5. Quantum Electrodynamics

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The Trouble with Higher Spin

Armed with our functional methods, we may now proceed to the quantization of electrodynamics. What is the origin of the technical problems that make functional methods so useful in treating QED?

So far, we have dealt with field theories in which the fields transform as a representation of the Lorentz group that transforms irreducibly under rotations (the little group for massive particles):

(Real) Scalars $(0,0) \longrightarrow$ Spin 0
(Weyl) Fermions $(\frac{1}{2},0) \longrightarrow$ Spin $\frac{1}{2}$

This means that the number of field components is right for the field to create particles that transform irreducibly under the Poincare group when they act on the vacuum. If the transformation law of the field is

$$U(a,\Lambda)\, \phi_\alpha(x)\, U(a,\Lambda)^{-1} = D_{\alpha\beta}(\Lambda^{-1})\, \phi_\beta(\Lambda x + a),$$

then consider $\tilde{\phi}_\alpha(t=0, \vec{p}=0)|0\rangle$; i.e. ϕ Fourier transformed in $\vec{x}$ at fixed time.

If R is a rotation (leaving $\vec{p}=0$ fixed), then

$$U(R)\,\tilde{\phi}_\alpha |0\rangle = D_{\alpha\beta}(R^{-1})\,\tilde{\phi}_\beta |0\rangle$$

Thus, the states $\tilde{\phi}_\alpha|0\rangle$ transform under the "little group" as the representation D restricted to the $SO(3)$ subgroup of the Lorentz group (this is the little group for massive particles). Under the full Poincare group, these states transform irreducibly if they transform irreducibly under the little group.

But consider, for example, a field that transforms as a Lorentz 4-vector.

$$A^\mu(x): \left(\tfrac{1}{2},\tfrac{1}{2}\right) \longrightarrow \text{Spin } 1 + \text{Spin } 0$$

The states created by A^μ acting on the vacuum are not purely spin 1; A^μ couples to both spin 0 and spin 1. If we desire to construct a covariant field theory for spin 1 particles, we are burdened with an unwanted spin 0 particle that we will need to remove from the quantum theory somehow.

For spin 2 things are even worse:

$$h^{\mu\nu}(x): (1,1) \longrightarrow \text{Spin } 2 + \text{Spin } 1 + \text{Spin } 0$$

symmetric traceless tensor

We are stuck in a dilemma. A field theory of (for example) spin 1 particles requires either --

- Loss of Covariance
- Unphysical Particles

For massless particles things are of course even worse. We want the 4-component field A_μ to create two physical helicity states $\lambda = \pm 1$. (This is a reducible rep, but we know we have to put them together to get a CPT-invariant theory) So in a covariant formulation we will have two unphysical degrees of freedom.

Because it is computationally much simpler, we will quantize electrodynamics by introducing unphysical degrees of freedom, rather than by breaking Lorentz invariance. Of course, if the theory really makes sense, the two methods should be equivalent. The apparent loss of covariance must be illusory, and not afflict physical quantities. Similarly, the unphysical degrees of freedom must "decouple" from physical processes. (E.g., they are not produced in scattering experiments!) Functional methods provide the most efficient way of dealing with such unphysical excitations and showing that they are harmless.

Note:

We could attempt to construct a field theory for spin 1 from an antisymmetric tensor field (self dual)

$B_{\mu\nu}$: $(1,0) \rightarrow$ Spin 1

↳ self-dual antisym tensor

But such a theory would not be parity invariant. To obtain a parity-invariant theory we would need the representation $(1,0)+(0,1)$ -- so this would be a theory of two spin-1 particles rather than one, unless some degrees of freedom are unphysical. (The selfdual and antiselfdual components of the electromagnetic field are $\vec{E}\pm i\vec{B}$ which, of course, are not independent.)

The Free Vector Field

We will now construct the free field theory for a vector field $A^\mu(x)$. Ultimately we will be interested in a massless vector (the photon) but we will not yet impose masslessness as a condition on the theory. We will consider the theory of a massive vector first, and treat the photon as the massless limit eventually

To begin, we construct the most general Lagrangian quadratic in A^μ that obeys the usual principles: it is local, Poincare invariant, hermitian, and depends on no derivatives higher than the first. Possible terms are --

No derivatives: $A_\mu A^\mu$

One derivative: $\partial_\mu A^\mu$ (since a total derivative, does not contribute to action)

Two derivatives: $\partial_\mu A^\nu \partial^\mu A_\nu$

(NOTE: $\epsilon_{\mu\nu\lambda\sigma}\partial^\mu A^\nu \partial^\lambda A^\sigma$ is also a total derivative)

$$\left[\begin{array}{l} \partial_\mu A^\mu \partial_\nu A^\nu \rightarrow -A^\mu \partial_\mu \partial_\nu A^\nu + \text{tot. deriv.} \\ \partial_\mu A^\nu \partial_\nu A^\mu \rightarrow -A^\nu \partial_\mu \partial_\nu A^\mu + \text{tot. deriv.} \end{array}\right.$$

These are not independent, since they are related by integration by parts

The most general quadratic action is

$$\mathcal{L} = \pm \frac{1}{2}\left[\partial_\mu A^\nu \partial^\mu A_\nu + a\, \partial_\mu A^\nu \partial_\nu A^\mu - \mu^2 A_\mu A^\mu\right],$$

where the field has been appropriately rescaled to fix the normalization of the first term.
The classical field equation is

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A^\nu} - \frac{\partial \mathcal{L}}{\partial A^\nu} = \partial_\mu\left(\partial^\mu A_\nu + a\, \partial_\nu A^\mu + \mu^2 A_\nu\right)$$

or $$0 = \partial_\mu \partial^\mu A_\nu + a\, \partial_\mu \partial_\nu A^\mu + \mu^2 A_\nu$$

What are the plane-wave solutions?

They have the form

$$A_\mu(x) = e_\mu e^{-ik\cdot x}$$

where

$$0 = -k^2 e_\nu - a\, e\cdot k\, k_\nu + \mu^2 e_\nu$$

We may distinguish two types of solutions:

- $e\cdot k = 0$ (transverse)

For these solutions we have $k^2 = \mu^2$. In the rest frame, $k = (\mu, \vec{0})$, there are three values for e_μ:

$$\left.\begin{aligned} e^{(1)} &= (0, 1, 0, 0) \\ e^{(2)} &= (0, 0, 1, 0) \\ e^{(3)} &= (0, 0, 0, 1) \end{aligned}\right\}$$ These transform as a spin 1 representation of the little group.

For other frames, we obtain a basis

$$e_\mu^{(r)}(k) \qquad r = 1, 2, 3$$

by boosting the above basis. In any frame, these satisfy

$$e^{(r)}(k)\cdot e^{(s)}(k) = -\delta^{rs} \quad \text{(invariant inner product)}$$

-- since this is true in the rest frame. We also have

$$\sum_r e_\mu^{(r)} e_\nu^{(r)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$$

-- this is true in the rest frame, and, since it is a covariant relation, is true in any frame.

- $e \propto k$ (longitudinal)

For these solutions, we have $(1+a)k^2 = \mu^2$

or

$$k^2 = \mu_L^2 \equiv \frac{\mu^2}{1+a}$$

In the rest frame $k = (\mu_L, \vec{0})$
we see that

$$\left. e^{(L)} = (1,0,0,0) = \frac{k}{\mu_L} \right\}$$ A spin 0 representation of the little group.

In an arbitrary frame, this evidently boosts to — $e^{(L)} = \frac{1}{\mu_L} k$

$$e^{(L)} \cdot e^{(r)} = 0, \quad e^{(L)2} = 1$$

$$e^{(L)}_\mu e^{(L)}_\nu = \frac{k_\mu k_\nu}{\mu_L^2}$$

As we anticipated, we see that the plane wave solutions correspond to a particle of spin 1 and mass μ, created (in the rest frame) by A^i acting on the vacuum; and a particle of spin 0 and mass μ_L, created by A^0 acting on the vacuum. (The masses can be different because they correspond to distinct irreducible representations of the little group.) But we are free to choose $a = -1$. This removes the spin zero longitudinal mode from the theory (pushes its mass to infinity.)

