^2) \right] \right]$$

The argument of the exponential has a quartic ultraviolet divergence. To make sense of it, we must <u>regulate</u>; that is, introduce an ultraviolet cutoff Λ. This is no big deal though, since the determinant drops out of $Z[\ell]$ anyway.

- The functional integral does not really exist. We applied our formula for a Gaussian integral. But the formula makes sense only if the eigenvalues of A have positive real parts. Here, all the eigenvalues $-i(k^2-m^2)$ are pure imaginary. We'll have to do something about this.

- What happened to the $i\epsilon$? The operator $i(\Box+m^2)$ does not have a well-defined inverse, because of the zero eigenvalue at $k^2=m^2$. We need a boundary condition to define an inverse. This amounts to a prescription for integrating around the singularity in i/k^2-m^2. We know that the right prescription is the $i\epsilon$ prescription. But shouldn't the path integral tell us so?

These last two problems are closely related, and it turns out that a solution to one also solves the other.

A sensible way of dealing with the problem that the path integral does not exist is to modify it gently in a way that forces it to converge; this modification then allows us to justify the formal manipulations that we used to derive the path integral expression for $Z[\rho]$. That is, it makes sensible the limiting procedure in the derivation of the path integral.

A convenient modification is to perform an infinitesimal "rotation" of time

$$t \to (1-i\epsilon)t$$

that gives the time a small imaginary part. Under such a rotation

$$\dot\phi^2 \to (1+2i\epsilon)\,\dot\phi^2$$

and

$$S = \int dt\, d^3x\, \left[\tfrac{1}{2}\dot\phi^2 - \tfrac{1}{2}(\vec\nabla\phi)^2 - U(\phi)\right]$$

includes mass, interaction and source term (We may include a constant so $U \geq 0$)

$$\to S + i\epsilon \int dt\, d^3x \left[\tfrac{1}{2}\dot\phi^2 + (\nabla\phi)^2 + U(\phi)\right]$$

and

$$e^{iS} \to e^{iS}\, e^{-\epsilon \delta S}$$

positive definite

The extra factor $\exp(-\epsilon\,\delta S)$ aids the convergence of the path integral -- it suppresses large gradients and large values of ϕ (for which U is large). We can <u>define</u> the path integral as the limit $\epsilon \to 0$ (there is surely no <u>harm</u> in doing this, if the integral makes sense even without the ϵ.)

Now,

$$S = \int dt\, d^3x \left[\tfrac{1}{2}(1+i\epsilon)\dot\phi^2 - \tfrac{1}{2}(1-i\epsilon)\left((\vec\nabla\phi)^2 + \phi^2\right)\right.$$

becomes when we Fourier transform

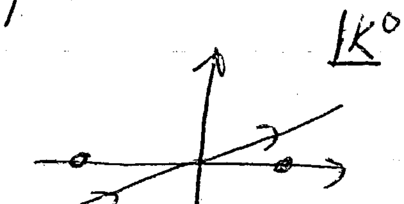
$$S = \int \frac{d^4k}{(2\pi)^4}\, \tfrac{1}{2}\tilde{\phi}(k)\left[k^{0\,2} - \vec{k}^2 - m^2 + i\epsilon\left(k^{0\,2} + \vec{k}^2 + m^2\right)\right]\tilde{\phi}(-k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\, \tfrac{1}{2}\tilde{\phi}(k)\left(k^2 - m^2 + i\epsilon\right)\tilde{\phi}(-k).$$

So rotating the time so as to encourage the convergence of the integral resolves the ambiguity in the inverse of the operator, and is in fact equivalent to the $i\epsilon$ prescription

An equivalent way of describing this is to say that k^0 is rotated

$$k^0 \to (1 + i\epsilon)\, k^0$$



so that the k^0 integral misses the poles. We rotate k^0 and t together so that $k^0 x^0$ stays fixed (this way, Fourier integrals do not blow up)

In free field theory, no singularities prevent us from rotating the k^0 integral to the imaginary axis (the Wick rotation)

$$k^0 \to i k^4$$

which corresponds to rotating

$$x^0 \to -i x^4$$

A functional integral prescription that gives Wick rotated amplitudes amounts to summing over "histories" in a space with a Euclidean (rather than Minkowski) metric

$$Z_E[e] = N\int(d\phi)\; e^{-S_E + \int e\phi}$$

where
$$S_E = \int d^4x_E \;\tfrac{1}{2}\,\phi(x)\,(-\Delta + m^2)\,\phi(x)$$

Euclidean Laplacian

is positive definite.
In Euclidean space, the functional integral is "maximally" convergent -- e^{-S} looks like the Boltzman factor of statistical mechanics, and quantum fluctuations are analgous to thermal fluctuations, with $\hbar \to 0$ being the limit of "zero temperature". We can recover the physical amplitudes by analytically continuing the Euclidean Green functions $G_E^{(n)}$ back from imaginary to real time. This is the same as defining amplitudes as an $\epsilon \to 0^+$ limit

We can expect these remarks about the connection between Euclidean and Minkowski field theory to hold not just for free field theory but in general. That is, we expect to be able to define the functional integral in Euclidean space, and obtain physical quantities by analytic continuation back to real time. In order for this to work, there must be no singularities that prevent the continuation -- we want Green functions to be analytic in t for $\mathrm{Arg}(t)$ between 0 and $-\pi/2$ (or analytic in k for $\mathrm{Arg}(k)$ between 0 and $\pi/2$)

This analyticity seems to be assured on very general grounds. Consider for example

$$\langle 0 | T[\phi_H(t_1)\,\phi_H(t_2)] | 0 \rangle$$
$$= \sum_n |\langle 0 | \phi_H(0) | n \rangle|^2 \, e^{-iE_n(t_1 - t_2)} \qquad (\text{for } t_1 > t_2)$$

A negative imaginary part for $t_1 - t_2$ enhances convergence (since $E_n > 0$) of the sum, so that it defines an analytic function of $t_1 - t_2$ for $\operatorname{Im}(t_1 - t_2) < 0$. The physical Green function can be obtained by letting $t_1 - t_2$ approach a <u>real</u> value, at the <u>boundary</u> of the domain of analyticity.

(Compare the discussion of Axiomatic Field Theory — page 1.85. There we noted that amplitudes can be defined as boundary values of analytic functions. The general axioms of field theory are enough to ensure that there is a unique analytic continuation between Euclidean and Minkowski space)

Although we will usually use a Minkowski space notation, all functional integral manipulations may be understood to be legitimized in Euclidean space, with a subsequent analytic continuation.

Semiclassical Expansion:

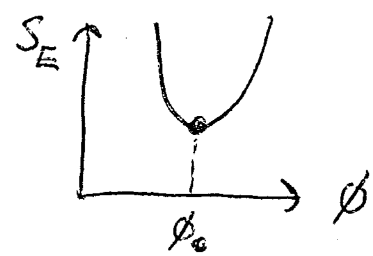
We've seen how the path integral reproduces classical physics in the limit $\hbar \to 0$. There is a systematic expansion about the classical limit in powers of $\hbar$ -- the semiclassical expansion. This expansion, as we will see, is just the Feynman diagram expansion.

The expansion in powers of $\hbar$ can be formulated as a "stationary phase" approximation (in Minkowski space) or a "steepest descent" approximation (in Euclidean space). We'll consider (at first) the Euclidean space formulation, which is a little more transparent.

