

6. Spontaneous Symmetry Breakdown

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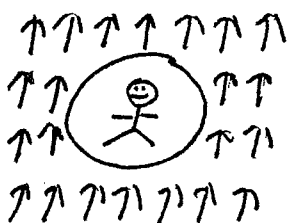
6. Spontaneous Symmetry Breakdown

The Man in the Magnet:

consider a magnet, a collection of spins (on a lattice). The spins have interactions that favor alignment of neighboring spins. E.g. nearest neighbors interact via

$$H_{\text{spin-spin}} = -J \vec{S}_1 \cdot \vec{S}_2$$

This interaction is rotationally invariant -- it depends only on the relative angle between the spins, and so is unchanged if we rotate both spins together.



But the ground state of the magnet is not rotationally invariant; all the spins align and point in some direction in space, so there is a preferred direction.

Now consider a little physicist who lives inside the magnet. The rotational invariance of the physics of the magnet is not at all obvious to him. To see it explicitly he would have to rotate all of the spins in the magnet by some angle, and thus verify that there is a physically identical magnet that has a magnetization that points in a different direction. This is not possible, even in principle, if the magnet is infinite in extent.

Thus, the underlying dynamics of the magnet has an $SO(3)$ symmetry, rotational invariance (if we ignore the breaking of the symmetry by the lattice). But the symmetry that is manifest to the little physicist is a smaller symmetry -- $SO(2)$ -- rotations about the axis picked out by the magnetization. The rest of the symmetry is hidden, because, while it is a symmetry of the dynamics, it is not a symmetry of the state of the magnet.

The symmetry of the underlying dynamics that is not a symmetry of the state is said to be spontaneously broken. In the above example, we say $SO(3)$ is spontaneously broken to $SO(2)$, or

$$SO(3) \rightarrow SO(2).$$

The word "spontaneous" emphasizes that the symmetry is really exact (not intrinsically broken) but it is not manifest.

This example raises an interesting question: Is it possible that the "Lagrangian of the World" has exact symmetries that are not manifest to us, because they are not symmetries of the vacuum state? This is the possibility that we will now pursue.

Example: Real Scalar Field

As a simple example of a field theory in which spontaneous symmetry breakdown occurs, consider a theory of a single real scalar field.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi)$$

$$U(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \quad (\lambda > 0)$$

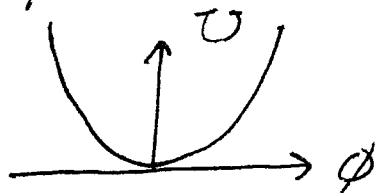
This theory has a (discrete) symmetry

$$\phi \rightarrow -\phi$$

Is this symmetry spontaneously broken?

Suppose, at first, that this theory is classical ($\hbar \rightarrow 0$) then the realization of the symmetry depends on the sign of μ^2 :

(i) $\mu^2 \geq 0$



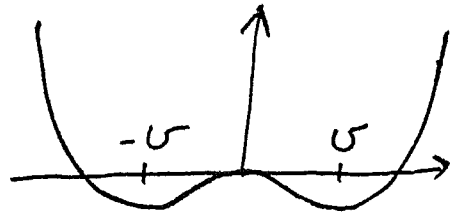
In this case, the minimum of U occurs at $\phi = 0$.
So

$$\phi(x) = 0$$

is the classical ground state. This state is invariant under the $\phi \rightarrow -\phi$ symmetry, so the symmetry is manifest. Oscillations about the minimum of the potential have a frequency determined by the mass μ .

(ii) $\mu^2 < 0$

In this case, the potential has the form



$$U(\phi) = \frac{1}{4} \lambda (\phi^2 - v^2)^2 + \text{constant}$$

where $\lambda v^2 = -\mu^2$

This potential has two minima, and there are two distinct classical ground states

$$\phi(x) = v$$

and $\phi(x) = -v$

Neither ground state is invariant under the $\phi \rightarrow -\phi$ symmetry; instead, the two states transform into one another under the symmetry. The symmetry is spontaneously broken.

We may consider small fluctuations of the field ϕ about either ground state. Choosing (arbitrarily) the ground state with $\phi = v$, we characterize small fluctuations about the ground state by writing

$$\phi = v + \phi' \quad \text{where } \phi' = \text{shifted field}$$

$$\begin{aligned} \text{Then } U(\phi') &= \frac{1}{4} \lambda [(v + \phi')^2 - v^2]^2 \\ &= \frac{1}{4} \lambda (2v\phi' + \phi'^2)^2 \end{aligned}$$

(6.5)

$$\text{or } U(\phi') = \frac{1}{4} \lambda (4v^2 \phi'^2 + 4v \phi'^3 + \phi'^4) \\ = \lambda v^2 \phi'^2 + \lambda v \phi'^3 + \frac{1}{4} \lambda \phi'^4$$

therefore, the mass of the ϕ' field, which determines the frequency of small oscillations about the minimum of U is

$$m^2 = 2\lambda v^2 = -2\mu^2.$$

Note also that $U(\phi')$ has a ϕ'^3 term; there is no hint of a $\phi \rightarrow -\phi$ symmetry when we study small fluctuations of ϕ' about the minimum (i.e. perturbation theory). To see the symmetry, we must perform the transformation

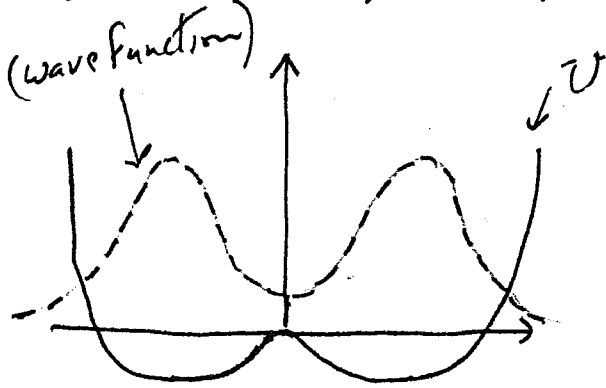
$$\begin{aligned} \phi' &\rightarrow -\phi', \\ v &\rightarrow -v, \end{aligned} \quad (\text{i.e. } \phi \rightarrow -\phi)$$

but as far as the ϕ' field is concerned, v is just a parameter in the potential, not a dynamical variable.

(Recall the little physicist in the magnet. We cannot change the ground state from $\phi=v$ to $\phi=-v$ by doing any local experiment. The potential $U(\phi)$ is left-right symmetric, but if we are sitting at one of the minima it does not look symmetrical.)

Quantum Mechanics

Now turn on $\hbar$ and consider quantum field theory instead of classical field theory. Is the symmetry still spontaneously broken?