If we have removed one of the plane wave solutions from the theory, this must correspond to a constraint on A^μ that removes one of the classical degrees of freedom from the theory. To see what the constraint is, consider again the classical equation of motion:

With $a=-1$, the equation of motion is

$$0 = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) + \mu^2 A_\nu$$

or $\quad 0 = \partial^\mu F_{\mu\nu} + \mu^2 A_\nu$

where $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ -- an antisymmetric tensor

But since $F_{\mu\nu}$ is antisymmetric

$$0 = \partial^\mu \partial^\nu F_{\mu\nu} = -\mu^2 \partial^\nu A_\nu ,$$

Thus, $\quad \partial^\nu A_\nu(x) = 0 \quad (\text{if } \mu^2 \neq 0)$

is required for consistency.

(and tells us that plane wave solutions must obey $e \cdot K = 0$.) If we plug this condition back into the equation of motion we have --

$$(\partial^\mu \partial_\mu + \mu^2) A^\nu = 0 , \qquad \partial_\mu A^\mu = 0$$

Each component of A^ν obeys the Klein-Gordon equation for a particle of mass μ. But the field components are not independent.

Canonical Formulation

We can understand better the nature of the constraints, and identify easily the physical degrees of freedom, in the canonical formulation of the theory. In the canonical (Hamiltonian) formulation, we have a set of p's and q's, and if the initial values of all independent p's and q's are specified, the subsequent trajectory is completely determined. If the system has n independent degrees of freedom (in the Lagrangian sense -- n q's) then 2n initial data are needed and the dimension of phase space (p's and q's) is 2n

If we wish to canonically quantize, we must, of course, impose canonical commutation relations on a complete set of independent p's and q's.

Consider again the Lagrangian

$$\mathcal{L} = \pm\left[\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\mu^2 A_\mu A^\mu\right]$$

$$= \pm\left[\tfrac{1}{2}F_{0i}F^{0i} + \tfrac{1}{4}F_{ij}F^{ij} - \tfrac{1}{2}\mu^2 A_i A^i - \tfrac{1}{2}\mu^2 A_0 A^0\right]$$

(where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$).

The conjugate momenta to A_i and A_0 are

$$\frac{\partial \mathcal{L}}{\partial \partial_0 A_i} = \pm F^{0i}, \quad \frac{\partial \mathcal{L}}{\partial \partial_0 A_0} = 0$$

A_0 has no conjugate momentum. What is going on? Since time derivatives of A_0 do not appear in $\mathcal{L}$, A_0 is not really a dynamical variable; rather, it is constrained. When the other dynamical variables are specified at a given time, A_0 is determined. To see this more clearly, construct $\mathcal{H}$ by Legendre transforming with respect to the three variables A_i that have conjugate momenta

$$\mathcal{H} = \pm F^{0i}\partial_0 A_i - \mathcal{L} = \pm F^{0i}F_{0i} \pm F^{0i}\partial_i A_0 - \mathcal{L}$$

$$= \pm\left[\tfrac{1}{2}F_{0i}F^{0i} - \tfrac{1}{4}F_{ij}F^{ij} + \tfrac{\mu^2}{2}A_i A^i + \tfrac{\mu^2}{2}A_0A^0 - \partial_i F^{0i}A_0\right]$$

(integration by parts, which does not affect H)

Now the Lagrangian "equation of motion" for A_0

$$0 = \partial_i \frac{\partial\mathcal{L}}{\partial\partial_i A_0} - \frac{\partial\mathcal{L}}{\partial A_0} = \partial_i F^{i0} + \mu^2 A^0$$

coincides with the Hamiltonian equation

$$\dot{\pi}_0 = 0 = \frac{\partial\mathcal{H}}{\partial A_0} = -\partial_i F^{0i} + \mu^2 A^0$$

This equation is nondynamical, it determines A^0 in terms of the conjugate momenta $F^{0i}(\vec{x})$. Thus, A^0 is a redundant variable, and we may eliminate it, substituting

$$A_0 = \frac{1}{\mu^2}\partial_i F^{0i}$$

Then

$$\boxed{\mathcal{H} = -\tfrac{1}{2}F_{0i}F^{0i} + \tfrac{1}{4}F_{ij}F^{ij} + -\tfrac{\mu^2}{2}A_iA^i + \tfrac{\mu^2}{2}A_0A^0 \quad \text{where} \quad A_0 = \frac{1}{\mu^2}\partial_i F^{0i}}$$

(We've chosen the minus sign in front of H, since with this choice, all terms in H are now positive definite)

The conjugate momentum is

$$-F^{0i} = F^{i0} \quad \text{-- conjugate to } A_i$$

And the Hamiltonian equations of motion are

$$\dot{A}_i = \frac{\partial H}{\partial F^{i0}} \Rightarrow \quad F_{i0} = -\partial_0 A_i + \partial_i A_0$$

$$A_0 = \frac{1}{\mu^2}\partial_i F^{0i}$$

$$\dot{F}^{i0} = -\frac{\partial H}{\partial A_i} \Rightarrow \quad 0 = \partial_\mu F^{\mu i} + \mu^2 A^i$$

For these first order equations, $A_i(\vec{x})$ and $F^{0i}(\vec{x})$ are a complete set of initial data -- they determine the subsequent evolution.
The absence of a conjugate momentum for A_0 should not be a surprise and is not a disaster. Only three of the field components are dynamical.

Canonical Quantization

Treating $A_i(\vec{x},t)$ as the independent dynamical coordinates, and recognizing F^{i0} as its conjugate momentum, we should quantize by choosing the canonical equal-time commutation relations

$$[A_i(\vec{x},t), F^{j0}(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})\,\delta_i^{\,j}.$$

$$[A, A]_{e.t} = [F, F]_{et} = 0$$

Let us expand A_μ in plane waves

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2\omega_k)^{\frac{1}{2}}} \sum_{r=1}^{3} \left[a_{\vec{k}}^{(r)} e_\mu^{(r)}(k) e^{-ik\cdot x} + a_{\vec{k}}^{(r)\dagger} e_\mu^{(r)} e^{ik\cdot x} \right]$$

The most general real vector field that satisfies $\partial^\mu A_\mu = 0$ has an expansion of this form (the $e^{(r)}$'s satisfy $k\cdot e = 0$). Now we would like to consider the $a_{\vec{k}}$'s and $a_{\vec{k}}^\dagger$'s to be the dynamical variables. Based on past experience, we conjecture that

$$[a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)\dagger}] = \delta^{rs}\delta^3(\vec{k}-\vec{k}')$$

$$[a, a] = 0$$

$$[a^\dagger, a^\dagger] = 0$$

are the commutation relations that result from canonical quantization. To verify this conjecture, we compute the equal-time commutators, to check that they are right (this is sufficient, because commutators of a's and $a^\dagger$'s are determined by equal-time commutators of the fields.)