We want to evaluate

$$\int d\phi \, e^{-S_E[\phi]/\hbar}$$

where S_E is the positive definite Euclidean action functional (perhaps including source terms). In the limit $\hbar \to 0$, this integral is completely dominated by the history ϕ_0 that minimizes S_E; deviations from ϕ_0 are very strongly exponentially damped. An asymptotic expansion in powers of $\hbar$ is formulated by expanding the field ϕ in terms of a small fluctuation around ϕ_0:



$$\cancel{S_E[\phi] = S_E[\phi_0] + \frac{1}{2}\int \frac{\delta^2 S_E}{\delta\phi(x_1)\,\delta\phi(x_2)}\bigg|_{\phi=\phi_0} \delta\phi(x_1)\,\delta\phi(x_2)\, dx_1\, dx_2}$$

$$S_E[\phi] \doteq S_E[\phi_0] + \frac{1}{2}\int d^4x_1 d^4x_2 \left.\frac{\delta^2 S_E}{\delta\phi(x_1)\delta\phi(x_2)}\right|_{\phi=\phi_0} \delta\phi(x_1)\delta\phi(x_2)$$

$$+ \frac{1}{6}\int d^4x_1 d^4x_2 d^4x_3 \left.\frac{\delta^3 S_E}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)}\right|_{\phi=\phi_0} \delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)$$

$$+ \cdots$$

where $\quad \delta\phi(x) = \phi(x) - \phi_0(x)$

(there is no term linear in $\delta\phi$, because ϕ_0 is a stationary point of S_E.) Now plug this expansion into the integral:

$$\int d\phi\, e^{-S_E[\phi]/\hbar}$$

$$= \int d\phi\, e^{-S_E[\phi_0]/\hbar} \exp\left[-\frac{1}{2\hbar}\int \delta\phi \left.\frac{\delta^2 S_E}{\delta\phi\,\delta\phi}\right|_{\phi=\phi_0}\delta\phi\right]$$

$$\left[1 - \frac{1}{\hbar 6}\int \left.\frac{\delta^3 S_E}{\delta\phi\delta\phi\delta\phi}\right|_{\phi=\phi_0}\delta\phi\delta\phi\delta\phi - \frac{1}{\hbar 4!}\int \left.\frac{\delta^4 S_E}{\delta\phi\delta\phi\delta\phi\delta\phi}\right|_{\phi=\phi_0}\delta\phi\delta\phi\delta\phi\delta\phi + \cdots\right]$$

-- Keeping the term quadratic in $\delta\phi$ in the exponent, and expanding the rest. Now we just have to evaluate integrals that a polynomial × Gaussian, which is easy (described in detail below). Only the terms even in $\delta\phi$ contribute; the others integrate to zero by symmetry. So --

$$\int d\phi\, e^{-S_E[\phi]/\hbar} = \left[\det\left(\frac{1}{\hbar}\left.\frac{\delta^2 S_E}{\delta\phi\,\delta\phi}\right|_{\phi=\phi_0}\right)\right]^{-\frac{1}{2}} e^{-S_E[\phi_0]/\hbar}$$

$$\times \left[1 + O(\hbar)\right]$$

the determinant is generated by the "harmonic" small fluctuations about the classical trajectory ϕ_0, and the order $\hbar$ corrections are induced by the "anharmonic" terms ("interactions").

Now, what does this expansion in powers of $\hbar$ have to do with the Feynman diagram expansion, which is an expansion in powers of the coupling constant. In fact, they are the same. Consider, for example, the self coupled scalar field:

$$S/\hbar = \frac{1}{\hbar}\int d^4x\left[\frac{1}{2}\phi(-\Box-m^2)\phi - \frac{\lambda}{4!}\phi^4 + e\phi\right]$$

Now rescale the field:

$$\phi = \frac{1}{\sqrt{\lambda}}\tilde{\phi}$$

From the point of view of the functional integral, this is just a trivial change in the variable of integration. Now

$$S/\hbar = \frac{1}{\hbar\lambda}\int d^4x\left[\frac{1}{2}\tilde{\phi}(-\Box-m^2)\tilde{\phi} - \frac{1}{4!}\tilde{\phi}^4 + \sqrt{\lambda}\, e\tilde{\phi}\right]$$

If we expand a <u>Green function</u> (a term of <u>given</u> <u>order</u> in e in $N\int d\phi\, e^{iS/\hbar}$)

in powers of $\hbar$, this is <u>the same</u> as expanding in powers of λ, since only the product $\hbar\lambda$ appears in the action (except for the coupling to the source e.) The weak coupling expansion of a Green function in powers of λ

and the semiclassical expansion in powers of $\hbar$ are the same. Thus, the path integral provides us with a nice picture of what the weak coupling expansion is: it is generated by expanding the fields in small "quantum fluctuations" about the classical trajectory.

The result that the expansion in $\hbar$ is the same as the Feynman diagram expansion can be derived in another way that makes it obvious that the conclusion applies to any theory. When we evaluate

$$\int d\phi \, e^{iS[\phi]/\hbar}$$

in perturbation theory, the propagator is the inverse of the quadratic part of S (this will soon be derived if it is not yet obvious) and so is order $\hbar$, while a vertex is a term in $S/\hbar$ and so is order $\hbar^{-1}$! Thus, a graph with I propagators and V vertices is

$$\text{Graph} \sim \hbar^{I-V}$$

But recall the topological identity that says (for connected graphs)

$$I - V + 1 = L \quad \text{-- the number of loops}$$

So --

$$\text{Connected Graph} \sim \hbar^{L-1}$$

"Tree" graphs are order $\hbar^{-1}$, and each loop costs an extra factor of $\hbar$.

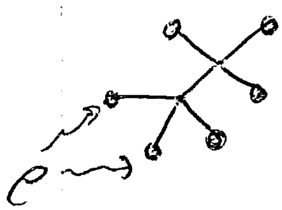
Applying this result to the diagram expansion for the generating functional:

$$Z[e] = \exp\Big[\underbrace{\Sigma(\text{connected diagrams})}_{\text{excluding vacuum bubbles}}\Big]$$

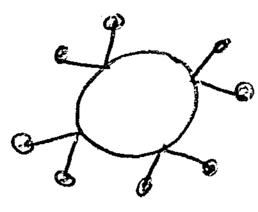
$$= \exp\Big[\underbrace{iS_0/\hbar}_{\substack{\text{tree}\\\text{graphs}}} + \underbrace{A}_{\substack{\text{one}\\\text{loop}}} + \underbrace{B\hbar}_{\substack{\text{two}\\\text{loop}}} + \cdots\Big]$$

$$= e^{A}\, e^{iS_0/\hbar}\,(1 + O(\hbar))$$

If we compare to the expansion on page 4.24, we see ~~that~~ the sum of all tree graphs is the action S_0 of the classical solution in the presence of the source e.



Tree Graph



One-Loop Graph

The one-loop graphs sum to produce the logarithm of the determinant generated by the Gaussian integral about the classical solution. Graphs with two or more loops are the "anharmonic" contribution to the semiclassical expansion.

Feynman Rules from the Path Integral

We saw that the functional integral successfully reproduced the correct expression for $Z[\rho]$ in free field theory. For interacting theories we cannot calculate $Z[\rho]$ exactly, but we can expand it in powers of the coupling; this reproduces the Feynman diagram expansion.

But the path integral formulation of the Feynman expansion has an advantage. For more complicated theories, the operator methods we used before are very awkward, while path integral methods are still quite efficient.

To begin, consider free field theory again

$$S_0 = \int d^4x \; \tfrac{1}{2} \phi(x) \left(-\Box - m^2 + i\epsilon\right) \phi(x)$$

or $$e^{iS_0} = \exp\left(-\tfrac{1}{2} \phi \Delta^{-1} \phi\right)$$

Here Δ^{-1} is the "matrix" $\Delta^{-1} = i(\Box + m^2 - i\epsilon)$

or $$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

(Note the change in notation; I called this function $i\Delta_F(x-y)$ before.) Doing a Gaussian functional integral gives

$$Z[\varrho] = \frac{\int [d\phi]\, e^{i(S_0 + \int \varrho \phi)}}{\int [d\phi]\, e^{iS_0}} = \frac{\int (d\phi) \exp(-\frac{1}{2}\phi \Delta^{-1} \phi + i \varrho \phi)}{\int (d\phi) \exp(-\frac{1}{2}\phi \Delta^{-1} \phi)}$$

(schematic notation)

$$= \exp\left(-\tfrac{1}{2} \varrho \Delta \varrho\right) \equiv \exp\left[-\tfrac{1}{2}\int d^4x\, d^4y\, \varrho(x)\, \Delta(x-y)\, \varrho(y)\right]$$

(Think of Δ as a matrix, ϱ as a vector)

The Green functions of free field theory are obtained by expanding $Z[\varrho]$ in powers of ϱ

$$G_0^{(2n)}(x_1 \ldots x_{2n}) = \langle 0 | T\, \phi(x_1) \ldots \phi(x_{2n}) | 0 \rangle_0$$

(no. of arguments must be even)

$$= (-i)^{2n} \frac{\delta}{\delta \varrho(x_1)} \cdots \frac{\delta}{\delta \varrho(x_{2n})} Z[\varrho] \Big|_{\varrho = 0}$$

$$(-i)^{2n} \frac{\delta}{\delta \varrho(x_1)} \cdots \frac{\delta}{\delta \varrho(x_{2n})} \frac{(-1)^n}{2^n n!} (\varrho \Delta \varrho)^n$$

(from expanding exponential)

Letting $\left(\frac{\delta}{\delta \varrho}\right)^{2n}$ act generates $(2n)!$ terms:

$$G_0^{(2n)}(x_1 \ldots x_{2n}) = \frac{1}{2^n n!} \sum_{\text{perms}} \Delta(x_{i_1} - x_{i_2}) \cdots \Delta(x_{i_{2n-1}} - x_{i_{2n}})$$

where the sum is over all $(2n)!$ permutations of the i_j's.

