Elementary Processes in (Spinor) Quantum Elektrodynamics

What are the $2 \to 2$ body scattering processes that occur in order e^2 (tree approximation) in QED?

① **Coulomb Scattering** (Møller) $e^-e^- \to e^-e^-$

$e^+e^+ \to e^+e^+$

Diagrams —

② **Bhabha Scattering** $e^+e^- \to e^+e^-$

(Related to ① by crossing)

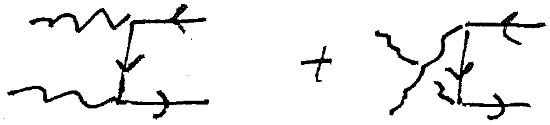
③ **Compton Scattering** $e^-\gamma \to e^-\gamma$

$e^+\gamma \to e^+\gamma$

④ **Pair Annihilation** $e^+e^- \to \gamma\gamma$

$\gamma\gamma \to e^+e^-$

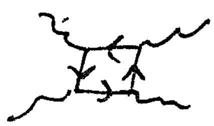
(Related to ③ by crossing)



The cross sections are computed in various standard textbooks, e.g.

Itzykson + Zuber, Quantum Field Theory
Bjorken + Drell, Relativistic Quantum Mech.
Berestetskii, Lifshitz, and Pitaevskii, QED

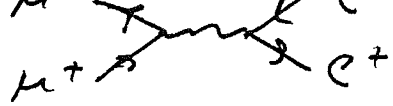
<u>Remark</u>: The other possible $2 \to 2$ scattering process in QED is $\gamma\gamma \to \gamma\gamma$. But this doesn't occur in tree approximation. The leading contribution is in one loop:

 + crossed graphs (Delbrück Scattering)

<u>Example</u>: $e^+e^- \to \mu^+\mu^-$

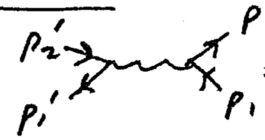
As a sample computation, we'll consider annihilation of an e^+e^- pair into a $\mu^+\mu^-$ pair, ignoring the electron mass (i.e. in the limit $E \gg m_e$). The muon has couplings identical to the electron, but is much heavier.

This is pretty easy, because (in contrast to Bhabha scattering) there is only one diagram:



5.54

$e^+e^- \to \mu^+\mu^-$

$$iA = \;$$ 

$$= (-ie)^2 \, \frac{-i}{s} \, \bar{u}_1' \gamma^\mu v_2' \; \bar{v}_2 \gamma_\mu u_1$$

Next -- square the amplitude, sum over final spins and average over initial spins, using

$$\sum_r u_1^{(r)} \bar{u}_1^{(r)} = \not{p}_1 \qquad \text{(ignore mass of electron)}$$

$$\sum_r v_2^{(r)} \bar{v}_2^{(r)} = \not{p}_2$$

$$\sum_r u_1'^{(r)} \bar{u}_1'^{(r)} = \not{p}_1' + m$$

$$\sum_r v_2'^{(r)} \bar{v}_2'^{(r)} = \not{p}_2' - m$$

Therefore,

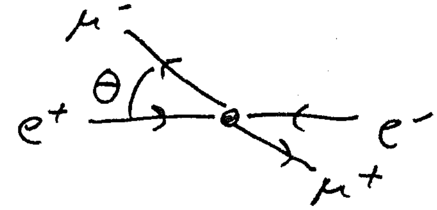
$$\langle |A|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |A|^2 = \frac{(e^2)^2}{4s^2} \, \mathrm{tr}\left[(\not{p}_1' + m)\gamma^\mu (\not{p}_2' - m)\gamma^\nu\right] \times \mathrm{tr}\left[\not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu\right]$$

$$= \frac{4(e^2)^2}{s^2} \left(p_1'^\mu p_2'^\nu + p_1'^\nu p_2'^\mu - \eta^{\mu\nu} p_1' \cdot p_2' - \eta^{\mu\nu} m^2 \right) \times \left(p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - \eta_{\mu\nu} p_1 \cdot p_2 \right)$$

$$= \frac{4(e^2)^2}{s^2} \Big[2(p_1 \cdot p_1')(p_2 \cdot p_2') + 2(p_1 \cdot p_2')(p_2 \cdot p_1') - 2(p_1 \cdot p_2)(p_1' \cdot p_2' + m^2) + (p_1 \cdot p_2)(-2 p_1' \cdot p_2' + 4 p_1' \cdot p_2' + 4m^2) \Big]$$

$$= \frac{4(e^2)^2}{s^2} \Big[2(p_1 \cdot p_1')(p_2 \cdot p_2') + 2(p_1 \cdot p_2')(p_2 \cdot p_1') + 2m^2 (p_1 \cdot p_2) \Big]$$

Now, consider kinematics of zero-momentum frame



$p_1 = (E, \vec{p})$
$p_2 = (E, -\vec{p})$ where $p = |\vec{p}| = E$ $(m_e = 0)$

$p_1' = (E, \vec{p}\,')$
$p_2' = (E, -\vec{p}\,')$ where $p' = |\vec{p}\,'| = \sqrt{E^2 - m^2}$

$$p_1 \cdot p_2 = E^2 + p^2 = 2E^2$$

$$p_1 \cdot p_1' = p_2 \cdot p_2' = E^2 - pp'\cos\theta$$
$$= E^2(1 - v\cos\theta)$$

$(v = p'/E$ is velocity of muon)

$$p_1 \cdot p_2' = p_2 \cdot p_1' = E^2 + pp'\cos\theta = E^2(1 + v\cos\theta)$$

Thus $$\langle |A|^2 \rangle = \frac{8(e^2)^2 E^4}{s^2}\left[(1 - v\cos\theta)^2 + (1 + v\cos\theta)^2 + 2\frac{m^2}{E^2}\right]$$

But $s = (2E)^2 = 4E^2$, $\frac{m^2}{E^2} = \gamma^{-2} = 1 - v^2$

$$\langle |A^2| \rangle = (e^2)^2 (1 + v^2\cos^2\theta + 1 - v^2)$$
$$= (e^2)^2 (2 - v^2 + v^2\cos^2\theta)$$

Now, to find the differential cross section, refer to page 2.56 of the notes:

$$\frac{d\sigma}{d\Omega} = \frac{k_{final}}{k_{initial}} \frac{1}{64\pi^2 (E_{total})^2} |A|^2$$

We have $k_{initial} = E$ $\quad k_{final} = vE$. So

$$\frac{d\sigma}{d\Omega} = \frac{(e^2)^2}{64\pi^2 (2E)^2} v\left(2 - v^2 + v^2\cos^2\theta\right)$$

or

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\substack{spin\\ave}} = \frac{\alpha^2}{16E^2} v\left(2 - v^2 + v^2\cos^2\theta\right)}$$

In the limit $E \gg m$, $v \to 1$, and this becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{spin\\ave}} \longrightarrow \frac{\alpha^2}{16E^2}\left(1 + \cos^2\theta\right)$$

Total cross section:

Since $\int d\Omega \cos^2\theta = 2\pi \int_{-1}^{1} d(\cos\theta) \cos^2\theta = 4\pi/3$,

we have

$$\sigma_{total}(e^+e^- \to \mu^+\mu^-)\Big|_{\substack{spin\\ave}} = \frac{\alpha^2}{16E^2} v \, 4\pi\left(2 - \tfrac{2}{3}v^2\right)$$

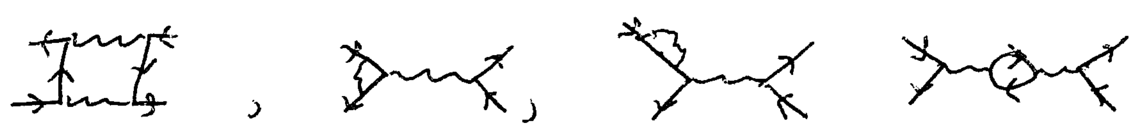
or $$\boxed{\sigma_{total}(e^+e^- \to \mu^+\mu^-)\Big|_{\substack{\text{spin}\\ \text{ave}}} = \frac{\pi\alpha^2}{6E^2}v(3-v^2)}$$

In the relativistic limit, $v \to 1$, we have

$$\sigma_{total}\Big|_{\substack{\text{spin}\\ \text{ave}}} \to \frac{\pi\alpha^2}{3E^2} \sim \frac{22 \text{ nanobarns}}{E^2(\text{GeV})}$$

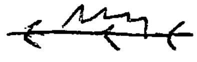
Loops in QED

As in any field theory, there are, of course, radiative corrections in QED. For example, in the next order in α, diagrams that contribute to the amplitude for $e^+e^- \to \mu^+\mu^-$ include

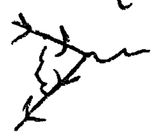


(+ counterterm insertions)

Some of these diagrams are divergent, and we will need to include counterterms to obtain finite results. E.g.

 induces e (or μ) mass and field renorm.

 induces A field (and perhaps mass) renorm

 induces coupling renormalization.

We can carry out these renormalizations order by order in perturbation theory in the coupling e^2, as we have in other theories. But in QED, there is a new issue that we did not need to face before:

Are the counterterms gauge-invariant?

Recall that gauge invariance was essential to our argument that QED was both unitary and covariant. We needed to show that the theory in axial gauge and covariant gauge describe the same physics. But the argument was very formal -- it did not take into account the existence of divergences and the need to renormalize. Now we must face the question -- Do divergences invalidate the argument?

We used the gauge invariance of the action. But what we really need is gauge invariance of the "bare" action, including counterterms. Only if we write

$$\int d\phi\, e^{-S} \ldots$$

where S includes counterterms does the path integral generate all Feynman diagrams, including insertions of counterterms. So, to justify our earlier arguments, we need to show that all infinities can be removed with gauge-invariant counterterms.

The difficulty is — while the QED action is the most general gauge-invariant action for photon fields and spin-$\frac{1}{2}$ fermions of dimension ≤ 4, it is not the most general action. Operators such as

$$A_\mu A^\mu \quad \text{(photon mass)}$$

$$\partial_\mu A_\nu(\partial^\nu A^\mu + \partial^\mu A^\nu)$$

(+, not minus)

$$(A_\mu A^\mu)^2$$

are all terms that we could have included in the action, but did not, because these terms are not gauge invariant. Power counting suggests that we will need counterterms of these types. If that is true, it is a disaster; gauge invariance will be lost, and we are unable to ensure that our theory is both unitary and covariant.

To argue that these counterterms do not arise, we need to argue that the counterterms must respect the symmetries (gauge invariance) of the classical action. But this is not at all obvious, especially because we had to break the symmetry (fix the gauge) in order to quantize the theory.

At this point, we find the Ward Identity (of page 4.57) extremely useful.

We showed that, if there is a symmetry transformation under which the action changes by an expression linear in fields, then

$$\delta S[\bar{\phi}] = \delta \Gamma[\bar{\phi}]$$

-- the change of the effective action is the same as the change in the action under the symmetry operation. (And this applies to local as well as global symmetries.)

Now consider QED with a gauge fixing term (in a covariant gauge):

$$S = S_{\substack{\text{gauge} \\ \text{invariant}}} - \frac{1}{2\alpha} \int (\partial_\mu A^\mu)^2$$

Under a gauge transformation,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x)$$

the change in the action is

$$\delta S = -\frac{1}{\alpha} \int (\partial_\mu A^\mu)(\partial^2 \omega)$$

This is linear in the fields. So the change in the effective action is

$$\delta \Gamma = -\frac{1}{\alpha} \int (\partial_\mu \bar{A}^\mu)(\partial^2 \omega)$$

under a gauge transformation. Or

$$\Gamma = \Gamma_{\substack{\text{gauge}\\ \text{invariant}}} - \frac{1}{2\alpha}\int (\partial_\mu A^\mu)^2$$

What have we learned? We have found that, order-by-order in perturbation theory in e^2, the effective action (which generates the 1PI diagrams) is gauge invariant, except for a gauge-fixing term that has exactly the same form as in the classical action. This means that, order-by-order, we encounter no cutoff dependent terms in Γ that are not gauge invariant, so we never need to introduce counterterms that are not gauge invariant. This justifies our formal arguments. (Actually, the argument is still somewhat formal, because we need to make sure that we can regulate the diagrams without spoiling gauge invariance. More about this below.)

To summarize, we have found:

- All divergences can be removed with gauge-invariant counterterms.

Therefore,
- In particular, the gauge-fixing term is unrenormalized.

- And furthermore, gauge invariance imposes relations among counter-terms.

