The charged Weak Current

One reason that the algebra of currents is useful is that the Lagrangian of the weak interactions can be expressed (at low energy) in terms of currents. For the charged weak interactions, it has the form

$$\mathcal{L}_{int} = 2\sqrt{2}\, G_F\, J_\mu J^{\mu +}$$

where
$$J^\mu = J^\mu_{Hadron} + J^\mu_{Lepton}$$

$$J^\mu_{Lepton} = \bar{\nu}_{eL}\gamma^\mu e_L + (e \to \mu) + (e \to \tau)$$

$$J^\mu_{Hadron} = \bar{u}_L \gamma^\mu d_L + \text{other flavors}$$

(setting $\cos\theta_{Cabibbo} = 1$; actually $\cos\theta_C = .975$)

Here the coupling constant G_F is Fermi's constant

$$G_F \sim 10^{-5} m_{proton}^{-2} \sim (300\,\text{GeV})^{-2}.$$

The weak interactions are weak because G_F is small (in units of the hadronic mass scale). Note that the charged weak currents are <u>Left-handed</u> or $V-A$ (first proposed by Feynman and Gellman). Hence the weak interactions violate parity in a <u>maximal</u> sense (all LH and no RH).

Charged weak interactions may be classified into three types:

$$\mathcal{L}_{int} = 2\sqrt{2}\, G_F \Big[J^{\mu}_{Lep} J^{+}_{\mu Lep} \qquad \text{"Leptonic"}$$

$$+ J^{\mu}_{Had} J^{+}_{\mu Lep} + J^{\mu}_{Lep} J^{+}_{\mu Had} \qquad \text{"Semileptonic"}$$

$$+ \quad J^{\mu}_{Had} J^{+}_{\mu Had} \Big] \qquad \text{"Hadronic"}$$

E.g. Leptonic -- $\mu^- \to e^- \nu_\mu \bar{\nu}_e$

semileptonic -- $\pi^- \to \mu^- \bar{\nu}_\mu$

Hadronic -- $K^- \to \pi^- \pi^0$

<u>Pion Decay</u>

The amplitude for π decay in lowest order in G_F is

$$\langle \mu^- \bar{\nu}_\mu | \, i\mathcal{L}_{int} \, | \pi^- \rangle$$

$$= i 2\sqrt{2}\, G_F \langle \mu^- \bar{\nu}_\mu | J^{+}_{\mu Lep} | 0 \rangle$$

$$\langle 0 | J^{\mu}_{Had} | \pi^- \rangle$$

$$= i\,2\sqrt{2}\,G_F \quad \bar{u}_\nu \gamma_\mu \tfrac{1}{2}(1-\gamma_5) u_\mu$$
$$\times \langle 0 | \bar{u}_L \gamma^\mu d_L | \pi^- \rangle$$

(ignoring Cabibbo angle)

(Note: u's here are Dirac spinors -- Not to be confused with a quark.)

But --
$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}(\bar{u}\gamma^\mu d - \bar{u}\gamma^\mu \gamma_5 d)$
and the pion is a Goldstone boson that couples to the axial current, so

$$\langle 0 | \bar{u}_L \gamma^\mu d_L | \pi^- \rangle = -\tfrac{1}{2} i f_\pi p^\mu$$

Comparing the observed rate for $\pi^- \to \mu^- \bar{\nu}_\mu$ with this calculated amplitude we can measure f_π, finding $f_\pi = 132$ MeV, as mentioned above.

Neutron Decay (the Goldberger-Treiman Relation)

Another semileptonic decay is

$$n \to p + e^- + \bar{\nu}_e$$

$$\text{Amplitude} = i\,2\sqrt{2}\,G_F \quad \bar{u}_\nu \gamma_\mu \tfrac{1}{2}(1-\gamma_5) u_e$$
$$\times \quad \langle p | \bar{u}_L \gamma_\mu d_L | n \rangle$$

What can we say about the hadronic matrix element in this case? We have

$$\langle p|\bar{u}_L\gamma_\mu d_L|n\rangle = \frac{1}{2}\langle p|\bar{u}\gamma_\mu d|n\rangle - \frac{1}{2}\langle p|\bar{u}\gamma_\mu\gamma_5 d|n\rangle$$

and we can use the parity invariance of the strong interaction to write these matrix elements in terms of invariant functions of $K = p_n - p_p$:

$$\langle p|\bar{u}\gamma^\mu d(x)|n\rangle = e^{-iK\cdot x}\,\bar{u}_p\left[f_1(K^2)\gamma^\mu + f_2(K^2)(-i\sigma^{\mu\nu}K_\nu) + f_3(K^2)K^\mu\right]u_n$$

$$\langle p|\bar{u}\gamma^\mu\gamma_5 d(x)|n\rangle = e^{-iK\cdot x}\,\bar{u}_p\left[g_1(K^2)\gamma^\mu\gamma_5 + g_2(K^2)(-i\sigma^{\mu\nu}K_\nu\gamma_5) + g_3(K^2)K^\mu\gamma_5\right]u_n$$

Now, the form of the functions $f_{1,2,3}$ and $g_{1,2,3}$ are restricted by the requirement that the currents are conserved, in the limit of exact chiral symmetry (massless quarks). Conservation of the vector current gives

$$0 = \bar{u}_p\left[f_1\ \gamma^\mu K_\mu + f_3 K^2\right]u_n$$

But $\bar{u}_p(\not{p}_n - \not{p}_p)\ u_n$

$= (m_n - m_p)\,\bar{u}_p u_n = 0$, because $m_p = m_n$ in the limit of exact isospin symmetry (p and n are an isospin doublet.) So we conclude

$$f_3(K^2) = 0.$$

Conservation of the axial current gives

$$\bar{u}_p [g_1 K_\mu \gamma^\mu \gamma_5 + g_3 K^2 \gamma_5] u_n$$

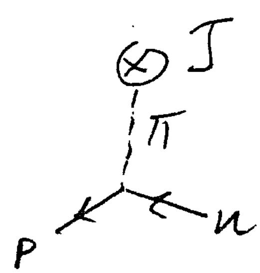
or
$$-2m\, g_1 + K^2 g_3 = 0$$

$$\Longrightarrow \quad g_3(K^2) = \frac{2m}{K^2} g_1(K^2)$$

(where $m = m_n = m_p$) In the isospin limit $K=0$ in neutron decay, and in reality we are quite close to this limit; $m_n - m_u \sim 1.2$ MeV $\ll 938$ MeV $= m_p$. Experimentally, it is found that