The $(2n)!$ permutations can be grouped into equivalence classes that generate the same product of Δ's. There are $2^n n!$ perms in each class, since there are $n!$ ways of permuting the n Δ's, and 2^n ways of interchanging the two arguments of any given Δ. So we may write

$$G_0^{(2n)}(x_1 \cdots x_{2n}) = \sum_{\text{pairings}} \Delta(x_{i_1} - x_{i_2}) \cdots \Delta(x_{i_{2n-1}} - x_{i_{2n}})$$

Now the sum is over all independent ways of Wick contracting the $2n$ points. Of course this is the right answer.

What we have learned is that performing a normalized Gaussian functional integral weighted by a polynomial in the fields is equivalent to performing all Wick contractions of the polynomial:

$$\frac{\int (d\phi)\, e^{-\frac{1}{2}\phi\Delta^{-1}\phi}\ \phi_1 \cdots \phi_{2n}}{\int d\phi\, e^{-\frac{1}{2}\phi\Delta^{-1}\phi}}$$

$$= \sum_{\text{pairings}} \Delta_{i_1, i_2} \cdots \Delta_{i_{2n-1}, i_{2n}}$$

This will be true for any quadratic form $\phi\Delta^{-1}\phi$, so we have justified the statement on p 4.26 that the Feynman propagator is just the inverse of the operator in the quadratic part of the action.

The point is that we can evaluate

$$Z[\rho] = N \int (d\phi)\, e^{iS_0} e^{iS_{int}} e^{iS\rho\phi}$$

by expanding $e^{i(S_{int} + S\rho\phi)}$ in powers of S_{int} and ρ. Then performing the integral amounts (up to normalization) to performing all possible contractions term-by-term.

We can write down a nice "closed-form" expression for $Z[\rho]$ that summarizes the Feynman rules in a very compact way. We have

$$G_0^{(2n)}(x_1, \ldots x_{2n}) = \frac{1}{2^n n!} \frac{\delta}{\delta\rho_1} \cdots \frac{\delta}{\delta\rho_{2n}} \left[\rho_i \Delta_{ij} \rho_j \right]^n$$

This can also be written --

$$= \frac{1}{2^n n!} \left(\frac{\delta}{\delta\phi_i} \Delta_{ij} \frac{\delta}{\delta\phi_j} \right)^n \phi_1 \cdots \phi_{2n}$$

since both terms generate all possible pairings. In other words:

$$\frac{\int (d\phi)\, e^{-\frac{1}{2}\phi\Delta^{-1}\phi}\, \phi_1 \cdots \phi_{2n}}{\int (d\phi)\, e^{-\frac{1}{2}\phi\Delta^{-1}\phi}} = \exp\left(\frac{1}{2} \frac{\delta}{\delta\phi_i} \Delta_{ij} \frac{\delta}{\delta\phi_j} \right) \phi_1 \cdots \phi_{2n} \Big|_{\phi=0}$$

Now, for any functional $F[\phi]$, we may show, by expanding F in powers of ϕ, integrating, and resumming, that

$$\frac{\int (d\phi) e^{-\frac{1}{2}\phi\Delta^{-1}\phi} F[\phi]}{\int (d\phi) e^{-\frac{1}{2}\phi\Delta^{-1}\phi}} = \exp\left(\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right) F[\phi] \Big|_{\phi=0}$$

Applying this to the generating functional gives

$$Z[e] = N \int e^{iS_0} e^{(iS_{int} + i\int e\phi)}$$

(norm constant)

$$= N' \exp\left[\frac{1}{2}\int d^4y_1 d^4y_2 \frac{\delta}{\delta\phi(y_1)} \Delta(y_1 - y_2) \frac{\delta}{\delta\phi(y_2)}\right]$$

$$\exp\left[i\int d^4x (\mathcal{L}_{int} + e\phi)\right] \Big|_{\phi=0}$$

(a different constant. N' just divides out all the vacuum bubbles.)

These are the Feynman rules in a nutshell.

The crucial thing we have learned is that the propagator is always the inverse of the operator that appears in the quadratic part of the action. This can be readily generalized to, for example, a theory with many scalar fields, and

$$iS_0 = -\frac{i}{2}\int \phi^a(x) \Delta^{-1}_{ab} \phi^b(x)$$

In this theory the contraction is

$$\phi^a(x)\phi^b(y) = \Delta^{ab}(x-y)$$

-- carries field indices as well as spacetime indices!

Derivative Interactions

An example of a type of theory in which path integral methods are very handy because canonical methods are extremely awkward is a theory with derivative interactions. Consider, for example, two scalar fields A and B with the coupling --

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}m_A^2 A^2 + \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}m_B^2 B^2$$
$$+ \tfrac{1}{4} g(\partial^\mu A)^2 B^2$$

(A and B are said to be "derivatively coupled.")

The kinetic terms for A is the funny term $\frac{1}{2}\dot{A}^2\left(1+\frac{g}{2}B^2\right)$;

so the conjugate momentum is

$$\pi_A = \frac{\partial \mathcal{L}}{\partial \dot{A}} = \dot{A}\left(1+\frac{g}{2}B^2\right).$$

The Hamiltonian density is therefore

$$\mathcal{H} = \pi_A \dot{A} + \pi_B \dot{B} - \mathcal{L}$$

$$= \tfrac{1}{2}\pi_A^2\left(1+\tfrac{g}{2}B^2\right)^{-1} + \tfrac{1}{2}\pi_B^2 + \tfrac{1}{2}(\vec{\nabla}A)^2 + \tfrac{1}{2}(\vec{\nabla}B)^2 + \tfrac{g}{4}(\vec{\nabla}A)^2 B^2$$
$$+ \tfrac{1}{2}m_A^2 A^2 + \tfrac{1}{2}m_B^2 B^2$$

To derive Feynman rules by the canonical method, we split $\mathcal{H}$ into free and interacting parts

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}$$

where $$\mathcal{H}_0 = \tfrac{1}{2}(\pi_A^{\,2} + (\vec{\nabla} A)^2 + m^2 A^2) + \tfrac{1}{2}(\pi_B^{\,2} + (\vec{\nabla} B)^2 + m^2 B^2)$$

$$\mathcal{H}_{int} = \frac{g}{4}(\vec{\nabla} A)^2 B^2 + \frac{1}{2}\pi_A^{\,2}\left(\left(1+\frac{g}{2}B^2\right)^{-1} - 1\right)$$
$$= \frac{g}{4}(\vec{\nabla} A)^2 B^2 + \frac{1}{2}\pi_A^{\,2}\left(\frac{g/2\, B^2}{1+g/2\, B^2}\right).$$

This is not a covariant object, so the vertices in our Feynman rules (aside from being complicated) are ugly and noncovariant. (The nonpolynomial term $(1+g/2\,B^2)^{-1}$ can be expanded in powers of B^2, generating an infinite number of interaction vertices.) But the theory we started with was covariant, so we expect to be able to obtain a relativistically invariant S-matrix. How does this come about?

