7. Nonabelian Gauge Fields

7. Nonabelian Gauge Fields and the Standard Model

The Nonabelian Minimal Coupling Prescription:

As described on page (5.39)ff, the "minimal coupling prescription" provides an algorithm for promoting a theory with an (abelian) global U(1) symmetry to a theory with a local U(1) symmetry. This is achieved, if the global symmetry acts by

$$\phi(x) \to e^{i\alpha}\phi(x),$$

by replacing

$$\partial_\mu \phi \to D_\mu \phi,$$

where

$$D_\mu = \partial_\mu + i g A_\mu$$

is the covariant derivative, and

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x)$$
$$\phi(x) \to e^{-ie\omega(x)}\phi(x),$$

under a local transformation

Now we wish to formulate a similar prescription that can be applied to a theory with a nonabelian global symmetry.

For example, consider the isospin symmetry discussed on page (6.36):

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \Omega^{-1} q$$

where $\Omega^{-1} \in SU(2)$. Can we construct a Lagrangian that has a *local* "isospin" symmetry.

Why should we try? Following Yang and Mills (1954), we note that the global SU(2) symmetry means that the names u and d are assigned to the two fields arbitrarily. We could just as well "rotate our axes" in isospace and redefine u, d by a unitary transformation. But if the symmetry is global, we can resolve this ambiguity once and for all by assigning the labels u, d at one point in spacetime. It seems contrary to the *spirit* of local field theory that we should not have the freedom to label u, d *differently* here than at, say, the Andromeda galaxy. Hence, we try to construct theories with *local* symmetry.

We may write an *infinitesimal* SU(2) transformation as

$$\Omega^{-1} = \mathbb{I} - i g \omega^a T^a$$

where $T^a = \frac{1}{2}\sigma^a \quad a = 1, 2, 3.$

As discussed on page (3.13), the generators T^a form a Lie Algebra closed under addition and commutation:

$$[T^a, T^b] = i \underbrace{\epsilon^{abc}} T^c$$

(totally antisymmetric $\epsilon^{123} = 1$)

The generators have the normalization ---

$$\text{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$$

Now consider a local transformation

$$q(x) \to (\mathbb{1} - ig\,\omega(x))\, q(x)$$

(where $\omega(x) \equiv \omega^a(x) T^a$ is hermitian, traceless).
under this transformation:

$$\partial_\mu q(x) \to \partial_\mu (\mathbb{1} - ig\omega)\, q$$

$$= \underbrace{(\mathbb{1} - ig\omega)\,\partial_\mu q}_{\text{"good" term}} - \underbrace{ig(\partial_\mu \omega)\, q}_{\text{"bad" term}}$$

To construct an invariant $\mathcal{L}$, we need to cancel the "bad" term.

So consider $D_\mu = \partial_\mu + ig A_\mu$,
where

$$A_\mu(x) = A^a_\mu(x)\, T^a$$

is hermitian and traceless. Under

$$q \to (\mathbb{1} - ig\omega)\, q$$

$$A \to A + \underline{\delta A} \quad (\text{order } \omega)$$

We have

$$D_\mu q \to (D_\mu + ig\,\delta A_\mu)(\mathbb{1} - ig\omega)\, q$$

$$= (\mathbb{1} - ig\omega)\, D_\mu q + ig(-[D_\mu, \omega] + \delta A_\mu)\, q$$

(to linear order in ω), so $D_\mu q$ transforms nicely if

$$\delta A_\mu = [D_\mu, \omega] = \partial_\mu \omega + ig[A_\mu, \omega];$$

we then have $\quad D_\mu q \to \Omega^{-1} D_\mu q.$

In terms of components, the transformation property for A_μ is

$$\delta A_\mu^a T^a = \partial_\mu \omega^a T^a + ig A_\mu^b \omega^c [T^b, T^c]$$

$$\text{or } \delta A_\mu^a = \partial_\mu \omega^a + g\, \epsilon^{abc} \omega^b A_\mu^c$$

This transformation law has a new feature, not seen in electrodynamics; the A_μ^a's transform even under a <u>global</u> transformation with ω = constant.

The components

$$A_\mu^a \qquad a = 1, 2, 3$$

transform as a <u>3-vector</u> representation of SU(2). (This property is required so that $D_\mu q$ transforms as desired, since that T^a's do not commute.)