In fact, we are accustomed to the idea that quantum mechanical ground states are much different from classical ground states, based on experience with particle mechanics. In particle mechanics ($0+1$ dimensional field theory) spontaneous symmetry breaking does not occur. The classical ground states can mix with each other by (classically forbidden) barrier penetration, and there is a unique ground state that shares the symmetry of the potential.

But quantum field theory is different. Spontaneous symmetry breakdown does not occur in a quantum system with a finite number of degrees of freedom (except in trivial cases; e.g., an infinitely high, impenetrable barrier), but it can occur in a system with an infinite number of degrees of freedom.

To appreciate this distinction, let us define more carefully what is meant

by spontaneous symmetry breakdown. Imagine that we introduce into our theory with a $\phi \rightarrow -\phi$ symmetry a symmetry-breaking perturbation

$$\mathcal{L} \rightarrow \mathcal{L} - h\phi$$

h = "external field"

Of course, the external field ϕ breaks the degeneracy of the two classical ground states; in the presence of the perturbation there is a unique ground state. Now, we will say that spontaneous symmetry breakdown occurs if

$$\lim_{h \rightarrow 0^+} \langle \phi(x) \rangle_h \neq 0$$

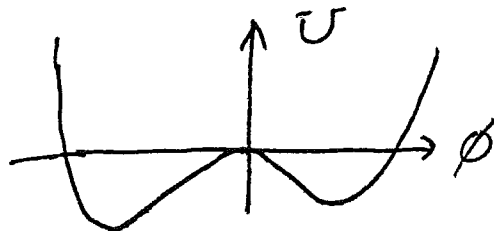
approaches zero
with definite sign

$\leadsto$ vacuum expectation value
in presence of external
field.

This is a sensible definition. In the case of the ferromagnet, we say the system has spontaneous magnetization if the magnetization remains non-zero when we turn off an external magnetic field.

By this definition, our classical field theory shows spontaneous symmetry breakdown. If $h > 0$, no matter how small, the ground state has $\phi < 0$, and

$$\lim_{h \rightarrow 0^+} \langle \phi \rangle_h = -v$$



But in particle quantum mechanics, as $\hbar \rightarrow 0$ the two classical ground states mix by tunneling. Although $\langle \phi \rangle \neq 0$ when $\hbar \neq 0$,

$$\lim_{\hbar \rightarrow 0^+} \langle \phi \rangle_{\hbar} = 0,$$

and there is no spontaneous symmetry breaking.

How is field theory different? In particle mechanics, with the external field on, the two classical ground states differ in energy by a finite amount, and so can mix with each other. But in field theory, in an infinite spatial volume, the splitting in energy between the states is infinite for arbitrarily small $\hbar$. Thus we can have

$$\lim_{\hbar \rightarrow 0} \lim_{\Omega \rightarrow \infty} \langle \phi \rangle_{\hbar, \Omega} \neq 0$$

although $\lim_{\Omega \rightarrow \infty} \lim_{\hbar \rightarrow 0} \langle \phi \rangle_{\hbar, \Omega} = 0$

(where Ω is the spatial volume). In other words, spontaneous symmetry breakdown can occur only in the infinite-volume limit.

(The same is true in statistical physics at any nonzero temperature. If the volume is finite, thermal fluctuations always remove long range order as the external field

is turned off. Indeed, we have seen that thermal fluctuations in classical statistical mechanics in D spatial dimensions behave the same way as quantum fluctuations in quantum field theory in D spacetime dimensions.)

Vacuum Degeneracy in Quantum Theory

So we have seen that for quantum field theory in the infinite volume limit, spontaneous symmetry breakdown is possible (quantum tunneling cannot occur globally in all of space), and there can be degenerate vacua. But quantum mechanics differs from classical mechanics in another important way -- there is a superposition principle. In our real scalar field theory, we have ground states $|+\rangle$ and $|-\rangle$, with

$$\begin{aligned} \langle + | \phi(x) | + \rangle &= v \\ \langle - | \phi(x) | - \rangle &= -v \end{aligned} \quad \left. \vphantom{\begin{aligned} \langle + | \phi(x) | + \rangle &= v \\ \langle - | \phi(x) | - \rangle &= -v \end{aligned}} \right\} \begin{array}{l} \text{(Not necessarily} \\ \text{the same } v \\ \text{as in the} \\ \text{classical} \\ \text{theory.)} \end{array}$$

But any linear combination of these is also a ground state. E.g.

$$|S\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$$

$$|A\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle)$$

6.10

These states are invariant under the symmetry operation that interchanges $|+\rangle$ and $|-\rangle$ (Actually, $|A\rangle \rightarrow -|A\rangle$, but $|A\rangle$ and $-|A\rangle$ describe the same state.) It appears, then, that it is possible to choose a vacuum state such that the symmetry is not spontaneously broken.

But there is an important distinction between the $|+\rangle, |-\rangle$ vacua and the $|S\rangle, |A\rangle$ vacua, that makes the $|+\rangle, |-\rangle$ preferable as physical ground states: Namely, the matrix elements of local operators are diagonal in the $|+\rangle, |-\rangle$ basis, but not in the $|S\rangle, |A\rangle$ basis.

We have

$$\langle - | \mathcal{O} | + \rangle = 0$$

if $\mathcal{O}$ is any local operator. Think of the physicist in the magnet. Acting locally inside the magnet, he can turn over the spins in only a finite volume of the infinite magnet. The state with a finite number of spins turned is orthogonal to the other vacuum, which has an infinite number of spins turned.

But, on the other hand,

$$\begin{aligned} \langle A | \phi(x) | S \rangle &= \frac{1}{2} (\langle + | - \langle - |) \phi(x) (| + \rangle + | - \rangle) \\ &= \frac{1}{2} \langle + | \phi(x) | + \rangle - \langle - | \phi(x) | - \rangle \\ &= 0. \end{aligned}$$

So $\phi(x)|5\rangle$ is not orthogonal to $|A\rangle$

We can build up two mutually orthogonal Hilbert spaces by allowing local operators to act on $|+\rangle$ or $|-\rangle$ respectively. These two Hilbert spaces cannot communicate with each other; no local observable has a matrix element between states of the two Hilbert spaces:

$$\langle 1 | \phi | 2 \rangle = 0.$$

It is simply perverse to take linear combinations of states $|1\rangle$ and $|2\rangle$ from these two distinct spaces.