There is another source of noncovariance in the Feynman rules, in the contractions themselves. To see this, note that

$$\partial_0^x\, T(A(x)B(y)) = \partial_0^x\left[\theta(x^0-y^0)A(x)B(y) + \theta(y^0-x^0)B(y)A(x)\right]$$
$$= T(\partial_0 A(x)\, B(y)) + \underbrace{\delta(x^0-y^0)}_{\text{from derivative of } \theta}\,[A(x), B(y)]$$

$$\cancel{\partial_0^x \partial_0^y T(A(x)B(y))}$$
$$\cancel{= \partial_0^y\left[T(\partial_0 A(x) B(y)) + \delta(x^0-y^0)[A(x),B(y)]\right]}$$
$$\cancel{= T(\partial_0 A(x)\partial_0 B(y)) + \delta(x^0-y^0)\left([B(y),\partial_0 A(x)] + [A(x),\partial_0 B(y)]\right)}$$
$$\cancel{- \delta'(x^0-y^0)[A(x),B(y)]}$$

If A and B are interaction picture fields:

$$\partial_0^x T(\phi_I(x)\,\phi_I(y)) = T(\partial_0 \phi_I(x)\,\phi_I(y))$$

(since fields commute at equal times) Now apply the identity again:

$$\partial_0^y T'(\,\phi_I(y)\,\partial_0 \phi_I(x)\,)$$

$$= T'(\,\partial_0 \phi_I(y)\,\partial_0 \phi_I(x)\,) + i\delta^4(y-x)$$

We've derived

$$\partial_0^x \partial_0^y T'(\phi_I(x)\,\phi_I(y)) = T'(\partial_0 \phi_I(x)\,\partial_0 \phi_I(y)) + i\delta^4(x-y)$$

Since spatial gradients commute with T,

$$\underbrace{\partial_\mu^x \partial_\nu^y (T'\phi_I(x)\,\phi_I(y))}_{\text{covariant}}$$

$$= \underbrace{T'(\partial_\mu \phi_I(x)\,\partial_\nu \phi_I(y))}_{\text{noncovariant}} + \underbrace{i\,\eta_{\mu 0}\eta_{\nu 0}\,\delta^4(x-y)}_{\text{noncovariant}}$$

We see that, canonically, the contraction

$$\overbrace{\partial_\mu \phi(x)\;\partial_\nu \phi(y)}$$

is not covariant. It must be that the noncovariance of the contractions compensates for the noncovariance of the vertices, so that we get covariant amplitudes. This really happens, but it is very complicated to show that it happens.

The path integral approach is much nicer because it generates manifestly covariant Feynman rules, with covariant vertices and covariant contractions. The covariance of the contractions is obvious:

$$\overbrace{\partial_\mu \phi(x)\, \partial_\nu \phi(y)} = \frac{\int d\phi\, e^{iS_0}\, \partial_\mu \phi(x)\, \partial_\nu \phi(y)}{\int d\phi\, e^{iS_0}}$$

$$= \partial_\mu^x \partial_\nu^y \frac{\int d\phi\, e^{iS_0}\, \phi(x)\,\phi(y)}{\int d\phi\, e^{iS_0}} = \partial_\mu^x \partial_\nu^y\, \overbrace{\phi(x)\,\phi(y)}$$

-- It is legitimate to take derivatives outside the integral because $\phi(x)\,\phi(y)$ are just functions. But one must be a bit careful to get the right vertices.

Reverting to a notation appropriate for a system with several degrees of freedom, consider

$$L = \frac{1}{2}\dot{q}^a M_{ab}(q)\,\dot{q}^b - V(q)$$

where $M_{ab}(q)$ is an invertible matrix (and depends on q, in general). Conjugate momentum is

$$p_a = \frac{\partial L}{\partial \dot{q}_a} = M_{ab}\,\dot{q}^b,$$

and

$$H = p_a \dot{q}^a - L = \frac{1}{2} p_a (M^{-1})^{ab} p_b + V$$

Now, since M^{-1} is a function of q but not p, the momenta can be "integrated out"-- they appear only quadratically:

$$\int (dp)\, e^{iS_H} = \int (dp) \exp i\left(p_a \dot{q}^a - \tfrac{1}{2} p_a (M^{-1})^{ab} p_b - V(q)\right)$$

$$= (\det iM)^{\frac{1}{2}} e^{iS_L[q]}$$

(up to an irrelevant normalization factor).

Applying this result to field theory, we see that if

$$\mathcal{L} = \tfrac{1}{2} \partial_0 \phi^a(x)\, M_{ab}(\phi(x))\, \partial_0 \phi^b(x)$$
$$+ (\text{terms without time derivatives})$$

then

$$Z[e] = N \int (d\phi) (\det iM)^{\frac{1}{2}} e^{i(S[\phi] + \int e\phi)}$$

is the generating functional for the Green functions. (The operator M may include spatial gradients.) We can now generate a Feynman diagram expansion of $Z[e]$ in the usual way. But for this purpose, it is quite convenient to express $(\det iM)^{1/2}$ as an exponential of something. We use the identity

$$\det A = \exp \operatorname{tr} \ln A$$

Therefore $(\det iM)^{\frac{1}{2}} = \exp[\tfrac{1}{2} \operatorname{tr} \ln(iM)]$,

and we have ...

$$Z[e] = N\int(d\phi)\ \exp i\left(S_{eff}[\phi] + \int e\phi\right)$$

where

$$S_{eff} = S[\phi] - \tfrac{i}{2}\,\mathrm{Tr}\ln(iM(\phi))$$

The Feynman rules for a field theory with derivative interactions are the "naive" ones with covariant contractions. But the subtlety is ... we must include a new interaction term in the action.

In the example described on page (4.33), with

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu A\,\partial^\mu A)(1+\tfrac{1}{2}gB^2)$$
$$-\tfrac{1}{2}m_A^2A^2 + \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}m_B^2B^2,$$

we have

$$M = 1+\tfrac{1}{2}gB^2(x)$$ (Not invertible for all B, but OK in perturbation theory.)

We'll see below a neat way of writing out the new effective interaction term $-\frac{i}{2}\,\mathrm{Tr}\ln(iM)$ in general. In this special case it is easy, since M is diagonal in the x-basis. We have $\mathrm{Tr}\ln M = \int d^4x\,\langle x|M|x\rangle$, where $|x\rangle$ is a δ-function localized at x

or $-\frac{i}{2}\,\mathrm{Tr}\ln(iM) = \delta^4(0)\left(\frac{-i}{2}\right)\int d^4x\,\ln\left(1+\frac{g}{2}B^2\right)$

(position space $\delta^4(0)$ -- or $\int\frac{d^4k}{(2\pi)^4}(1)$.)

Fermionic Functional Integration

So far, our derivation of Feynman rules using path integral methods applies to only scalar field theories. We want to find a suitable generalization that also can be applied to theories involving fermions -- theories that are quantized with canonical anticommutators rather than canonical commutators. This generalization is bound to involve subtleties; analogy suggests that quantum amplitudes might be expressible as integrals over "classical" fermi fields, but these classical fields are anticommuting numbers, so such an integral will have to be carefully defined.

We'll use a pragmatic approach; we know what the right Feynman rules are (for an action quadratic in spinor fields). We will search for the right rules for the functional integral that reproduce this known Feynman graph expansion. Although it only reproduces previously known results, the fermionic path integral representation of the amplitudes will prove to be quite useful.