<u>Finite Transformations:</u>

Under finite SU(2) transformations, we have

$$q \to \Omega^{-1} q$$

$$ig A_\mu \to \Omega^{-1} ig A_\mu \Omega - (\partial_\mu \Omega^{-1}) \Omega$$

$$D_\mu q \to \Omega^{-1} D_\mu q$$

Kinetic Term

If we want A_μ to be a dynamical field (as in electrodynamics), we need to construct a gauge-invariant kinetic term. Can we find an analog of the field $F_{\mu\nu}$?

We note that $q \to \Omega^{-1} q$

$$\Rightarrow \quad D_\mu q \to \Omega^{-1} D_\mu q$$
$$D_\mu D_\nu q \to \Omega^{-1} D_\mu D_\nu q,$$

or

$$D_\mu D_\nu \to \Omega^{-1} D_\mu D_\nu \Omega$$

Similarly, $[D_\mu, D_\nu] \to \Omega^{-1}[D_\mu, D_\nu]\Omega$.

But-- $\frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$.

This is the desired generalization of $F_{\mu\nu}$; we will define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu].$$

This field is not gauge-invariant, but transforms simply:

$$F_{\mu\nu} \to \Omega^{-1} F_{\mu\nu} \Omega$$

Note also that it contains a term quadratic in A. $F_{\mu\nu}$, like A, is a hermitian traceless matrix, and can be expanded

$$F_{\mu\nu} = F_{\mu\nu}^a T^a.$$

Now we can construct gauge-invariant Lorentz invariant operators from F. The simplest is

$$-\frac{1}{2}\operatorname{tr} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}^{a}F^{\mu\nu a}.$$

Expanding in terms of components,

$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a} - g\,\epsilon^{abc}A_\mu^{b}A_\nu^{c},$$

and

$$-\frac{1}{4}F_{\mu\nu}^{a}F^{\mu\nu a} = -\frac{1}{4}(\partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a})(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})$$

$$+\frac{g}{2}\epsilon^{abc}(\partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a})A^{\mu b}A^{\nu c}$$

$$-\frac{g^2}{4}\epsilon^{abc}\epsilon^{ade}A_\mu^{b}A_\nu^{c}A^{\mu d}A^{\nu e}$$

So our "kinetic term" actually contains a (conventionally normalized) "photon" kinetic term for each of A_μ^a, $a=1,2,3$, plus trilinear and quadrilinear interaction terms (of order g and g^2 respectively)

What is going on? The gauge field Lagrangian is nonlinear because the A_μ^a's themselves carry the charges to which they couple. E.g., we may think of the A_μ^3 field as a photon that couples to T^3 charge, but $A_\mu^{1,2}$ carry this charge (since $T^{1,2}$ do not commute with T^3)
So $A^{1,2}$ have a coupling to A^3 reminiscent

of scalar electrodynamics (page 5.41):

$$e A_\mu (\phi_1 \partial^\mu \phi_2 - \phi_2 \partial^\mu \phi_1)$$
$$+ \tfrac{1}{2} e^2 A_\mu A^\mu (\phi_1^2 + \phi_2^2)$$

Generalization to Other (Continuous) Symmetry Groups

The above construction is easily generalized to arbitrary compact symmetry groups (those for which the generators are hermitian matrices). We may write an infinitesimal transformation as

$$\Omega = \mathbb{1} + i g \omega^a T^a \qquad (a = 1, 2, \ldots, n)$$

-- where n is the dimension of the symmetry group G. The generators satisfy a Lie algebra

$$[T^a, T^b] = i C^{abc} T^c$$

"structure constants" (generalization of ϵ)

We note that $\mathrm{tr}\, T^a T^b = \Delta^{ab}$ is positive definite if T's are hermitian (and linearly independent). By choosing suitable linear combinations

$$\mathrm{tr}\, T^a T^b = c^a \delta^{ab},$$

where $c^a > 0$. Now we can rescale generators so that

$$\mathrm{tr}\, T^a T^b = \tfrac{1}{2} \delta^{ab}.$$

With this normalization

$$c^{abc} = -2i\, \mathrm{tr}([T^a, T^b]\, T^c),$$

and, by the cyclic property of the trace, the c^{abc}'s are totally antisymmetric

Given a theory invariant under this group of global symmetries, we can construct a theory with <u>local</u> gauge symmetry just as in the $SU(2)$ case considered above; we just replace ϵ^{abc} by c^{abc}.