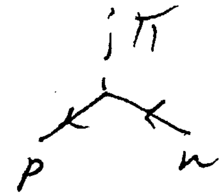
$$g_1(K^2)\Big|_{K^2=0} \equiv g_A \cong 1.26 \neq 0,$$

so $g_3(K^2)$ has a <u>pole</u> at $K^2 = 0$. This pole is due to the massless pion, and arises from the process:



$$= \bar{u}_p \, i f_\pi K^\mu \frac{i}{K^2} (-\gamma_5 \sqrt{2}\, g_{\pi NN}) u_n,$$

where $g_{\pi NN}$ is the π-nucleon coupling at $k=0$; this coupling is defined by



$$= -\gamma_5\, g_{\pi NN}(k^2)\sqrt{2}$$

Thus
$$g_3(k^2)\Big|_{k^2=0} = \frac{\sqrt{2}\, f_\pi\, g_{\pi NN}}{k^2},$$

And therefore,
$$2M g_A = \sqrt{2}\, f_\pi\, g_{\pi NN} = 2F_\pi\, g_{\pi NN},$$

or
$$\boxed{m_{Nucleon}\, g_A = F_\pi\, g_{\pi NN}}$$

This is the Goldberger-Treiman relation, which would be exact if chiral symmetry were exact. Experimentally,

$$(938)(1.26) \stackrel{?}{=} (93)(13.5)$$
$$1181 \stackrel{?}{=} 1256$$

It works to 6% accuracy. This is another way of quantifying how close the real world is to the limit of exact chiral symmetry.

The Interactions of Goldstone Bosons

The observation that pions (and similarly Kaons) are the Goldstone bosons of spontaneously broken chiral symmetry enables us to understand many features of the weak interactions of π's. And in fact, even the strong interactions of pions at low energy are determined by mere symmetry considerations, once we we recognize that pions are Goldstone bosons.

This is truly remarkable. The dynamics of the strong interactions, which bind pions from quark-antiquark pairs, are very complicated, and not terribly well understood. The pion has a size $R \sim 10^{-13}$ cm and a complex structure. But I claim that at energies $E \ll \Lambda \sim R^{-1}$, the pion can be regarded as effectively pointlike (we can ignore its structure) and its couplings are determined by chiral symmetry. (The main idea of Current Algebra.)

To see how this works, we'll continue to consider the fictitious world with two exactly massless quarks. The real world is remarkably close to this limit. If we wish, we may include (small) intrinsic quark masses and treat them perturbatively ("chiral perturbation theory").

We want to construct an effective field theory that describes the self-couplings of pions at $E \ll \Lambda$. That is, we will regard the inverse size of the pion as a cutoff at which our description of pions as effectively pointlike breaks down. For $E \gtrsim \Lambda$, the structure of the pion becomes important, but in processes with $E \ll \Lambda$, this structure cannot be resolved.

The key idea is that a "soft" pion is equivalent to a slowly varying orientation of the vacuum. Recall, as discussed on p. 6.38, that the spontaneous symmetry breakdown

$$SU(2)_L \times SU(2)_R \to SU(2)_V$$

is driven by a composite order parameter:

$$\langle \bar{q}_{Li}\, q_{Rj} \rangle_\Sigma = -v^3\, \Sigma_{ij} .$$

An SU(2) matrix that labels degenerate vacua

To understand the details of this symmetry breakdown (e.g., to calculate v) requires a detailed understanding of strong-interaction dynamics. But the low energy consequences of the symmetry breakdown follow from symmetry alone.

For the purpose of describing low-energy π-π scattering, we construct a "phenomenological Lagrangian" (or "chiral Lagrangian") in terms of a field $\Sigma(x)$ -- which characterizes the "orientation of the vacuum" locally in spacetime. The only information we have about this Lagrangian is that it is invariant under global $SU(2)_L \times SU(2)_R$. That is,

$$\Sigma(x) \longrightarrow V_L^{\dagger}\, \Sigma(x)\, V_R$$

is just a rotation of the vacuum, which does not change the pion physics. Note that

$$\Sigma^{\dagger}\Sigma = \mathbb{1}$$

is a trivial invariant (independent of Σ). Hence, the terms in $\mathcal{L}$ must involve derivatives of Σ. This is a restated form of our earlier conclusion that Goldstone bosons always have derivative couplings; because a global vacuum rotation is a symmetry, couplings approach zero as $E \to 0$

The pions themselves are associated with the three independent infinitesimal chiral rotations of the vacuum; we may express Σ in terms of pions as

$$\Sigma = \exp\left(i\pi^a\sigma^a/F\right)$$

Here F has the dimensions of mass -- it will actually turn out to be $F_\pi \sim 93$ MeV. Note that the transformation

$$\Sigma \to V_L^+ \Sigma V_R,$$

while simple when expressed in terms of the Σ field, is a complicated (nonlinear) transformation acting on the π's. Hence, chiral symmetry (more precisely, the broken chiral symmetries with $V_L \neq V_R$) are said to be "nonlinearly realized" in the chiral Lagrangian. The reason this is necessary is that the three π^a's $a=1,2,3$ do not comprise a complete multiplet under chiral $SU(2)_L \times SU(2)_R$, but only under $SU(2)_V$ isospin.

If restricted by only chiral symmetry, $\mathcal{L}$ contains an infinite number of terms. But at low momentum, $p \ll \Lambda$, we expect that all but a few of these terms become irrelevant. Low-energy scattering becomes dominated by terms with the minimal number of derivatives (and hence minimal number of powers of p).

The minimal number of derivatives in a Lorentz invariant operator is two. Chiral invariant terms with two derivatives are:
(Recall $\Sigma^+\Sigma$, with zero derivatives, is trivial.)

① $(\mathrm{tr}\,\Sigma^{+}\partial_\mu \Sigma)^2$

② $\mathrm{tr}(\partial^\mu \Sigma^{+} \partial_\mu \Sigma)$

③ $\mathrm{tr}(\Sigma^{+}\partial_\mu \Sigma)^2$

But in fact, there is really just one term.
① = 0, because $\Sigma^{+}\partial_\mu\Sigma$ is traceless, and
③ can be transformed to ② by an integration by parts.