To begin, consider the case of a free complex scalar field, with

$$iS = -\bar{\phi} A \phi \qquad (\bar{\phi} \text{ is cx conjugate of } \phi)$$

(in our schematic matrix notation)

This complex scalar field can be expressed in terms of two real scalar fields

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

So that $iS = -\frac{1}{2}\phi_1 A \phi_1 - \frac{1}{2}\phi_2 A \phi_2$

Now, if we define the measure for the complex scalar field by

$$(d\phi)(d\bar{\phi}) = (d\phi_1)(d\phi_2),$$

we see that

$$\int (d\phi)(d\bar{\phi})\, e^{-\bar{\phi}A\phi} = \underbrace{(\det A)^{-\frac{1}{2}}}_{\text{integral over } \phi_1} \underbrace{(\det A)^{-\frac{1}{2}}}_{\text{integral over } \phi_2}$$

$$= (\det A)^{-1}$$

Now, suppose we couple a source J bilinearly to the complex scalar field ϕ:

$$iS = -\bar{\phi}A\phi + \bar{\phi}J\phi.$$

And compute:

$$Z[J] = \left(\int d\phi\, d\bar{\phi}\, e^{-\bar{\phi}A\phi + \bar{\phi}J\phi}\right) \Big/ \left(\int d\phi\, d\bar{\phi}\, e^{-\bar{\phi}A\phi}\right)$$

$$= \frac{\int d\phi\, d\bar{\phi}\, e^{-\bar{\phi}(A-J)\phi}}{\int d\phi\, d\bar{\phi}\, e^{-\bar{\phi}A\phi}} = \frac{[\det(A-J)]^{-1}}{(\det A)^{-1}}$$

$$= [\det(1 - A^{-1}J)]^{-1}$$

So this generating functional is

$$Z[J] = [\det(1 - A^{-1}J)]^{-1} = \exp[-\operatorname{tr}\ln(1 - A^{-1}J)]$$

$$= \exp[\operatorname{tr} A^{-1}J + \tfrac{1}{2}\operatorname{tr}(A^{-1}J)^2 + \tfrac{1}{3}\operatorname{tr}(A^{-1}J)^3 + \cdots]$$

As expected, the argument of the exponential is just the sum of the connected Feynman diagrams in the presence of the source J:

$Z[J] = \exp($  $+ \cdots),$

with propagator $\leftarrow\ = A^{-1}$
and vertex $\leftarrow\!\circ\!\leftarrow\ = J$

(the propagator has an arrow because ϕ and $\bar{\phi}$ are distinguishable.) The factor $\frac{1}{n}$ in front of the one loop graph with n vertices is a symmetry factor -- a rotation of the diagram does not give rise to a new contraction.

Now, let's consider what the sum of Feynman diagrams would be if we coupled a source to a bilinear constructed from <u>spinor</u> fields

$$iS = -\bar{\psi}A\psi + \bar{\psi}J\psi$$

(In the Dirac theory, we have $A = (-i)(\not{p} - m)$ in momentum space and $A^{-1} = i/(\not{p} - m)$ is the propagator. We'll take it for granted that A^{-1} is the right propagator in general.)

The field ψ is a complex field (with spinor indices that have not been written out explicitly). We can express $\ln Z[J]$ as a Feynman diagram expansion as before, but with one essential difference -- there is a Fermi (-1) associated with each closed loop. So

$$Z[J] = \exp\left(-\mathrm{tr}\,A^{-1}J - \frac{1}{2}\mathrm{tr}(A^{-1}J)^2 - \frac{1}{3}\mathrm{tr}(A^{-1}J)^3 + \cdots\right)$$

$$= \frac{\det(A-J)}{\det A}$$

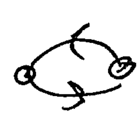
This is just the inverse of what we had for complex scalars -- the sign of the argument of the exponential is changed.

If we want to express $Z[J]$ as a functional integral over Fermi field, we should define this integral so that

$$\int d\psi\, d\bar\psi\, e^{\bar\psi A \psi} = \det A$$

(or $\det(-A)$ -- but the sign doesn't matter because it just gets absorbed into the normalization) then --

$$\frac{\int d\psi\, d\bar\psi\, e^{\bar\psi(A-J)\psi}}{\int d\psi\, d\bar\psi\, e^{\bar\psi A\psi}} = \frac{\det(A-J)}{\det A} = \det(1 - A^{-1}J)$$

$= \exp[$  $+$  $+$  $+ \cdots]$

To find a concrete way of implementing this identity, consider first an integral over a single (complex) fermionic variable η. We want

$$\int d\eta\, d\bar\eta\; e^{\lambda\bar\eta\eta} = \lambda$$

Expanding the exponential in powers of λ, we have

$$\left.\begin{aligned} &\int d\eta\, d\bar\eta\;\; 1 = 0 \\ &\int d\eta\, d\bar\eta\, (\bar\eta\eta) = 1 \\ &\int d\eta\, d\bar\eta\, (\bar\eta\eta)^2 = 0 \\ &\quad \text{etc.} \\ &\quad\;\; \vdots \end{aligned}\right]$$

Here $\int d\eta\, d\bar\eta$ must be regarded as an abstract operation (assumed linear), that we need to <u>define</u>.

We can summarize this infinite sequence of identities by a few simple rules: Consider η and $\bar\eta$ to be "anticommuting c-nos."

$$\eta\bar\eta + \bar\eta\eta = 0 \qquad \eta^2 = \bar\eta^2 = 0$$

(Also called Grassmann variables.)

And define

$$\int d\eta \begin{pmatrix} 1 \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\int d\bar\eta \begin{pmatrix} 1 \\ \bar\eta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now we can integrate any function of η and $\bar\eta$ that can be expanded in power series.

When we expand, terms higher than linear order in $\eta, \bar{\eta}$ vanish. Integrating then picks out the term linear in both η and $\bar{\eta}$.

It is easy now to do "Gaussian" integrals over many fermionic variables. A "vector" of anticommuting variables has the form

$$\eta = \sum_n \eta_n \hat{e}_n ;$$

it can be expanded in a basis like any vector, but the coefficients are anticommuting numbers. We define

$$\int d\eta \, d\bar{\eta} = \prod_n \int d\eta_n \, d\bar{\eta}_n$$

And therefore:

$$\int d\eta \, d\bar{\eta} \, e^{\bar{\eta} A \eta} = \int d\eta \, d\bar{\eta} \, e^{\sum_n \lambda_n \bar{\eta}_n \eta_n}$$

$$= \prod_n \left[\int d\eta_n d\bar{\eta}_n \, e^{\lambda_n \bar{\eta}_n \eta_n} \right] = \prod_n \lambda_n = \det A,$$

where A is a matrix and the λ_n's are its eigenvalues. So our definition of fermionic integration gives the answer we wanted. In the field theory case, we will write

$$\eta(x) = \sum_n \eta_n f_n(x) ;$$

An anticommuting classical field $\eta(x)$ has an expansion in terms of c-no. functions $f_n(x)$, but with anticommuting coefficients.

Feynman Rules

We can also do a fermionic functional integral with a linear source term.

$$\int d\psi d\bar{\psi}\ e^{\bar{\psi}A\psi + \bar{\xi}\psi + \bar{\psi}\xi}$$

(ξ and $\bar{\xi}$ are two independent sources)

$$= \int d\psi d\bar{\psi}\ \exp\left[(\bar{\psi} + \bar{\xi}A^{-1})A(\psi + A^{-1}\xi) - \bar{\xi}A^{-1}\xi\right]$$

-- completing the square. Notice that the rules for fermionic integration imply

$$\int d\eta\ f(\eta + \xi) = \int d\eta\, f(\eta).$$

We just note that

$$\int d\eta\ (\eta + \xi) = \int d\eta\ \eta = 1,$$

and this is sufficient, since only the linear term in η survives when we integrate.
So we can "shift" the variable of integration.

$$= e^{-\bar{\xi}A^{-1}\xi} \int d\psi d\bar{\psi}\ e^{\bar{\psi}A\psi} = e^{-\bar{\xi}A^{-1}\xi} \det A$$

In free field theory, we have

$$Z[\xi, \bar{\xi}] = \frac{\int d\psi d\bar{\psi}\, e^{\bar{\psi}A\psi + \bar{\xi}\psi + \bar{\psi}\xi}}{\int d\psi d\bar{\psi}\, e^{\bar{\psi}A\psi}} = e^{-\bar{\xi}A^{-1}\xi}$$

If we expand the exponential, we see that

$$G^{(n,n)} = \langle \underbrace{\psi \cdots \psi}_{n}\ \underbrace{\bar{\psi} \cdots \bar{\psi}}_{n} \rangle$$

is given by all possible contractions of ψ's with $\bar{\psi}$'s. (The combinatorics is essentially the same as that in the scalar case, considered previously.) In other words, it is a consequence of our rules for fermionic integration that

$$\frac{\int d\psi d\bar{\psi}\, e^{-\bar{\psi}\Delta^{-1}\psi}\, \psi_1 \cdots \psi_n \bar{\psi}_1 \cdots \bar{\psi}_n}{\int d\psi d\bar{\psi}\, e^{-\bar{\psi}A\psi}} = \sum_{\text{pairings}} (-1)^P \Delta_{1,i_1} \cdots \Delta_{n,i_n}$$

(sign of perm -- origin of Fermi (-1))

-- the sum is over all Wick contractions of the ψ's with the $\bar{\psi}$'s. For a theory of coupled scalars and fermions, it follows (as in the scalar case considered before) that

$$Z[\rho, \xi, \bar{\xi}] = N \int (d\phi)(d\psi)(d\bar{\psi})\, e^{i[S + \int(\rho\phi + \bar{\xi}\psi + \bar{\psi}\xi)]}$$

is given by the usual Feynman diagram expansion (obtained by expanding $e^{iS_{int}}$ in powers of S_{int}, and performing all possible contractions, using as propagators the inverses of operators in the quadratic part of the action.)