Another way to see that $|A\rangle$ and $|5\rangle$ are inappropriate vacuum states is to consider, e.g.

$$\begin{aligned} \langle 5 | \phi(x) \phi(y) | 5 \rangle &= \frac{1}{2} \langle + | \phi(x) \phi(y) | + \rangle \\ &\quad + \frac{1}{2} \langle - | \phi(x) \phi(y) | - \rangle \end{aligned}$$

Now, take the limit $|x-y| \rightarrow \infty$. In this limit

$$\langle + | \phi(x) \phi(y) | + \rangle \rightarrow \langle + | \phi(x) | + \rangle \langle + | \phi(y) | + \rangle$$

("cluster property") -- We proved this on page 3.36. In the proof we used uniqueness of the vacuum. But that is valid in the theory built on the $|+\rangle$ vacuum; it has a Hilbert space with just one vacuum, and knows nothing about the other, orthogonal Hilbert space.

We have $\langle S | \phi(x) \phi(y) | S \rangle \xrightarrow{|x-y| \rightarrow \infty} V^2$

But $\langle S | \phi(x) | S \rangle = 0$. So the cluster property is not satisfied. (The proof fails because $|A\rangle$ and $|S\rangle$ are two ground states, and the Hilbert spaces spanned by local operators acting on these states are not orthogonal.) The failure of the cluster property shows that the $|S\rangle, |A\rangle$ basis is not appropriate for a description of local physics. Probability distributions for the values of the fields $\phi(x), \phi(y)$ ought to factorize as x and y are widely separated, assuming physics is local.

To summarize, we have learned that:

- Spontaneous symmetry breakdown can occur in quantum field theory, but only in the infinite-volume limit.
- A consequence of spontaneous symmetry breakdown is that the vacuum is not unique. But the Hilbert space of the theory can then be divided into mutually orthogonal sectors, each containing a single vacuum, such that each sector is preserved by all local operators.

This latter statement can be elevated to the status of a theorem. Let

$$|n\rangle \quad n=1,2,3,\dots$$

denote a set of degenerate vacuum states that are mutually orthogonal

$$\langle n|m\rangle = \delta_{n,m}$$

then it is always possible to choose a basis for the vacua that is diagonal in local operators. That is

$$\langle n|A|m\rangle = 0 \text{ for } n \neq m,$$

where A is any local operator
Hermitian (i.e. observable)

Proof:

Suppose A and B are any two local operators (constructed from fields smeared with functions that are localized in space time) then

$$\lim_{|\vec{a}|\rightarrow\infty} [A, U(\vec{a}) B U(\vec{a})^{-1}] = 0.$$

Here $U(\vec{a})$ generates a spatial translation by $\vec{a}$. The statement is true because, for $\vec{a}$ large enough A and UBU^{-1} are spacelike separated, and hence commute

thus,

$$0 = \lim_{|\vec{a}|\rightarrow\infty} \langle n| [A, U(\vec{a}) B U(\vec{a})^{-1}] |m\rangle$$

But also, --

(6.14)

$$\lim_{|\vec{a}| \rightarrow \infty} \langle n | A U(\vec{a}) B U(\vec{a})^{-1} | m \rangle$$

$$= \sum_{\text{states}} \lim_{|\vec{a}| \rightarrow \infty} \langle n | A | \text{state} \rangle \langle \text{state} | B | m \rangle e^{i \vec{L}_{\text{state}} \cdot \vec{a}}$$

By the standard Riemann-Lebesgue argument, only the translation invariant states -- the vacua -- contribute in the limit:

$$= \sum_K \langle n | A | K \rangle \langle K | B | m \rangle$$

Therefore, $A_{nK} B_{Km} - B_{nK} A_{Km} = 0$
 where $A_{nK} = \langle n | A | K \rangle$ etc.

thus, all local ^(Hermitian) operators are mutually commuting ^(Hermitian) matrices in the basis of the vacua, and they can all be simultaneously diagonalized. That is what we wanted to show.

Continuous Symmetry (Goldstone Theorem)

The $\phi \rightarrow -\phi$ symmetry considered above is an example of a discrete symmetry. Next we wish to contemplate the consequences of spontaneous breakdown of a continuous symmetry.

Example: Complex Scalar Field

Consider a theory of a complex scalar field ϕ :

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - U(|\phi|)$$

$$U(|\phi|) = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \quad (\lambda > 0)$$

The potential is a function of only the modulus of ϕ , not its phase. So the Lagrangian is invariant under the global symmetry operation:

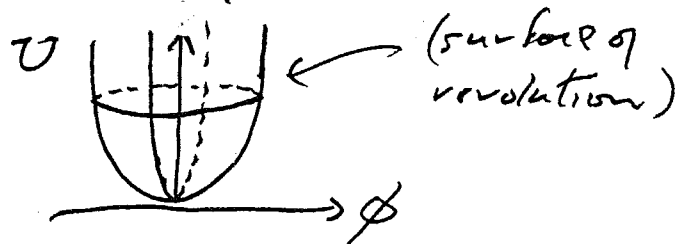
$$\phi(x) \rightarrow e^{i\alpha} \phi(x)$$

(This is a continuous $U(1)$ or $SO(2)$ symmetry.)

How is this symmetry realized in the classical ($\hbar \rightarrow 0$) limit?

(i) $\mu^2 \geq 0$

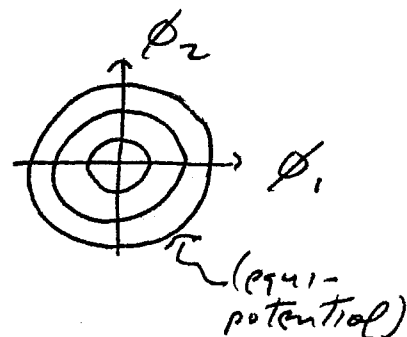
The minimum of the potential occurs at $\phi(x) = 0$



-- this is the classical ground state.
 this state is preserved by $\phi \rightarrow e^{i\alpha}\phi$,
 so the symmetry is manifest.

Because of the symmetry,
 there is a degeneracy in the
 spectrum. If we write

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2),$$

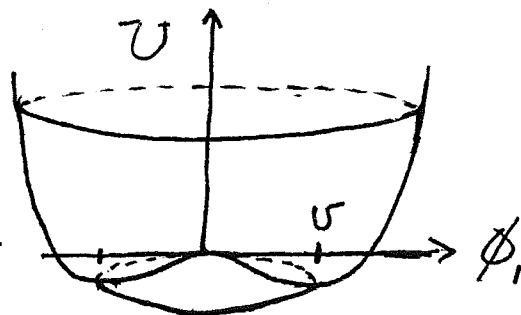


where $\phi_{1,2}$ are real, then the
 symmetry is a rotation in the ϕ_1 - ϕ_2 plane.
 If we consider small oscillations about
 the classical ground state, there are two
 independent modes -- oscillations of ϕ_1 and ϕ_2 .
Both modes have frequency μ . This
 degeneracy is enforced by the symmetry
 that relates ϕ_1 and ϕ_2 .