To see that ① is traceless:

$$\epsilon^\mu \Sigma^{+}\partial_\mu \Sigma = \Sigma^{+}(x)\left(\Sigma(x+\epsilon) - \Sigma(x)\right)$$
$$= \Sigma^{+}(x)\,\Sigma(x+\epsilon) - \mathbb{1}$$

But any SU(2) matrix can be obtained from any other one by right multiplication by some SU(2) matrix,

$$\Sigma(x+\epsilon) = \Sigma(x)\, V(\epsilon)$$

where $V(\epsilon) \to \mathbb{1}$ as $\epsilon \to 0$

So

$$\epsilon^\mu \Sigma^{+}\partial_\mu \Sigma = \underbrace{V(\epsilon) - \mathbb{1}}$$

an SU(2) generator in order ϵ, which is traceless.

To relate ③ and ②:

Since $\Sigma^{+}\Sigma = \mathbb{1}$,

$$0 = \partial_\mu \Sigma^{+}\Sigma = (\partial_\mu \Sigma^{+})\Sigma + \Sigma^{+}\partial_\mu \Sigma,$$

or $\Sigma^{+}\partial_\mu \Sigma = -(\partial_\mu \Sigma^{+})\Sigma$.

Thus, $(\Sigma^{+}\partial_\mu \Sigma)^2 = -(\partial_\mu \Sigma^{+})(\Sigma\,\Sigma^{+})\partial_\mu \Sigma = -\partial_\mu \Sigma^{+}\partial^\mu \Sigma$

We conclude that the most general phenomenological Lagrangian invariant under Lorentz transformations and chiral symmetry is

$$\mathcal{L} = \tfrac{1}{4} F^2 \, tr(\partial^\mu \Sigma^+ \partial_\mu \Sigma)$$
$$+ (\text{terms with more derivatives})$$

Let's express this in terms of pion fields

$$\Sigma = e^{i\mathcal{M}/F} \qquad \mathcal{M} = \pi^a \sigma^a$$

We'll expand to quartic order in $\mathcal{M}$, which is sufficient for $2 \to 2$ scattering.

$$\Sigma = 1 + \frac{i\mathcal{M}}{F} - \frac{\mathcal{M}^2}{2F^2} - \frac{i\mathcal{M}^3}{6F^3} + \cdots$$

$$\mathcal{L} = \tfrac{1}{4} F^2 tr\Big[\Big(-\frac{i}{F}\partial^\mu \mathcal{M} - \frac{1}{2F^2}\partial^\mu \mathcal{M}^2 + \frac{i}{6F^3}\partial^\mu \mathcal{M}^3 + \cdots\Big)$$
$$\Big(\frac{i}{F}\partial_\mu \mathcal{M} - \frac{1}{2F^2}\partial_\mu \mathcal{M}^2 - \frac{i}{6F^3}\partial_\mu \mathcal{M}^3 + \cdots\Big)$$

$$= \underbrace{\tfrac{1}{4}\, tr\, \partial^\mu \mathcal{M} \partial_\mu \mathcal{M}}_{\frac{1}{2}\partial^\mu \pi^a \partial_\mu \pi^a} + \mathcal{L}_{int}$$

$(tr\, \sigma^a \sigma^b = 2\delta^{ab})$ -- So π's are conventionally normalized scalars

$$\mathcal{L}_{int} = \frac{1}{16F^2} tr\Big[(\partial^\mu \mathcal{M}^2)(\partial_\mu \mathcal{M}^2) - \tfrac{4}{3}(\partial^\mu \mathcal{M})(\partial_\mu \mathcal{M}^3)\Big]$$

or

$$\mathcal{L}_{int} = \frac{1}{16F^2}\,\mathrm{tr}\Big[\big((\partial^\mu M)M + M(\partial^\mu M)\big)^2 - \frac{4}{3}\partial^\mu M\big((\partial_\mu M)M^2 + M(\partial_\mu M)M + M^2\partial_\mu M\big)\Big]$$

$$= \frac{1}{16F^2}\,\mathrm{tr}\Big[\left(2-\frac{4}{3}\right)(\partial^\mu M)M(\partial_\mu M)M + \left(2-\frac{8}{3}\right)(\partial^\mu M)(\partial_\mu M)M^2\Big]$$

$$= \frac{1}{24F^2}\,\mathrm{tr}\big[(\partial^\mu M)M(\partial_\mu M)M - (\partial^\mu M)(\partial_\mu M)M^2\big]$$

$$= \frac{1}{48F^2}\,\mathrm{tr}\big([M,\partial^\mu M][M,\partial_\mu M]\big)$$

Now, $M = \pi^a\sigma^a \Rightarrow$

$$[M,\partial^\mu M] = \pi^a\partial^\mu\pi^b\, 2i\epsilon^{abc}\sigma^c,$$

and

$$\mathcal{L}_{int} = \frac{1}{48F^2}\,\pi^a\partial^\mu\pi^b\pi^c\partial_\mu\pi^d(2i\epsilon^{abe})(2i\epsilon^{cdf})(2\delta^{ef})$$

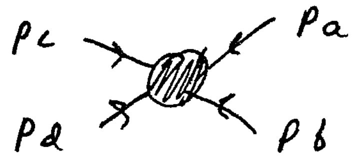
$$= \frac{-1}{6F^2}\,\pi^a\partial^\mu\pi^b\pi^c\partial_\mu\pi^d\left(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}\right)$$

$$\boxed{\mathcal{L}_{int} = \frac{1}{6F^2}\Big[(\vec{\pi}\cdot\partial^\mu\vec{\pi})^2 - \vec{\pi}^2(\partial^\mu\vec{\pi})^2\Big] + (\text{higher order in } \pi) + (\text{more derivatives})}$$

There is a single free parameter, F, in this expression. Chiral symmetry not only ensures that π couples derivatively, but also relates the coefficients of the two terms quadratic in derivatives that are consistent with isospin symmetry. (Note: vector in $\vec{\pi}$ denotes an isotriplet) In fact, all the terms in $\mathcal{L}$ quadratic in derivatives, involving arbitrary numbers of pion fields, can be expressed in terms of the one parameter F. So chiral symmetry determines not just the leading behavior of the 4-body pion scattering amplitude in the limit $E \ll \Lambda$, but also the n-body amplitude. (The terms higher order in derivatives, however, involve additional free parameters.)