<u>Question</u>: Can we construct other dimension 4 invariants besides $\mathrm{tr}\, F^2$?
The answer is <u>yes</u>, if the generators T^a transform <u>reducibly</u> under the symmetry group G.

That is, suppose the generators T^a split into two sets, $T^a_{(1)}$, $T^b_{(2)}$, that are mutually commuting.

$$[T^a_{(1)}, T^b_{(2)}] = 0$$

Then

$$\Omega = e^{iT^a\omega^a} = \Omega_1 \Omega_2$$

where

$$\Omega_2^{-1}\, T^a_{(1)}\, \Omega_2 = T^a_{(1)}$$
$$\Omega_1^{-1}\, T^a_{(2)}\, \Omega_1 = T^a_{(2)}$$

Then $F = F^a T^a$ can be split into two pieces,

$$F = F_1 + F_2,$$

and transforms as

$$F \to \Omega_1^{-1} F_1 \Omega_1 + \Omega_2^{-1} F_2 \Omega_2.$$

Therefore, we can construct two independent dimension 4 invariants:

$$\operatorname{tr} F_1^2, \quad \operatorname{tr} F_2^2$$

The most general kinetic term is

$$-\tfrac{1}{2} a_1 \operatorname{tr} F_1^2 - \tfrac{1}{2} a_2 \operatorname{tr} F_2^2.$$

We may rescale A_1 and A_2 so as to absorb the constants a_1, a_2, and thus obtain conventionally normalized kinetic terms, but then the covariant derivative becomes:

$$D_\mu = \partial_\mu + \frac{ig}{a_1} A_\mu^{(1)} + \frac{ig}{a_2} A_\mu^{(2)}.$$

So there are now <u>two</u> independent coupling constants g_1, g_2

If the Lie algebra splits into mutually commuting subalgebras, we say that it is <u>not</u> <u>simple</u>. (E.g., an example of a Lie group whose Lie algebra is not simple is $SU(3) \times SU(2)$.) Then we have as many independent kinetic terms (and coupling constants) as the number of mutually commuting subalgebras (or <u>factors</u> in the group).

The Geometry of Gauge Fields

We remarked on the geometric interpretation of the gauge field in the abelian case on page (5.42). Here we briefly describe the corresponding interpretation in the nonabelian case.

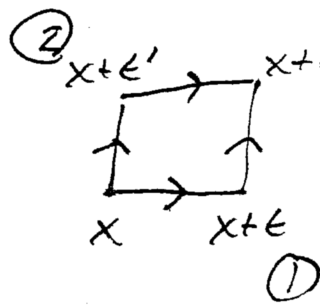
If, for example, the identity u, d of a quark varies in spacetime, we can compare the flavors of two quarks at different points only by introducing a notion of parallel transport. The gauge potential provides a notion of parallel transport for a quark field $q(x)$:

$$x \longrightarrow x+\epsilon$$

we say that $q(x)$ is parallel transported from x to $x+\epsilon$ if

$$\epsilon^\mu D_\mu q(x) = 0.$$

Now, what is the effect of parallel transport about an infinitesimal closed path? Alternatively, what is the difference between parallel transport of $q(x)$ to $x+\epsilon+\epsilon'$ along the two paths ϵ, ϵ' shown?



It is convenient to introduce another field $\phi(x)$, and construct a gauge-invariant composite field

$$q \cdot \phi(x) = q^i(x)\, \phi_i(x),$$

(where i is the index on which the matrix A_μ acts). Now, to linear order in ϵ

$$q \cdot \phi(x+\epsilon) = (I + \epsilon^\mu \partial_\mu)\, q \cdot \phi$$

And, since $q \cdot \phi$ is gauge invariant,

$$\partial_\mu (q \cdot \phi) = D_\mu (q \cdot \phi),$$

so we have

$$q \cdot \phi(x+\epsilon) = (I + \epsilon^\mu D_\mu)\, q \cdot \phi.$$

Now, the difference between transport along path ① and ② of $q \cdot \phi$ is

$$\delta(q \cdot \phi) = [(I + \epsilon'^\nu D_\nu)(I + \epsilon^\mu D_\mu) - (I + \epsilon^\mu D_\mu)(I + \epsilon'^\nu D_\nu)]\, q \cdot \phi$$

$$= \epsilon^\mu \epsilon'^\nu [D_\nu, D_\mu]\, q \cdot \phi$$

Suppose q is parallel transported. Then $D_\mu q = 0$, and we have

$$\delta(q \cdot \phi) = \epsilon^\mu \epsilon'^\nu\, q \cdot [D_\nu, D_\mu]\, \phi$$