(ii) $\mu^2 < 0$

The potential is

$$U = \lambda \left(\phi^\dagger \phi - \frac{v^2}{2} \right)^2 + \text{constant}$$



where $\lambda v^2 = -\mu^2$

this is a "Mexican hat" potential with
 minima at

$$|\phi(x)| = \frac{1}{\sqrt{2}} v$$

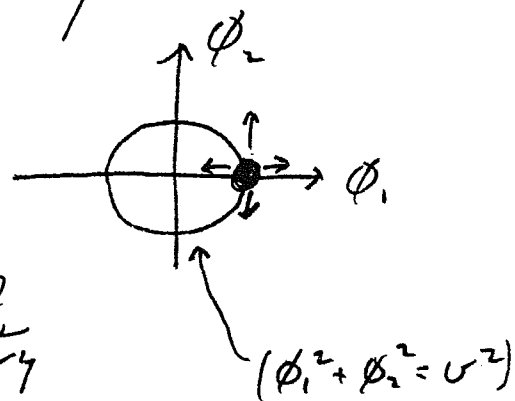
Thus, there are many classical ground
 states -- as many as the number of points
 on a circle.

The classical ground states are

$$\phi(x) = \frac{1}{\sqrt{2}} v e^{i\theta},$$

where θ can take any (constant) value in $[0, 2\pi)$. None of these classical ground states is invariant under the symmetry; rather the symmetry, acting on one ground state, yields another. The symmetry is spontaneously broken.

If we choose one of these states arbitrarily (say $\phi_1 = v$, $\phi_2 = 0$) and then consider small oscillations of the fields about the classical ground state, then the $U(1)$ symmetry is not at all manifest.



To study these small oscillations, we perform a shift of the fields. We make the change of variables.

$$\phi(x) = \frac{1}{\sqrt{2}} \rho(x) e^{i\theta(x)}$$

$$\text{then } \partial_\mu \phi = \frac{1}{\sqrt{2}} (\partial_\mu \rho + i \rho \partial_\mu \theta) e^{i\theta}$$

$$\text{so } \partial_\mu \phi \partial^\mu \phi^* = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta$$

$$\text{and } U = \frac{1}{4} \left(\rho^2 - \frac{v^2}{2} \right)^2 = \frac{1}{4} (\rho^2 - v^2)^2$$

$$\text{so } \mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{1}{4} (\rho^2 - v^2)^2 + \frac{1}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta$$

In these coordinates (which are singular when $\rho = 0$) the $U(1)$ symmetry has the action

$$\phi \rightarrow \phi + \alpha$$

Invariance under the symmetry requires that only derivatives of ϕ , and not ϕ itself, appear in $\mathcal{L}$.

If we expand around the classical ground state, we have

$$\rho(x) = v + \rho'(x)$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \rho')^2 - \frac{1}{4}(\rho'^2 + 2v\rho')^2 \\ &\quad + \frac{1}{2}(v^2 + 2v\rho' + \rho'^2)(\partial_\mu \phi)^2 \\ &= \frac{1}{2}(\partial_\mu \rho')^2 - \frac{1}{4}v^2\rho'^2 - \frac{1}{2}v\rho'^3 - \frac{1}{4}\rho'^4 \\ &\quad + \frac{1}{2}v^2(\partial_\mu \phi)^2 + v\rho'(\partial_\mu \phi)^2 + \frac{1}{2}\rho'^2(\partial_\mu \phi)^2 \end{aligned}$$

If we want to replace ϕ by a canonically normalized scalar field, we may write

$$v\phi = \tilde{\phi},$$

and then

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \rho')^2 - \frac{1}{4}\rho'^2\rho'^2 - \frac{1}{2}v\rho'^3 - \frac{1}{4}\rho'^4 \\ &\quad + \frac{1}{2}(\partial_\mu \tilde{\phi})^2 + \frac{1}{v}\rho'(\partial_\mu \tilde{\phi})^2 + \frac{1}{2v^2}\rho'^2(\partial_\mu \tilde{\phi})^2 \end{aligned}$$

where $\mu'^2 = 2\lambda v^2 = -2\mu^2$.

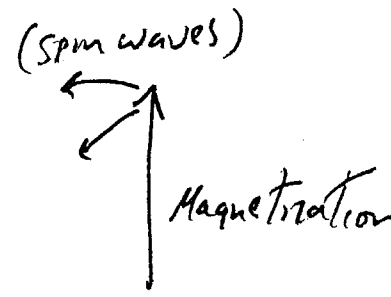
We find no degeneracy here. The ϕ' oscillations have frequency μ' , but the ϕ oscillations have zero frequency. When a continuous symmetry is spontaneously broken, the symmetry does not result in a degeneracy in the spectrum, but instead in a massless particle (called a "Goldstone boson"). This is easy to understand (classically). If the classical ground state is not invariant under the symmetry, then the potential at its minimum must have a flat direction in field space. Thus there is a massless excitation.

It should be obvious that this conclusion is very general (at least classically). There are always as many Goldstone bosons as there are spontaneously broken symmetries (i.e. the number of infinitesimal symmetry operators that do not leave the classical ground state invariant). This is the number of flat directions -- the number of directions in field space in which the potential has no curvature.

To return to our ferromagnet example again, we found there

$$SO(3) \rightarrow SO(2).$$

Rotational invariance broken to invariance under rotations about the $\hat{z}$ axis (the direction in which the magnetization points). There are two spontaneously broken symmetries and hence two Goldstone bosons. These Goldstone bosons are the two linear polarizations of the spin waves in the magnet--small oscillations of the magnetization about the homogeneous ground state.



Invariance under the symmetry

$$\phi \rightarrow \phi + \alpha$$

requires not only that ϕ is massless but also that all interactions of ϕ are derivative interactions. Goldstone bosons have only derivative couplings.

This result is also completely general for any Goldstone boson field, $\phi \rightarrow \phi + \alpha$, where α is a constant, is a symmetry. (It corresponds to a global rotation of the classical ground state) Thus, a ϕ excitation with wavelength $\lambda \rightarrow \infty$ has zero energy -- in other words, ϕ is massless.

To summarize, then, the results of our classical analysis of continuous symmetries, we have found that there are two possible realizations of such a symmetry these are:

• The Wigner-Weyl Realization --

the symmetry is manifest. It results in degeneracy among the vibrational excitations of the field theory.