To appreciate this last point, we note that

$$\bar{\psi}(i\not{\partial} - e\not{A})\psi$$

has the invariance

$$\psi(x) \to e^{-ie\omega(x)}\psi(x)$$
$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x)$$

So, order-by-order in perturbation theory, the counterterm that arises has the form

$$(Z_\psi - 1)\ \bar{\psi}(i\not{\partial} - e\not{A})\psi$$

Gauge invariance thus restricts the bare action (including counterterms) to have the form

$$\mathcal{L} = -\frac{1}{4} Z_A F_{\mu\nu}F^{\mu\nu} + Z_\psi\ \bar{\psi}(i\not{\partial} - e\not{A})\psi$$
$$- m_0 Z_\psi \bar{\psi}\psi - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

where ψ, A are renormalized fields, e is the renormalized charge, and m_0 is the bare mass. If we rewrite this in terms of <u>bare</u> fields

$$\psi_B = Z_\psi^{\frac{1}{2}}\psi$$
$$A_B = Z_A^{\frac{1}{2}} A,$$

$$\mathcal{L} = -\frac{1}{4} F_{B\mu\nu}F_B^{\mu\nu} + \bar{\psi}_B(i\not{\partial} - e Z_A^{-\frac{1}{2}}\not{A}_B)\psi_B$$
$$- m_0 \bar{\psi}\psi - \frac{1}{2\alpha} Z_A^{-1}(\partial_\mu A^\mu)^2 ,$$

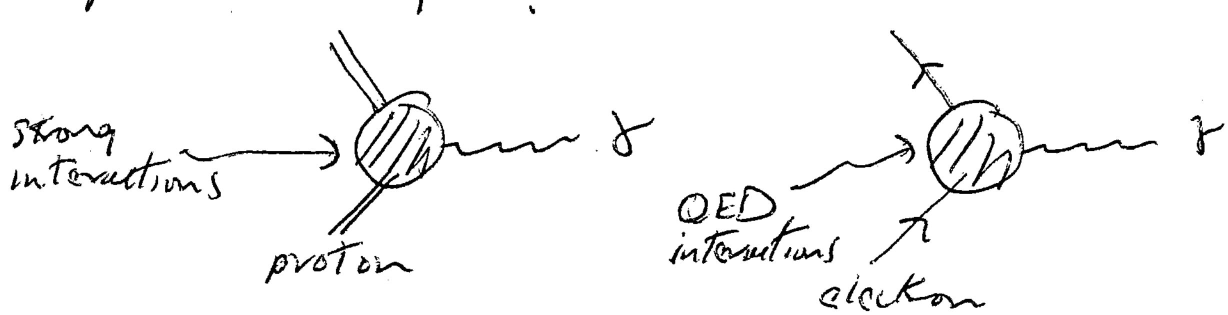
we find the relation between renormalized and bare e, α:

$$e = Z_A^{\frac{1}{2}} e_0$$ bare coupling

$$\alpha = Z_A^{-1} \alpha_0$$ bare gauge-fixing parameter

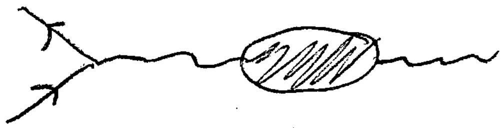
So the renormalization of e and α are determined by the photon field renormalization.

It is a remarkable and important observation that renormalization of the charge e is completely determined by the renormalization of the field A_μ. This observation is the key to understanding why the proton and electron electric charges are the same to fantastic accuracy. Even if the bare charges are the same exactly, wouldn't you expect the strong interactions to renormalize the proton charge?



But gauge invariance ensures that this does not happen.

that charge renormalization is completely determined by photon field renormalization



may be interpretted to mean that charge renormalization is entirely an effect of the "dielectric properties of the vacuum." Virtual pairs of charged particles tend to screen the bare charge. The relation between the bare and renormalized charge varies (logarithmically) with the cutoff Λ because, as Λ increases,

the photon probes more and more deeply into the "vacuum polarization" cloud surrounding the bare charge. (the bare charge e_0 actually becomes infinite as $\Lambda \to \infty$.)

Now, it may seem odd that charge renormalization and A_μ field renormalization are related, because they arise from completely different diagrams. E.g.,

what the Ward identity tells us is that these apparently independent diagrams are actually related.

Indeed, not just the infinite parts, but the finite parts too of such diagrams are related.

What precisely does the Ward identity tell us? In an abstract notation, it says

$$\int d^4x\, d^4x' \,\delta\omega(x) \frac{\delta \bar\phi(x')}{\delta\omega(x)} \frac{\delta\Gamma}{\delta\bar\phi(x')} = \delta S[\bar\phi]$$

(sum over field components understood). And, specifically in QED, where

$$\delta A_\mu(x) = \partial_\mu \omega(x)$$
$$\delta \psi(x) = -ie\,\omega(x)\,\psi(x)$$
$$\delta \bar\psi(x) = ie\,\omega(x)\,\bar\psi(x),$$

we have

(integration by parts)

$$\int d^4x\ \omega(x)\left[-ie\,\psi(x)\frac{\delta\Gamma}{\delta\psi(x)} + ie\,\bar\psi(x)\frac{\delta\Gamma}{\delta\bar\psi(x)} - \partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)}\right]$$

$$= \delta S = -\frac{1}{\alpha}\int d^4x\ \omega(x)\,\partial^2\partial_\mu A^\mu(x)$$

(I stopped writing bars over the fields, to avoid confusion) This has to be true for any $\omega(x)$. So we have

$$\boxed{ie\left[\bar\psi(x)\frac{\delta\Gamma}{\delta\bar\psi(x)} - \psi(x)\frac{\delta\Gamma}{\delta\psi(x)}\right] - \partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)} = -\frac{1}{\alpha}\partial^2\partial_\mu A^\mu(x)}$$

This functional identity concisely summarizes an infinite number of relations among one-particle irreducible Green functions. We can obtain any one of these identities by expanding Γ to the appropriate order in A_μ, ψ, and $\bar{\psi}$. For example, to obtain the term in the identity of order

$$\psi \bar{\psi},$$

we note that —

$$\Gamma = \int d^4y\, d^4z\, \Gamma^{\psi\bar{\psi}}(y,z)\, \psi(y)\, \bar{\psi}(z)$$

$$+ \int d^4y\, d^4z\, d^4x\, \Gamma^{\psi\bar{\psi}A}_{\mu}(y,z,x)\, \psi(y)\, \bar{\psi}(z)\, A^\mu(x)$$

$$+ \text{ other terms}$$

One of the components of the functional Ward identity is therefore:

$$ie\left[\Gamma^{\psi\bar{\psi}}(y,z)\,\delta^4(x-z) - \Gamma^{\psi\bar{\psi}}(y,z)\,\delta^4(x-y)\right]$$

$$= \partial^\mu_x\, \Gamma^{\psi\bar{\psi}A}_{\mu}(y,z,x)$$

If we Fourier transform this equation by integrating

$$\int d^4x\, d^4y\, d^4z\; e^{iqx} e^{ipy} e^{ip'z},$$

we obtain

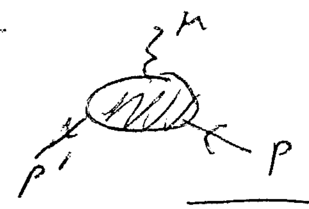
$$ie\left[\tilde{\Gamma}^{\psi\bar{\psi}}(p) - \tilde{\Gamma}(-p')\right] = -iq^\mu\, \tilde{\Gamma}^{\psi\bar{\psi}A}(p,p';q)$$

Now, we recall that $\Gamma^{\psi\bar{\psi}}$ is just an <u>inverse</u> <u>propagator</u>:

$$\Gamma^{(2)}(p) = \frac{-1}{D_c^{(2)}(p)}$$ (page 4.51)

Our identity, in graphical form, is

$ie\Big[$  $\Big]^{-1} - ie\Big[$  $\Big]^{-1}$

$= i(p+p')^{\mu}\Big[$  $\Big]$

(All momenta directed inward)
This is the original identity derived by Ward.

Let's check the identity in lowest order:

$$ie\left[(-i)(\not{p}-m) - (-i)(-\not{p}'-m)\right]$$

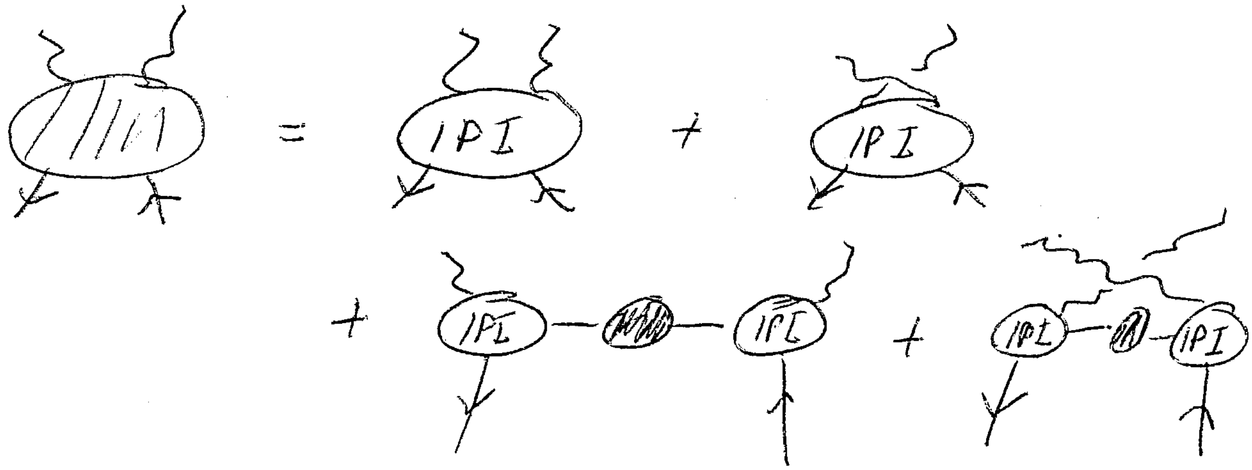
$$= i(p+p')^{\mu}(-ie\gamma_{\mu})$$

It checks.

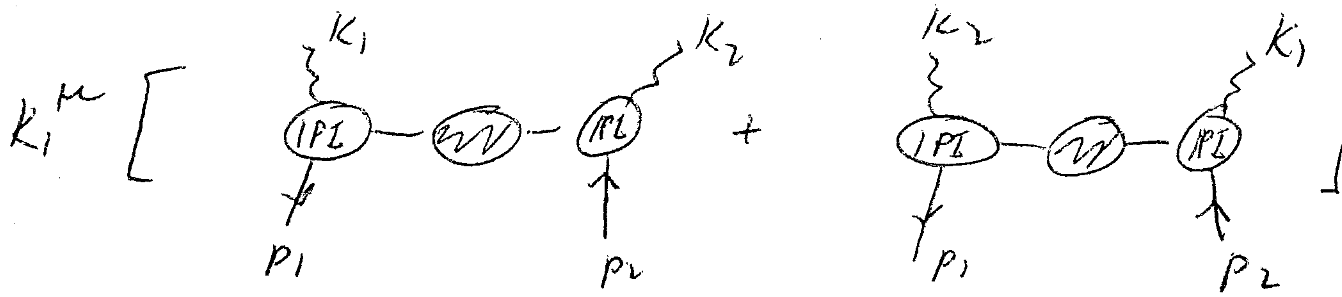
The Ward identity can be used to justify our claim back on page (5.49) that an amplitude for emission of a photon has the form ---

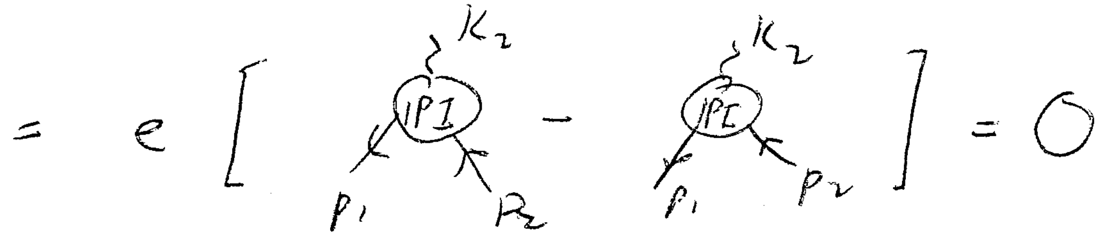
$$A = e_\mu(K)\, M^\mu(K)\,, \text{ where } K_\mu M^\mu(K) = 0$$

To illustrate how this goes, consider the amplitude ---



We see that --



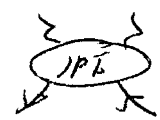


(the other two terms vanish because, on the <u>mass shell</u>, the inverse propagator is

$(-i)(\not{p} - m)$, and

$$\bar{u}(p_1)(\not{p}_1 - m) = (\not{p}_2 - m)\, u(p_2) = 0.)$$

Similarly, the ward identity ensures a cancellation between

k_1^μ 

and k_1^μ 

,

and this argument can be extended to arbitrary processes.

Gauge-Invariant Regularization

We used the Ward identity to show that non-gauge-invariant counterterms are not needed. But this argument was still not airtight, because our derivation of the Ward identity itself was merely "formal." We derived it without paying careful attention to the fact that we were blithely manipulating divergent quantities. We used the path integral method to derive the Ward identity. But the path integral is not <u>defined</u> unless we regulate the ultraviolet divergences of the Feynman diagrams. Our derivation of the Ward identity therefore implicitly <u>assumes</u> that the theory can be regulated without spoiling the gauge invariance.