Ghost Fields

Our discussion of derivative interactions was rather formal, since we did not give an explicit prescription for computing the extra interaction term $-\frac{i}{2}\,\mathrm{tr}\ln(iM(\phi))$.

This situation may now be remedied, with the help of our rules for fermionic integration.

We wanted to write $\det iM$ in a form from which Feynman rules are readily deduced. A convenient method is to introduce a ghost field that obeys fermi statistics.

$$(\det iM)^{1/2} = \int d\eta\, d\bar{\eta}\; e^{\bar{\eta}(iM)^{1/2}\eta}$$

The field η is called a "ghost" because it does not correspond to a physical degree of freedom of the theory -- it is just a technical trick for rewriting the determinant. In fact, η does not even obey the spin-statistics connection -- it is a scalar (zero-spin) field that obeys fermi statistics. This suggests that this field theory would be sick if we considered correlation functions of the η's. But we won't; we are interested in correlation functions of the other fields -- the η's will appear only in loops, with a fermi (-1) accompanying each η loop.

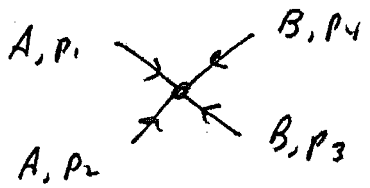
In the theory discussed earlier, we have

$$S_{eff} = S[A,B] + \int \bar{\eta}\left(1 + \tfrac{1}{2} g B^2\right)^{1/2} \eta \, d^4x$$

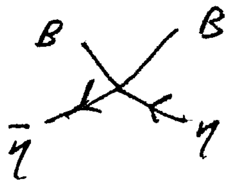
The η propagator is $(-i)^{-1} = i$ in momentum space, or

$$\bar{\eta}(x)\,\eta(y) = i\,\delta^4(x-y)$$

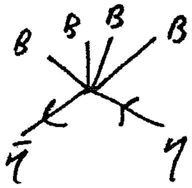
in position space. The vertices are



$$= (ig)(-p_1 \cdot p_2)$$



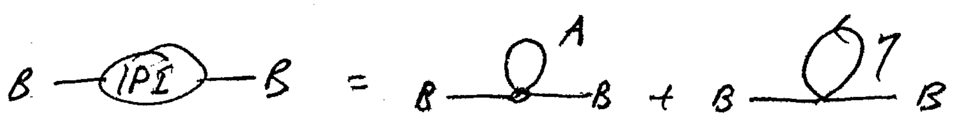
$$= \tfrac{1}{2} i g$$



$$= -\tfrac{3}{4} i g^2$$

etc.

For example, compute the leading propagator correction for B:



$$= (ig)\int \frac{d^4k}{(2\pi)^4}\Big[\underbrace{\left(\tfrac{1}{2}\right)}_{\text{(sym factor)}} \frac{i}{k^2 - m_A^2}\, k^2 + \underbrace{(-1)}_{\text{(Fermi)}}\underbrace{\left(\tfrac{1}{2}\right) i}_{\text{(η propagator)}}\Big]$$

Notice that the leading (quartic) divergence cancels. (Such cancellations are easier to keep track of when we compute with ghosts, instead of expanding $\ln M$ as on page 4.38.)

The Effective Action

We are often interested in the consequences of symmetry in quantum field theory. We know that there is a Noether current associated with a symmetry of the classical action. But we are especially interested in the consequences of a symmetry for the Green functions of a field theory. These consequences can be stated in the form of what are generically refered to as "Ward identities", which are especially easy to formulate using the functional integral language.

Ward identities are especially useful when expressed as constraints on the one-particle-irreducible (1PI) Feynman diagrams. Therefore it is quite handy to work with a generating functional for the 1PI Green functions (just as $Z[\varphi]$ is a generating functional for the complete Green functions. This object is called the "effective action" -- We'll work out how it is related to $Z[\varphi]$.

First, we recall that

$$Z[\varphi] = \frac{\int(d\phi)\, e^{i(S+\int\varphi\phi)}}{\int d\phi\, e^{iS}} = \sum_{n=0}^{\infty} \frac{(i)^n}{n!}\int d^4x_1 \cdots d^4x_n\, \varphi(x_1)\cdots\varphi(x_n)\, G^{(n)}(x_1,\dots x_n)$$

where $G^{(0)} = 1$

and $G^{(n)}(x_1 \dots x_n) = \langle 0|\, T(\phi(x_1)\cdots\phi(x_n))\,|0\rangle\ , \quad n>0$

(For notational simplicity, we'll use a notation appropriate for a scalar field theory, but it will be clear that all the manipulations below can also be applied to a theory involving fermions.)

We also know that:

All Diagrams = Exp[All connected Diagrams]

Therefore, we may write

$$Z[\rho] = e^{iW[\rho]}$$

and W is the generating functional of connected diagrams

$$iW[\rho] = \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int d^4x_1 \cdots d^4x_n\, \rho(x_1) \cdots \rho(x_n)\, G_c^{(n)}(x_1 \cdots x_n),$$

where $G_c^{(n)}$, the connected n-point Green function, is the sum of all connected Feynman diagrams with n external lines.

Now, let us denote by

$$\Gamma^{(n)}(x_1 \cdots x_n)$$

the sum of all one-particle irreducible Feynman diagrams with n external lines. (At least, we define $\Gamma^{(n)}$ this way for $n = 1, 3, 4 \cdots$; we'll define $\Gamma^{(2)}$ in a minute.) From the $\Gamma^{(n)}$'s we may construct a generating functional:

$$i\Gamma[\bar{\phi}] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \bar{\phi}(x_1) \cdots \bar{\phi}(x_n) \, \Gamma^{(n)}(x_1, \cdots x_n)$$

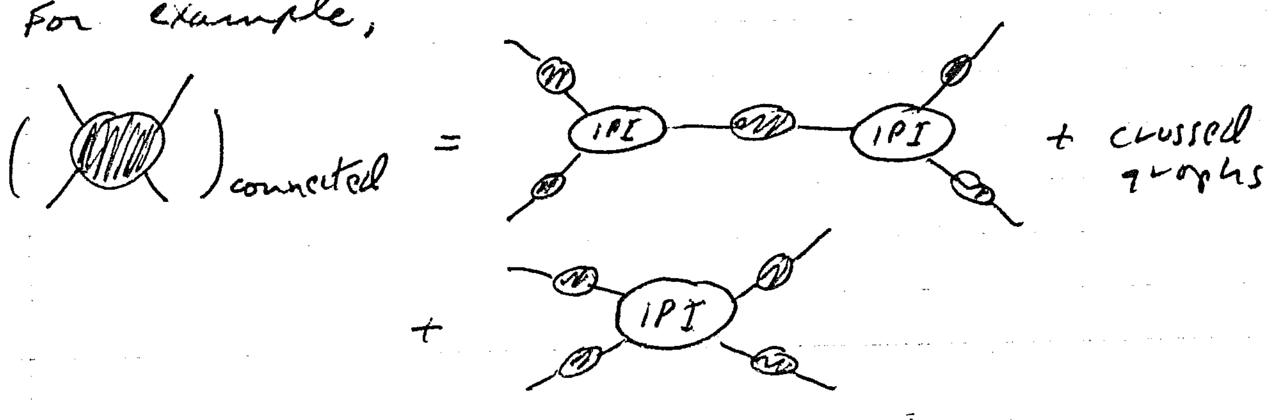
(Note that we have chosen to expand Γ in terms of a new dummy variable denoted $\bar{\phi}$, instead of φ)
For $\Gamma^{(2)}$ we will use the definition

$$\Gamma^{(2)}(p) = \frac{-1}{G_c^{(2)}(p)}$$ (includes matrix inverse if there are many fields)