So consider

$$[A_\mu(x), A_\nu(y)] = [A_\mu^{(-)}(x), A_\nu^{(+)}(y)] + [A_\mu^{(+)}(x), A_\nu^{(-)}(y)]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=1}^{3} \Big[e_\mu^{(r)}(k) e_\nu^{(r)}(k) e^{-ik\cdot(x-y)} - e_\nu^{(r)}(k) e_\mu^{(r)}(k) e^{-ik(y-x)} \Big]$$

But $$\sum_{r=1}^{3} e_\mu^{(r)}(k)\, e_\nu^{(r)}(k) = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2},$$

as we've seen, so

$$[A_\mu(x), A_\nu(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}\right)\left(e^{-ik\cdot(x-y)} - e^{-ik\cdot(y-x)}\right)$$

$$= \left(-\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\mu^2}\right) i\Delta(x-y)$$

where $$i\Delta(x-y) = [\phi(x), \phi(y)]$$

Since we already know that Δ vanishes at equal times for all values of $(\vec{x}-\vec{y})$, we have

$$[A_i(\vec{x},t), A_j(\vec{y},t)] = 0$$

Also $$[F_{\mu\nu}(x), A_\lambda(y)] = \partial_\mu^x [A_\nu(x), A_\lambda(y)] - \partial_\nu^x [A_\mu(x), A_\lambda(y)]$$

$$= \left[-\eta_{\nu\lambda}\partial_\mu - \frac{\partial_\mu\partial_\nu\partial_\lambda}{\mu^2} + \eta_{\mu\lambda}\partial_\nu + \frac{\partial_\mu\partial_\nu\partial_\lambda}{\mu^2}\right] i\Delta(x-y)$$

$$= (-\eta_{\nu\lambda}\partial_\mu + \eta_{\mu\lambda}\partial_\nu)\, i\Delta(x-y)$$

So -- $$[F_{i0}(x), A_j(y)] = \eta_{ij}\, \partial_0\, i\Delta(x-y)$$

or $$[F_{i0}(\vec{x},t), A_j(\vec{y},t)] = -i\eta_{ij}\, \delta^3(\vec{x}-\vec{y})$$

which also checks. (Since $\partial_0 i\Delta(x-y) = [\dot{\phi}(x), \phi(y)]$)

Finally, consider:

$$[F_{\mu\nu}(x), F_{\lambda\sigma}(y)] = \partial^y_\lambda [F_{\mu\nu}(x), A_\sigma(y)] - \partial^y_\sigma [F_{\mu\nu}(x), A_\lambda(y)]$$

$$= \left[-\partial_\lambda(\eta_{\nu\sigma}\partial_\mu - \eta_{\mu\sigma}\partial_\nu) + \partial_\sigma(\eta_{\nu\lambda}\partial_\mu - \eta_{\mu\lambda}\partial_\nu)\right] (i\Delta(x-y))$$

$$= -(\eta_{\mu\lambda}\partial_\nu\partial_\sigma + \eta_{\nu\sigma}\partial_\mu\partial_\lambda - \eta_{\nu\lambda}\partial_\mu\partial_\sigma - \eta_{\mu\sigma}\partial_\nu\partial_\lambda)\, i\Delta(x-y)$$

So

$$[F_{i0}(x), F_{j0}(y)] = -\eta_{ij}\partial_0^2\, i\Delta(x-y) - \partial_i\partial_j\, i\Delta(x-y)$$

But $-\partial_0^2 i\Delta(x-y) = [\dot\phi(x), \dot\phi(y)]$, and we have

$$[F_{i0}(\vec x, t), F_{j0}(\vec y, t)] = 0$$

So we have succeeded in checking the conjectured commutations.

We can also calculate a propagator:

$$\langle 0| T\, A_\mu(x) A_\nu(y) |0\rangle$$

$$= \theta(x^0-y^0)[A^{(-)}_\mu(x), A^{(+)}_\nu(y)] + \theta(y^0-x^0)[A^{(-)}_\nu(y), A^{(+)}_\mu(x)]$$

$$= \theta(x^0-y^0)\left(-\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\mu^2}\right)\Delta_+(x-y)$$
$$+ \theta(y^0-x^0)\left(-\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\mu^2}\right)\Delta_+(y-x)$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}\right)}{k^2-\mu^2+i\epsilon}$$

We know that the poles of this propagator are associated with one particle intermediate states that couple to the field A_μ
This pole occurs at $K^2=\mu^2$, as expected, and has residue

$$i\left(-\eta_{\mu\nu}+\frac{K_\mu K_\nu}{\mu^2}\right)$$

this residue is a 4×4 matrix, but it has a vanishing eigenvalue (for $K^2=\mu^2$); namely, it annihilates K^ν.
This reflects the fact that the vector particle has only three physical polarization states -- there is no "longitudinal" state with $e \propto K$.
E.g., for $K=(\mu,\vec{0})$, this matrix is

$$\begin{pmatrix} 0&0&0&0\\ 0&i&0&0\\ 0&0&i&0\\ 0&0&0&i \end{pmatrix}$$

$\longleftarrow$ three particles, with standard normalization of the residue

A related point: We can plug the mode expansion of $A_\mu(x)$ into the expression for $\mathcal{H}=-\frac{1}{2}F_{0i}F^{0i}+\frac{1}{4}F_{ij}F^{ij}-\frac{1}{2}\mu^2A_iA^i+\frac{1}{2}\mu^2A_0A^0$

to express H in terms of creation and annihilation operators. Find

$$H=\int d^3K\sum_{r=1}^{3}\omega_K\left(a_{\vec K}^{\dagger(r)}a_{\vec K}^{(r)}\right)+\text{consta}$$

there is one oscillator for each of the three polarn states of the physical vector meson.

The $\mu \to 0$ limit (Photons)

Something unpleasant happens to, e.g., our expression for the propagator when we allow μ to approach zero -- it blows up. Apparently something goes wrong with canonical quantization in this limit. But this does not surprise us. A photon ($\mu = 0$) has just two physical helicity states, not three. So we ought to expect that the number of independent dynamical variables is different for $\mu = 0$ than for $\mu \neq 0$. In that case, we cannot expect to be able to quantize the massless theory by first quantizing the massive theory and then taking the $\mu \to 0$ limit at the end. We need to think through canonical quantization again from scratch.

Before reformulating the quantization of the theory, though, we might look further at the physics of the limit $\mu \to 0$. Surely a theory with $\mu = \epsilon > 0$ ought to closely resemble a theory with $\mu = 0$ if ϵ is small enough. But how is this possible if there are three helicity states for $\mu \neq 0$ and two for $\mu = 0$? It must be that the helicity-0 state "decouples" from the physics as μ gets small. (Otherwise, it would be easy to tell that a photon has a mass, no matter how small. There is a big difference in the specific heat of a black body if three photon helicity states are in equilibrium, instead of two.)

Suppose we couple the field A_μ to a source

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\mu^2 A_\mu A^\mu - J_\mu A^\mu$$

then the equation of motion becomes

$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu = J_\nu$$

and therefore

$$\mu^2 \partial^\nu A_\nu = \partial^\nu J_\nu .$$

In the limit $\mu^2 \to 0$, we must have $\partial^\nu J_\nu = 0$ -- A_μ is coupled to a conserved current.

Let us suppose that $\partial^\nu J_\nu = 0$ even for $\mu^2 \neq 0$. Then we have the constraint $\partial^\nu A_\nu = 0$, as we assumed before, even when the source is turned on.

We have the Feynman rule (in momentum space)

$$k, r \sim\sim\sim \otimes \quad = \quad -i\, e_\mu^{(r)} \tilde{J}^\mu(k)$$

for the emission of a particle with momentum k and polarization (r) induced by the source. Suppose

$$k = \left(\sqrt{k^2+\mu^2},\, 0,\, 0,\, k\right) \quad \text{-- Momentum along } +\hat{z}\text{-axis}$$

The three polarization vectors obeying $e \cdot K = 0$ are

$$e^{(1)} = (0,1,0,0)$$
$$e^{(2)} = (0,0,1,0)$$
helicity ± 1

$$e^{(3)} = \frac{1}{\mu} K = \frac{1}{\mu}(k, 0, 0, \sqrt{k^2+\mu^2})$$

The amplitude for the source to emit a helicity zero photon is

$$iA^{(3)} = -i e_\mu^{(3)} \tilde{J}^\mu(k)$$
$$= \frac{-i}{\mu}\left(k \tilde{J}^0 - \sqrt{k^2+\mu^2}\, \tilde{J}^3\right)$$

But $\partial_\mu J^\mu = 0$, or $K_\mu \tilde{J}^\mu = 0$

$$\Rightarrow \tilde{J}^3 = \frac{1}{k}\sqrt{k^2+\mu^2}\, \tilde{J}^0$$

Thus $$iA^{(3)} = \frac{-i}{\mu}\left(k - \left(k + \frac{\mu^2}{k}\right)\right)\tilde{J}^0$$
$$= i\frac{\mu}{k}\tilde{J}^0$$

We see that the amplitude for emitting the helicity zero state is suppressed by (μ/k), and hence is small if the wavelength is short compared to the Compton wavelength. This explains why helicity-0 photons would not be in equilibrium with a black body even for $\mu_\gamma \neq 0$, unless we waited a very long time.