Using the effective Lagrangian we may derive the amplitude for soft elastic pion scattering

$$\pi^a + \pi^b \to \pi^c + \pi^d$$



$$= \frac{i}{6F^2}\Big[\delta^{ab}\delta^{cd}(2)(-1)(p_a\cdot p_c + p_a\cdot p_d + p_b\cdot p_c + p_b\cdot p_d)$$
$$- \delta^{ab}\delta^{cd}(4)(-1)(p_a\cdot p_b + p_d\cdot p_c)$$
$$+ (2 \text{ crossed channels})\Big]$$

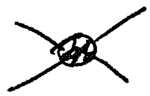
Since pions are massless, we have --

$p_a\cdot p_b = s/2 = p_d\cdot p_c$

$p_a\cdot p_c = t/2 = p_b\cdot p_d$

$p_a\cdot p_d = u/2 = p_b\cdot p_c$

$s+t+u = 0$

So  $= \frac{i}{6F^2}\Big[\delta^{ab}\delta^{cd}(4s - 2t - 2u)$
$+ (\text{crossed channels})\Big]$

or  $\boxed{= \frac{i}{F^2}\left(\delta^{ab}\delta^{cd}s + \delta^{ac}\delta^{bd}t + \delta^{ad}\delta^{bc}u\right)}$

A famous formula, first derived by Weinberg. It is exact in the chiral limit (pion mass = 0) and $s, t, u \to 0$.

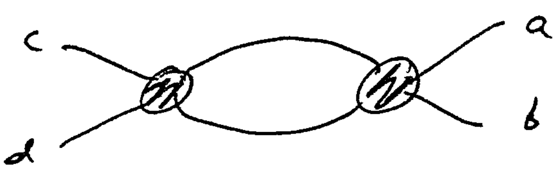
What is remarkable is that this formula applies to any model with $SU(2)_L \times SU(2)_V$ symmetry spontaneously broken to $SU(2)_V$. In particular, it must apply to the (linear) σ-model described in a homework exercise, in the limit $E \to 0$. This is a bit surprising, since in individual diagrams the π's of the (linear) σ-model appear to have nonderivative couplings. But the π scattering amplitudes in the $E \to 0$ limit agree with those obtained from our derivatively coupled phenomenological Lagrangian, as they must.

Corrections

There are two types of corrections to the above formula. The first type is due to higher dimension operators with more derivatives. These corrections are obviously suppressed by more powers of E/Λ.

But there are also loop corrections. How do we deal with loop corrections in a field theory like this one that is not renormalizable? We must recall that this is an effective field theory with an explicit cutoff Λ. (Higher dimension operators are suppressed by powers of Λ^{-1}, where Λ^{-1} is of order the size of the pion.)

For example, consider the one-loop correction to elastic π-π scattering:



+ crossed diagrams

(Here  is the 4-pion vertex in $\mathcal{L}_{int}$, considered earlier) This diagram has an ultraviolet divergence. As always, the ultraviolet divergent part of the diagram is a polynomial in external momenta that can be absorbed into a renormalization of parameters in $\mathcal{L}$ (can be cancelled by counterterms). Since the interaction is not renormalizable, we eventually will generate (in higher orders) an infinite number of such counterterms. This is really no big deal, though; we already have an infinite number of terms in $\mathcal{L}$ anyway.

The Ward identities of chiral symmetry ensure that the counterterms respect chiral invariance, and so can be absorbed into our most general chiral-invariant $\mathcal{L}$. There are no counterterms zeroth order in momentum (chiral symmetry) so counterterms are of two types

- Quadratic in momenta

  order $\left(\frac{1}{F^4} p^2 \Lambda^2 \right)$, by power counting

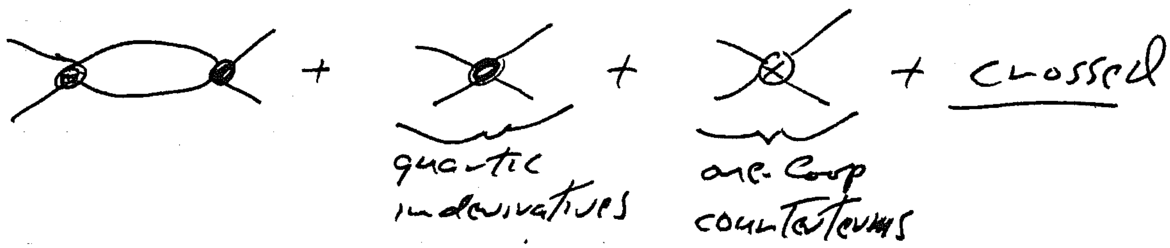
  These may be absorbed into a renormalization of the $\partial_\mu \Sigma^\dagger \partial^\mu \Sigma$ term; i.e., a rescaling of F.

- Quartic in momenta

  order $\left(\frac{1}{F^4} p^4 \ln(\Lambda) \right)$, by power counting

  These may be absorbed into renormalization of terms 4th order in derivatives, and in any case are suppressed by order p^2/F^2 relative to tree approximation.

We see that the tree approximation is actually exact, up to corrections suppressed by more powers of E_{pion}. Including counterterms and the order (p^4) tree contribution, the leading corrections to Weinberg's formula have the form



$$\frac{1}{F^2}\,\frac{1}{16\pi^2 F^2}\Big[\left(\begin{array}{c}\text{Polynomial Quartic}\\ \text{in momenta}\end{array}\right)\times \ln\left(S/\mu^2\right)$$

$$+\ (\text{Quartic Polynomial}) + \text{cross terms}\ \Big]$$

Here μ is a subtraction point that enters when we define renormalized couplings; A change in μ can be absorbed into a rescaling of coefficients in the polynomial piece. Note that, since the polynomial piece of the loop graph is suppressed by

$$\text{order } \left(\frac{p}{4\pi F}\right)^2$$

relative to tree approximation, it seems sensible to regard $p^2/(4\pi f)^2$ as the small expansion parameter in the problem; i.e.

$$\Lambda \sim 4\pi F$$

Note also that, for $s, t, u \to 0$, the log term coming from the loop diagram dominates the pure polynomial piece. To this extent, the (leading) corrections to Weinberg's formula are actually calculable -- since the log term is related (by unitarity) to the tree approximation diagram.

Before leaving the subject of the chiral Lagrangian, we should establish that the parameter F appearing in $\mathcal{L}$ is actually the same as F_π defined as on page (6.43):

$$\langle 0 | A^{\mu a} | \pi^b \rangle = \delta^{ab}\, i F_\pi\, p^\mu$$

To do this, we must find an expression for the (properly normalized) axial current in terms of the pion fields. First we will find the chiral symmetry currents of the chiral Lagrangian by applying the Noether procedure; then we will check the normalization by computing canonical commutators.