It is not completely obvious that this assumption is correct. And indeed, if we compute, say, the diagram

 $\sim e^2 \Lambda^2$

using the same momentum space cutoff that we have used before, then it actually appears that we need a photon mass counterterm after all!

Apparently, our naive momentum space cutoff regulator does not preserve gauge invariance. Therefore, if we use this regulator, our naive arguments cannot be justified, and proving that QED is unitary and covariant, although it can be done, is a technical nightmare.

Fortunately, there is another, more clever, method of regularization that does preserve gauge invariance, and therefore does allow us to justify the naive arguments. Furthermore, this method of regularization is by far the most convenient for actually doing calculations in gauge theories.

The new regularization method we will use is called "dimensional regularization". The idea of the method is that the degree of divergence of a diagram depends on the dimension D of spacetime. And, in fact, the lower the dimension D, the better the ultraviolet behavior of the diagrams.

Furthermore, although it is not clear what is meant by a field theory in a dimensionality D that is not an integer, the Feynman diagrams themselves are smooth, in fact, analytic (actually meromorphic) functions of D. So we can imagine starting out at a small value of D where the integrals converge, and then extending the integrals to larger values of D by analytic continuation in D. If the Feynman integral is formally divergent for $D=4$, then something must go wrong as D approaches 4. It turns out that the (analytically continued) integrals have (isolated) poles in D at $D=4$. To regulate the diagrams we continue them to

$$D = 4 - \epsilon$$

dimensions. So the dimensionless number ϵ takes the place of our ultraviolet cutoff Λ. Removing the regulator will now mean, rather than allowing $\Lambda \to \infty$, to take the limit $\epsilon \to 0$.

An important observation is that, if a theory is gauge invariant, it is gauge-invariant in any space-time dimension D. Therefore, we may expect regularization by continuation in D to preserve gauge invariance, and for diagrams so regulated to satisfy the Ward identities.

How does this dimensional regularization of the Feynman diagrams work? Let us consider an integral such as

$$I(a^2) = \int \frac{d^4K}{(2\pi)^4} \frac{1}{(K^2+a^2)^2}, \quad \text{(already Wick rotated to Euclidean space)}$$

that is formally (logarithmically) ultraviolet divergence. We want to give a mathematical meaning to the expression

$$I_D(a^2) = \int \frac{d^DK}{(2\pi)^D} \frac{1}{(K^2+a^2)^2}$$

which is formally convergent by power counting for $D<4$, and hence can be regarded as a regularized form of $I(a^2)$.
Let us consider, in fact, a somewhat more general expression:

$$I_D^{r,q}(a^2) = \int \frac{d^DK}{(2\pi)^D} \frac{(K^2)^r}{(K^2+a^2)^q}$$

We need to define $\int d^DK$. For this purpose, we can use the rotational invariance of the integrand, and write

$$\int d^DK = \Omega_{D-1} \int_0^\infty dK \, K^{D-1}$$

where Ω_{D-1} is the volume of a $(D-1)$-dimensional sphere. For integer values of D, we can evaluate Ω_{D-1} as on page(2.128),

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}$$

We will use this expression for Ω_{D-1} for all D, including noninteger values. Now we have a sensible expression for I_D:

$$I_D^{r,q}(a^2) = \frac{1}{(4\pi^2)^{D/2}\,\Gamma(\frac{D}{2})} \int_0^\infty d(k^2)(k^2)^{\frac{D}{2}+r-1}(k^2+a^2)^{-q}$$

The remaining integral is one that we can do

$$\int_0^\infty dk^2 (k^2)^p (k^2+a^2)^{-q}$$

$$= \frac{1}{\Gamma(q)} \int_0^\infty ds \int_0^\infty dk^2 (k^2)^p s^{q-1} e^{-s(k^2+a^2)}$$

(using integral representation of the Γ function)

Now we can do k^2 integral:

$$= \frac{1}{\Gamma(q)}\Gamma(p+1)\int_0^\infty ds \; s^{q-1} s^{-1-p} e^{-sa^2}$$

$$= \frac{\Gamma(p+1)\,\Gamma(q-p-1)}{\Gamma(q)} (a^2)^{p-q+1}$$

All together, then, we have

$$\boxed{\begin{aligned} I_D^{r,q}(a^2) &= \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^r}{(k^2+a^2)^q} \\ &= \frac{\Gamma(\frac{D}{2}+r)\,\Gamma(q-r-\frac{D}{2})}{(4\pi^2)^{D/2}\,\Gamma(\frac{D}{2})\,\Gamma(q)} (a^2)^{\frac{D}{2}+r-q} \end{aligned}}$$

This is what we wanted -- A continuation in D of the Feynman integral that is analytic except for isolated poles. (The function $\Gamma(-z)$ is meromorphic with poles at the nonnegative integers.) Poles occur for

$$\frac{D}{2} = q - r + (\text{nonneg integer}).$$

If q and r are integers, then these are the (even) integer values of D for which the integral is formally divergent.

Any Feynman diagram can be continued in this way. If there are loops of fermions, we need to be able to take a trace of a product of γ matrices in D dimensions. This is done by the same methods as in 4 dimensions, except that

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}\,\mathbb{I}, \text{ where } \eta_{\mu\nu}\eta^{\mu\nu} = D$$

(instead of 4)

And $\quad \mathrm{tr}\,\mathbb{1} = 2^{D/2}$

(actually, any smooth function of D that equals 4 at $D=4$ will do here.)

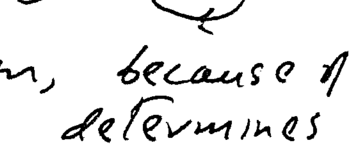
Actually, one γ matrix identity is a bit of a problem! In 4 dimensions we have

$$\mathrm{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma) = 4i\,\epsilon_{\mu\nu\lambda\sigma}$$

How do we continue this away from 4 dimensions? (In D dimensions, it is trace of "γ_{D+1}" times D γ's that gives an ϵ tensor.) This is not just a technical nuisance, but a <u>deep</u> problem, it turns out. Ward identities involving γ_5's cannot be properly regulated. And as a result Ward identity "anomolies" (violations) arise. But this problem does not afflict the Ward identities of QED, in which no γ_5's appear.

<u>Example: One-Loop Vacuum Polarization</u>

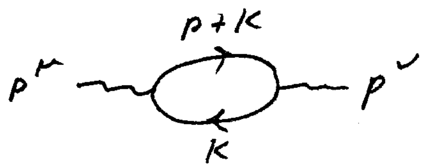
Let us see how dimensional regularization works in a particular calculation -- the one-loop photon propagator correction:



As we've seen, because of the Ward identity, this diagram determines both the photon field renormalization in $O(e^2)$ and the coupling renormalization in $O(e^3)$. (It is a calculation of the "polarizability" of the vacuum)

we have --

(fermi)



$$= (-ie)^2(-1)\int \frac{d^Dk}{(2\pi)^D}\, tr\left[\frac{[(\not p+\not k)+m]\gamma^\mu(\not k+m)\gamma^\nu (i)^2}{((p+k)^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}\right]$$

We take the trace using the identity

$$tr(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu) = 4[\eta^{\alpha\mu}\eta^{\beta\nu}+\eta^{\alpha\nu}\eta^{\beta\mu}-\eta^{\alpha\beta}\eta^{\mu\nu}]$$

(I'll take $tr\,\mathbb{1} = 4$ in D dimensions)

$$tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$$

Thus

$$= -4e^2\int\frac{d^Dk}{(2\pi)^D}\left(\frac{num}{denom}\right)$$

$$num = (p+k)^\mu k^\nu + k^\mu(p+k)^\nu - \eta^{\mu\nu}(p+k)\cdot k + m^2\eta^{\mu\nu}$$

$$\frac{1}{denom} = \frac{1}{((p+k)^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

[use identity $\frac{1}{ab} = \int_0^1 dx\,\frac{1}{[ax+b(1-x)]^2}$

$$= \int_0^1 dx\,\frac{1}{[k^2+2xp\cdot k+xp^2-m^2+i\epsilon]^2}$$

$$= \int_0^1 dx\,[(k+xp)^2+x(1-x)p^2-m^2+i\epsilon]^{-2}$$

Now, if we shift the integration variable,

$$k \to k - xp,$$

we obtain

$$
\text{(loop)} = -4e^2 \int_0^1 dx \int \frac{d^D K}{(2\pi)^D} \frac{(\text{num})'}{[K^2 + x(1-x)p^2 - m^2 + i\epsilon]^2}
$$

where

$$
\begin{aligned}
(\text{num})' &= [(1-x)p + K]^\mu (K - xp)^\nu + (K - xp)^\mu [(1-x)p + K]^\nu \\
&\quad - \eta^{\mu\nu} [(1-x)p + K]\cdot(K - xp) + m^2 \eta^{\mu\nu} \\
&= 2K^\mu K^\nu - 2x(1-x)p^\mu p^\nu - \eta^{\mu\nu}[K^2 - x(1-x)p^2 - m^2] \\
&\quad + \text{terms linear in } K
\end{aligned}
$$

By symmetry, terms in the numerator linear in K can be dropped (they integrate to zero). Furthermore "rotational invariance" allows us to replace

$$
K^\mu K^\nu \rightarrow \frac{1}{D} \eta^{\mu\nu} K^2
$$

(Remember $\eta^{\mu\nu}\eta_{\mu\nu} = D$.) We have

$$
\text{(loop)} = -4e^2 \int_0^1 dx \int \frac{d^D K}{(2\pi)^D} \frac{(\frac{2}{D} - 1) K^2 \eta^{\mu\nu} + x(1-x)[\eta^{\mu\nu} p^2 - 2p^\mu p^\nu] + m^2 \eta^{\mu\nu}}{[K^2 + x(1-x)p^2 - m^2 + i\epsilon]^2}
$$

Now -- rotate the K^0 integral, $K^0 \rightarrow iK_E^0$,

$$
\text{(loop)} = -4ie^2 \int_0^1 dx \int \frac{d^D K}{(2\pi)^D} \frac{(1 - \frac{2}{D}) K^2 \eta^{\mu\nu} + x(1-x)(\eta^{\mu\nu} p^2 - 2p^\mu p^\nu) + m^2 \eta^{\mu\nu}}{[K^2 - x(1-x)p^2 + m^2 - i\epsilon]^2}
$$

To evaluate the D-dimensional Euclidean integrals, we use the master formula on page 5.74
It tells us

$$\int \frac{d^D K}{(2\pi)^D} \frac{K^2}{(K^2+a^2)^2} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(1+\frac{D}{2})}{\Gamma(\frac{D}{2})} \frac{\Gamma(1-\frac{D}{2})}{\Gamma(2)} (a^2)^{\frac{D}{2}-1}$$

If we use the identity $\Gamma(x+1) = x\,\Gamma(x)$, we have

$$= \frac{1}{(4\pi)^{D/2}} \left(\frac{D}{2}\right) \frac{\Gamma(2-\frac{D}{2})}{1-D/2} a^2 (a^2)^{D/2-2}$$

or
$$\int \frac{d^D K}{(2\pi)^D} \frac{K^2}{(K^2+a^2)^2} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2-\frac{D}{2})}{(\frac{2}{D}-1)} (a^2)(a^2)^{\frac{D}{2}-2}$$

and
$$\int \frac{d^D K}{(2\pi)^D} \frac{1}{(K^2+a^2)^2} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2})} \frac{\Gamma(2-\frac{D}{2})}{\Gamma(2)} (a^2)^{D/2-2}$$

$$= \frac{1}{(4\pi)^{D/2}} \Gamma(2-\tfrac{D}{2}) (a^2)^{\frac{D}{2}-2}$$

Plugging this in, we get

$$= -4ie^2 \int_0^1 dx \left(m^2 - x(1-x)p^2 - i\epsilon\right)^{D/2-2} \frac{\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}} \times$$

$$\left[-\eta^{\mu\nu}\left(m^2 - x(1-x)p^2 - i\epsilon\right) + x(1-x)\left(\eta^{\mu\nu}p^2 - 2p^\mu p^\nu\right) + m^2\eta^{\mu\nu}\right]$$

$$= \frac{-8ie^2}{(4\pi)^{D/2}} \left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right) \Gamma(2-\tfrac{D}{2}) \int_0^1 dx \frac{x(1-x)}{\left[m^2 - x(1-x)p^2 - i\epsilon\right]^{2-\frac{D}{2}}}$$

Now we have a regularized expression for the diagram that is well-defined for $\frac{D}{2}-2 \neq$ nonneg int. but blows up as $D \to 4$. As we anticipated, the regulated diagram is gauge-invariant (annihilated by contracting with p_μ.)