Electrodynamics $(\mu = 0)$

Let us consider further now the theory of massless photons. For photons coupled to a source, the Lagrange density is

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu,$$

and, as we have seen, the field equations are

$$\partial_\mu F^{\mu\nu} = J^\nu$$

The other Maxwell equation, $\partial_\mu {}^*F^{\mu\nu} = 0$, is a trivial identity

Since $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is antisymmetric in $\mu \leftrightarrow \nu$, we see that consistency of the field equation requires

$$\partial_\nu \partial_\mu F^{\mu\nu} = 0 = \partial_\nu J^\nu$$

the current J^ν must be conserved.

Now, if this theory, when quantized, is to describe massless photons that have 2 helicity states, it ought to have fewer dynamical degrees of freedom than the massive vector theory we considered before, since massive vectors have three helicity states. How does this arise?

Notice that the massless theory has a local symmetry not shared by the theory with $\mu \neq 0$. Consider the transformation...

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x)$$

where $\omega(x)$ is any function of spacetime (assumed to approach zero at spatial and temporal infinity). Under this transformation, $F_{\mu\nu}$ is unchanged

$$F_{\mu\nu}(x) \to F_{\mu\nu}(x)$$

and the coupling to the source changes by

$$J_\mu A^\mu \to J_\mu A^\mu + J_\mu \partial^\mu \omega$$
$$= J_\mu A^\mu + \partial^\mu (J_\mu \omega) - \omega \partial^\mu J_\mu$$
$$= J_\mu A^\mu + \partial^\mu (J_\mu \omega);$$

the change is a total derivative, that does not affect the action. So the action $S = \int d^4x \, \mathcal{L}$ is invariant under the local transformation. This property is called gauge invariance. The local transformation is called a gauge transformation.

Because of gauge invariance, this field theory does not have a well-defined initial value problem. If $A_\mu(x)$ solves the classical field equations (leaves action stationary) then so does $A_\mu(x) + \partial_\mu \omega(x)$. (The field equation can be expressed in terms of $F_{\mu\nu}(x)$, which is gauge invariant) But we can choose $\partial_\mu \omega(x)$ to vanish on an initial time surface, so there are many histories $A_\mu(x)$ with the same initial data.

This ambiguity in the initial value problem means that the number of dynamical degrees of freedom is smaller than naive counting would suggest. We can think of this reduction of the number of degrees of freedom from either of two points of view:

- "Geometrical" Viewpoint:

  The classical dynamics is well-defined on gauge equivalence classes. Two trajectories should be regarded as equivalent if they differ by a gauge transformation. Then we have a well-defined initial value problem -- A gauge equivalence class of initial data uniquely specifies a gauge equivalence class of trajectories.

- "Gauge-Fixed" Viewpoint:

  Instead of working directly with equivalence classes, we can pick out a representative of each equivalence class. Given any trajectory $A_\mu(x)$, we have the freedom to perform a gauge transformation to impose a local condition on $A_\mu(x)$. For example, suppose we choose the condition

$$A_3(x) = 0.$$

Then given $A_\mu(x)$, we must find $\omega(x)$ so that

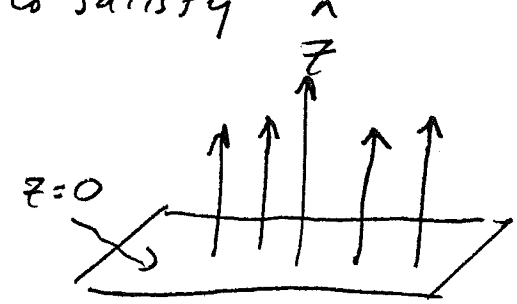
$$A_3'(x) = A_3(x) + \partial_3 \omega(x) = 0$$

So we must choose $\omega(x)$ to satisfy

$$\partial_3 \omega = -A_3(x),$$

or

$$\omega(x) = -\int dx^3 A_3(x)$$



(that is for $x=(x^0, x^1, x^2, x^3)$, integrate along the straight line path from $(x^0, x^1, x^2, 0)$ to x.)

So we see that it is possible to find a gauge transformation that transforms $A_\mu(x)$ to a history that satisfies the condition $A_3=0$. Furthermore -- the history gauge equivalent to our original history that satisfies this condition is <u>unique</u>. If ω is a gauge transformation that preserves the condition $A_3=0$, then

$$\partial_3 \omega(x) = 0.$$

So ω is independent of x^3. But if ω vanishes at spatial infinity, and in particular at $x^3 \to \pm\infty$, we have $\omega=0$.

The condition $A_3(x)=0$ is called the "axial gauge" condition, and the procedure described above for associating each trajectory with a uniquely defined gauge equivalent trajectory that satisfies $A_3=0$ is called "fixing axial gauge."

Once we have fixed the gauge (by imposing $A_3(x)=0$ or some other appropriate local condition) the initial value problem is unambiguous, because there is a *unique* representative of each gauge equivalence class that satisfies the gauge condition.

The geometrical viewpoint is more "natural", but the gauge-fixed viewpoint is easier to implement when we canonically quantize the theory. From either viewpoint, we see that gauge invariance reduces the number of dynamical degrees of freedom. For example, consider the axial gauge. Of the fields A_0, A_1, A_2, A_3, we have already seen that A_0 is *nondynamical* -- a constrained variable. And A_3 can be set to zero by gauge fixing. So we are left with A_1, A_2 as the genuine dynamical variables, the right number for describing the two helicity states of the photon.

One should appreciate that the "local symmetry" is not a symmetry in the usual sense. When the action is invariant under a usual (global) symmetry, this means that there are different points in the configuration space of the system that are physically *equivalent*. But when the action is invariant under a gauge (local) symmetry, this means that there are *apparently* different points in the configuration space that are physically *identical*. Suppose we compute the Noether current associated with

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x)$$

in the source-free theory. It is

$$K^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \partial_\nu \omega = -F^{\mu\nu}(x)\partial_\nu \omega(x),$$

and the conserved "charge" is

$$Q = \int d^3x\, F^{\nu 0}(x)\partial_\nu \omega(x) = \int d^3x\, F^{i0}(x)\partial_i \omega(x)$$

$$(\text{since } F^{00} = 0)$$

But if we integrate by parts:

$$Q = -\int d^3x\, \omega(x)\partial_i F^{i0}(x) = 0$$

(by the equation of motion). So the Noether charge vanishes. The charge can be regarded as a generator of the gauge transformation. It vanishes because a gauge transformation of a state does not change the state; it just gives us a different description of the same state of the dynamical system. In contrast, if the system has a global symmetry (e.g., translation invariance), the symmetry transformation produces a different state (translated) that is physically equivalent but not identical to the original state.

(For A_μ coupled to a source, we still have $Q = \int d^3x\, \omega(x)\left[-\partial_i F^{i0} + J^0\right] = 0$)

Canonical Quantization in Axial Gauge

The axial gauge $A_3 = 0$ is not a convenient gauge in which to do calculations, because it is noncovariant. But it is a good gauge in another respect; it is especially easy to carry out canonical quantization in axial gauge (at least formally). We will carry out this procedure now, to the point of deriving a formal expression for the Feynman rules of electrodynamics in axial gauge. We will subsequently see how to transform these Feynman rules to other gauges that are more convenient for calculational purposes.