We will implement the Noether procedure in the form on page (4.59). There we showed that if

$$\delta \phi = \epsilon A \phi \qquad (\epsilon = \text{constant})$$

is a global symmetry, and we make the transformation

$$\delta \phi(x) = \epsilon(x) A \phi(x),$$

then

$$\delta \mathcal{L} = \partial_\mu \epsilon\, J^\mu$$

where J^μ is the conserved Noether current

To construct the $SU(2)_L$ currents, we apply this procedure to

6.63

$$\mathcal{L} = \frac{F^2}{4} \mathrm{tr}(\partial^\mu \Sigma^\dagger \partial_\mu \Sigma) + \cdots$$

where $SU(2)_L$ acts as $\Sigma \to V_L^\dagger \Sigma$, or, in infinitesimal form

$$\Sigma \to \left(\mathbb{1} - i\frac{\epsilon^a}{2}\sigma^a\right)\Sigma,$$

so that

$$\delta\mathcal{L} = \frac{F^2}{4}\mathrm{tr}\Big[\partial^\mu\Big(\Sigma^\dagger\big(i\frac{\epsilon^a}{2}\sigma^a\big)\Big)\partial_\mu \Sigma + \partial^\mu\Sigma^\dagger\partial_\mu\Big(\big(-i\frac{\epsilon^a}{2}\sigma^a\big)\Sigma\Big)\Big] + \cdots$$

$$= \frac{iF^2}{4}\partial_\mu\epsilon^a\,\mathrm{tr}\left[\Sigma^\dagger\frac{\sigma^a}{2}\partial^\mu\Sigma - \partial^\mu\Sigma^\dagger\frac{\sigma^a}{2}\Sigma\right]$$

$$+ (\text{no derivatives acting on } \epsilon) + \cdots$$

therefore

$$J_L^{\mu a} = \frac{iF^2}{4}\mathrm{tr}\left[\Sigma^\dagger\frac{\sigma^a}{2}\partial^\mu\Sigma - \partial^\mu\Sigma^\dagger\frac{\sigma^a}{2}\Sigma\right]$$

(+ terms with more derivatives)

Now we expand in powers of the pion field

$$\Sigma = \exp(i\pi^a\sigma^a/F) = I + \frac{i}{F}\pi^a\sigma^a + \cdots$$

Thus $\boxed{J_L^{\mu a} = -\frac{1}{2}F\partial^\mu\pi^a + \cdots}$

And, since π is parity odd, we obtain J_R from J_L by reversing the sign of terms odd in π:

$$\boxed{J_R^{\mu a} = \frac{1}{2}F\partial^\mu\pi^a + \cdots}$$

The axial current is

$$A^{\mu a} = J_R^{\mu a} - J_L^{\mu a} = F\partial^\mu \pi^a + \cdots$$

And so we have

$$\langle 0 | A^{\mu a} | \pi^b \rangle = -iF\delta^{ab} p^\mu$$

Thus $F = F_\pi$ (up to sign).

But to check that our normalization convention for the currents is the correct one, we should compute the current algebra. For this purpose we could expand $J_L^{\mu a}$ out to the next order. But it is just as good to note that, since the conserved charges

$$Q_L^a = \int d^3x \, J_L^{0a}$$

implement the infinitesimal symmetry transformation

$$\Sigma \to \left(\mathbb{1} - \frac{i\epsilon^a}{2}\sigma^a\right)\Sigma,$$

or $$[Q^a, \Sigma] = (-i)\left(\frac{i\sigma^a}{2}\right)\Sigma = \frac{\sigma^a}{2}\Sigma .$$

(as on page 4.63) And, by the Jacobi identity --

$$[[Q^a, Q^b], \Sigma] = -[[Q^b, \Sigma], Q^a] - [[\Sigma, Q^a], Q^b]$$

$$= -\left[\frac{\sigma^b}{2}\Sigma, Q^a\right] + \left[\frac{\sigma^a}{2}\Sigma, Q^b\right]$$

$$= \left(\frac{\sigma^b}{2}\frac{\sigma^a}{2}\right)\Sigma - \left(\frac{\sigma^a}{2}\frac{\sigma^b}{2}\right)\Sigma$$

$$= -i\,\epsilon^{abc}\frac{\sigma^c}{2}\Sigma$$

We conclude that $[Q_L^a, Q_L^b] = -i\epsilon^{abc} Q_L^c$

This is the same normalization as on page (6.42), except for the sign change. (With this sign difference, we have $F = F_\pi$, without a relative minus sign.)

The Higgs Mechanism

Let us return to the Goldstone model (page (6.15)) in which a global $U(1)$ symmetry is spontaneously broken:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \lambda \left(\phi^*\phi - \tfrac{1}{2}v^2\right)^2$$

Now we will ask -- what happens if the $U(1)$ symmetry is gauged?

Recall the minimal coupling prescription (page (5.39)). To promote the global symmetry

$$\phi \to e^{i\alpha}\phi,$$

to a local symmetry

$$\phi \to e^{-ie\omega}\phi$$
$$A_\mu \to A_\mu + \partial_\mu \omega,$$

we replace $\partial_\mu \phi$ by a covariant derivative

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$$

Then...

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - \lambda(\phi^*\phi - \tfrac{1}{2}v^2)^2$$

is gauge invariant.

We first study this theory in the classical limit. What is the classical ground state? As before, it is convenient to change variables

$$\phi = \frac{1}{\sqrt{2}}\rho e^{i\sigma}$$

In these variables, the scalar field kinetic term of the Goldstone model is

$$\frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}\rho^2\partial_\mu\sigma\,\partial^\mu\sigma,$$

and the global symmetry is

$$\sigma \to \sigma + \alpha.$$

Promoting this to a local symmetry

$$\sigma \to \sigma - e\omega$$
$$A_\mu \to A_\mu + \partial_\mu\omega$$

by the minimal coupling prescription, we have

$$D_\mu\sigma = \partial_\mu\sigma + eA_\mu,$$

and

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}\rho^2(\partial_\mu\sigma + eA_\mu)^2 - \frac{\lambda}{4}(\rho^2 - v^2)^2$$

(which is what we would obtain by substituting $\phi = \frac{1}{\sqrt{2}} \rho e^{i\sigma}$ into our previous expression for $\mathcal{L}$).