We can isolate the ultraviolet divergence by considering the behavior of the diagram as D approaches 4. The Γ function is

$$\Gamma(z) = \frac{1}{z} - \gamma + O(z) \text{ for small } z$$

$$\gamma = .57721\ldots \quad \text{"Euler's Constant"}$$

Also $a^\epsilon = e^{\epsilon \ln a} = 1 + \epsilon \ln a + \ldots$, expanding in powers of ϵ. So let us write

$$D = 4 - \epsilon$$

and expand the diagram in powers of ϵ:

$$\frac{1}{(4\pi)^{D/2}} = \frac{1}{16\pi^2}\left(1 + \frac{\epsilon}{2}\ln 4\pi + \ldots\right)$$

$$\Gamma\left(2 - \frac{D}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{1}{\epsilon/2} - \gamma + \ldots$$

$$(a^2)^{\frac{D}{2}-2} = (a^2)^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln a^2 + \ldots$$

So...

$$= \frac{-ie^2}{2\pi^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)$$

$$\times\left(1 + \frac{\epsilon}{2}\ln 4\pi + \ldots\right)\left(\frac{1}{\epsilon/2} - \gamma + \ldots\right)\int_0^1 dx\, x(1-x)\left(1 - \frac{\epsilon}{2}\ln\left(m^2 - x(1-x)p^2\right)\right)$$

$$= \frac{-ie^2}{2\pi^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)\Bigg[\frac{1}{6}\left(\frac{1}{\epsilon/2} - \gamma + \ln 4\pi\right)$$

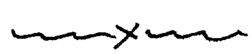
$$- \int_0^1 dx\, x(1-x)\ln\left(m^2 - x(1-x)p^2\right) + O(\epsilon)\Bigg]$$

We have found:

 $= \dfrac{-ie^2}{6\pi^2\epsilon}(\eta^{\mu\nu}p^2 - p^\mu p^\nu)$ + finite part

The ultraviolet divergence turns up as a pole in ϵ.

We can remove this divergence with an appropriate counterterm. We require

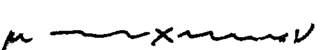
 +  = finite

The counterterm is

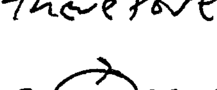
$$(Z_A - 1)\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) = (Z_A - 1)\tfrac{1}{2}A_\mu(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu$$

(integrating by parts)

so the insertion of the counterterm gives

μ  ν $= (Z_A - 1)(-i)(p^2\eta^{\mu\nu} - p^\mu p^\nu)$

Therefore,

 +  $= (-i)(p^2\eta^{\mu\nu} - p^\mu p^\nu)\left(\dfrac{e^2}{6\pi^2\epsilon} + Z_A - 1 + \text{finite}\right)$

and we have

$$\boxed{Z_A = 1 - \frac{e^2}{6\pi^2\epsilon} + \text{finite part} + \text{higher order}}$$

The finite part can be chosen to satisfy our field renormalization condition.

In the past, we have chosen our field renormalization convention so that the pole in the propagator would have a properly normalized residue. This was convenient, but it was not <u>necessary</u> to renormalize this way. No physical quantity will depend on how we renormalize fields. When we use dimensional regularization, a different renormalization scheme is usually the most convenient.

This renormalization scheme is called <u>minimal subtraction</u>, and in this scheme

$$Z_A = 1 - \frac{e^2}{6\pi^2 \epsilon} + \text{higher order}$$

In other words, we choose --

$$\begin{array}{l}\text{finite part}\\ \text{of counterterm}\end{array} = 0$$

In minimal subtraction, we choose our counterterms order by order in perturbation theory to subtract away the poles in ϵ, and nothing else. One reason this scheme is so convenient is that it is very easy to identify in each order of perturbation theory what the proper counterterm is.

After the appropriate subtraction, we can finally take the $\epsilon \to 0$ limit. Thus, we have, in minimal subtraction.

 + 

$$= \frac{-ie^2}{2\pi^2}\left(\eta^{\mu\nu}p^2 - p^\mu p^\nu\right)$$

$$\times \left[\tfrac{1}{6}(-\gamma + \ln 4\pi) - \int_0^1 dx\, x(1-x)\ln\left(m^2 - x(1-x)p^2 - i\epsilon\right)\right]$$

Unfortunately, this result makes no sense, because the argument of the logarithm has dimensions. We have apparently done something wrong.

To understand the origin of the problem, notice that, in D dimensions, the integral  has dimension $D/2 - 2$. We tried to remove the infinity in this diagram with a <u>dimensionless</u> counterterm, which does not make sense.

To cure this problem, we observe that the coupling constant e, while dimensionless in 4 dimensions, is not dimensionless in D dimensions. Considering

$$\mathcal{L} = -\tfrac{1}{4}F^2 + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi$$

we see that, if the action is dimensionless, then

$$eA \sim \mu \quad \text{(dimensions of mass)}$$

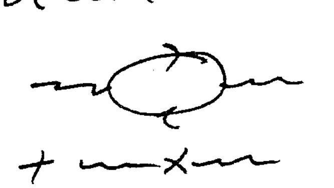
$$A \sim \mu^{(D-2)/2}$$

$$e \sim \mu^{2-D/2} \sim \mu^{\epsilon/2}$$

If we want e to be dimensionless in any dimension D, then we should write the coupling constant as

$$\mu^{\epsilon} e^2$$

where μ is an arbitrarily chosen mass parameter. When we replace e^2 by $e^2 \mu^{\epsilon} \sim e^2 (1 + \frac{\epsilon}{2} \ln \mu^2)$, our expression for one-loop vacuum polarization becomes



$$= \frac{ie^2}{2\pi^2} (\eta^{\mu\nu} p^2 - p^{\mu} p^{\nu}) \int_0^1 dx\, x(1-x) \ln \left[\frac{m^2 - x(1-x)p^2 - i\epsilon}{C\mu^2} \right]$$

(where $C = 4\pi e^{-\gamma}$), which is at least dimensionally consistent.

But what is the meaning of the mass parameter μ (called a "subtraction point")? Since it was chosen arbitrarily, physics must not depend on it. Yet the amplitude we have computed appears to depend on it explicitly. What is going on?

Remember that the Ward identity told us (page 5.63) how field and coupling renormalization are related, namely

$$\underbrace{e_0^2}_{\text{bare coupling in } 4-\epsilon \text{ dimensions}} = Z_A^{-1} \mu^{\epsilon} \underbrace{e^2}_{\text{dimensionless renormalized coupling}}$$

Z_A is a function of e^2 and ϵ, computable in perturbation theory. We found

$$Z_A^{-1}(e^2;\epsilon) = 1 + \frac{e^2}{6\pi^2\epsilon} + \text{higher order}$$

in minimal subtraction. So we have also determined the one-loop coupling renormalization

$$e_0^2 = \mu^\epsilon\left(e^2 + \frac{e^4}{6\pi^2\epsilon} + \text{higher order}\right)$$

But μ is arbitrary -- it enters only when we express the (dimensionful) bare coupling in terms of the (dimensionless) renormalized coupling.

The physics of the theory, in particular all scattering amplitudes, is independent of μ. This means that the <u>bare</u> <u>action</u> is independent of μ; in particular e_0^2 is μ-independent.
But then the relation

$$e_0^2 = \mu^\epsilon e^2 Z_A^{-1}(e^2;\epsilon)$$

tells us that e^2 must have an <u>implicit</u> μ dependence. Thus, the apparent dependence of vacuum polarization on μ is illusory. The explicit μ-dependence must cancel against the implicit μ-dependence of e^2

In fact, we can <u>calculate</u> the μ-dependence of the renormalized coupling

Define $\beta(e,\epsilon) = \mu \frac{d}{d\mu} e$

then if $e_0^2 = \mu^\epsilon f(e^2)$, the μ-independence of e_0^2 tells us that

$$\mu \frac{d}{d\mu} e_0^2 = 0 = \mu^\epsilon \left[\epsilon f(e^2) + f'(e^2)\, \mu \frac{d}{d\mu} e^2 \right]$$

Therefore, $\epsilon f(e^2) + 2e\,\beta(e) f'(e^2) = 0$

Now, to one loop order we found

$$f = e^2 + \frac{e^4}{6\pi^2 \epsilon} + \cdots$$

So $$\left(\epsilon e^2 + \frac{e^4}{6\pi^2} + \cdots\right) = -2e\,\beta(e)\left(1 + \frac{e^2}{3\pi^2\epsilon} + \cdots\right)$$

Solving perturbatively in powers of e,

$$\beta(e) = -\frac{\epsilon}{2} e + \frac{e^3}{12\pi^2} + \cdots$$

The first term arises from trivial dimensional analysis, and disappears in four dimensions. As $\epsilon \to 0$, we have

$$\boxed{\beta(e) = \mu \frac{d}{d\mu} e = \frac{e^3}{12\pi^2} + \cdots}$$

-- A famous result of Gell-Mann + Low

What does it *mean*?

The Running Coupling Constant

We have found that the renormalized coupling e defined by dimensional regularization and minimal subtraction has an implicit dependence on a mass scale μ, given by

$$\mu \frac{d}{d\mu} e = \frac{e^3}{12\pi^2} + \text{higher order.}$$

In fact, e obeys the same equation even if we define the coupling constant by a "nonminimal" subtraction. In performing field renormalization, we might include in the counterterm an extra finite part

$$Z_A = 1 - \frac{e^2}{6\pi^2\epsilon} - \underbrace{Ce^2}_{\text{(finite part)}} + \text{higher order}$$

So then

$$e_0^2 = \mu^\epsilon e^2 Z_A^{-1}$$
$$= \mu^\epsilon e^2 \left(1 + \frac{e^2}{6\pi^2\epsilon} + Ce^2 + \cdots\right)$$

This freedom to make a finite subtraction corresponds to our freedom, for a given bare coupling, to define a renormalized coupling in a different way; to this order,

$$e^2 \longrightarrow e^2 + Ce^4 + \cdots .$$

As long as C is independent of μ ("mass-independent subtraction"), this redefinition of the renormalized coupling does not change the leading term in $\mu \frac{d}{d\mu} e$.

This equation is easy to solve:

$$\mu \frac{d}{d\mu} e = b e^3 \Rightarrow \frac{de}{e^3} = b \frac{d\mu}{\mu}$$

$$\Rightarrow -\frac{1}{2}\left[\frac{1}{e_{\mu'}^2} - \frac{1}{e_\mu^2}\right] = b \ln \frac{\mu'}{\mu}$$

$$\Rightarrow \frac{1}{e_{\mu'}^2} = \frac{1}{e_\mu^2} - 2b \ln \mu'/\mu$$

$$\Rightarrow e_{\mu'}^2 = \frac{e_\mu^2}{1 - b e_\mu^2 \ln \mu'^2/\mu^2}$$

or $$\boxed{e_{\mu'}^2 = \frac{e_\mu^2}{1 - \frac{e_\mu^2}{12\pi^2} \ln\left(\mu'^2/\mu^2\right)}}$$ (valid if e_μ and $e_{\mu'}$ are small.)

What interpretation can we give to this formula?

To understand what it means, let us consider the implications of the finite part of the one-loop vacuum polarization that we computed. We found

$$= i(g^{\mu\nu} p^2 - p^\mu p^\nu)\, \Pi(p^2)$$

where

$$\pi(p^2) = \frac{e_\mu^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - x(1-x)p^2 - i\epsilon}{C\mu^2}\right]$$

($C = 4\pi e^{-\gamma}$), and e_μ is the renormalized coupling defined by minimal subtraction.

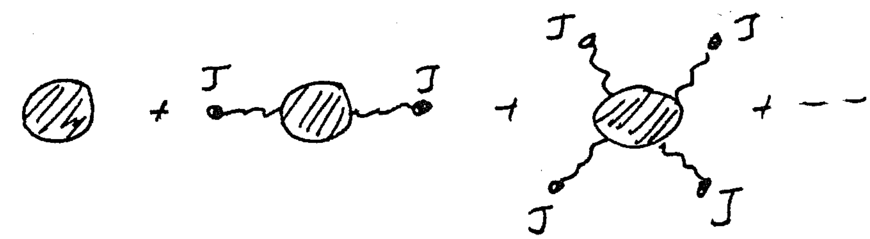
The quantity $\pi(p^2)$ is said to measure the "polarizability" of the vacuum; it tells us how the electric charge of the electron is screened by virtual e^+e^- pairs in the vacuum, as a function of distance from the electron $\sim (p^2)^{-1/2}$. To see this recall (page 2.23), that if the photon field A_μ is coupled to an external charge distribution J_μ,

$$\mathcal{L}' = -e\, J^\mu(x) A_\mu(x),$$

then the dependence of the energy $E(J)$ of the vacuum on the source J (assumed static) is

$$Z[J] \equiv \langle 0|T\exp\left(-ie\int d^4x\, J^\mu(x) A_\mu(x)\right)|0\rangle \xrightarrow[T\to\infty]{} e^{-iE(J)T},$$

where T is the time that the source is "on". In other words, $-iE(J)T$ is given by the sum of connected vacuum diagrams in the presence of the source:

$-iE(J)T =$ 

$= \text{(J-independent)}$

(photon propagator)

$$+ \frac{(-ie)^2}{2}\int d^4x\, d^4y\; J^\mu(x)\, \Delta_{\mu\nu}(x-y)\, J^\nu(y)$$

+ higher-order in J

or

$$E(J)T = \text{constant}$$
$$-\frac{ie^2}{2}\int \frac{d^4p}{(2\pi)^4}\, \tilde{J}^\mu(p)\, \hat{\Delta}_{\mu\nu}(p)\, \tilde{J}^\nu(-p)$$
$$+ \cdots$$

Now, if the source is <u>static</u>, then $J^\mu(x)$ is independent of x^0, and

$$\tilde{J}^\mu(p) = 2\pi\,\delta(p^0)\, \tilde{J}^\mu(\vec{p})$$

3-dimensional (spatial) Fourier transform

So, $\int \frac{d^4p}{(2\pi)^4}\, \tilde{J}^\mu(p)\, \tilde{\Delta}_{\mu\nu}(p)\, \tilde{J}^\nu(-p)$

$$= 2\pi\,\delta(0) \int \frac{d^3p}{(2\pi)^3}\, \tilde{J}^\mu(\vec{p})\, \tilde{\Delta}_{\mu\nu}(\vec{p})\, \tilde{J}^\nu(-\vec{p})$$

(i.e. $p^0 = 0$)

And $2\pi\delta(0) = T$, the time the source is "on".