• The Nambu-Goldstone Realization --

the symmetry is spontaneously broken. It results in Goldstone Bosons, as many zero frequency excitations as the number of spontaneously broken symmetries. The Goldstone bosons have derivative couplings.

of course, a mixed realization is possible, e.g.

$$G \rightarrow H \subset G$$

(a global symmetry group G is spontaneously broken to a subgroup H). E.g., in the ferromagnet,

$$G = SO(3)$$

$$H = SO(2)$$

Because of the manifest $SO(2)$ symmetry,

The two types of massless spin waves have identical couplings.

Continuous Symmetry in Quantum Field Theory.

How is the above classical analysis modified when we include the effects of quantum fluctuations?

If a continuous symmetry is spontaneously broken, we may argue as before that (in the infinite volume limit) the Hilbert space splits into sectors that cannot communicate through the action of any local observables. We should therefore do physics in one such sector, which has a unique vacuum corresponding to one of the classical ground states.

Furthermore, at least order-by-order in perturbation theory, if spontaneous symmetry breaking occurs classically, it also occurs quantum mechanically. Small quantum fluctuations cannot wipe out long range order of the vacuum. And perturbation theory is an expansion in $\hbar$, so for the purpose of doing perturbation theory, we should regard $\hbar$ as arbitrarily small.

Actually, the above conclusion is

correct only if the dimension D of spacetime is $D > 2$. For $D = 2$, quantum fluctuations actually destroy long range order for any $t \neq 0$. The analogous statistical mechanical statement is that in 2 dimensions continuous symmetry can be spontaneously broken only at zero temperature. This is the Mermin-Wagner-Coleman theorem. Recall that we also found that discrete symmetry could not be spontaneously broken for $t > 0$ unless $D > 1$. Thus $D = 1$ is said to be a "lower critical dimension" (of spacetime) for discrete symmetry, and $D = 2$ is a "lower critical dimension" for continuous symmetry.

The idea of the Mermin-Wagner-Coleman theorem is very simple. If ϕ is a Goldstone boson field (classically) consider a ϕ field configuration in which ϕ varies by an amount of order one over a region of size R in D dimensional Euclidean space. This configuration has an action

$$S \sim \int d^D x \frac{1}{2} v^2 (\partial_\mu \phi)^2 \\ \sim v^2 R^{D-2}.$$

Such a configuration occurring as a

quantum fluctuation is suppressed by a factor

$$e^{-S/\hbar} \sim e^{-(v^2/\hbar R^{D-2})}$$

Thus, for $D > 2$, such fluctuations are suppressed as $\hbar \rightarrow 0$ for R large. But $D=2$ is a marginal case. Long wavelength fluctuations can occur that tend to wipe out the long range order. If we try to do perturbation theory in $\hbar$ about a classical ground state for which spontaneous symmetry breakdown occurs, these long wavelength fluctuations give rise to

$$\int \frac{d^2 k}{k^2}$$

infrared divergences that signal that our choice of ground state is not appropriate quantum mechanically.

Anyway, suppose $D > 2$ and that SSB occurs classically, so that it also occurs order-by-order in $\hbar$. Classically, there is a conserved Noether current associated with each global symmetry

$$\partial_\mu J^\mu(x) = 0,$$

and a conserved charge

$$Q = \int d^3x J^0(x), \quad \frac{d}{dt} Q = 0$$

Quantum mechanically,

$$U(\alpha) = e^{iQ\alpha}$$

is the unitary operator that represents the symmetry acting on the Hilbert space of the theory. Since

$[Q, H] = 0$, we have also $[U, H] = 0$,
 thus

$$|state'\rangle = U(\alpha) |state\rangle$$

is degenerate with $|state\rangle$. Thus, the symmetry results in degeneracy.

What goes wrong, then, if there is SSB?
 (Why doesn't the above argument require the states form degenerate multiplets that transform irreducibly under the symmetry group) In a sense, the argument is correct, $|state'\rangle$ is degenerate with $|state\rangle$, because

$$Q |state\rangle = |state + 0\text{-momentum Goldstone boson}\rangle$$

Q generates the global rotation of the vacuum that corresponds to a Goldstone boson excitation with wavelength $\lambda \rightarrow \infty$.
 Technically, Q does not really exist, because a momentum eigenstate is not a properly normalizable state, so $Q |state\rangle$ is not in the Hilbert space. (Another way to describe this is -- Q , a nonlocal operator, takes one sector of Hilbert space to another.)

One wonders though, whether our classical intuition really applies to the quantum theory. In particular, we can argue on general grounds that a classical field theory with SSB has a Goldstone boson (massless particle). But does the Goldstone boson survive to all orders in $\hbar$? It seems conceivable that quantum corrections will generate a mass for the would-be Goldstone boson.

Fortunately, though, we can argue that this does not happen order-by-order in $\hbar$. The point is that the masslessness of the Goldstone boson arises from a symmetry. The Ward identity tells us that the effective action has the symmetries of the classical action. So

(E.g. in our extension field theory) $\Gamma[\phi, \phi^*] = \Gamma[\phi, \phi]$

is invariant under $\phi \rightarrow \phi + \alpha$. This is enough to ensure that only derivatives of ϕ appear in Γ , and hence that the exact ϕ propagator has a pole at $p^2 = 0$ to all orders in $\hbar$. This argument is evidently quite general.

(Incidentally, note that our expression for $Z(\phi, \tilde{\phi})$ after the shift -- page (6.18) -- appears to be nonrenormalizable. This is surely an illusion, because we obtained $\mathcal{L}$

from a manifestly renormalizable Lagrangian by a simple change of variable. It is the Ward identity that will ensure that all of the counterterms needed to cancel the infinities encountered in perturbation theory in λ have the form of the terms already included in $\mathcal{L}$. All renormalizations can be absorbed into rescalings of e' , $\tilde{e}$, μ'^2 , λ .

Goldstone's Theorem - - More general proof

The statement that spontaneous breakdown of a continuous symmetry always results in a massless particle is called Goldstone's Theorem. We will show that this conclusion can be reached by a very general argument.

Recall that on page (4.60) we used functional methods to derive a version of the Ward identity. Consider a linear symmetry operation of the form

$$\phi(x) \rightarrow \phi(x) + \epsilon A\phi(x)$$

under which the Lagrangian changes by

$$\mathcal{L} \rightarrow \mathcal{L} + \epsilon \partial_\mu f^\mu$$

then there is a conserved Noether current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} A\phi - f^\mu, \quad \partial_\mu j^\mu = 0.$$

And this current obeys a Ward identity

$$\begin{aligned} \partial_\mu^x \langle 0 | T (J^\mu(x) \phi(y)) | 0 \rangle \\ = -i \delta^4(x-y) \langle 0 | A \phi(y) | 0 \rangle \end{aligned}$$

We would like to show that it follows from this identity that spontaneous symmetry breakdown implies a massless particle.