If this is to be satisfied for any ϕ, we must have

$$\delta q^i = \epsilon^\mu \epsilon'^\nu q^j [D_\nu, D_\mu]_j{}^i$$

or $$\delta q = \epsilon^\mu \epsilon'^\nu (-igF_{\mu\nu}) q$$

We conclude that $-igF_{\mu\nu}$ is the Yang-Mills "curvature", just as $-ieF_{\mu\nu}$ was found to be the "curvature" of electrodynamics.

Nonabelian Higgs Mechanism:

We have considered two very interesting ideas about physics that took a long time to find their proper place in particle theory. The first is the Nambu-Goldstone idea: Nature may be more symmetrical than it appears to be. Exact symmetries of the Lagrangian of the world may not be manifest to us, because they are not symmetries of the vacuum. This idea runs into trouble, because if it is true there must be massless spin zero particles (Goldstone bosons) and these are not observed.

The second idea is the Yang-Mills idea: the exact symmetries of nature should be local symmetries. The trouble with this idea is that if it is true there must be a massless spin-one particle associated with each such local symmetry, and the only observed massless spin-one particle is the photon.

It took a while to appreciate that when these ideas are combined together, these objections can be overcome. The point is that the Higgs mechanism discussed previously for an abelian $U(1)$ gauge symmetry can be generalized. Whenever a local symmetry is "spontaneously broken" the Goldstone boson is eaten, and the spin-one gauge boson acquires mass.

To see how this works in general, let us first give a general discussion of the Nambu-Goldstone phenomenon, at the level of classical field theory.

Suppose G is a compact global symmetry group, of dimension n. An infinitesimal group element may be expanded

$$g(\omega) = (\mathbb{1} - i\omega^a T^a) \quad \text{where } a = 1, \ldots, n$$

and the T^a are the group generators. The statement that G is compact means that the T^a's are Hermitian. Finite transformations may be written

$$g(\omega) = e^{-i\omega^a T^a}$$

(and are unitary)

Now suppose that

$$\phi(x) = \langle \phi \rangle = \text{constant}$$

is one of the classical ground states in the theory of a (in general complex) scalar field ϕ, where the theory respects the global symmetry G, under which ϕ transforms nontrivially. The symmetry G is then spontaneously broken to a subgroup H consisting of the elements $h \in G$ such that

$$h\langle\phi\rangle = \langle\phi\rangle.$$

If h has dimension m, then we may identify m generators T^a, $a=1,\dots,m$, among the generators of G that are the generators of H; these must satisfy

$$T^a\langle\phi\rangle = 0, \quad a=1,\dots,m.$$

Of the remaining $n-m$ generators of G, no linear combination annihilates $\langle\phi\rangle$; in other words,

$$\alpha^b T^b\langle\phi\rangle = 0, \quad b=m+1,\dots,n$$
$$\Longrightarrow \alpha^b = 0.$$

There are thus $n-m$ independent motions of ϕ about $\langle\phi\rangle$ generated by the action of the symmetry group G. These motions correspond to the $n-m$ massless excitations -- the Goldstone bosons.

Now suppose that the symmetry group G is gauged. Then we can fix the unitary gauge by "gauging away" the Goldstone bosons. By performing a local G transformation, we can impose the condition on ϕ:

$$\phi(x) \cdot T^b \langle \phi \rangle = 0$$

-- ϕ is orthogonal to all directions in field space generated by G acting on $\langle \phi \rangle$. In this gauge, the gauge bosons acquire masses, from the

$$D_\mu \phi^* D^\mu \phi$$

term in the Lagrangian, where

$$D_\mu = \partial_\mu + i g_a A_\mu^a T^a$$

(g_a can depend on a if G is not simple), that are of the form:

$$\frac{1}{2}(\mu^2)_{ab} A^{\mu a} A_\mu^b = g_a g_b (T^a \langle \phi \rangle \cdot T^b \langle \phi \rangle) A^{\mu a} A_{\mu b},$$

or

$$(\mu^2)_{ab} = 2 g_a g_b (T^a \langle \phi \rangle \cdot T^b \langle \phi \rangle)$$