Now, what is the classical ground state? Clearly we must choose

$$\rho = v.$$

In the Goldstone model (global symmetry) the σ particle was then a massless scalar -- the Goldstone boson. But now, fluctuations in σ are not physical, they are just gauge transformations. Hence, by gauging the symmetry we remove the Goldstone boson from the spectrum of the theory. This trick is called the Higgs mechanism.

In fact, though one often speaks of "spontaneous breakdown" of the gauge symmetry, this is really a misnomer. Different values of σ, which in the Goldstone model represented distinct but equivalent classical ground states, now in the gauged model are just different (gauge-equivalent) mathematical descriptions of the same unique classical ground state. This distinction recalls the key difference between global and local symmetries, emphasized on page (5.23). Global symmetries relate different states of a system that are physically equivalent. But local symmetry (gauge invariance) relates different mathematical descriptions of the same field configuration. (The configurations should be taken to be gauge equivalence classes, in order that the initial value problem be well-defined.)

In spite of the observation that spontaneous symmetry breakdown does not occur in the gauged $U(1)$ model (the "Abelian Higgs model") there is a qualitative difference between the properties of the model for $v^2 \geq 0$ and $v^2 < 0$ (classically). We can see the difference by analyzing the spectrum of the model. This is best done by fixing the gauge, and then studying the small fluctuations around the classical ground state.

One way to fix the gauge is to "gauge σ away" -- to perform a gauge transformation so that $\sigma = 0$. In this gauge,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \rho \, \partial^\mu \rho + \frac{1}{2} \rho^2 (e A_\mu)^2 - \frac{\lambda}{4} (\rho^2 - v^2)^2.$$

To identify the spectrum, we expand around the ground state

$$\rho = v + \rho'.$$

Then $\quad \frac{1}{2}(\rho'+v)^2 e^2 A_\mu A^\mu - \frac{\lambda}{4}((\rho'+v)^2 - v^2)^2$

$$= \tfrac{1}{2}e^2 v^2 A_\mu A^\mu + e^2 v \rho' A_\mu A^\mu + \tfrac{1}{2}e^2 \rho'^2 A_\mu A^\mu$$
$$- \lambda v^2 \rho'^2 - \lambda v \rho'^3 - \tfrac{\lambda}{4}\rho'^4$$

$$= \tfrac{1}{2} m_V^2 A_\mu A^\mu - \tfrac{1}{2} m_S^2 \rho'^2 + \text{interaction terms}$$

where $\quad m_V^2 = e^2 v^2 \quad$ -- vector mass
$\qquad\quad m_S^2 = 2\lambda v^2 \quad$ -- scalar mass

We have a <u>massive</u> spin-1 field with mass m_V and a massive spin-0 field with mass m_S.

For $v^2 > 0$, the classical ground state has $\rho = 0$ (and we can't use our change of variable, since it is singular at $\rho = 0$). Then the spectrum consists of a massless spin-1 field (photon) and degenerate spin-0 fields with mass $m_S^2 = \lambda v^2$. The two scalars have electric charges $\pm e$ -- they are particle and antiparticle.

The spectrum of the theory should behave smoothly in the limit $e \to 0$, in which the gauging "turns off." Now we can understand how this works.

Spectrum of the Abelian Higgs Model:

$v^2 < 0$	$v^2 > 0$
Massless Photon ($\lambda = \pm 1$) Massive Complex Scalar (two $\lambda = 0$ states)	Massive ($m_V^2 = e^2 v^2$) Spin-1 ($\lambda = \pm 1, 0$) Massive Real Scalar ($\lambda = 0$)

$\Downarrow$ $e \to 0$

(U(1) Goldstone model decoupled from photon) $\longrightarrow$

Massless Spin-1 ($\lambda = \pm 1$)
Massless ($\lambda = 0$)
Massive ($\lambda = 0$)

For all values of e, v, the spectrum contains 4 helicity states (states do not appear or disappear as we smoothly change parameters). For $v^2 > 0$, one of the scalar particles becomes the helicity $\lambda = 0$ component of a <u>vector</u> particle. As $e \to 0$, the vector mass $\to 0$, and the $\lambda = 0$ component decouples from the $\lambda = \pm 1$ states (page 3.16) and becomes the Goldstone boson of the U(1) Goldstone model.

One says that for $e \neq 0$, the photon "eats" the Goldstone boson, and upon digesting it, gains a mass.

Quantization

We emphasized on page (6.26) that spontaneous symmetry breakdown does not spoil the Ward identities associated with a global symmetry. Hence, the counterterms generated by radiative corrections will respect the symmetry. The ultraviolet behavior (e.g., renormalizability) is unaffected by SSB.

This remark becomes quite important in the context of a gauge theory that undergoes the Higgs mechanism. The gauge $\sigma = 0$ (the "unitary gauge") is quite convenient for analyzing the spectrum of the Abelian Higgs model, but not at all convenient for studying the ultraviolet behavior. It is not at all obvious in this gauge that the model is renormalizable.

The problem is the massive vector particle. Recall that we found by canonical methods that a massive vector has a propagator (page 5.14)

$$\sim\sim\sim = \Delta_{\mu\nu} = \frac{-i}{k^2-\mu^2}\left[\eta_{\mu\nu} - \frac{k_\mu k_\nu}{\mu^2}\right].$$

Hence for large k ($k^2 \gg \mu^2$):

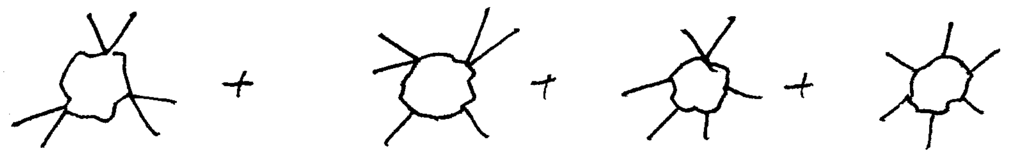
$$\Delta_{\mu\nu} \longrightarrow \frac{i}{\mu^2}\,\frac{k_\mu k_\nu}{k^2} + O\left(\frac{1}{k^2}\right)$$

This propagator does not fall off for large K.