Here $G_c^{(2)}(p)$ is the exact propagator, the sum of all connected diagrams with two external lines, and $\Gamma^{(2)}(p)$ is (up to a sign) the inverse propagator. In position space:

$$\Gamma^{(2)}(x_1, x_2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1 - x_2)} \frac{-1}{G_c^{(2)}(p)}$$

Now, we correlate $W[\varphi]$ to $\Gamma[\bar{\phi}]$ by making the following observation:
The sum of all connected diagrams can be obtained by constructing all "tree diagrams" with the exact connected two-point function used as a propagator, and the complete 1PI Green functions used as vertices.
For example,



All the loop corrections can be absorbed into the propagator or the 1PI parts of the diagrams. The exact Green functions are thus given by the tree approximation to an effective field theory with a (very complicated and nonlocal) effective action

$$S_{eff}[\bar\phi] = \Gamma[\bar\phi]$$

We've seen that the tree approximation can be described as the classical (small $\hbar$) approximation. In the limit of small $\hbar$

$$\int d\bar\phi \, e^{i(\Gamma[\bar\phi] + \int e\bar\phi)/\hbar}$$
$$= \exp\left[i W[e]/\hbar \, (1 + O(\hbar))\right]$$

sum of tree diagrams of the effective field theory with action $\Gamma[\bar\phi]$

(Compare page 4.27)

But in the limit of small $\hbar$, this functional integral can be evaluated by the stationary phase method (as on page 4.23 ff)
The leading term for $\hbar \to 0$ in the exponent on the right-hand side is found by evaluating the integrand in the functional integral when the argument of the exponential is stationary. Thus

$$W[e] = \Gamma[\bar\phi] + \int d^4x \, e(x)\bar\phi(x) \Big|_{\substack{\text{stationary} \\ \text{point (in } \bar\phi)}}$$

The stationary point is found by solving the equation

$$\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)} = -\varphi(x)$$

(We'll assume that $\Gamma[\bar\phi]$ is a "convex" functional, so that the solution to this equation is unique.)

What we have found is that $W[\varphi]$ and $\Gamma[\bar\phi]$ are related by a <u>Legendre transformation</u>, an operation that is familiar in mechanics and thermodynamics. E.g. recall how the functions $U(S)$ and $F(T)$ are related in thermodynamics

$$U(S) = (F(T) + TS)\Big|_{\substack{\text{stationary}\\ \text{in } T}}$$

or $\quad U(S) = F(T) + TS$, where T is chosen so that $dF/dT = -S$

It follows that $\quad \dfrac{dU}{dS} = T \quad$ -- the value of the "dummy" variable T at which stationary condition is satisfied

To invert this transform, we write

$$F(T) = U(S) - TS\Big|_{\text{stationary in } S}$$

Because the condition for stationarity is $\dfrac{dU}{dS} = T$, and we saw above that

$$F(T) = U(S) - TS \quad \text{when } \frac{dU}{dS} = T.$$

Generalizing to a functional Legendre transform, we have...

$$\Gamma[\bar{\phi}] = W[\rho] - \int d^4x\, \rho(x)\bar{\phi}(x) \Big|_{\substack{\text{stationary} \\ \text{in } \rho}}$$

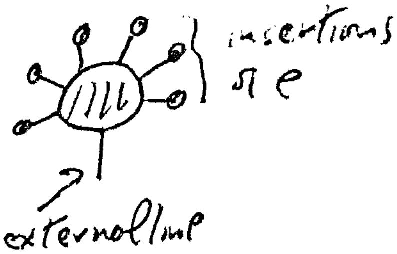
This inverts the functional Legendre transform. The value of ρ at the stationary point is chosen to satisfy

$$\frac{\delta W[\rho]}{\delta \rho(x)} = \bar{\phi}(x)$$

But $\frac{\delta W}{\delta \rho}$ is just the one point function

$$\langle 0|\, \phi(x)\, |0\rangle_\rho$$

in the presence of the source ρ. Our procedure for computing $\Gamma[\bar{\phi}]$ can be described this way. We "turn on" a source $\rho(x)$ coupled to $\phi(x)$ that is chosen so that the expectation value of $\phi(x)$ is the specified function $\bar{\phi}(x)$:



$$\langle \phi(x)\rangle_\rho = \bar{\phi}(x)$$

Then, with this source ρ present, calculate

$$\Gamma[\bar{\phi}] = W[\rho] - \int \rho\bar{\phi}$$

-- this is the effective action evaluated at $\bar{\phi}$.

Symmetries and Ward Identities

Now we are prepared to discuss the connection between symmetries of the classical action S and symmetries of the effective action Γ, or, in other words, the consequences of symmetries of S for the structure of 1PI diagrams.

Consider an infinitesimal transformation of the fields ϕ,

$$\delta\phi(x) = A(\phi)\,\delta\omega(x)$$

where $\delta\omega(x)$ is an infinitesimal function of x and $A(\phi)$ is an expression linear in fields. (There may be many field components, in which case A may be a matrix acting on field components.) Suppose that the change of the action under this transformation is also linear

$$\delta S = \int d^4x\, B(\phi)\,\delta\omega(x)$$

Now, we can show that the change $\delta\Gamma$ in the effective action has exactly the form of the change δS in the classical action (provided the transformation $\delta\phi$ does not change the functional measure $(d\phi)$ -- i.e., it has trivial Jacobian.)

To see this, perform a change of variable in the functional integral.

$$e^{iW[\rho]} = \int(d\phi)\, e^{i(S[\phi] + \int \rho\phi)}$$

$$= \int(d\phi)\, e^{i(S[\phi+\delta\phi] + \int \rho(\phi+\delta\phi))}$$

(trivial Jacobian)

$$= \int(d\phi) e^{i(S[\phi] + \int\rho\phi)}\Big[1 + i\int \delta\phi\left(\frac{\delta S}{\delta\phi} + \rho\right) + \text{order}\,(\delta\phi)^2\Big]$$

But the left hand side is independent of $\delta\phi$, so matching powers of $\delta\phi$ gives

$$0 = \int(d\phi)\, e^{i(S[\phi] + \int\rho\phi)}\Big[\int \delta\phi\left(\frac{\delta S}{\delta\phi} + \rho\right)\Big]$$

~~Substituting in the assumed (linear) form for δS and $\delta\phi$, we have~~

$$\cancel{0 = \int(d\phi)\, e^{i(S[\phi]+\int\rho\phi)}\Big[\int d^4x\, \delta\phi(x)\,(\ldots)\Big]}$$

Now, suppose we choose the source ρ so that

$$\langle\phi\rangle_\rho = \bar\phi$$

where $\bar\phi$ is some specified function of x then we know that

$$\rho(x) = -\frac{\delta\Gamma}{\delta\bar\phi(x)}$$

where Γ is the effective action. The above identity becomes:

$$\int d^4x \, \frac{\delta\Gamma}{\delta\bar{\phi}(x)} \langle \delta\phi(x) \rangle_e = \langle \delta S \rangle_e$$

So far, we have not used the linearity of $\delta\phi$ and δS. Now we do so, by noting that

$$\langle F[\phi] \rangle_e = F[\langle \phi \rangle_e] = F[\bar{\phi}]$$

for any linear functional. (The mean value of a linear function is the function evaluated at the mean value of its argument) Therefore, if $\delta\phi$ and δS are linear, we have

$$\delta S[\bar{\phi}] = \int \frac{\delta\Gamma}{\delta\bar{\phi}} \delta\bar{\phi} \equiv \delta\Gamma[\bar{\phi}]$$

-- This is the Ward identity; it says that the effective action varies as the classical action does under the transformation.

In particular, suppose that $\phi \to \phi + \delta\phi$ is a classical symmetry -- it leaves S invariant. Then as long as $\delta\phi$ is linear in fields, and leaves the function space measure unchanged, we have

$$\delta\Gamma = 0$$

-- the effective action is also invariant. We see that the effective action has all the symmetries of the classical action. One should note, though, that this derivation has been rather "formal". The path integral defines a Feynman graph expansion, but the Feynman graphs are typically ill-defined, because of ultraviolet divergences. So we need to impose an ultraviolet cutoff ("regulator") in order to make sense of the path integral. It is implicitly assumed above that this regulator does not spoil the classical symmetries.