Consider

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

In order to make the initial value problem well-defined, so that we have a theory suited for canonical quantization, we impose the axial gauge condition

$$A_3(x) = 0$$

Then

$$\mathcal{L} = -\frac{1}{4}\left(F_{01} F^{01} + F_{02} F^{02} \right) \quad \text{] Depend on time derivatives}$$

$$-\frac{1}{4}\left(F_{03} F^{03} + F_{ij} F^{ij} \right). \quad \text{] No time derivatives}$$

Now $F_{\mu 3} = -\partial_3 A_\mu$. No time derivatives of A_0 appear in $\mathcal{L}$ -- it is a constrained variable.

The dynamical variables are A_1, A_2, and their conjugate momenta are

$$\frac{\partial \mathcal{L}}{\partial \partial_0 A_1} = -F^{01} \qquad \frac{\partial \mathcal{L}}{\partial \partial_0 A_2} = -F^{02}$$

Thus, the Hamiltonian density is --

$$\mathcal{H} = -\partial_0 A_1 F^{01} - \partial_0 A_2 F^{02} - \mathcal{L}$$

where it is understood that the constrained variable A_0 has been eliminated by solving the equation

$$\partial_i F^{i0} = \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03}$$

$$\text{or} \quad -\partial_3^2 A^0 = \partial_1 F^{01} + \partial_2 F^{02}.$$

$\mathcal{H}$ is expressed in terms of the canonical variables

$$A_1, A_2, -F^{01}, -F^{02}$$

Now, we learned in chapter 4 how to express the Green functions of a canonically quantized theory as functional integrals --

$$\langle 0 | T \underbrace{A(x_1) \cdots A(x_n)}_{\text{must be } A_1 \text{ or } A_2} | 0 \rangle = \underbrace{N}_{\text{(normalization constant)}} \int dA_1 dA_2 dF_{01} dF_{02}\, e^{iS_H} A(x_1) \cdots A(x_n)$$

where S_H is the action in Hamiltonian form

$$S_H = \int d^4x \left(\dot{A}_1 F^{10} + \dot{A}_2 F^{20} - \mathcal{H} \right)$$

Or, the generating functional for the Green functions is

$$Z[J] = N \int dA_1 dA_2 dF_{01} dF_{02}\, e^{i(S_H + \int JA)}$$

-- the source term is

$$\int d^4x \left(J_1(x) A^1(x) + J_2(x) A^2(x) \right)$$

Now, we would like to "integrate out" the variables F_{01} and F_{02} in order to obtain the Feynman rules for electrodynamics. This seems to give a complicated expression (and complicated Feynman rules) but we can simplify things considerably by a trick.

The basic trick is to notice that integrating over a dynamical variable is equivalent to solving the equation of motion for the variable, and substituting the solution back into the action, under the following conditions:

- the variable appears quadratically in the action (and not higher order).
- the coefficient of the quadratic term is a constant (independent of the other dynamical variables).

For example, suppose that q is a dynamical variable and y is a constrained variable (no time derivatives of y appear in the action). The action is

$$S = \int dt \; L_0(q, \dot{q}) + \tfrac{1}{2} a y^2 - b(q) y$$

Then the integral

$$\int dy \; e^{iS}$$ is gaussian, and

$$\int dy \; e^{i\left[\frac{1}{2} a (y - b/a)^2 - b^2/2a\right]}$$

$$= \det(-ia) \; e^{i b^2/2a}$$

If a is independent of q, then $\det(-ia)$ is just a constant that can be absorbed into the normalization of the functional integral.

$$\int (dq)(dy) e^{iS} = N \int (dq) \exp i \int dt \left[L_0 - b(q)^2 / 2a \right]$$

Integrating over y is equivalent (up to the overall normalization) to solving

$$a y - b(q) = 0$$

(the y eqn of motion), and substituting:

$$\tfrac{1}{2} a (b/a)^2 - b(b/a) = - b^2/2a$$

Using this rule we can write a simple expression for $Z[J]$. The idea is to reintroduce A_0 (which we had eliminated before canonically quantizing) before we integrate out F_{01} and F_{02}. The Hamiltonian density is

$$\mathcal{H} = -\tfrac{1}{2} F_{01}F^{01} - \tfrac{1}{2} F_{02}F^{02} - \tfrac{1}{2}\partial_3 A_0 \partial^3 A^0$$
$$+ A_0(\partial_1 F^{01} + \partial_2 F^{02}) + \tfrac{1}{4} F_{ij}F^{ij}$$

but where A_0 has been eliminated by solving

$$0 = \partial_3\partial^3 A_0 + \partial_1 F^{01} + \partial_2 F^{02}.$$

Since eliminating A_0 is equivalent to integrating over it, we may write

$$Z[J] = N'\int dA_0 dA_1 dA_2 dF_{01} dF_{02}\, e^{i(S_H + \int JA)},$$

where now A_0 has not yet been eliminated in the integrand. We have

$$S_H = \int d^4x \Big[-\partial_0 A_1 F^{01} - \partial_0 A_2 F^{02}$$
$$+ \tfrac{1}{2}(F_{01}F^{01} + F_{02}F^{02}) + \tfrac{1}{2} F_{03}F^{03}$$
$$+ \partial_1 A_0 F^{01} + \partial_2 A_0 F^{02} - \tfrac{1}{4} F_{ij}F^{ij} \Big]$$

(where $F_{03} = -\partial_3 A_0$ and $F_{ij} = \partial_i A_j - \partial_j A_i$). Now F^{01} and F^{02} appear quadratically, and the coefficients of their quadratic terms are constants. So they are readily integrated out, and we obtain

$$\boxed{Z[J] = N'' \int dA_0\, dA_1\, dA_2\, e^{i(S_L + \int JA)}}$$

where S_L is the Lagrangian form of the action

$$S_L = \int d^4x \; -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(with $A_3 = 0$). In spite of constraints, gauge fixing, etc., we have in the end a remarkably simple expression for $Z[J]$

The Faddeev-Popov "Ansatz"

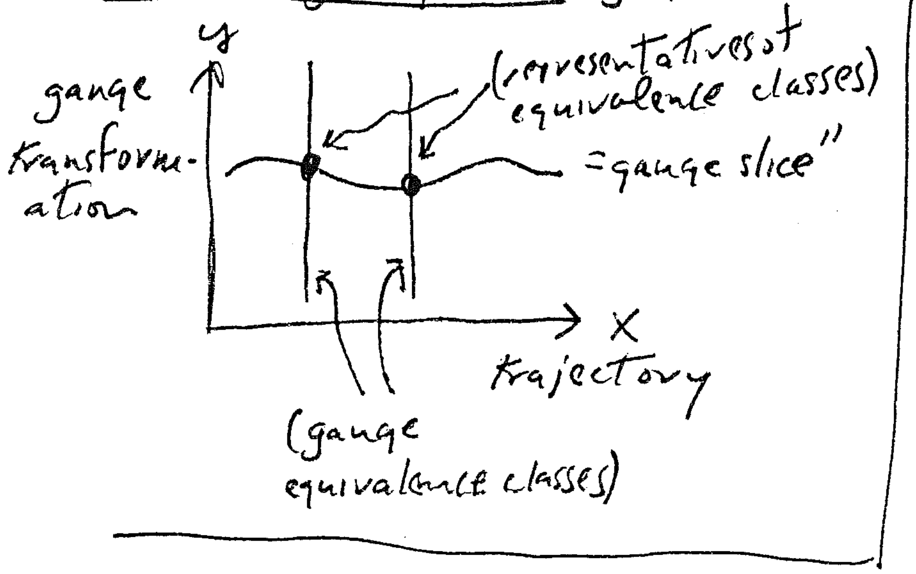
Having found a path integral expression for the canonically quantized theory in axial gauge, we now consider how to "transform" this expression to other gauges (which are more convenient for calculation). First, we rewrite our expression for $Z[J]$:

$$Z[J] = N \int dA_0\, dA_1\, dA_2\, dA_3\, e^{i(S + \int JA)}\, \delta[A_3(x)]$$

where $\delta[A_3]$ is a "δ-functional" with support on $A_3 = 0$, and normalized so that

$$\int dA_3\, \delta[A_3] = 1$$

This formula has a nice interpretation, which suggests how it can be generalized. We integrate over all four components of $A_\mu(x)$, but subject to the constraint, imposed by the δ functional, that fixes the axial gauge $A_3 = 0$.