In the unitary gauge, there are couplings (page 6.69)

$$\mathcal{L}' = e^2 v \rho' A_\mu A^\mu + \tfrac{1}{2} e^2 \rho'^2 A_\mu A^\mu + \cdots,$$

and hence, at one loop, diagrams such as



Because of the bad behavior of the A_μ propagator, all of these diagrams are quartically divergent by power counting

$$\sim (e^2)^3 \int^\Lambda \frac{d^4K}{(\mu^2)^3} \sim \left(\frac{e}{\mu}\right)^6 \Lambda^4 \sim \frac{\Lambda^4}{v^6}$$

(vector mass)

Unless quite delicate cancellations occur among the diagrams, it appears that we will need to introduce a ρ'^6 counterterm, and indeed, counterterms of all orders in ρ'. If so, the theory has an infinite number of parameters and loses predictive power. The failure of renormalizability means that this theory is not an adequate description of physics at short distances; it is useful only for momenta $p \lesssim v$.

In fact, miraculous cancellations of the ultraviolet divergences *do* occur when we sum diagrams, but it is virtually impossible to verify that this works to all orders in the unitary gauge. Fortunately, it is possible to make alternative gauge choices in which the good ultraviolet behavior of the theory is manifest diagram-by-diagram. In these gauges, fictitious particles must be introduced, and the unitarity of the theory is obscured. Once again, it is gauge-invariance that ensures that the theory is at once unitary and renormalizable.

Instead of the unitary gauge, we may fix the gauge with a covariant gauge-fixing term (page 5.35)

$$\mathcal{L}_{g.f} = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$$

Actually, this gauge choice is not the most convenient one in the Higgs model. The reason is that the action of the model contains the term

$$\frac{1}{2}v^2 D_\mu\sigma D^\mu\sigma$$

$$= \frac{1}{2}v^2\left(\partial_\mu\sigma\partial^\mu\sigma + 2e\partial_\mu\sigma A^\mu + e^2 A_\mu A^\mu\right)$$

The coupling $\qquad e v^2 \partial_\mu \sigma A^\mu$

is a source of σ-A mixing:

We have to sum up insertions of this mixing vertex to get the right σ and A propagators. This is not a problem of principle, but it is a nuisance.

We can remove the mixing, though, by making a clever choice for the gauge-fixing term (= "t Hooft gauges"). Actually, for the purpose of studying the ultraviolet behavior of the theory, it is more convenient to use different variables. So in

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - \lambda\left(\phi^*\phi - \frac{v^2}{2}\right)^2,$$

we substitute $\phi = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2)$.
Then

$$\begin{aligned} D_\mu\phi^* D^\mu\phi = {} & \frac{1}{2}(\partial_\mu - ieA_\mu)(\phi_1 - i\phi_2)(ieA^\mu v) \\ & + \frac{1}{2}(-ieA_\mu v)(\partial^\mu + ieA^\mu)(\phi_1 + i\phi_2) \\ & + \frac{1}{2}(\partial_\mu - ieA_\mu)(\phi_1 - i\phi_2)(\partial^\mu + ieA^\mu)(\phi_1 + i\phi_2) \\ = {} & \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 \\ & + ev A_\mu\partial^\mu\phi_2 \\ & + e^2 v A_\mu A^\mu\phi_1 \\ & + eA_\mu(\phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1) \\ & + \frac{1}{2}e^2 A_\mu A^\mu(\phi_1^2 + \phi_2^2) \end{aligned}$$

In these variables, the mixing term is

$$e v A_\mu \partial^\mu \phi_2$$

The 't Hooft gauges are defined by

$$\partial_\mu A^\mu - \xi v \phi_2 = f, \text{ or}$$

$$\mathcal{L}_{g.f.} = -\frac{1}{2\alpha}(\partial_\mu A^\mu - \xi v \phi_2)^2$$

(Since $\phi \to e^{-ie\omega}\phi$ or $\phi_2 \to \phi_2 - e\omega v$ under an infinitesimal gauge transformation, the variation of the gauge-fixing condition with respect to ω is a constant, and we don't need a Faddeev-Popov "ghost"!)

$$\mathcal{L}_{g.f.} = -\frac{1}{2\alpha}\left((\partial_\mu A^\mu)^2 - \xi v \partial_\mu A^\mu \phi_2 - \xi^2 v^2 \phi_2^2\right)$$

After integrating by parts, the cross term is

$$-\frac{\xi v}{\alpha} A^\mu \partial_\mu \phi_2 ;$$

therefore, if we choose $\xi = e\alpha$, the unwanted mixing does not occur.

We now have ...

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{g.f.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$
$$-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2$$
$$-\frac{1}{2}m_S^2 \phi_1^2 - \frac{1}{2}\alpha\mu^2\phi_2^2 + (\text{couplings})$$

Therefore, in these gauges, the ϕ_2 particle has mass

$$m_{\phi_2}^2 = \alpha \mu^2$$

where $\mu = ev$ is the mass of the vector A_μ.

What is the A_μ propagator in these ('t Hooft) gauges? We must invert

$$(\Delta^{-1})^{\mu\nu} = i\left[\eta^{\mu\nu}k^2 - k^\mu k^\nu + \frac{1}{\alpha}k^\mu k^\nu - \mu^2\eta^{\mu\nu}\right],$$

which may be rewritten as

$$(\Delta^{-1})^{\mu\nu} = i\left[(k^2-\mu^2)\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + (k^2-\mu^2)\frac{k^\mu k^\nu}{k^2} - k^2\frac{k^\mu k^\nu}{k^2} + \frac{k^2}{\alpha}\frac{k^\mu k^\nu}{k^2}\right]$$

$$= i\left[(k^2-\mu^2)\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \left(\frac{k^2}{\alpha} - \mu^2\right)\frac{k^\mu k^\nu}{k^2}\right].$$

In terms of the orthogonal projections

$$T^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

$$L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2},$$

this is

$$(\Delta^{-1})^{\mu\nu} = i\left[(k^2-\mu^2)\, T^{\mu\nu} + \frac{k^2-\alpha\mu^2}{\alpha} L^{\mu\nu}\right],$$

which is inverted by

$$\Delta^{\mu\nu} = (-i)\left[\frac{1}{k^2-\mu^2} T^{\mu\nu} + \frac{\alpha}{k^2-\alpha\mu^2} L^{\mu\nu}\right]$$

$$= \frac{-i}{(k^2-\mu^2)}\left[\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \frac{\alpha(k^2-\mu^2)}{k^2-\alpha\mu^2}\frac{k^\mu k^\nu}{k^2}\right]$$

or

$$\boxed{\Delta^{\mu\nu} = \frac{-i}{(k^2-\mu^2)}\left[\eta^{\mu\nu} + (\alpha-1)\frac{k^\mu k^\nu}{k^2-\alpha\mu^2}\right]}$$