We see that

$$(\mu^2)_{ab} = 0 \qquad a, b = 1, \ldots, m$$

-- the H (unbroken) gauge bosons remain massless. But the remaining gauge bosons have acquired a mass matrix:

$$(\mu^2)_{ab} = 2\,(g_a T^a \langle\phi\rangle)\cdot(g_b T^b \langle\phi\rangle)$$

$a, b = m+1, \ldots, n$

(not summed). This mass matrix is evidently positive definite, since the T's here are hermitian generators that are not in H.

Some Remarks about Quantization of Nonabelian Gauge Fields

I do not have time for an in depth discussion of the quantization of nonabelian gauge fields. Fortunately, though, quantization in the nonabelian case involves no new concepts that we have not already encountered in the abelian case.

As in the abelian case, a nonabelian gauge theory has an ill-defined initial value problem, so we must fix the gauge in order to canonically quantize. To formulate path integral quantization, we show that the manifestly gauge-independent Faddeev-Popov Ansatz is equivalent to canonical quantization in a particular convenient gauge. It turns out (I won't go through it) that this is easy to establish (as in the abelian case) in the axial gauge $A_3^a = 0$.

Covariant gauges are constructed by introducing a gauge fixing term

$$\mathcal{L}_{g.f.} = -\frac{1}{2\alpha}\partial_\mu A^{\mu a}\partial_\nu A^{\nu a}$$

Since the change in A under an infinitesimal gauge transformation is

$$\delta A_\mu = [D_\mu, \omega]$$

or $$\delta A_\mu^a = \partial_\mu \omega^a + g\, C^{abc}\omega^b A_\mu^c,$$

we must, according to the Faddeev-Popov Ansatz, include in the path integral the factor

$$\det\left(\frac{\delta}{\delta\omega^b}\partial_\mu A^{\mu a}\right)$$

$$= \det\left(\partial^\mu\partial_\mu \delta^{ab} + g\, C^{abc}\partial^\mu A_\mu^c\right)$$

And, as discussed on page (4.46) ff, such a factor can be incorporated into the Feynman rules by introducing a spinless fermionic "ghost" field (the Faddeev-Popov ghost) with Lagrangian

$$\mathcal{L}_{ghost} = \partial^\mu \bar{\eta}^a \partial_\mu \eta^a - g\,\partial^\mu \bar{\eta}^a C^{abc} A_\mu^b \eta^c$$

Feynman Rules

With a path integral formulation in hand, it is easy to derive Feynman rules. The gauge boson propagator in a covariant α gauge is (compare page (5.37))

$$\nu, b \;\overset{k}{\sim\sim\sim}\; \mu, a \;=\; \frac{-i\delta^{ab}}{k^2+i\epsilon}\left[\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+\alpha\frac{k_\mu k_\nu}{k^2}\right]$$

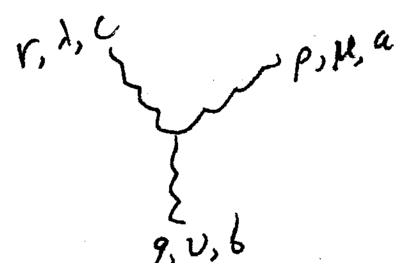
and the ghost propagator is

$$a \;\overset{k}{\longleftarrow}\; b \;=\; \frac{i\,\delta^{ab}}{k^2+i\epsilon}$$

Now consider the vertices. From the interaction term

$$g\,C^{abc}\,\partial_\mu A_\nu^a A^{\mu b} A^{\nu c},$$

we obtain:

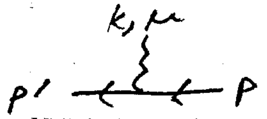


$$= (ig)(-i)\,C^{abc}$$

$$\Big[\,p_\nu\,\eta_{\lambda\mu} - p_\lambda\,\eta_{\mu\nu} + q_\lambda\,\eta_{\mu\nu} - q_\mu\,\eta_{\lambda\nu} + r_\mu\,\eta_{\nu\lambda} - r_\nu\,\eta_{\mu\lambda}\Big]$$

There are six terms; ∂A can annihilate any of the 3 gluons, and $A\cdot A$ can annihilate the remaining two in two ways. Minus signs arise because of the antisymmetry of C^{abc}.