The δ-functional restricts the integration over histories $A_\mu(x)$ to a "surface" that intersects each gauge equivalence class exactly once (in this case, picking out the representative of each gauge equivalence class that satisfies $A_3 = 0$).
This suggests how to do the path integral in other gauges. We simply pick out a different surface ("gauge slice") in the space of trajectories $A_\mu(x)$. But we must define the integral (by including the appropriate Jacobian factor) so that it is independent of the choice of the gauge slice.

That is, if we want to impose the gauge condition

$$G(A) = 0,$$

we write (Faddeev-Popov Ansatz)

$$Z[J] = N \int (dA_\mu)\, e^{i(S + \int JA)}\, \delta[G(A)] \det\left(\frac{\delta G}{\delta \omega}\right)\Big|_{G=0}$$

The δ-functional restricts the integration to A_μ satisfying the gauge condition.

The factor $\det\left(\frac{\delta G}{\delta \omega}\right)$ (where ω is the gauge parameter in the transformation law

$$A_\mu \to A_\mu + \partial_\mu \omega)$$

is inserted to ensure that the integral is independent of the choice of gauge slice.

E.g., suppose variables x_a parametrize the trajectories that we want to integrate over, and y_b is parametrize the elements of a gauge equivalence class. We want to evaluate integrals

$$I = \int dx_a \; F(\vec{x}) = \int dx_a dy_b \; F(\vec{x}) \prod_b \delta(y_b)$$

But, because

$$\int dy_b \; \prod_b \delta(G_b(\vec{y})) \det\left(\frac{\partial G_b}{\partial y_c}\right)\Bigg|_{G=0} = 1,$$

we may also express the integral as

$$I = \int dx_a \, dy_b \; F(\vec{x}) \prod_b \delta(G_b) \det\left(\frac{\partial G_b}{\partial y_c}\right)\Bigg|_{G=0}$$

-- Here the integral is evaluated by picking out a different "gauge slice", but with the same result. Our expression for $Z[J]$ is of exactly this form, and so is independent of the gauge condition $G(A)=0$ chosen.

Notice that, to reach this conclusion, we must demand that

$$S[A] + \int d^4x\, J^\mu A_\mu$$

is gauge-invariant. Thus, the source J^μ must be conserved,

$$\partial_\mu J^\mu = 0.$$

However, conservation of the source, though it reduces the number of independent components of the source J^μ, does not prevent J^μ from having enough independent components to generate all Green functions. For example, in axial gauge

$$-\partial_3 J^3 = \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2.$$

Conservation $\Rightarrow$ J^3 is determined by the other components, but J^3 doesn't couple to anything in axial gauge.

Note that $Z[J]$ = (gauge-independent) for $\partial_\mu J^\mu = 0$ does not mean that Green functions such as

$$\Delta_{\mu\nu}(x-y) = \langle T^* A_\mu(x) A_\nu(y) \rangle$$

are gauge independent. Rather, we have

$$\int d^4k\, \tilde{J}^\mu(k)\, \tilde{\Delta}_{\mu\nu}(k)\, \tilde{J}^\nu(k)$$

where $k_\mu \tilde{J}^\mu(k) = 0$. So $\Delta_{\mu\nu}$ can have terms proportional to $k_\mu k_\nu$ that are gauge-dependent. When we consider the interacting theory, then, our Feynman rules will be different in different gauges. But physics is independent of gauge, and all calculations of measurable quantities will give a result independent of our gauge choice.

The Faddeev-Popov "ansatz"--

$$Z[J] = N\int (dA_\mu)\, e^{i(S+\int JA)}\, \delta[G(A)] \det\left(\frac{\delta G}{\delta \omega}\right)\Big|_{G=0}$$

is an expression that is independent of the choice of gauge condition $G(A)=0$. To show that it is correct, we need only show that it agrees with our expression obtained from canonical quantization in the particular gauge <u>axial</u> gauge. But if

$$G(A) = A_3$$

then, under a gauge transformation

$$G(A) \to A_3 + \partial_3 \omega$$

so $\left(\delta G/\delta \omega\right) = \partial_3$

And $\det \delta G/\delta\omega = \det \partial_3$ is just an (infinite) normalization factor than can be absorbed into the normalization constant N (it has no effect on the Green functions). So the two expressions for $Z[J]$ agree, up to normalization. This <u>justifies</u> the ansatz.

Covariant Gauges

Covariant gauge choices are the most convenient for perturbation theory. Imagine we choose

$$G(A) = \partial_\mu A^\mu(x) - f(x),$$

where f is some specified function. This choice is independent of frame, and because it is linear in A,

$$\det \frac{\delta G}{\delta \omega} = \det(\partial_\mu \partial^\mu)$$

is a mere constant that can be absorbed into normalization of path integral. In this gauge:

$$Z[J] = N\int(dA_\mu)\, e^{i(S+\int JA)}\, \delta[\partial_\mu A^\mu - f]$$

We want to rewrite $Z[J]$ in a way that makes it possible to derive simple Feynman rules. Since $Z[J]$ is independent of $f(x)$, we may insert the constant

$$\int df\, H[f]$$

in the integral, where H is a functional we are free to choose. Then

$$Z[J] = N\int(dA_\mu)\, e^{i(S+\int JA)}\, H[\partial_\mu A^\mu]$$

For the purpose of obtaining simple Feynman rules, we choose --

$$H[\partial_\mu A^\mu] = \exp\left[\frac{-i}{2\alpha}\int d^4x\,(\partial_\mu A^\mu)^2\right]$$

(The parameter α is arbitrary) In Euclidean space, and for the proper choice of the sign of α, we can think of this term as a "convergence factor" that damps the integral over $A_\mu(x)$ in the same gauge equivalence class. (Then we define path integral for other sign of α by analytic continuation.)

The effect of gauge-fixing then, is to replace the action by

$$S_{eff} = S - \frac{1}{2\alpha}\int d^4x\,(\partial_\mu A^\mu)^2$$

or

$$e^{iS_{eff}} = \exp\left[-i\int d^4x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right)\right]$$

$$= \exp\left[\frac{i}{2}\int d^4x\, A_\mu(x)\left[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + \frac{1}{\alpha}\partial^\mu\partial^\nu\right]A_\nu(x)\right]$$

(after an integration by parts.) We can now find the propagator by inverting

$$(\Delta^{-1}(k))^{\mu\nu} = i\left(\eta^{\mu\nu}k^2 - k^\mu k^\nu + \frac{1}{\alpha}k^\mu k^\nu\right)$$

To perform the inversion, note that

$$T^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

and

$$L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}$$

are the "transverse" and "longitudinal" projection operators respectively, satisfying

$$T^{\mu\nu} T_\nu{}^\lambda = T^{\mu\lambda}$$
$$L^{\mu\nu} L_\nu{}^\lambda = L^{\mu\lambda} \qquad T^{\mu\nu} + L^{\mu\nu} = \eta^{\mu\nu}$$
$$T^{\mu\nu} L_\nu{}^\lambda = 0$$

So $$(\Delta^{-1})^{\mu\nu} = i k^2 \left(T^{\mu\nu} + \tfrac{1}{\alpha} L^{\mu\nu} \right)$$

$$\Rightarrow \quad \Delta_{\mu\nu} = \frac{-i}{k^2} \left(T_{\mu\nu} + \alpha L_{\mu\nu} \right)$$

or $$\boxed{\Delta_{\mu\nu} = \frac{-i}{k^2 + i\epsilon} \left[\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \alpha \frac{k_\mu k_\nu}{k^2} \right]}$$

Notice that this blows up as $\alpha \to \infty$ (when the gauge fixing term "turns off") the infinity arises from the integration over all the "gauge copies" of each trajectory.