Classically, spontaneous symmetry breakdown means that the ground state is not invariant under the symmetry. The corresponding (formal) quantum mechanical statement is

$$U(x)|0\rangle = e^{i\alpha Q}|0\rangle \neq |0\rangle$$

$$\text{or } Q|0\rangle \neq 0$$

(The unbroken charge does not annihilate the vacuum.) This statement is merely formal because we do not expect Q to exist ($Q|0\rangle$ is not a state) in the event of spontaneous symmetry breakdown. We should instead define SSB in a different way.

If Q does exist (e.g. in the case of manifest symmetry) then Q has a canonical commutator with ϕ :

$$[Q, \phi(x)] = -i A \phi(x)$$

(Q generates the symmetry) So in the case of manifest symmetry

$$Q|0\rangle = 0 \quad \Rightarrow$$

$$0 = i \langle 0|[Q, \phi(x)]|0\rangle = \langle 0|A\phi(x)|0\rangle$$

So we may use $\langle 0|A\phi(x)|0\rangle \neq 0$ as a criterion for SSB. This makes sense even if Q does not exist.

Thus, a statement of Goldstone's Theorem is

$$\langle 0|A\phi(x)|0\rangle \neq 0 \Rightarrow \text{there is a massless particle.}$$

The proof is simple: Define

$$\begin{aligned} G^M(x-y) &= \langle 0|T(\mathcal{J}^M(x) \phi(y))|0\rangle \\ &= \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \tilde{G}^M(p) \end{aligned}$$

Then, the Ward identity says

$$\begin{aligned} \partial_\mu^x G^M(x-y) &= \int \frac{d^4 p}{(2\pi)^4} -ip_\mu \tilde{G}^M(p) e^{-ip \cdot (x-y)} \\ &= \langle 0|A\phi|0\rangle \int \frac{d^4 p}{(2\pi)^4} (-i) e^{-ip \cdot (x-y)} \end{aligned}$$

or --

$$p_\mu \tilde{G}^\mu(p) = \langle 0 | A \phi(0) | 0 \rangle$$

By covariance, $\tilde{G}^\mu(p) = p^\mu f(p^2)$,
 so we see that $p^2 f(p^2) = \langle 0 | A \phi(0) | 0 \rangle$, or

$$\boxed{\tilde{G}^\mu(p) = \frac{p^\mu}{p^2} \langle 0 | A \phi(0) | 0 \rangle}$$

Since the Green function $\tilde{G}^\mu(p)$ has a pole at $p^2 = 0$, we conclude that there is a massless particle in the spectrum, which incidentally couples to both J^μ and ϕ . This proves Goldstone's theorem.

In fact, we can say a bit more about the coupling of the Goldstone boson to J^μ and ϕ . Translation invariance and covariance implies

$$\langle 0 | T^\mu(x) | p \rangle = -i f p^\mu e^{-i p \cdot x}$$

Goldstone boson with 4-mom. p
 (where f is real -- a phase convention for the state $|p\rangle$)
 Thus, inserting a sum over one-particle states,

$$\begin{aligned} G^\mu(x-y) &= \langle 0 | T^\mu J^\mu(x) \phi(y) | 0 \rangle \\ &= \theta(x^0 - y^0) \int \frac{d^3 p}{(2\pi)^3 2\omega_p} (-i f) p^\mu \langle p | \phi(0) | 0 \rangle e^{-i p \cdot (x-y)} \\ &\quad + \theta(y^0 - x^0) \int \frac{d^3 p}{(2\pi)^3 2\omega_p} (+i f) p^\mu \langle 0 | \phi(0) | p \rangle e^{i p \cdot (x-y)} \end{aligned}$$

If we choose the phase of ϕ so that
 $\langle 0 | \phi(0) | p \rangle$ is real,

then

$$\begin{aligned} G^{\mu}(x-y) &= -if \langle 0 | \phi(0) | p \rangle \left(i \frac{\partial}{\partial x^{\mu}} \right) \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 + i\epsilon} e^{-ip \cdot (x-y)} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{p^{\mu}}{p^2} f \langle 0 | \phi(0) | p \rangle e^{-ip \cdot (x-y)} \end{aligned}$$

or --

$$f \langle 0 | \phi(0) | p \rangle = \langle 0 | A \phi(0) | 0 \rangle$$

If ϕ is a conventionally normalized scalar field, $\langle 0 | \phi | p \rangle = 1$, then

$$f = \langle 0 | A \phi(0) | 0 \rangle$$

-- the strength of the coupling of the Goldstone boson to g^{μ} is determined by the expectation value of $= i[A, \phi]$.
 In this sense f (a mass scale) quantifies how "big" the spontaneous symmetry breaking is.

Note that the coupling

$$\langle 0 | J_{\mu}(0) | p \rangle = -if p_{\mu} \quad \text{required by the theorem}$$

is characteristic of a spin-zero particle.
 Goldstone bosons are necessarily spinless.
 Furthermore, a coupling of a spin-zero

particle to a conserved 4-vector current is possible only if the particle is massless:

$$\begin{aligned}\partial^\mu \langle 0 | J_\mu(x) | p \rangle &= \langle 0 | \partial^\mu J_\mu(x) | p \rangle = 0 \\ &= -i f(i) p_\mu p^\mu \Rightarrow p^2 = 0\end{aligned}$$

Note also that Goldstone's Theorem can be run in reverse. If there is a massless spin-zero particle that couples to both J_μ and ϕ , then the symmetry associated with the current J_μ is spontaneously broken.

Symmetries of the Vacuum

We have seen that, if a field theory has a continuous symmetry, there is a conserved charge Q . Either this charge annihilates the vacuum (manifest symmetry -- Wigner-Weyl) or not (spontaneously broken symmetry -- Nambu-Goldstone).

Now, suppose that the vacuum has a symmetry -- that is

$$Q|0\rangle = 0,$$

where $Q = \int d^3x J^0$, and

J^μ is a four-vector.

Is this then a symmetry of the dynamics.
Does Q commute with the Hamiltonian H ?