Green Function Identities

We can also derive relations satisfied by Green functions as a consequence of symmetry

Consider a global symmetry transformation

$$\delta\phi(x) = \epsilon A \phi(x),$$

where A is a matrix acting on field components (perhaps including linear differential operators). "Global" means that the same transformation is performed at each spacetime point. This is a symmetry if S is invariant, or $\mathcal{L}$ changes by a total divergence:

$$\delta\mathcal{L} = \epsilon\, \partial_\mu f^\mu(x).$$

Now suppose the transformation is "local":

$$\delta\phi(x) = \epsilon(x) A \phi(x)$$

(ϵ is a function of spacetime.) Then $\mathcal{L}$ changes by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\mu(\epsilon A\phi) + \frac{\partial\mathcal{L}}{\partial\phi}(\epsilon A\phi)$$

or $\delta\mathcal{L} = \underbrace{(\partial_\mu \epsilon(x))\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} A\phi}_{\text{vanishes if } \epsilon \text{ is a constant}} + \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} A \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial \phi} A \phi\right]}_{\partial_\mu f^\mu} \epsilon$

We have $\delta\mathcal{L} = \partial_\mu(f^\mu \epsilon) + (\partial_\mu \epsilon)\left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} A\phi - f^\mu\right]$,

and the action changes by

$$\delta S = \int \delta\mathcal{L} = -\int \epsilon(x)\, \partial_\mu J^\mu(x)$$

(We may choose ϵ to vanish on the "boundary", so that there are no surface terms) Here

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} A\phi - f^\mu$$

is the same current that we constructed in proving Noether's theorem.

Now, perform the change of variable

$$\phi(x) \to \phi(x) + \epsilon(x) A\phi(x)$$

in the functional integral

$$\int d\phi \; e^{i(S[\phi] + \int e\phi)}$$

$$= \int d\phi \; e^{i(S[\phi] + \int e\phi)}\left[1 + i\int \epsilon\,(-\partial_\mu J^\mu + A\phi e) + \cdots\right]$$

(assuming that the Jacobian is trivial) In particular, the order ϵ term on the right hand side must vanish, and we have (since $\epsilon(x)$ may be chosen to be any function):

$$\boxed{0 = N \int d\phi \, e^{i(S[\phi] + S e\phi)} \left(-\partial_\mu J^\mu(x) + A\phi(x) e(x) \right)}$$

This identity, which holds if $\delta\phi = \epsilon A \phi$ is any symmetry operation, and J^μ is the associated Noether current, is also called the Ward identity.

This Ward identity is actually an infinite number of identities satisfied by the Green functions, as we may see by expanding in powers of the source e.

<u>Zeroth Order:</u>

$$\partial_\mu \quad \langle 0 | J^\mu(x) | 0 \rangle = 0$$

<u>1st-order:</u> $\partial_\mu^x \langle 0 | T\left[\left(i\int d^4y \, e(y) \phi(y)\right) \quad J^\mu(x)\right] | 0 \rangle$

$$= \quad \langle 0 | \, A \, \phi(x) e(x) \, | 0 \rangle$$

or --

$$\int d^4y \, e(y) \left[\partial_\mu^x \langle 0 | T \, J^\mu(x) \, \phi(y) | 0 \rangle + i \delta^4(x-y) \langle 0 | A\phi(x) | 0 \rangle \right]$$

Since this holds for any function $e(x)$, we have

$$\boxed{\partial_\mu^x \langle 0 | T\left[J^\mu(x) \, \phi(y)\right] | 0 \rangle = -i \delta^4(x-y) \, \langle 0 | A \phi(x) | 0 \rangle}$$

<u>nth-order:</u>

The generalization is obviously --

$$\partial_\mu^x \langle 0|T[\; J^\mu(x)\, \phi(y_1) \cdots \phi(y_n)]|0\rangle$$

$$= -i\Big[\, \delta^4(x-y_1)\langle 0|T[A\phi(y_1)\,\phi(y_2)\cdots\phi(y_n)]|0\rangle$$
$$+\, \delta^4(x-y_2)\langle 0|T[\phi(y_1)\,A\phi(y_2)\,\phi(y_3)\cdots\phi(y_n)]|0\rangle$$
$$+ \cdots$$
$$+\, \delta^4(x-y_n)\langle 0|T[\phi(y_1)\cdots\phi(y_{n-1})\,A\phi(y_n)]|0\rangle\Big]$$

We may also derive these Ward identities by canonical methods. If J^μ is a conserved current, $\partial_\mu J^\mu(x) = 0$, we have

$$\partial_\mu^x \langle 0|T[J^\mu(x)\,\phi(y)]|0\rangle$$
$$= \partial_\mu^x \langle 0|\big[\theta(x^0-y^0)\, J^\mu(x)\phi(y) \;+\; \theta(y^0-x^0)\; \phi(y) J^\mu(x)\big]|0\rangle$$
$$= \langle 0|\delta(x^0-y^0)\,[\,J^0(x),\, \phi(y)]\,|0\rangle$$

In the special case where the symmetry transformation leaves $\mathcal{L}$ invariant,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} A\phi \quad \text{or} \quad J^0(x) = \pi(x)\, A\phi(x)$$

The equal-time commutation relations imply that

$$[J^0(x^0,\vec{x}),\; \phi(x^0,\vec{y})] = -i\,\delta^3(\vec{x}-\vec{y})\; A\phi(x^0,\vec{x})$$

So we have

$$\partial_\mu^x \langle 0|T[\,J^\mu(x)\,\phi(y)]|0\rangle = -i\delta^4(x-y)\; \langle 0|A\phi(x)|0\rangle$$

— which agrees with our previous Ward identity

The functional derivation of the Ward identity is convenient because it is more general. In fact, it is important to understand that the functional Ward identity and canonical Ward identity actually apply to different objects. The Green functions appearing in the functional Ward identity are evaluated in perturbation theory using <u>covariant</u> contractions; in these Green functions a derivative can be commuted through the T-ordered product. But in the canonical Green functions, contractions of derivatives of fields are <u>noncovariant</u>, and a derivative of a field inside the T symbol is different than a derivative outside T. (Sometimes the time-ordered products defined by functional integration are called T^* products to distinguish them from the canonical T products.) In fact, the canonical derivation above is often wrong, because interactions modify the commutator of the current with the fields. But the functional derivation is always correct, as long as the Jacobian of the transformation is really trivial.

<u>Low Energy Theorem</u>

If we integrate the Ward identity —

$$\int d^4x\, \partial_\mu^x \langle 0|T[J^\mu(x)\,\phi(y)]|0\rangle = -i\,\langle 0|A\phi(y)|0\rangle$$

<u>If there are no massless particles</u>, we can justify neglecting the surface term on the left-hand-side

(the propagator falls off exponentially in Euclidean space). So we have

$$\langle 0|\, A\phi(y)\,|0\rangle = 0$$

this identity (which is obviously closely related to the identity that says that the effective action is invariant $\Gamma[\phi+\epsilon A\phi] = \Gamma[\phi]$) also has a canonical interpretation.
The charge

$$Q = \int d^3x\; J^0(x)$$

has commutator with the field

$$[Q, \phi(x)] = -i\,A\phi(x)$$

and is conserved. According to the argument of page 1.76 ff, this means that Q annihilates the vacuum. In brief, $e^{i\epsilon Q}$ commutes with H, and so $e^{i\epsilon Q}|0\rangle$ is degenerate with $|0\rangle$ -- but if the vacuum is unique (nondegenerate)

$$e^{i\epsilon Q}|0\rangle = \underbrace{e^{i\alpha(\epsilon)}}_{\text{a phase}}|0\rangle$$

or, expanding to order ϵ $\quad Q|0\rangle = \underbrace{c}_{\text{a real constant}}|0\rangle$, and we can redefine Q by an additive constant so that $Q|0\rangle = 0$. But even if we perform no such redefinition:

$$0 = \langle 0|\,[Q, \phi(x)]\,|0\rangle = -i\,\langle 0|A\,\phi(x)|0\rangle$$

(Note: the qualifications "if there are no massless particles" and "if the vacuum is unique" are important, as we'll see.)