Two standard choices for α are

$\alpha = 1$ $\qquad \Delta_{\mu\nu} = \dfrac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}$ $\qquad$ "Feynman Gauge" (usually the simplest for calculations)

$\alpha = 0$ $\qquad \Delta_{\mu\nu} = \dfrac{-i}{k^2 + i\epsilon} \left(\eta_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2} \right)$ $\qquad$ "Landau Gauge" (only transverse states propagate)

Remember that the $k_\mu k_\nu$ terms in $\Delta_{\mu\nu}$ do not affect $Z[J]$, for $\partial_\mu J^\mu = 0$, so it is clear that Z is really independent of the gauge parameter α.

Note that in Feynman gauge, the residue of the pole in Δ_{00} at $k^2=0$ has the <u>wrong</u> sign, the opposite of that generated by a physical one particle state, according to the spectral representation — page 2.148! This is the problem of <u>ghosts</u>. There appear to be states in the Hilbert space of the theory for which the inner product $\langle k|k\rangle$ is <u>negative</u>.

In the loop diagrams of <u>interacting</u> theories, the ghosts play the important role of cancelling unwanted contributions from the third helicity state of the photon. But, because of the ghosts, the <u>unitarity</u> of the theory in a covariant gauge is far from obvious (Amplitudes for producing ghost states in scattering processes are required to vanish.)

So we have two ways (at least) of quantizing electrodynamics. In axial gauge, unitarity is assured, because we obtained the quantum theory by canonical quantization. But Lorentz covariance is not at all obvious. In Feynman gauge, covariance is manifest, but unitarity is not. However, we have seen that the two quantum theories are equivalent -- they give the same $Z[J]$ and therefore the same S-matrix. Hence, the theory, in either gauge, must be both covariant and unitarity.

Gauge-Invariant Interactions

Next we want to consider theories in which photons are coupled to matter. But in view of the above remarks, if we want such a theory to make sense, we better choose the interactions to be gauge-invariant. Otherwise, we have no way to assure that the quantum theory will be both covariant and unitary.

Suppose we have a theory

$$\mathcal{L}(\phi, \partial_\mu \phi)$$

that respects the global $U(1)$ symmetry

$$\phi \to e^{i\lambda} \phi$$

then there is a prescription (the "minimal coupling" prescription) for coupling this theory to photons in a gauge invariant way.

The prescription is, replace $\mathcal{L}$ by

$$\mathcal{L}' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}(\phi, D_\mu \phi)$$

where we have replaced $\partial_\mu \phi$ by the "covariant derivative"

$$D_\mu = \partial_\mu + ie A_\mu .$$

This Lagrangian $\mathcal{L}'$ is then invariant under the <u>local</u> transformation:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu w(x)$$

$$\phi(x) \to e^{-iew(x)} \phi(x)$$

The key observation is that $D_\mu \phi$ then transforms as

$$D_\mu \phi(x) \to e^{-iew(x)} D_\mu \phi(x)$$

So, if $\mathcal{L}(\phi, \partial_\mu \phi)$ is <u>local</u> and invariant under global transformations $\phi \to e^{i\lambda}\phi$, then $\mathcal{L}(\phi, D_\mu \phi)$ is invariant under local transformations.

<u>Examples</u>

- <u>Dirac</u> <u>Theory</u>

Consider $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

The minimal coupling prescription gives

$$\mathcal{L} \to -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\, i(\partial_\mu + ieA_\mu)\gamma^\mu \Psi - m\bar{\Psi}\Psi$$

$$= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m)\Psi - \underline{eA_\mu \bar{\Psi}\gamma^\mu\Psi}$$

The photon couples to the current $e\bar{\Psi}\gamma^\mu\Psi$. The parameter e enters as a coupling constant.

- Complex Scalars

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

becomes

$$\mathcal{L} \to -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^* (\partial^\mu + ieA^\mu)\phi^* - m^2 \phi^*\phi$$

$$= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^*\phi - ieA_\mu(\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi) + e^2 A_\mu A^\mu \phi^*\phi$$

In this theory, there is a derivative coupling, and a term quadratic in A. It has a quite different structure from the spinor theory.

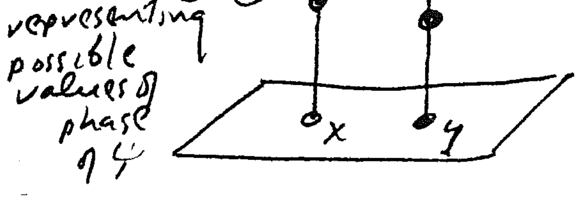
Of course, the minimal coupling prescription provides just one gauge-invariant way to couple photons to other particles; nonminimal couplings are possible. For example,

$$F_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}\psi$$

is gauge-invariant. But it happens that the nonminimal couplings to scalars and Dirac particles are all of dimension greater than 4 (the above is dimension 5), and so do not occur in the most general renormalizable theory.

Geometry of Gauge Fields

The important justification for gauge invariance in physics is that it enables us to construct interacting theories of massless spin 1 particles that are both covariant and unitary. But it is also interesting to see that gauge invariance has a geometrical interpretation.



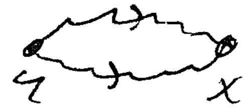
Consider a classical electron field $\psi(x)$. Gauge invariance means that the phase of ψ has no physical meaning; that is,

$\psi(x)$ and $e^{-ie\omega(x)}\psi(x)$ are physically equivalent.

On the other hand, the relative phase of two electrons at the same point in spacetime does have a meaning. Now, suppose I want to compare the phases of two electrons at different points x and y. I propose to do this by transporting the electron at y to x and then comparing the phases. But this procedure suffers from two ambiguities:

1. I need a notion of parallel transport that tells me how to rotate the phase as I move the electron.

2. The outcome of the comparison may depend on the path taken from y to x.



To deal with problem (1), I need to define parallel transport. Given a gauge field $A_\mu(x)$, I will define parallel transport of ψ from x to $x+\epsilon$ (ϵ infinitesimal)

by

$$\psi(x+\epsilon) = \psi(x) - \epsilon_\mu ieA^\mu(x)\,\psi(x)$$

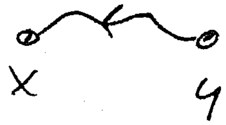
or, in other words

$$0 = \epsilon_\mu(\partial^\mu + ieA^\mu(x))\psi(x) = \epsilon_\mu D^\mu \psi(x)$$

(the covariant derivative along ϵ vanishes)

Thus, a gauge field $A_\mu(x)$ may be regarded as a connection that defines the notion of parallel transport of the phase of the electron.

For a finite path, we compose



$$\psi(x+\epsilon) = e^{-ieA^\mu(x)\epsilon_\mu}\,\psi(x)$$

many times to obtain

$$\psi(x) = \exp\left[-ie\int_y^x dx'_\mu\, A^\mu(x')\right]\psi(y)$$

the result of parallel transporting $\psi(y)$ to the point x

integrated along path.

Now we can compare the phases of electrons at x and y, but this comparison is not independent of the gauge. Since I have the freedom to change the basis with respect to which the phase is measured independently at x and y (i.e., perform a local gauge transformation) the comparison has no meaning that is independent of this local choice of coordinates. (It does not tell us about the "intrinsic geometry" of the phase.)

On the other hand, consider the difference between the phases obtained by parallel transport of an electron along two different paths from y to x, or equivalently, the amount by which the phase of the electron is rotated upon parallel transport around a closed path.

this is $\exp[-ie \oint A^\mu(x) dx_\mu]$

and it is <u>gauge</u> <u>invariant</u>.