So we have derived --

$$E(J) = -\frac{ie^2}{2}\int\frac{d^3p}{(2\pi)^3}\,\tilde{J}^\mu(\vec{p})\,\tilde{\Delta}_{\mu\nu}(\vec{p})\,\tilde{J}^\nu(-\vec{p}).$$

Or, if $\vec{J}=0$ (no currents flowing)

$$E(J) = \frac{1}{2}\int d^3x\,d^3y\; J^0(\vec{x})\,V(\vec{x}-\vec{y})\,J^0(\vec{y})$$

where $$\boxed{V(\vec{r}) = -ie^2\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec{p}\cdot\vec{r}}\,\tilde{\Delta}_{00}(\vec{p})}$$

This is the interaction energy of two static charges with separation $\vec{r}$.

The expression for V is not Lorentz invariant. This is not a surprise, because a charge distribution is static only in a particular frame; if we had considered an arbitrary current distribution, we would have obtained the Liénard-Wiechert potential. More surprising is that V is expressed in terms of $\tilde{\Delta}_{00}$, which is not gauge-invariant. But, in fact, gauge transformations change only the $p_\mu p_\nu$ term in $\tilde{\Delta}_{\mu\nu}$ -- which is no change at all in $\tilde{\Delta}_{00}$ when $p^0=0$. The static potential is determined by the 00 part of the propagator, which is gauge-independent.

Now, if $\Pi(p^2)$ is defined by

$$= i(\eta^{\mu\nu}p^2 - p^\mu p^\nu)\,\Pi(p^2),$$

then we can express the exact propagator $\tilde{\Delta}_{\mu\nu}$ in terms of $\Pi(p^2)$ by summing up the insertions of 1PI parts:

or (in Feynman gauge)

$$\tilde{\Delta}_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2} + \frac{-i\eta_{\mu\alpha}}{p^2}\, i(\eta^{\alpha\beta}p^2 - p^\alpha p^\beta)\Pi(p^2)\,\frac{-i\eta_{\beta\nu}}{p^2} + \cdots$$

$$= -i\left(\frac{\eta_{\mu\nu}}{p^2} + \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi(p^2) + \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi(p^2)^2 + \cdots\right)$$

$$= \frac{-i\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)}{p^2(1-\Pi(p^2))} - \frac{i p_\mu p_\nu}{(p^2)^2}$$

gauge dependent term

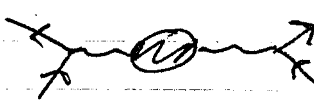
We have found therefore, that

$$V(\vec{r}) = \int \frac{d^3p}{(2\pi)^3}\, \tilde{V}(\vec{p})\, e^{i\vec{p}\cdot\vec{r}}, \text{ where } \tilde{V}(\vec{p}) = \frac{e_\mu^2}{\vec{p}^2}\left[1 - \Pi(-\vec{p}^2)\right]^{-1}$$

or --

$$\boxed{\tilde{V}(\vec{p}) = \frac{e_\mu^2}{\vec{p}^{\,2}}\left[1 - \frac{e_\mu^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 + x(1-x)\vec{p}^{\,2} - i\epsilon}{c\mu^2}\right] + \cdots\right]^{-1}}$$

In tree approximation, this is just the Coulomb potential. What we have computed is a one-loop correction to the Coulomb potential. This correction can actually be measured, e.g. in elastic ee scattering (at spacelike momentum transfer).



The form of the correction to the Coulomb potential is especially simple in the short-distance limit $\vec{p}^{\,2} \gg m^2$. In this limit,

$$\ln\left(\frac{m^2 + x(1-x)\vec{p}^{\,2}}{c\mu^2}\right) = \ln\frac{\vec{p}^{\,2}}{c\mu^2} + \ln x(1-x) + O\left(\frac{m^2}{\vec{p}^{\,2}}\right)$$

Since $\int_0^1 dx\, x(1-x)\left(\ln x + \ln(1-x)\right)$

$$= 2\int_0^1 dx\,(x - x^2)\ln x = 2\left(\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 - \tfrac{1}{3}x^3\ln x + \tfrac{1}{9}x^3\right)\Big|_0^1$$

$$= 2\left(-\tfrac{1}{4} + \tfrac{1}{9}\right) = -5/18,$$

we have

$$\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2 + x(1-x)\vec{p}^{\,2}}{c\mu^2}\right) = \frac{1}{6}\left(\ln\frac{\vec{p}^{\,2}}{c\mu^2} - 5/3\right)$$

$$= \frac{1}{6}\ln\left(\vec{p}^{\,2}/c'\mu^2\right), \text{ where } c' = c\,e^{5/3}$$

and ...

$$\boxed{\tilde{V}(\vec{p}) = \frac{1}{\vec{p}^{\,2}}\,\frac{e_\mu^2}{1 - \frac{e_\mu^2}{12\pi^2}\ln\left(\vec{p}^{\,2}/c'\mu^2\right)} + O\left(\frac{m^2}{\vec{p}^{\,2}}\right)}$$

If we compare to our expression for the "running" coupling constant, obtained by integrating the Gell-Mann Low equation, (p. 5.87)

$$e_{\mu'}^2 = \frac{e_\mu^2}{1 - \frac{e_\mu^2}{12\pi^2} \ln\left(\mu'^2/\mu^2\right)},$$

we see that the one-loop correction to the Coulomb potential can be absorbed into a rescaling of the mass scale μ:

$$\tilde{V}(\vec{p}) = \frac{e_{\mu'}^2}{\vec{p}^{\,2}} \quad \text{where } \mu'^2 = \vec{p}^{\,2}/C'$$

-- $\tilde{V}$ has the same form as the tree approx. Coulomb potential, but with e^2 replaced by the <u>running</u> coupling constant at a mass scale characteristic of the momentum transfer $\vec{p}$. So e_μ is an <u>effective</u> value of the charge that is appropriate to use in a description of physics at a distance scale of order μ^{-1}. The running coupling e_μ gets larger and larger as μ increases because at short distances the <u>bare</u> charge of the electron is screened <u>less</u>; we are probing further into the e^+e^- polarization cloud surrounding the bare charge.

Now consider the behavior of the potential in the long-distance limit $\vec{p}^{\,2} \ll m^2$; i.e., at distances much larger than the electron Compton wavelength. In this limit,

$$\ln \frac{m^2 + x(1-x)\vec{p}^{\,2}}{c\mu^2} = \ln \frac{m^2}{c\mu^2} + O\left(\frac{\vec{p}^{\,2}}{m^2}\right)$$

and $\tilde{V}(\vec{p}) = \dfrac{e_{\mu'}^2}{\vec{p}^{\,2}}$ where $\mu'^2 = m^2/c$

Thus, at distances much longer than m^{-1}, the effective value of the charge stops running. This happens because the polarization cloud has a characteristic size of order m^{-1}, and the screening of the charge is cut off at this distance scale. The charge that we measure in low-momentum-transfer experiments is actually —

$$\frac{1}{4\pi} e^2_{\mu' = m/\sqrt{c}} = (137.0360\ldots)^{-1}$$

<u>Does QED "exist"?</u>

We might imagine that electrodynamics is a theory with a cutoff Λ. (We <u>ought to</u>, since we don't expect QED to be a correct description of physics at arbitrarily short distances.) Then the "bare" coupling is the appropriate coupling to use at the mass scale $\mu = \Lambda$

$$e_0 = e_\Lambda .$$

The *physical* charge measured at low momentum transfer is

$$e_P = e_m .$$

The physical and bare charges are related by

$$e_P^2 = \frac{e_0^2}{1+\frac{e_0^2}{12\pi^2}\ln \Lambda^2/m^2} \quad \text{or} \quad e_0^2 = \frac{e_P^2}{1-\frac{e_P^2}{12\pi^2}\ln(\Lambda^2/m^2)}$$

Now "taking the continuum limit" corresponds to *removing* the cutoff, or allowing Λ/m to become arbitrarily large. Notice what happens if we take $\Lambda/m \to 0$ with e_0 fixed -- the physical charge goes to *zero*. Equivalently, if we keep e_P fixed at a nonzero value and increase Λ/m, then e_0 *blows up* at a finite value of Λ. There is no reason to expect that higher order radiative corrections that we have left out will cure this problem.

We have thus discovered the dirty little secret about QED -- it (probably) doesn't exist! There seems to be no consistent way of removing the cutoff unless the theory is *free* (physical charge vanishes).

This is not a disaster, though. First of all, the value of the cutoff at which the bare charge blows up is

$$1 - \frac{e_B^2}{12\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right) \sim 0 \quad \text{or} \quad \ln \frac{\Lambda^2}{m^2} \sim \frac{3\pi}{\alpha}$$

$$\Rightarrow \Lambda \sim m e^{3\pi/2\alpha} \sim m e^{646} \sim 10^{280} \text{ MeV}$$

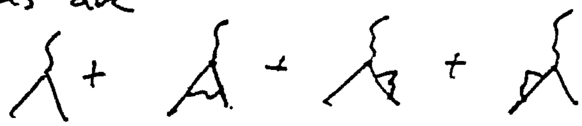
Although we cannot send Λ to ∞, there is no problem taking the cutoff to $\Lambda^{-1} \sim 10^{-290}$ cm. More importantly, it isn't necessary that a limit $\Lambda \to \infty$ exist in order for QED calculations to agree with experiment to high accuracy. QED does not describe physics accurately above some cutoff Λ anyway. But, up to corrections going like $(m/\Lambda)^P$, all of our ignorance about physics at the scale Λ can be absorbed into the definition of the renormalized parameters m and e_R (see page 2.124)

The Anomalous Magnetic Moment

Lecture by Prof. Zachariasen

Infrared divergences in QED

Consider scattering of an electron off an external electromagnetic field. To one-loop order, the graphs are



The infrared divergence is in

since the electron propagators are (in the denominator)

$$\frac{1}{(k+p)^2-m^2}=\frac{1}{k^2+2p\cdot k},$$

the loop integration behaves like $\int \frac{d^4k}{k^4}$ for small k (logarithmic infrared divergence). Let's calculate the IR divergent part of the graph by giving the photon a fictitious mass μ, and finding the part which diverges as $\mu\to 0$. To keep things simple, we'll calculate only the leading term for

$$|q^2| = |(p-p')^2| \gg m^2 \gg \mu^2$$

Note here

$$(ie)^2\int\frac{d^4k}{(2\pi)^4}\,\bar u(p')\gamma^\nu \frac{i(\not k+\not p'+m)}{k^2+2k\cdot p'}\gamma^\mu\frac{i(\not k+\not p+m)}{k^2+2k\cdot p}\gamma_\nu u(p)\,\frac{-i}{k^2-\mu^2}$$

We'll drop k in the numerator, since we're only interested in the IR divergent part, and we'll drop m, since we're keeping only the leading term for large $|q^2|$.

$$=(ie)^2 i\int\frac{d^4k}{(2\pi)^4}\, N/[(k^2+2k\cdot p')(k^2+2k\cdot p)(k^2-\mu^2)] + \text{IR finite}$$

$$N=\bar u(p')\gamma^\nu\not p'\gamma^\mu\not p\gamma_\nu u - 4(p\cdot p')\,\bar u(p')\gamma^\mu u(p)$$

To make Feynman parameter integrals easier, combine denominators as follows:

$$\frac{1}{(K^2+2p\cdot K)}\frac{1}{(K^2+2p'\cdot K)} = \int_0^1 dx \frac{1}{[K^2+2p_x\cdot K]^2} \qquad p_x = xp+(1-x)p'$$

$$\frac{1}{K^2-\mu^2}\times \text{above} = \int_0^1 dy\, 2y \int_0^1 dx \frac{1}{[(K+yp_x)^2-y^2p_x^2-(1-y)\mu^2]^3}$$