This expression should be compared to the Feynman rule of scalar electrodynamics (page 5.51)



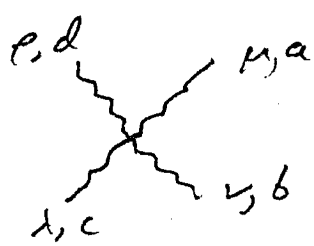
$$= (-ie)(p+p')_\mu$$

The rule for  is a sum of three pieces, each of the form of the rule for scalar electrodynamics, in which each gluon takes a turn behaving like the photon while the other two behave like charged scalars.

From the interaction

$$-\frac{1}{4} g^2 c^{abc} c^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}$$

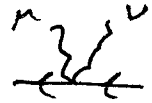
we obtain



$$= \frac{-i}{4} g^2$$
$$c^{eab} c^{ecd} (4)(\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda})$$
$$+ c^{ebc} c^{eda} (4)(\eta_{\nu\rho}\eta_{\mu\lambda} - \eta_{\mu\nu}\eta_{\rho\lambda})$$
$$+ c^{eac} c^{edb} (4)(\eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\rho\lambda})$$

There are 24 terms; we can group the color indices into two pairs in three different ways. There is a twofold choice of which pair of A's annihilates which pair of gluons, and then each pair of A's can annihilate a pair of gluons in two ways.

Again, this expression should be compared to the rule



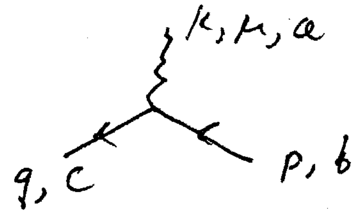
$$= 2ie^2\eta_{\mu\nu}$$

of scalar electrodynamics. The rule for  contains the tensor structures

$$\eta_{\mu\nu}\eta_{\rho\lambda}$$
$$\eta_{\mu\lambda}\eta_{\nu\rho}$$
$$\eta_{\mu\rho}\eta_{\nu\lambda}$$

Again, pairs of gluons take turns behaving like photons while the remaining pair behave like charged scalars.

From the interaction $-g\,\partial_\mu\bar{\eta}^a C^{abc} A^{\mu b}\eta^c$, we have --



$$= (-ig)(-i)\, q_\mu C^{abc}.$$

Renormalizability:

As in the abelian case, we can prove that the counterterms do not spoil gauge invariance. The proof is a little more involved than before, though. In the nonabelian case the variation of the gauge fixing term under a gauge transformation

$$\delta\left(-\frac{1}{2\alpha}\partial_\mu A^{\mu a}\partial_\nu A^{\nu a}\right)$$

$$= -\frac{1}{\alpha}\partial_\mu A^{\mu a}\partial_\nu[D^\nu, \omega]^a$$

is not linear in fields (and similarly for the ghost term), so our old proof of the Ward identity does not go through. This is a mere technical problem, though, and can be overcome.

For nonabelian gauge theories that undergo the Higgs mechanism, a generalization of 't Hooft's renormalizable gauges can be constructed. I won't go through the details of that either.

A Model of Leptons

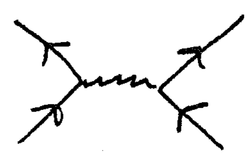
We will attempt to construct a nonabelian gauge theory of the weak interactions. At first, we will consider this problem in the simplest possible context -- we will ignore hadrons (which we'll introduce later) and will consider purely *leptonic* weak interactions.

The Fermi Lagrangian for leptonic weak interactions is

$$\mathcal{L}' = 2\sqrt{2}\, G_F\, J^{\mu\,+}_{Lep} J_{\mu\, Lep},$$

where $J^{\mu}_{Lep} = \bar{e}_L \gamma^\mu \nu_{eL}$ (+ more generations) this interaction is, of course, nonrenormalizable (dimension 6), and so cannot give an adequate description of weak interaction physics at energies above $G_F^{-1/2} \sim 300\,\text{GeV}$.