Remarkably, yes. (Coleman's Theorem)

To see this, consider first a state $\rightarrow$
~~that~~ is an eigenstate of three-momentum $\vec{P}$
with zero momentum

$$|\vec{P}=0\rangle.$$

Then $\langle \vec{P}=0 | J^0(\vec{x}, t) | 0 \rangle = 0$,
because

$$= \langle \vec{P}=0 | J^0(\vec{0}, t) | 0 \rangle \quad (\text{translation invariance}),$$

$$\text{and } 0 = \langle \vec{P}=0 | Q | 0 \rangle = \langle \vec{P}=0 | J^0(\vec{0}, t) | 0 \rangle \left(\int d^3x \right)$$

Furthermore, since $0 = \langle \vec{P}=0 | J^0(\vec{x}, t) | 0 \rangle$ is
clearly time-independent, we have

$$0 = \langle \vec{P}=0 | \partial_0 J^0(\vec{x}, t) | 0 \rangle$$

Also $\langle \vec{P}=0 | J^i(\vec{x}, t) | 0 \rangle = \langle \vec{P}=0 | J^i(\vec{0}, t) | 0 \rangle$
is $\vec{x}$ -independent, so

$$0 = \langle \vec{P}=0 | \partial_\mu J^\mu(\vec{x}, t) | 0 \rangle$$

Now, since J^μ is a 4-vector, $\partial_\mu J^\mu$ is a
Lorentz scalar. By performing a boost
we have

$$0 = \langle \vec{P} | \partial_\mu J^\mu(x) | 0 \rangle$$

where $|\vec{p}\rangle$ is any momentum eigenstate. Since the momentum eigenstates are a basis for the Hilbert space,

$$\partial_\mu J^\mu(x) |0\rangle = 0.$$

Finally, we recall the theorem on page (3.34) -- If a local operator annihilates the vacuum, then it must be zero. Thus

$$\partial_\mu J^\mu(x) = 0.$$

The current is conserved, and $[Q, H] = 0$ by the usual argument.

"Current Algebra" (The Pions a Goldstone Boson)

Are there spontaneously broken global symmetries in Nature? If so, where are the Goldstone bosons?

Consider strong-interaction physics. There are no massless spin-zero hadrons. But there is one hadron that is conspicuously light compared to all the others:

Lightest pseudoscalar ($J^P = 0^-$)	$m_\pi \sim 140 \text{ MeV}$
Lightest vector ($J^P = 1^-$)	$m_\rho \sim 770 \text{ MeV}$

$$\frac{m_\pi^2}{m_\rho^2} = .033$$

Hence, there is a small parameter in hadron physics. Why?

One is led to speculate (following Nambu) that the strong interactions have an approximate symmetry. And that in the exact symmetry limit, the pion would be exactly massless; it would be a Goldstone boson. The parameter m_π^2/m_ρ^2 quantifies how close the real world is to this fictitious symmetry limit. (We say that π is a "pseudo-Goldstone boson.") The technology surrounding the idea that the pion is a Goldstone boson is called "current algebra."

Free Quark Model

What are these approximate symmetries of the strong interactions? To understand them, we need only know that the elementary constituents of hadrons are quarks -- spin- $\frac{1}{2}$ particles. We'll pretend that quarks come in only two "flavors" -- up (u) and down (d) (We ignore strange, charm, bottom, --) Let us also ignore the interactions between quarks that hold hadrons together. Then the Lagrangian of the strong-interaction is just that of two free Dirac particles:

(6.36)

$$\mathcal{L} = \bar{u}(i\not{\partial} - m_u)u + \bar{d}(i\not{\partial} - m_d)d$$

This Lagrangian respects a large group of symmetries in the limit $m_{u,d} \rightarrow 0$.

$$\begin{aligned}\mathcal{L} &= \bar{u}_L i\not{\partial} u_L + \bar{u}_R i\not{\partial} u_R \\ &\quad + \bar{d}_L i\not{\partial} d_L + \bar{d}_R i\not{\partial} d_R \\ &= \bar{q}_L i\not{\partial} q_L + \bar{q}_R i\not{\partial} q_R\end{aligned}$$

where $q = \begin{pmatrix} u \\ d \end{pmatrix}$

In the absence of mass terms, nothing couples together left-handed and right-handed quark fields. The Lagrangian is invariant under

$$q_L \rightarrow V_L q_L$$

$$q_R \rightarrow V_R q_R$$

where $V_{L,R} \in U(2)$

(independent unitary transformations acting on q_L and q_R). The massless free quark model with two flavors has the symmetry

$$G_{\text{free}} = U(2)_L \times U(2)_R$$

(6.37)

Now, what happens when we include the strong interactions. It turns out (I won't explain this), that the interactions preserve the symmetry group

$$G_{\text{int}} = SU(2)_L \times SU(2)_R \times U(1)_V$$

that is, the transformations above with $\det V_L = \det V_R$.
the symmetry $U(1)_A$,

$$q_L \rightarrow e^{i\alpha} q_L$$

$$q_R \rightarrow e^{-i\alpha} q_R$$

is spoiled by the interaction. the $U(1)_V$ symmetry

$$q \rightarrow e^{i\alpha} q$$

is just quark number. It is a good symmetry even when the quarks have masses. But the symmetry

$$G_{\text{chiral}} = SU(2)_L \times SU(2)_R$$

is exact for $m_{u,d} = 0$, and broken only by the quark masses. It is called a "chiral" (hand) symmetry, because it acts on LH and RH fermions independently.

(6.38)

How is this symmetry realized in Nature? Or how would it be realized in a hypothetical world with $m_u = m_d = 0$? It turns out (again, I won't explain -- sorry) that a composite operator gets a vacuum expectation value and breaks the symmetry spontaneously. The form of this expectation value is

$$\langle 0 | \bar{q}_{Li} q_{Rj} | 0 \rangle = -v^3 \delta_{ij}$$

(where i, j are flavor indices).

What subgroup of G_{chiral} is preserved by this expectation value?

under

$$\begin{aligned} q_L &\rightarrow V_L q_L \\ q_R &\rightarrow V_R q_R, \end{aligned}$$

We have

$$\bar{q}_{Li} q_{Rj} \rightarrow V_{Li}^* V_{Rj} \bar{q}_{Lk} q_{Re},$$

$$\begin{aligned} \text{or } \langle 0 | \bar{q}_{Li} q_{Rj} | 0 \rangle &\rightarrow v^3 \delta_{ke} V_{Lki}^* V_{Rje} \\ &= v^3 (V_R V_L^\dagger)_{ji} \end{aligned}$$

The expectation value is invariant if

$$V_R V_L^\dagger = \mathbb{1} \quad \text{or} \quad V_R = V_L$$

The unbroken symmetry is the "diagonal"

$$H = SU(2)_V,$$

under which

$$q \rightarrow Vq.$$

(RH and LH quarks rotate together)
This symmetry is called isospin.

So the realization of the symmetry is --

$$SU(2)_L \times SU(2)_R \longrightarrow SU(2)_V$$

There are three spontaneously broken symmetries and hence three Goldstone bosons. We will identify these as the pions π^+ , π^- , π^0 .

Since the $SU(2)_V$ isospin symmetry is manifest, we may classify the states according to how they transform under isospin. The 3 pions form an isotriplet (see below).

Algebra of Currents

Why is all this called "Current Algebra"?