Shift doesn't change numerator,

$$\hat{A} \simeq -ie^2\int_0^1 2y\,dy \int_0^1 dx \int \frac{d^4K}{(2\pi)^4}[K^2-y^2p_x^2-(1-y)\mu^2]^{-3}\,[\bar{u}\gamma^\mu u \times (4p\cdot p')]$$

$$= -ie^2\int_0^1 dy\, 2y\int_0^1 dx \frac{(i)(-1)\frac{1}{2}}{16\pi^2}\frac{1}{y^2p_x^2+(1-y)\mu^2}(\bar{u}\gamma^\mu u)(4p\cdot p')$$

To take $\mu^2\to 0$ limit of y integral, note that log divergence at $y^2=0$ in

$$\frac{1}{p_x^2}\int_0^1 \frac{dy^2}{y^2+(1-y)(\mu^2/p_x^2)}$$

is cutoff by μ^2. For $\mu^2\to 0$, we can replace $(1-y)$ by 1, and evaluate

$$\sim \frac{1}{p_x^2}\ln\left[\frac{1+\mu^2/p_x^2}{(\mu^2/p_x^2)}\right] + \underset{\text{as } \mu^2\to 0}{\text{vanishing}}$$

Now we have

$$\hat{A} = \frac{-e^2}{32\pi^2}\bar{u}(p')\gamma^\mu u(p)(4p\cdot p')\int_0^1 dx \frac{1}{p_x^2}\ln\left(\frac{p_x^2}{\mu^2}\right)$$

$$+ (\text{finite for } \mu^2\to 0)$$

$$p_x^2 = xp+(1-x)p' = [x^2+(1-x)^2]m^2 + x(1-x)(2m^2-q^2)$$
$$= m^2 - x(1-x)q^2$$

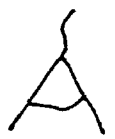
So we need to evaluate

$$\int_0^1 dx \frac{1}{m^2-x(1-x)q^2}\ln\frac{m^2-x(1-x)q^2}{\mu^2} \sim \ln\frac{-q^2}{\mu^2}\int_0^1 dx \frac{1}{m^2-x(1-x)q^2}$$

for $q^2 \lesssim 0$ and $|q^2| \gg m^2$

We could evaluate this integral explicitly, but it is clear that, as $m^2 \to 0$, there are log divergences at $x \to 0$ and $x \to 1$, cut off by m^2. So we have

$$\sim \frac{-2}{q^2} \ln -q^2/m^2 \ln -q^2/\mu^2 \quad , \text{ or}$$



$$\sim \underbrace{(\bar{u}\gamma^\mu u)}_{\text{zeroth order result}}\left(\frac{-e^2}{8\pi^2} \ln(-q^2/m^2) \ln(-q^2/\mu^2)\right)$$

(The double log, which takes the above form for $q^2 \gg m^2 \gg \mu^2$, would become $\ln^2(-q^2/\mu^2)$ + subleading log if we took $m^2 \to 0$ limit with μ^2 fixed.)

We see that the $O(\alpha)$ interference term in — $\Big|$  +  $\Big|^2$ is infrared divergent.

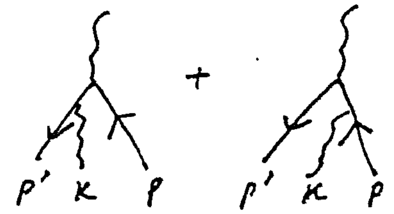
$$\boxed{(d\sigma)_{\text{coulomb}} = (d\sigma_0)_{\text{coulomb}}\left[1 - \frac{\alpha}{\pi} \ln(-q^2/\mu^2) \ln(-q^2/m^2) + \cdots\right]}$$

What is going on? How can coulomb scattering cross section be infinite in $O(\alpha)$? A key hint comes from the realization that the divergence is due to arbitrarily soft virtual photons.

Bremsstrahlung

The problem is, we have asked an unphysical question. In an experiment with nonzero energy resolution, we cannot tell whether the recoiling electron is accompanied by an arbitrarily soft photon; i.e., a real photon emitted during the scattering process.

Amplitude for emission of one photon is



$$= (ie)\,\bar u(p')\gamma^\nu \frac{i(\not p'+\not k+m)}{k^2+2p'\cdot k}\gamma^\mu u(p)\,\epsilon_\nu$$
$$+ (ie)\,\bar u(p')\gamma^\mu \frac{i(\not p-\not k+m)}{k^2-2p\cdot k}\gamma^\nu u(p)\,\epsilon_\nu$$

To isolate IR sensitive term (very soft real photon), ignore k in numerator

$$= e\,\bar u(p')\gamma^\mu u(p)\left(\frac{p\cdot\epsilon}{p\cdot k}-\frac{p'\cdot\epsilon}{p'\cdot k}\right)$$

therefore

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\;\underbrace{e^2\sum_\lambda\left(\frac{\epsilon\cdot p}{k\cdot p}-\frac{\epsilon\cdot p'}{k\cdot p'}\right)^2}_{\text{matrix element squared, summed over photon polarizations}}\;\underbrace{\frac{d^3k}{(2\pi)^3\,2k}}_{\text{photon phase space}}$$

Polarization sum $\quad \sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)} = -\eta_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}$

Ignoring m^2 compared to q^2

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\,\frac{2e^2\,p\cdot p'}{(k\cdot p)(k\cdot p')}\,\frac{d^3k}{(2\pi)^3\,2k}$$

Integrating over photon phase space,

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\,\frac{e^2\,p\cdot p'}{(2\pi)^3}\int dk\,k\int d\Omega_{\hat k}\,\frac{1}{(k\cdot p)\,k\cdot p'}$$

The angular integral can be done by the Feynman trick:

$$\int d\Omega_{\hat k}\int_0^1 dx\,\frac{1}{(k\cdot p_x)^2}\qquad p_x = xp+(1-x)p'$$

$$=\int_0^1 dx\;2\pi\int_{-1}^1 d\cos\theta\,\frac{1}{k^2[E_x-|\vec p_x|\cos\theta]^2}$$

$$=\frac{2\pi}{k^2}\int_0^1 dx\,\frac{1}{|\vec p_x|}\left[\frac{1}{E_x-|\vec p_x|}-\frac{1}{E_x+|\vec p_x|}\right]$$

$$=\frac{4\pi}{k^2}\int_0^1 dx\,\frac{1}{p_x^2} = \frac{4\pi}{k^2}\int_0^1\frac{dx}{m^2-x(1-x)q^2}\simeq\frac{8\pi}{-k^2q^2}\ln\left(-q^2/m^2\right)$$

or

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\, \frac{e^2}{2\pi^2}\left(\int \frac{dK}{K}\right) \ln(-q^2/m^2)$$

or

$$\boxed{(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\, \frac{\alpha}{\pi} \ln\left(\frac{K_{max}^2}{\mu^2}\right) \ln\left(-q^2/m^2\right)}$$

is the cross section for emission of a bremsstrahlung photon with momentum $|\vec{k}| < K_{max}$ (the photon mass μ has been reinstated here.)

The cross section for soft photon emission is also infrared divergent, but the sum

$$(d\sigma)_{coulomb} + (d\sigma)_{brem} = (d\sigma_0)_{coul}\left[1 - \frac{\alpha}{\pi} \ln(-q^2/m^2) \ln(-q^2/K_{max}^2)\right]$$

is finite in $O(\alpha)$, for a finite photon resolution K_{max}.

The cancellation of IR divergences due to real and virtual photons works to all orders (S. Weinberg, Phys. Rev. 140, B516 (1965).) The leading (infrared) logs of q^2/K_{max}^2 and q^2/m^2 can be summed to all orders. They exponentiate (D.R. Yennie, S. Frautschi, and H. Suura, Ann. Phys. 13, 379 (1961)):

$$(d\sigma)_{coul} + (d\sigma)_{brem} = (d\sigma_0)_{coul} \exp\left[-\frac{\alpha}{\pi} \ln(-q^2/m^2) \ln(-q^2/K_{max}^2)\right]$$

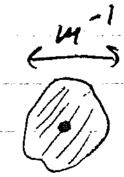
So the cross section does not really blow up as the resolution shrinks to zero

If we wish to take also the limit $m \to 0$, we must exercise more care to obtain finite quantities. Beginning in next order the process

 can occur, $\gamma \to e^+e^-$.

The final state can contain an arbitrary number of soft real e^+e^- pairs. ("Colinear divergence": one massless particle cannot be distinguished from several colinear massless particles.)

The Anomolous Magnetic Moment



We have seen that an electron is surrounded by a polarization cloud that screens its bare charge. As a result, the charge of the electron is not really a point charge; it is smeared out in space over a region with a size of order m^{-1}, the electron Compton wavelength. The distribution of electric charge surrounding the electron (or any charged particle) can be described by a form factor.

Let us present the theory of the Form factor in a general context. We can probe the electromagnetic structure of the electron by observing how the electron responds to an external electromagnetic field.

That is, we may divide the gauge field A_μ into a "classical" background term (which we control in the Laboratory) and a fluctuating "quantum term",

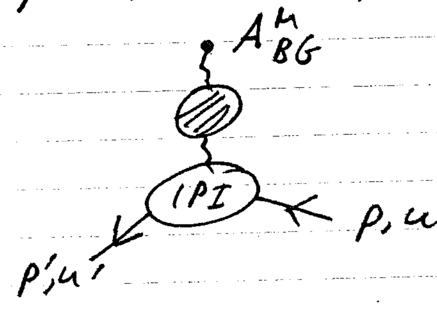
$$(A^\mu)_{total} = A^\mu_{BG} + \underbrace{A^\mu}_{\text{"quantum" field}}$$

the interaction term of (spinor) QED is then

$$\mathcal{L}_I = -e\,\bar\psi\gamma_\mu\psi\,(A^\mu_{BG} + A^\mu)$$

where only the quantum field A^μ "propagates"; that is A^μ_{BG} does not appear as an internal line in Feynman diagrams.

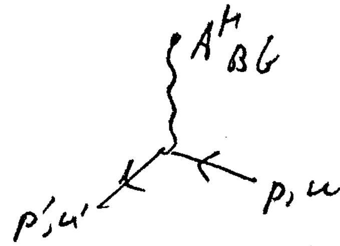
The scattering of an electron (on-shell) off of an external electromagnetic field is described by Feynman diagrams of the form:



Here A^ν ~ ~ A^μ_{BG} is the exact photon propagator, but with one factor of

$$\frac{-i\eta}{k^2}$$

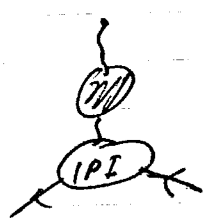
(the free propagator) "amputated". The lowest order diagram is



$$= (-ie)\tilde{A}^\mu_{BG}(q)\,\bar{u}'\gamma^\mu u$$

(where $q = p' - p$ is the momentum transfer)

When higher order corrections are included, the result may be written



$$= (-ie)\tilde{A}^\mu_{BG}(q)\bar{u}'F_\mu u$$

where $F_\mu = \gamma_\mu + \mathcal{O}(e^2)$

The form of F_μ is restricted by the symmetries of electrodynamics; the amplitude is a Lorentz scalar, invariant under parity and charge conjugation. Thus, F_μ is a 4-vector, odd under P and C.

Further, F_μ depends only on the momentum q that flows through the diagram the four-vectors that we can construct are:

$$q_\mu,\ q_\mu\gamma_5,\ \gamma_\mu,\ \gamma_\mu\gamma_5,\ \sigma_{\mu\nu}q^\nu,\ \epsilon_{\mu\nu\lambda\sigma}\sigma_{\lambda\sigma}q^\nu$$

but only γ_μ and $\sigma_{\mu\nu}q^\nu$ have the right transformation properties under C and P. Hence, on general grounds, F_μ can be expressed in terms of two Lorentz-invariant functions of q^2. We have

$$F_\mu(q) = \gamma_\mu F_1(q^2) - \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)$$

The functions $F_{1,2}$ are the <u>form factors</u> of the electron; F_1 is called the <u>Dirac</u> form factor and F_2 is the <u>Pauli</u> form factor.

The coupling of the background field to the electron thus has the form

$$iA = -ie\tilde{A}^\mu_{BG}(q)\Big[\bar{u}'\gamma_\mu u F_1(q^2) - \frac{i}{2m}\bar{u}'\sigma_{\mu\nu}u\, q^\nu F_2(q^2)\Big]$$

The Dirac factor $F_1(q^2)$ can be interpreted as giving the <u>shape</u> of the charge distribution, since it describes how the electron recoils when hit by a photon of momentum q. Furthermore

$$F_1(q^2=0) = 1,$$

if e is to be interpreted as the <u>physical</u> charge of an electron, as measured at low momentum transfer. (The same is true for any spin-$\frac{1}{2}$ charged particle, if e is the physical charge. E.g., even though the proton feels nonelectromagnetic interactions, it obeys $F_1(0) = 1$.)