It is an old idea (going back to Yukawa in the '30's), that the Fermi interaction is the



low-energy limit of an exchange of a massive vector particle. This softens the ultraviolet behavior of the theory some. But as we've already seen, a theory of massive vector particles is not *typically* renormalizable, unless it arises from a gauge theory by the Higgs mechanism. So we are led to formulate a gauge theory of the weak interactions.

But why a <u>nonabelian</u> gauge theory? Because we want

$$J^{\mu}_{Lep} = \bar{e}_L \gamma^{\mu} \nu_{eL}$$

to be one of the currents to which a gauge field couples. Hence, the gauge symmetry generators must be able to change the "flavor" of a lepton. Gauge transformations rotate e's into ν's, and so are nonabelian.

There is a gauge group G that becomes broken by the Higgs mechanism to U(1)em. That is, only one massless vector survives the Higgs symmetry breaking -- the photon:

$$G \rightarrow U(1)_{em}$$

Now, what is G? There will be a gauged current with the structure of J^{μ}_{Lep} if we choose G to contain SU(2),

$$G \supset SU(2),$$

where

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \text{transform as a doublet under this } SU(2)$$

(and e_R is a singlet). The simplest possibility would be to choose

$$G = SU(2)$$

Does this work?

If $SU(2) \to U(1)_{em}$,
then electric charge is one of the $SU(2)$ generators. Thus electric charge Q is traceless -- the sum of the Q's of the members of an $SU(2)$ multiplet must be zero. However,

$$Q_\nu + Q_e = 0 + (-1) = -1 \neq 0$$

Electric charge is not traceless, so this doesn't work. If we are to construct an electroweak theory with $G = SU(2)$, we need new leptons. For example, we could put ν_L, e_L in an $SU(2)$ triplet

$$\begin{pmatrix} E_L \\ \nu_L \\ e_L \end{pmatrix}$$

where $Q_E = 1$. Then the leptonic current would include a piece

$$\bar{\nu}_{eL} \gamma^\mu E_L .$$

Experiment shows that this is not the option chosen by Nature. (The E cannot be a muon, which has its own neutrino.)

If we are not to introduce new leptons, we need to enlarge the gauge group. The simplest choice (smallest number of gauge bosons) that can work is

$$G = SU(2)_L \times U(1)_Y$$

"weak hypercharge"

where electric charge Q is a linear combination of an $SU(2)_L$ generator (which we can choose to be T_3 by an appropriate choice of basis) and the $U(1)_Y$ generator Y. The $U(1)_Y$ charge assignment of the leptons is determined by

$$Q = T_3 + Y$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + Y\mathbb{1} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

charges of $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$

Thus $\begin{pmatrix} \nu \\ e \end{pmatrix}_L$ carries $Y = -\frac{1}{2}$.

This model has 4 gauge bosons, of which one is the photon and three must be heavy. We must introduce a scalar field that will give mass to the three heavy vector bosons by the Higgs mechanism.

Once again, we make the simplest possible choice. We'll introduce a doublet (under $SU(2)_L$) scalar field -- the "Higgs doublet" -- that carries weak hypercharge Y (to be determined in a moment). The scalar field Lagrangian is

$$\mathcal{L} = D^\mu \phi^\dagger D_\mu \phi - \frac{1}{2}\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)^2$$

where $$D_\mu \phi = \partial_\mu \phi + i\underbrace{g}_{\substack{SU(2)\\ \text{gauge}\\ \text{coupling}}}\underbrace{W_\mu^a}_{\substack{SU(2)\\ \text{gauge field}}} T^a \phi + i\underbrace{g'}_{\substack{U(1)_Y\\ \text{gauge}\\ \text{coupling}}}\underbrace{B_\mu}_{\substack{U(1)_Y\\ \text{gauge}\\ \text{field}}} Y\phi$$

The minimum of the classical potential occurs at
$$\phi^\dagger\phi = \frac{1}{2}v^2.$$

We have the freedom to perform a a gauge transformation so that
$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} v + \phi' \\ 0 \end{pmatrix}$$

where v and ϕ' are real. This fixes "unitary gauge". Of the four real scalar fields making up the complex Higgs doublet, three have been "gauged away", and one remains as a physical scalar excitation (the Higgs scalar boson). The three scalars that disappear are "eaten" by the three massive vector bosons.