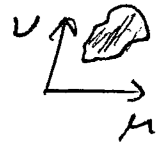
In Riemannian geometry, a rotation of tangent vectors (or of a locally inertial frame in general relativity) upon parallel transport about a closed path is said to be due to <u>curvature</u>. The curvature is an intrinsic geometrical property, independent of how we choose <u>coordinates</u>. Analogously, in electromagnetic theory, parallel transport of an electron around a closed path in spacetime results in a rotation of the phase of the electron. The phase changes by

$$\exp(-ie \oint A^\mu dx_\mu) = \exp[-ie \int F^{\mu\nu} \underbrace{dS_{\mu\nu}}_{\text{element of area}}]$$

according to Stokes' theorem.

Or ---

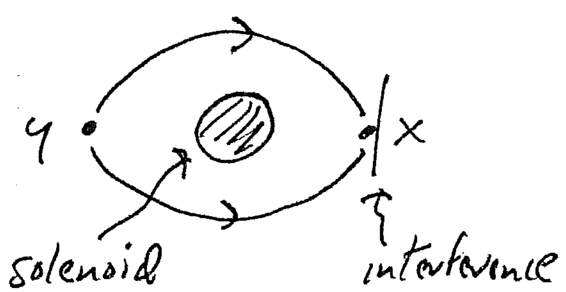
$$\psi(x) \to (1 - ie(da) F^{\mu\nu}) \psi(x)$$



upon parallel transport about an infinitesimal path in the $\mu\nu$ plane that encloses area (da).

We see that $F^{\mu\nu}(x)$ is a "curvature", closely related to the curvature tensor of Riemannian geometry.

The idea that the field A_μ is a connection that describes how the phase of an electron changes under parallel transport is vividly demonstrated by the Aharonov-Bohm effect.



Electrons propagate in a region with $F_{\mu\nu}=0$, passing on either side of a solenoid carrying nonzero magnetic flux Φ. As the flux through the solenoid changes, the relative phase of electrons that have travelled from y to x on either side of the solenoid varies as

$$\exp(-ie\Phi)$$

Thus, the interference pattern of the electron changes as Φ does, even though no electron has encountered a nonzero electromagnetic field $F_{\mu\nu}$. This shows that $F_{\mu\nu}$ alone does not completely specify the dynamics of electromagnetism. The electron is responding to the local value of $A_\mu(x)$, which tells it how to rotate its phase as it moves.

Feynman Rules

Consider spinor electrodynamics, with Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial}-m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi$$

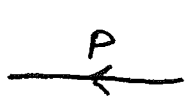
Because the theory is gauge invariant, we can quantize it the way we quantized the free theory -- carry out canonical quantization in axial gauge, and then use the Faddeev - Popov procedure to transform to a covariant gauge. This is the same as evaluating the path integral with an effective Lagrangian:

$$\mathcal{L}_{eff} = \mathcal{L} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

Now we can write down Feynman Rules by inspection.

Feynman Rules for (Spinor) Quantum Electrodynamics
(in Feynman gauge, $\alpha = 1$)

$$\frac{-i\eta_{\mu\nu}}{K^2+i\epsilon}$$



$$\frac{i}{\not{p}-m+i\epsilon}$$

$$-ie\gamma_\mu$$

$\bullet\!\leftarrow p, r$	$u_p^{(r)}$	incoming
$\circ\!\rightarrow p, r$	$\bar{v}_p^{(r)}$	
$p, r \leftarrow\!\circ$	$\bar{u}_p^{(r)}$	outgoing
$p, r \rightarrow\!\bullet$	$v_p^{(r)}$	

$\bullet\!\sim\!\sim\, k, \mu, r$	$e_\mu^{(r)}(k)$
$k, \mu, r \sim\!\sim\!\circ$	$e_\mu^{(r)}(k)^*$

Remark:

Our path integral methods provide us with rules for computing Green functions. To compute S-matrix elements, we must use the reduction formula. The factors for external particles arise from

$$\langle 0 |\, \psi(x)\, | \vec{p}, r \rangle = e^{-ip\cdot x}\, u_{\vec{p}}^{(r)}, \quad \text{etc.}$$

$$\langle 0 |\, A_\mu(x)\, | k, r \rangle = e^{-ik\cdot x}\, e_\mu^{(r)}(k)$$

For photons, there are two possible transverse polarizations. E.g., if

$$k^\mu = \omega(1\,0\,0\,1),$$

a basis for the two polarizations is

$$e^{(1)} = (0\,1\,0\,0) \qquad \text{(Linear}$$
$$e^{(2)} = (0\,0\,1\,0) \qquad \text{polarizations)}$$

Or, we may choose circular polarizations

(the helicity eigenstates)

$$e^{(\pm)} = \frac{1}{\sqrt{2}}\left(e^{(1)} \pm i e^{(2)}\right)$$

When we need to sum (or average) over polarizations we may use

$$\sum_r e_\mu^{(r)} e_\nu^{(r)*} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is an unpleasant and inconvenient expression. But we can see that it is equivalent to something less awkward if we recall that photons couple to a <u>conserved</u> <u>current</u>.

$$\text{Amplitude } A = e_\mu(K)\, \mathcal{M}^\mu(K)$$

$$\text{where} \quad K_\mu \mathcal{M}^\mu(K) = 0$$

-- is just the statement that scattering amplitudes are not changed by a gauge transformation (See page 5.68 for further discussion.)
Thus

$$\sum_r |A|^2 = \sum_r e_\mu^{(r)}(K)\, e_\nu^{(r)*}(K)\, \mathcal{M}^\mu(K)\, \mathcal{M}^\nu(K)^*$$

$$= \mathcal{M}^1 \mathcal{M}^{1*} + \mathcal{M}^2 \mathcal{M}^{2*}$$

$$= -\eta_{\mu\nu} \mathcal{M}^\mu \mathcal{M}^{\nu*},$$

because

$$K_\mu \mathcal{M}^\mu = \omega(\mathcal{M}^0 - \mathcal{M}^3) = 0 \Rightarrow \mathcal{M}^0 = \mathcal{M}^3$$

So -- in this particular frame --

we may write

$$\sum_{\nu} e_\mu^{(\nu)}(k)\, e_\nu^{(\nu)*}(k) = -\eta_{\mu\nu}$$

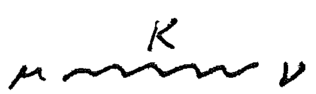
Now we have a covariant expression, which will hold in <u>all</u> frames.

Now consider <u>scalar</u> electrodynamics, with Lagrange density

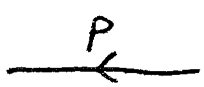
$$\begin{aligned}\mathcal{L}_{eff} = & -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\alpha}(\partial_\mu A^\mu)^2 \\ & + \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \\ & - ie A_\mu (\phi^* \partial_\mu \phi - \partial^\mu \phi^* \phi) \\ & + e^2 A_\mu A^\mu \phi^* \phi\end{aligned}$$

Again, write down Feynman rules by inspection

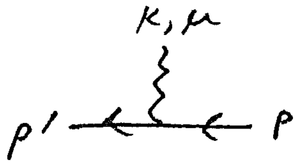
<u>Feynman Rules for (Scalar) Quantum Electrodynamics</u>
(in Feynman gauge, $\alpha = 1$)



$$\frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}$$



$$\frac{i}{p^2 - m^2 + i\epsilon}$$



$(-ie)\,(p+p')_\mu$

$2ie^2\,\eta_{\mu\nu}$

$e_\mu^{(r)}(k)$

$e_\mu^{(r)}(k)^*$

Remark:

In scalar electrodynamics, a renormalizable self coupling of the scalar can also occur (even if we retain the U(1) gauge symmetry). In fact, we'll need a $(\phi^*\phi)^2$ counterterm to remove divergences in diagrams such as

So we might modify the Lagrangian,

$$\mathcal{L} \to \mathcal{L} - \frac{\lambda}{4}(\phi^*\phi)^2$$

and add the Feynman rule

$= -i\lambda$