It is instructive to split this amplitude into a spin-independent and spin-dependent part, in the nonrelativistic limit, $\vec{p}, \vec{p}\,' \ll m$. First, note that

$$\tilde{F}^{\mu\nu}_{BG} = -i\left(q^\mu \tilde{A}^\nu_{BG} - q^\nu \tilde{A}^\mu_{BG}\right)$$

so that $\quad \sigma_{\mu\nu}\, q^\nu \tilde{A}^\mu_{BG}(q) = -\frac{i}{2}\sigma_{\mu\nu}\tilde{F}^{\mu\nu}_{BG}(q)$,

and

$$iA = -i\Big[\tilde{A}^\mu_{BG}\, e\,\bar{u}'\gamma_\mu u\, F_1(q^2) - \frac{e}{2m}\bar{u}'\sigma_{\mu\nu} u\, \tilde{F}^{\mu\nu}_{BG}\, F_2(q^2)\Big].$$

Next, we rewrite the vector part as

$$\bar{u}'\gamma_\mu u = \frac{1}{2m}\bar{u}'\left(\not{p}'\gamma_\mu + \gamma_\mu \not{p}\right)u$$

where

$$\not{p}'\gamma_\mu = \tfrac{1}{2}[\not{p}';\gamma_\mu]_- + \tfrac{1}{2}[\not{p}';\gamma_\mu]_+$$
$$= -i\sigma_{\mu\nu}p'^\nu + p'_\mu$$
$$\gamma_\mu \not{p} = i\sigma_{\mu\nu}p^\nu + p_\mu$$

So, $\quad \bar{u}'\gamma_\mu u = \frac{1}{2m}(p+p')_\mu \bar{u}'u - \frac{i}{2m}\bar{u}'\sigma_{\mu\nu}u\, q^\nu$

(where $q = p' - p$)

(This identity is called the <u>Gordon Decomposition</u>.)

Therefore,

$$iA = -i\Big[\frac{e}{2m}(p+p')_\mu \, \bar{u}'u \, \tilde{A}^\mu_{BG}(q) \, F_1(q^2)$$

$$- \frac{e}{4m} \tilde{F}^{\mu\nu}_{BG}(q) \left(F_1(q^2) + F_2(q^2) \right) \bar{u}' \sigma_{\mu\nu} u \Big]$$

Now, in the extreme nonrelativistic limit $\vec{p}, \vec{p}\,' \ll m$, the spinors, in the standard basis have the form

$$u = \begin{pmatrix} \omega_1 \\ \omega_2 \\ 0 \\ 0 \end{pmatrix}$$ (see page 3.58)

And, in this basis, $\sigma_{ij} = -\epsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

$$\sigma_{0i} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Thus,

$$\bar{u}'u = \omega'^+\omega$$

$$\bar{u}'\sigma_{0i} u = 0$$

$$\bar{u}'\sigma_{ij} u = -\epsilon_{ijk} \, \omega'^+ \sigma^k \omega$$

And the spin-dependent term in the amplitude has the form

$$iA_{spin} = i \frac{e}{2m} \omega^+ \vec{\sigma} \cdot \vec{B} \, \omega \left(F_1(q^2) + F_2(q^2) \right)$$

This can be compared to the matrix element

$$\langle - | -iH | - \rangle$$

of a Hamiltonian

$$H = -\mu \frac{1}{2} \vec{\sigma} \cdot \vec{B}$$

where $$\mu = \frac{e}{m}(1 + F_2(0))$$

is the magnetic moment of the electron (in the limit $\vec{q} \to 0$). The term

$$\frac{e}{m} \quad \text{or} \quad g\frac{e}{2m} \quad (\text{with } g = 2)$$

is called the Dirac magnetic moment. $F_2(0)$ is a correction of order e^2, called the anomalous magnetic moment.

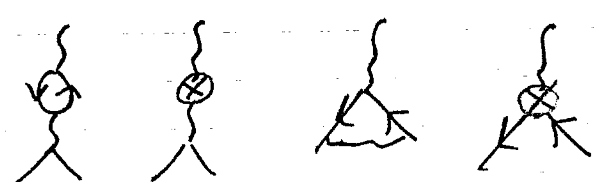
Actually, for a charged particle (like the proton) that is subject to strong nonelectrodynamic interactions, the anomalous moment need not be small.

Computation of the anomalous moment in order e^2

Let us compute the one-loop contribution to $F_2(0)$, and hence to

$$\left(\frac{g-2}{2}\right)_e = 1 + F_2(0).$$

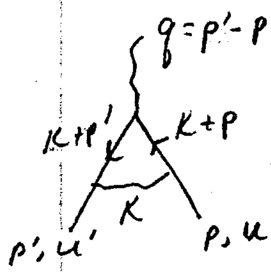
Diagrams contributing to scattering of an electron by an external field in one loop are



all except this one contribute to $F_1(q^2)$ only.

5.108

So we need to compute this diagram:



$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}'\gamma^\nu \frac{i(\not{k}+\not{p}'+m)}{(k+p')^2-m^2}\gamma^\mu \frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\gamma^\lambda u \;\frac{-i\eta_{\nu\lambda}}{k^2}$$

$$= -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\text{Num}}{\text{Denom}},$$

where

$$\text{Num} = \bar{u}'\gamma^\nu(\not{k}+\not{p}'+m)\gamma^\mu(\not{k}+\not{p}+m)\gamma_\nu u$$

To simplify the numerator, we use the identities:

$$\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$$

$$\gamma^\nu\gamma^\mu\gamma^\lambda\gamma_\nu = -\gamma^\nu\gamma^\mu\gamma_\nu\gamma^\lambda + 2\gamma^\lambda\gamma^\mu = 4\eta^{\mu\lambda}$$

$$\begin{aligned}\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\sigma\gamma_\nu &= -\gamma^\nu\gamma^\lambda\gamma^\mu\gamma_\nu\gamma^\sigma + 2\gamma^\sigma\gamma^\lambda\gamma^\mu \\ &= -4\eta^{\lambda\mu}\gamma^\sigma - 2\gamma^\sigma\gamma^\mu\gamma^\lambda + 4\gamma^\sigma\eta^{\lambda\mu} \\ &= -2\gamma^\sigma\gamma^\mu\gamma^\lambda\end{aligned}$$

Hence,

$$\text{Num} = \bar{u}'\left[-2(\not{k}+\not{p})\gamma^\mu(\not{k}+\not{p}') - 2m^2\gamma^\mu + 4m(2k+p+p')^\mu\right]u$$

The denominator is

$$\frac{1}{\text{denom}} = \frac{1}{(k+p')^2-m^2}\,\frac{1}{(k+p)^2-m^2}\,\frac{1}{k^2}$$

To simplify this, use the identity

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy\,\left[ax+by+c(1-x-y)\right]^{-3}$$

Since $p^2 = p'^2 = m^2$ (electron is "on shell"), we have

$$\frac{1}{\text{denom}} = 2\int_0^1 dx \int_0^{1-x} dy \; [k^2 + 2x\, k\cdot p' + 2y\, k\cdot p]^{-3}$$

Now, complete the square:

$$[\quad] = (k + xp' + yp)^2 - x^2 m^2 - y^2 m^2 - 2xy\, p\cdot p'$$

and $\quad 2p\cdot p' = -(p'-p)^2 + 2m^2 = -q^2 + 2m^2$, so

$$[\quad] = (k + xp' + yp)^2 - (x+y)^2 m^2 + xy\, q^2.$$

Next, we shift the loop momentum

$$k \to k - xp' - yp,$$

and then

$$\boxed{\frac{1}{\text{denom}} = 2\int_0^1 dx \int_0^{1-x} dy \; [k^2 - (x+y)^2 m^2 + xy\, q^2]^{-3},}$$

while the numerator becomes

$$\text{Num} = \bar{u}'\Big[-2\, \not{k}\gamma^\mu \not{k} - 2[(1-y)\not{p} - x\not{p}']\gamma^\mu[(1-x)\not{p}' - y\not{p}]$$
$$-2m^2\gamma^\mu + 4m[(1-2x)p' + (1-2y)p]^\mu\Big]u$$
$$+ \text{ terms linear in } k$$

(and terms linear in k will integrate to zero)

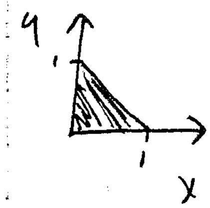
To simplify further, write, in the second term

$\not{p} = \not{p}' - \not{q}$ (on left side of γ^μ) $\quad$ and use $\quad \bar{u}'\not{p}' = \bar{u}' m$

$\not{p}' = \not{p} + \not{q}$ (on right side of γ^μ) $\quad\quad\quad\quad\quad\; \not{p}u = mu$

and so obtain

$$\text{Num} = \bar{u}'\Big[-2\not{k}\gamma^\mu\not{k}$$
$$-2[-(1-y)\not{q} + (1-x-y)m]\gamma^\mu[(1-x)\not{q} + (1-x-y)m]$$
$$-2m^2\gamma^\mu + 4m[(1-2x)p' + (1-2y)p]^\mu\Big]u$$



Now notice the integration region in the x-y plane is symmetric under the interchange $x \leftrightarrow y$, as is the denominator. So only the piece of the numerator invariant under $x \leftrightarrow y$ survives integration.

So the term

$$-2\Big(-(1-y)(1-x)\not{q}\gamma^\mu\not{q} + (1-x-y)(1-x)m\gamma^\mu\not{q}$$
$$-(1-x-y)(1-y)m\not{q}\gamma^\mu + (1-x-y)^2 m^2\gamma^\mu\Big)$$

can be simplified by symmetrizing in $x \leftrightarrow y$

$$(1-x)\gamma^\mu\not{q} - (1-y)\not{q}\gamma^\mu \to (2-x-y)\tfrac{1}{2}[\gamma^\mu, \gamma^\nu]q_\nu$$
$$= (2-x-y)(i\sigma^{\mu\nu})q_\nu$$

Also $\not{q}\gamma^\mu\not{q} = -\gamma^\mu q^2 + 2q^\mu\not{q}$, and $\bar{u}'\not{q}u = \bar{u}'(\not{p}' - \not{p})u = 0$, so

$$\text{term} = -2\Big((1-y)(1-x)q^2\gamma^\mu + (1-x-y)(2-x-y)m(i\sigma^{\mu\nu})q_\nu$$
$$+ (1-x-y)^2 m^2\gamma^\mu\Big)$$

Likewise, if we symmetrize

$$(1-2x)\,p' + (1-2y)\,p$$

we get $\quad (1-x-y)(p+p')$.

So now we have

$$\begin{aligned} Num = -2\bar{u}'\big[\, & \not{k}\gamma^\mu\not{k} + (1-y)(1-x)q^2\gamma^\mu \\ & + (1-x-y)(2-x-y)\,m(i\sigma^{\mu\nu})q_\nu + (1-x-y)^2 m^2\gamma^\mu \\ & + m^2\gamma^\mu - 2m(1-x-y)(p+p')^\mu \big]\, u \end{aligned}$$

When we integrate, we may replace

$$\begin{aligned} \not{k}\gamma^\mu\not{k} = k_\nu k_\lambda \gamma^\nu\gamma^\mu\gamma^\lambda \quad & \text{by} \\ \frac{1}{D}\eta_{\nu\lambda} k^2 \gamma^\nu\gamma^\mu\gamma^\lambda & = -\frac{2}{D}k^2\gamma^\mu \end{aligned}$$

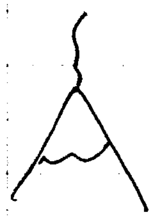
(in D dimensions)

To the $p+p'$ term, we may apply the Gordon decomposition:

$$\bar{u}'(p+p')^\mu u = 2m\bar{u}'\gamma^\mu u + i\bar{u}'\sigma^{\mu\nu}u\,q_\nu$$

$$\boxed{\begin{aligned} Num = -2\bar{u}'\big[& -\tfrac{2}{D}k^2\gamma^\mu + (1-y)(1-x)q^2\gamma^\mu \\ & + (1-x-y)^2 m^2\gamma^\mu + m^2\gamma^\mu - 4m^2(1-x-y)\gamma^\mu \\ & - (1-x-y)(x+y)\,m\,(i\sigma^{\mu\nu})q_\nu \big]\, u \end{aligned}}$$

As we anticipated, all terms are proportional to either γ^μ or $\sigma^{\mu\nu}q_\nu$. To compute the contribution to $F_2(q^2)$, we consider the $\sigma^{\mu\nu}q_\nu$ terms



$$= 2e^3(2)\int_0^1 dx \int_0^{1-x} dy \int \frac{d^4k}{(2\pi)^4}\, \bar{u}' \frac{[(1-x-y)(x+y)m(-i\sigma^{\mu\nu}q_\nu)]u}{[k^2-(x+y)^2m^2+xyq^2]^3}$$

$+ \ \gamma^\mu$ terms

Or -- since

$$iA = (-ie)\left(\frac{-i}{2m}\right)\bar{u}'\sigma^{\mu\nu}u\,q_\nu F_2(q^2)$$

We have

$$F_2(q^2) = 8e^2 i m^2 \int_0^1 dx \int_0^{1-X} dy \int \frac{d^4k}{(2\pi)^4}\frac{(1-x-y)(x+y)}{[k^2-(x+y)^2m^2+xyq^2]^3}$$

+ higher order

Now we can rotate to Euclidean space and do the k integral. Of course, the result is finite. Otherwise we would need to introduce a dimension 5 counterterm, and the theory would not be renormalizable.