We see that these gauges have good ultraviolet behavior -- the propagator falls off like $1/k^2$. For $\mu^2 \to 0$ (or $k^2 \gg \mu^2$) they approach our old α gauges of page (5.37). But for $\mu^2 \neq 0$, the propagator has a new pole at

$$k^2 = \alpha\mu^2.$$

The price of achieving good UV behavior is the introduction of a fictitious particle with mass $\alpha\mu^2$ -- degenerate with the ϕ_2 particle in these gauges. Obviously, this particle is purely a gauge artifact. The pole contributions at $k^2 = \alpha\mu^2$ must cancel in the calculation of any measurable quantity. In general, this cancellation will involve summing graphs with virtual 's and graphs with virtual A_μ's, and it is not so obvious that such cancellations have to occur to all orders of perturbation theory. But fortunately this is assured by gauge invariance. S-matrix elements computed in the renormalizable 't Hooft gauges will agree with those computed in unitary gauge, where the fictitious particles are absent. Since the two gauges agree, we are assured that both are actually renormalizable and unitary.

Special Gauges

- $\alpha = \infty$ In this gauge, the "bad" pole in the propagator goes away

$$\Delta^{\mu\nu} = \frac{-i}{k^2-\mu^2}\left[\eta^{\mu\nu} - \frac{k^\mu k^\nu}{\mu^2}\right]$$

-- A massive vector propagator

Also $m^2_{\phi_2} = \alpha\mu^2 \to \infty$, so the ϕ_2 particle decouples. This is the <u>unitary</u> gauge.

- <u>$\alpha = 0$</u> In this gauge, the "bad" pole occurs at $k^2 = 0$

$$\Delta^{\mu\nu} = \frac{-i}{k^2-\mu^2}\left[\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right]$$

-- the <u>'t Hooft-Landau gauge</u>

- <u>$\alpha = 1$</u> In this gauge, the bad pole <u>goes away</u>. (But there is still, of course, a fictitious ϕ_2-particle with mass μ, and a negative metric vector coupling to A_0)

$$\Delta^{\mu\nu} = \frac{-i\eta^{\mu\nu}}{k^2-\mu^2}$$

-- <u>'t Hooft-Feynman gauge</u>

-- this is the simplest gauge in which to perform explicit calculations.

The most important thing we have learned from the Higgs mechanism is that it is possible for a theory of interacting massive spin-1 particles to make sense at short distances. Naively, one expects a breakdown of the description of the physics at energies

$$E \geq \frac{\mu}{e}$$

But if the theory is obtained from a gauge theory that undergoes the Higgs mechanism, then nice ultraviolet behavior is ensured, and the description does not break down.

Incidentally, what happened to Goldstone's theorem; why doesn't it require the theory to contain a massless particle? We can explain how the theorem is evaded in either of two ways:

- If we do not fix the gauge, then the field ϕ cannot have an expectation value,

$$\langle \phi \rangle = 0.$$

  Taking the infinite volume limit is irrelevant; because the symmetry is local, $\phi(x)$ can fluctuate locally at each point x, at no cost in energy. Averaging over gauges gives $\langle \phi \rangle = 0$ ("Elitzur's theorem") -- As remarked above, spontaneous symmetry breaking does not really occur -- the vacuum is unique.

- If we do fix the gauge (e.g. unitary gauge) then there is no symmetry, and Goldstone's theorem is inapplicable.

But even though a gauge system cannot be "magnetized," we are able to distinguish two possible "phases" for the abelian Higgs model:

<u>Coulomb Phase</u>: $m_V = 0$

There is a massless photon, and a long range force between electric charges.

<u>Higgs Phase (Superconductor)</u>: $m_V > 0$

Electric and magnetic fields are <u>screened</u> with screening length m_V^{-1} (the "Meissner effect").

Thus, we can speak of two possible <u>realizations</u> of a $U(1)$ gauge symmetry, just as for a $U(1)$ global symmetry.

Note:

contrary to the statement on page (6.75), a Faddeev-Popov ghost is needed in 't Hooft's α gauges in the abelian Higgs model. From the discussion on page (5.34), we see that, in the gauge

$$G(A) = \partial_\mu A^\mu - \alpha\mu\phi_2 = 0,$$

the path integral prescription for quantization requires us to include the factor

$$\det(\delta G/\delta\omega)|_{G=0}.$$

The gauge transformation

$$\phi \to e^{-ie\omega}\phi$$

becomes in infinitesimal form

$$\delta(\phi_1 + i\phi_2) = -ie\omega(v + \phi_1 + i\phi_2)$$

or

$$\delta\phi_1 = e\omega\phi_2$$
$$\delta\phi_2 = -e\omega(v+\phi_1).$$

Hence $\delta G = \partial_\mu\partial^\mu\omega + \alpha\mu e(v+\phi_1)\omega$

$$\Rightarrow \delta G/\delta\omega = \partial_\mu\partial^\mu + \alpha\mu^2 + \frac{\alpha\mu^2}{v}\phi_1.$$

The factor

$$\det\left(\partial_\mu\partial^\mu + \alpha\mu^2 + \frac{\alpha\mu^2}{v}\phi_1\right)$$

has a nontrivial dependence on the field ϕ_1; it is not a mere constant that can be absorbed into the integration measure.

As discussed on page (4.46) ff, this factor can be incorporated into the Feynman rules by introducing a Faddeev-Popov ghost -- a spinless particle η that obeys Fermi statistics, with

$$\mathcal{L}_{ghost} = \bar{\eta}\left(-\partial_\mu\partial^\mu - \alpha\mu^2 - \frac{\alpha\mu^2}{v}\phi_1\right)\eta$$

$$= \partial_\mu\bar{\eta}\,\partial^\mu\eta - \alpha\mu^2\bar{\eta}\eta - \frac{\alpha\mu^2}{v}\bar{\eta}\eta\phi_1 .$$

The ghost has mass $m^2_{ghost} = \alpha\mu^2$, the same as the mass of ϕ_2 and the fictitious pole in the A_μ propagator. The ghost needs to be included to ensure the cancellation of all the unphysical singularities at $k^2 = \alpha\mu^2$.