One way to characterize the Gchiral symmetry is to construct the six conserved currents associated with Gchiral. These are

$$\begin{array}{l|l} J_L^\mu = \bar{u}_L \gamma^\mu u_L & \text{and} \\ J_L^3 = \frac{1}{2} \bar{u}_L \gamma^\mu u_L - \bar{d}_L \gamma^\mu d_L & L \rightarrow R \\ J_L^- = \bar{u}_L \gamma^\mu d_L & \end{array}$$

The information that these currents are associated with an $SU(2)_L \times SU(2)_R$ symmetry is encoded in the their canonical commutators. Let's work out these commutators.

Recall that we found on page (3.64) that the canonical equal time anticommutators of Dirac fermions can be written

$$[\bar{\Psi}, \bar{\Psi}]_+ = 0 = [\Psi, \Psi]_+ \quad (\text{equal time})$$

$$[\Psi_\alpha(\vec{x}, t), \bar{\Psi}_\beta(\vec{y}, t)]_+ = \delta_{\alpha\beta} \delta^{(3)}(\vec{x} - \vec{y})$$

If we multiply by $\frac{1}{2}(1 \pm \gamma_5)$, we find

$$[\Psi_L, \bar{\Psi}_L]_+ = \frac{1}{2}(1 - \gamma_5) \delta^0 \delta^{(3)}(-)$$

$$[\Psi_R, \bar{\Psi}_R]_+ = \frac{1}{2}(1 + \gamma_5) \delta^0 \delta^{(3)}(-)$$

$$[\Psi_L, \bar{\Psi}_R]_+ = [\Psi_R, \bar{\Psi}_L]_- = 0$$

(all at equal times).

If we include also flavor indices, then the nonvanishing equal time anticommutators are

$$[\Psi_{Li}, \bar{\Psi}_{Lj}]_+ = \frac{1}{2}(1 - \gamma_5) \delta^0 \delta_{ij} \delta^{(3)}(-)$$

$$[\Psi_{Ri}, \bar{\Psi}_{Rj}]_+ = \frac{1}{2}(1 + \gamma_5) \delta^0 \delta_{ij} \delta^{(3)}(-)$$

Now, we want to calculate commutators of fermion bilinears --

(6.41)

Consider -- $[\bar{A}B, \bar{C}D]$

$$= \bar{A}B\bar{C}D - \bar{C}D\bar{A}B$$

$$= \bar{A}(B\bar{C} + \bar{C}B)D - \bar{C}(\bar{D}\bar{A} + \bar{A}\bar{D})B - \underbrace{\bar{A}\bar{C}BD + \bar{C}\bar{A}DB}_{\cancel{=0}}$$

$$= \bar{A}[B, \bar{C}]_+ D - \bar{C}[D, \bar{A}]_+ B$$

Therefore,

$$[\bar{q}_L \gamma^0 1^a q_L, \bar{q}_R \gamma^0 1^b q_R] = 0$$

$$[\bar{q}_L \gamma^0 1^a q_L, \bar{q}_L \gamma^0 1^b q_L]$$

$$= (\bar{q}_L \gamma^0 1^a 1^b q_L - \bar{q}_L \gamma^0 1^b 1^a q_L) \delta^3(-)$$

$$\text{or } [\bar{q}_L \gamma^0 1^a q_L(\vec{x}, t), \bar{q}_L \gamma^0 1^b q_L(\vec{y}, t)]$$

$$= \delta^3(\vec{x} - \vec{y}) \bar{q}_L \gamma^0 [1^a, 1^b] q_L(\vec{x}, t)$$

(and $L \rightarrow R$), where $1^a, 1^b$ are matrices acting on flavor indices.

If we define currents

$$J_L^{\mu a} = \bar{q}_L \gamma^\mu \frac{\sigma^a}{2} q_L$$

$$J_R^{\mu a} = \bar{q}_R \gamma^\mu \frac{\sigma^a}{2} q_R,$$

then, at equal times, the currents form two mutually commuting $SO(2)$ algebras

$$[J_L^{0a}, J_L^{0b}] = 0$$

$$[J_L^{0a}, J_L^{0b}] = \delta^{3c} i \epsilon^{abc} J_L^{0c}$$

$$[J_R^{0a}, J_R^{0b}] = \delta^{3c} i \epsilon^{abc} J_R^{0c}$$

(At equal times). Formally then, the conserved charges satisfy

$$[Q_L, Q_R] = 0$$

$$[Q_L^a, Q_L^b] = i \epsilon^{abc} Q_L^c$$

$$[Q_R^a, Q_R^b] = i \epsilon^{abc} Q_R^c$$

But we have noted that these charges might not exist (if symmetries are spontaneously broken) so it is safer to speak of an algebra of currents instead of the algebra satisfied by the charges.

If $G_{\text{chiral}} \rightarrow H = SU(2)_V$, then the currents associated with the manifest symmetries are the vector currents.

$$V^\mu{}^a = \bar{\psi} \gamma^\mu \frac{\sigma^a}{2} \psi,$$

while those associated with spontaneously broken currents are axial currents

$$A^\mu{}^a = \bar{\psi} \gamma^\mu \gamma_5 \frac{\sigma^a}{2} \psi$$

In this basis, the equal time current commutators

are --

$$[V^{0a}, V^{0b}] = \delta^3(\epsilon) i \epsilon^{abc} V^{0c}$$

$$[V^{0a}, A^{0b}] = \delta^3(\epsilon) i \epsilon^{abc} A^{0c}$$

$$[A^{0a}, A^{0b}] = \delta^3(\epsilon) i \epsilon^{abc} V^{0c}$$

Thus, under the manifest $SU(2)_V$ symmetry, the axial currents transform as a triplet. This justifies our earlier claim that the three Goldstone bosons are a $SU(2)_V$ triplet.

Since the pions are Goldstone bosons, they couple to the spontaneously broken currents:

$$\langle 0 | A^\mu a | \pi^b \rangle = \delta^{ab} i F_\pi p^\mu$$

(SU(2)_V-invariance)

The constant F_π is measured (see below)

$$F_\pi = 93 \text{ MeV.}$$

Warning: Sometimes the f of the pion is defined differently --

$$\begin{aligned} \langle 0 | \bar{u} \gamma^\mu \gamma_5 d | \pi^- \rangle &= i f_\pi p^\mu \\ &= \langle 0 | (A^{\mu 1} + i A^{\mu 2}) | \frac{\pi_1 - i \pi_2}{\sqrt{2}} \rangle = i \sqrt{2} F_\pi p^\mu \end{aligned}$$

$$\text{So } f_\pi = \sqrt{2} F_\pi = 132 \text{ MeV.}$$