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-a^2)^3} = i\int\frac{d^4k_E}{(2\pi)^4}\frac{(-1)}{(k_E^2+a^2)^3} = \frac{-i}{32\pi^2a^2}$$

Thus,

$$F_2(q^2) = \frac{e^2}{4\pi^2}\int_0^1 dx\int_0^{1-x}dy\,\frac{(1-x-y)(x+y)m^2}{(x+y)^2m^2-xyq^2}$$

For the purpose of computing the anomalous magnetic moment, we are interested in $F_2(0)$, which now becomes an elementary integral

$$F_2(0) = \frac{e^2}{4\pi^2}\int_0^1 dx \int_0^{1-x} dy \; \frac{(1-x-y)}{x+y}$$

This integral is

$$\int_0^\infty dx\,dy\,dz\;\delta(x+y+z-1)\;\frac{z}{1-z}$$

$$= \int_0^1 dz \int_0^{1-z} dx \;\frac{z}{1-z} = \int_0^1 dz\, z = \frac{1}{2},$$

and therefore

$$F_2(0) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}$$

where $\alpha = e^2/4\pi = (137.036)^{-1}$ is the fine structure constant. We conclude

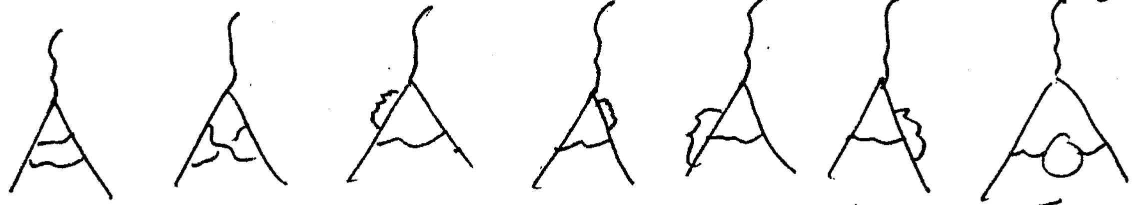
$$\boxed{\left(\frac{g-2}{2}\right)_{\text{electron}} = \frac{\alpha}{2\pi} + O(\alpha^2) \cong .00116}$$

This is a famous result derived by Schwinger (1948).

Schwinger's calculation was done soon after $g-2$ was first measured. Since then heroic effort has gone into improving both the theoretical calculation and the measured value.

In order α^2:

There are seven diagrams that contribute to $F_2(0)$.



(+ counterterms)

These were computed by Sommerfield and Petermann (1957) with the result:

$$\left(\frac{g-2}{2}\right)_{\text{electron}} \sim \frac{\alpha}{2\pi} - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2$$

In order α^3:

There are 72 diagrams that contribute. These were computed by many authors (1968-82) with the result

$$+1.1765(13)\left(\frac{\alpha}{\pi}\right)^3$$

In order α^4:

There are 891 diagrams that contribute. These were computed by Kinoshita and Lindquist (1981). It took four years of hard work, with the result

$$-(0.8)(2.5)\left(\frac{\alpha}{\pi}\right)^4$$

(The accuracy has been improved somewhat recently.)

The best current theoretical value is

$$\left(\frac{g_e}{2}\right)_{theory} = 1.001,159,652,263\,(22)(104)$$

(Kinoshita, 1986)

The first error is due to uncertainty in the calculation. The second (dominant) error is actually due to uncertainty in the measured value of α

$$\alpha^{-1} = 137.0359815\,(123)$$

(from AC Josephson effect experiments).

The best current experimental value is

$$\left(\frac{g_e}{2}\right)_{experiment} = 1.001,159,652,188\,(4)$$

(Van Dyck, Schwinberg, Dehmelt, Phys. Rev. Lett. 59 (1987) 26.)

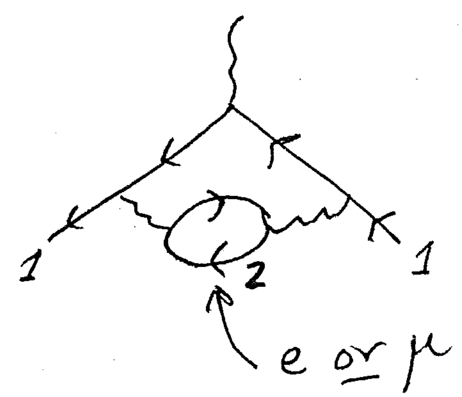
The accuracy of this measurement is truly astonishing. And it agrees with theory, within the errors. In order to improve the accuracy of this very stringent test of QED, we will need a more accurate measured value for α. (As things stand now, the $g_e - 2$ experiment can be regarded as the most accurate determination of α. What is desired is another determination of comparable accuracy.)

The measured and calculated values of the anomalous magnetic moment of the muon are also in agreement, but the accuracy of the experiment and calculation is not so spectacular as for the electron. To understand why the calculated value for the muon is less accurate than for the electron, consider how g_μ and g_e differ.

The muon has all the same (electromagnetic and weak) couplings as the electron, but it has a different mass

$$\frac{m_\mu}{m_e} = \frac{105.6591 \text{ MeV}}{.511003 \text{ MeV}} = 206.768$$

The order (α) contribution to g does not depend on the mass, but there is mass dependence in order (α^2) arising from the diagram:



If fermion 2 appearing in the loop is not the same as fermion 1 on the external line, then the diagram gives a contribution to g_1 that is a function of m_2/m_1. (It depends on this dimensionless ratio, because g_1 is dimensionless.)

Thus, the diagram with the μ loop contributes to g_e

$$\left(\frac{\alpha}{\pi}\right)^2 f\left(\frac{m_\mu}{m_e}\right),$$

while the diagram with the e loop contributes to g_μ

$$\left(\frac{\alpha}{\pi}\right)^2 f\left(\frac{m_e}{m_\mu}\right).$$

We can estimate how the function $f(x)$ behaves for $x \gg 1$ and $x \ll 1$. To do this, note that

 is just the one loop photon propagator correction, and the full photon propagator satisfies a spectral representation

$\Delta_{\mu\nu}$ 
$$= -i\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{k^2} + \int \frac{da^2\, \rho(a^2)}{k^2 - a^2}\right)$$
+ (gauge-dependent terms)

(This is derived as on page (2.148).) The electron contribution to the spectral function $\rho(a^2)$ begins at the e^+e^- threshold $a^2 = 4m_e^2$, and the muon contribution begins at $a^2 = 4m_\mu^2$.

If we were to include a mass a^2 in the photon propagator of the one loop diagram

, then our expression for $F_2(0)$ (page 5.113)

would become

$$F_2(q^2=0;a^2) = \frac{e^2}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy \; \frac{(1-x-y)(x+y)\,m^2}{m^2(x+y)^2+(1-x-y)a^2}$$

$$= \frac{e^2}{4\pi^2}\int_0^1 dz \; \frac{z(1-z)^2 m^2}{(1-z)^2 m^2 + z a^2}$$

For $a^2 \gg m^2$, we have

$$F_2(0) = \frac{e^2}{4\pi^2}\frac{m^2}{a^2}\int_0^1 dz\,(1-z)^2 = \frac{e^2}{12\pi^2}\frac{m^2}{a^2}$$

Therefore, the muon loop contribution to g_e is

$$(g_e)_{muon} \sim \left(\frac{\alpha}{\pi}\right)^2\left(\frac{m_e}{m_\mu}\right)^2$$

in this order. A careful calculation gives

$$(g_e)_{muon} \sim 2.8\times 10^{-12}$$

which is small (even compared to $(\alpha/\pi)^4$ contribution $\sim 2\times 10^{-11}$).

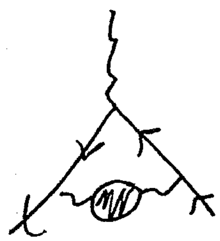
For $a^2 \ll m^2$, $F_2(q^2=0;a^2=0) = e^2/8\pi^2$, as we calculated before. But then we have a contribution from the electron loop to g_μ going like

$$(g_\mu)_{electron} \sim \left(\frac{\alpha}{\pi}\right)^2\int_{m_e^2}^{m_\mu^2} da^2\,\rho(a^2)$$

By dimensional analysis, $\rho(a^2) \sim \frac{1}{a^2}$ for $m_e^2 \ll a^2 < m_\mu^2$ so

$$(g_\mu)_{electron} \sim \left(\frac{\alpha}{\pi}\right)^2 \ln\left(m_\mu^2/m_e^2\right).$$

So the electron loop contribution to g_μ is actually the <u>dominant</u> contribution in order α^2.



Perhaps a better way to understand the logarithm is to note that the diagram shown (where is exact propagator) begins to converge (or is <u>peaked</u>) when more the loop momentum K is of order m_μ (if the external fermion is μ) or when the loop momentum is m_e (if the external fermion is e).

The effect of

(as we saw on page 5.93) is to renormalize the charge to e_μ with $\mu \sim \sqrt{K^2}$.

So the logarithmic term is

$$\frac{1}{2\pi}\alpha(\mu = m_\mu) - \frac{1}{2\pi}\alpha(\mu = m_e)$$

But $e^2_{\mu'} = e^2_\mu + \frac{e_\mu^4}{12\pi^2}\ln \mu'^2/\mu^2$ (we saw on page 5.87) so this is

$$\frac{1}{2\pi}\,\frac{\alpha^2}{3\pi}\ln m_\mu^2/m_e^2 = \frac{1}{6}\left(\frac{\alpha}{\pi}\right)^2 \ln m_\mu^2/m_e^2$$

Here we have even computed the coefficient of the logarithm. This argument also makes it clear that higher powers of $\log(m_\mu/m_e)$ will arise with higher powers of α.

There is also a hadronic contribution to

and hence to g_μ and g_e. The threshold for hadronic charged particles is at $4m_\pi^2$, so our spectral function argument shows that is is of order

$$(g_e)_{\text{hadron}} \sim \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_e}{m_\pi}\right)^2$$

(which is at the 10^{-12} level again)

The hadronic contribution to g_μ is much bigger; it is estimated as

$$\left(\frac{g_\mu}{2}\right)_{\text{hadrons}} \sim 70(2) \times 10^{-9}$$

Its error is comparable to the order α^4 terms, which is

$$140(6)\left(\frac{\alpha}{\pi}\right)^4 \sim 4 \times 10^{-9}.$$

The measurement is sensitive enough (accuracy $\sim 11 \times 10^{-9}$) to test the predicted hadronic contribution to about 15%. It is not sensitive enough to test the <u>electroweak</u> contribution, which is of order 2×10^{-9}.

The anomalous magnetic moment (of the muon in particular) is an excellent place to look for <u>new</u> physics. For example suppose that the muon is actually a composite particle, with structure on the distance scale Λ^{-1}

Then if we write down an effective field theory that describes the electromagnetic interactions of muons at momenta $p^2 < \Lambda^2$, it may contain higher dimension operators such as

$$C\, \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu},$$

which would contribute of order mC to $F_2(0)$. Dimensionally, since C has dimension $(\text{mass})^{-1}$, we might guess $C \sim \Lambda^{-1}$. We should be more careful though:

$$\bar{\psi} \sigma^{\mu\nu} \psi = \bar{\psi}_R \sigma^{\mu\nu} \psi_L + \bar{\psi}_L \sigma^{\mu\nu} \psi_R$$

is a chirality-violating operator that couples ψ_R to ψ_L. Since a source of chirality violation is necessary to generate a magnetic moment, a more reasonable estimate is that C is proportional to the fermion mass,

$$C \sim \frac{m_\mu}{\Lambda^2},$$

which would contribute $\sim \left(\frac{m_\mu}{\Lambda}\right)^2$ to $F_2(0)$.

That the measured $g_\mu/2$ agrees with theory to an accuracy of 10^{-8} then places a limit on Λ

$$\left(\frac{m_\mu^2}{\Lambda^2}\right) \lesssim 10^{-8} \Rightarrow \Lambda \gtrsim 10^4 m_\mu \sim 1\,\text{TeV}$$

Thus, the measured g_μ provides estimate that the muon is pointlike down to a distance scale

$$\Lambda^{-1} \sim (1\,\text{TeV})^{-1} \sim 2 \times 10^{-17}\,\text{cm}.$$