Since we want $Q = T^3 + Y$ to be the unbroken generator, we can determine the Y assignment of ϕ by demanding:

$$0 = Q\langle\phi\rangle = \begin{pmatrix} (\frac{1}{2}+Y) & 0 \\ 0 & (-\frac{1}{2}+Y) \end{pmatrix}\begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \Rightarrow Y = -\frac{1}{2}.$$

Gauge Boson Masses

Now we can work out the masses and couplings of the three massive vector bosons. In unitary gauge,

$$D_\mu \phi^\dagger D^\mu \phi =$$
$$\frac{1}{2}(v, 0)\left(g W_\mu^a T^a - \frac{1}{2} g' B_\mu\right)\left(g W^{\mu a} T^a - \frac{1}{2} g' B^\mu\right)\begin{pmatrix} v \\ 0 \end{pmatrix}$$
$$+ \cdots$$

We may write

$$W_\mu^a T^a = \frac{1}{\sqrt{2}} W_\mu^+ T^+ + \frac{1}{\sqrt{2}} W_\mu^- T^- + W_\mu^3 T^3$$

where

$$T^+ = T_1 + iT_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad W^+ = \frac{1}{\sqrt{2}}(W^1 - iW^2)$$
$$T^- = T_1 - iT_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad W^- = \frac{1}{\sqrt{2}}(W^1 + iW^2)$$

Then the gauge boson mass term is evidently

$$\frac{1}{2} g^2 v^2 W_\mu^+ W^{\mu -}$$
$$+ \frac{1}{4} v^2 (g W_\mu^3 - g' B_\mu)(g W^{\mu 3} - g' B^\mu)$$

We see that $W^\pm$ have mass $\mu_W^2 = \frac{1}{4} g^2 v^2$, and that the linear combination

$$Z = \frac{1}{\sqrt{g^2 + g'^2}} (g W^3 - g' B)$$

gets mass

$$\mu_Z^2 = \frac{1}{4}(g^2 + g'^2) v^2$$

We may define a mixing angle θ by

$$\sin\theta = \frac{g'}{\sqrt{g^2+g'^2}}, \quad \cos\theta = \frac{g}{\sqrt{g^2+g'^2}}.$$

Then $$Z = \cos\theta\, W^3 - \sin\theta\, B$$

acquires mass, but the orthogonal linear combination

$$A = \sin\theta\, W^3 + \cos\theta\, B$$

remains massless -- it is the <u>photon</u>

<u>Couplings of the Gauge Bosons</u>

To determine the charges to which the gauge bosons couple, we rewrite

$$ig W_\mu{}^a T^a + ig' B_\mu Y$$

in terms of the mass eigenstates

$$W^3 = \cos\theta\, Z + \sin\theta\, A$$
$$B = -\sin\theta\, Z + \cos\theta\, A$$

So we have

$$\frac{ig}{\sqrt{2}} W_\mu^+ T^+ + \frac{ig}{\sqrt{2}} W_\mu^- T^-$$
$$+ ig(\cos\theta\, Z_\mu + \sin\theta\, A_\mu) T^3$$
$$+ ig'(-\sin\theta\, Z_\mu + \cos\theta\, A_\mu) Y$$

$$= \frac{ig}{\sqrt{2}} \left(W_\mu^+ T^+ + W_\mu^- T^- \right)$$
$$+ i(g\cos\theta\, T^3 - g'\sin\theta\, Y) Z_\mu$$
$$+ i(g\sin\theta\, T^3 + g'\cos\theta\, Y) A_\mu$$

Thus, the photon couples to

$$\frac{gg'}{\sqrt{g^2+g'^2}} (T^3 + Y) = \frac{gg'}{\sqrt{g^2+g'^2}} Q$$

Thus $\boxed{\frac{gg'}{\sqrt{g^2+g'^2}} = e}$ -- the charge of the electron

that is, A_μ has the coupling

$$e A_\mu \left(\bar{e}_L \gamma^\mu e_L + \bar{e}_R \gamma^\mu e_R \right)$$

Note: This is parity conserving, even though underlying theory is parity violating

And we may write $\boxed{g = \frac{e}{\sin\theta}, \quad g' = \frac{e}{\cos\theta}}$

The Z couples to

$$g\cos\theta\, T^3 - g'\sin\theta\, (Q - T^3)$$
$$= e(\cot\theta + \tan\theta)\, T^3 - e\tan\theta\, Q$$
$$= \frac{e}{\cos\theta\sin\theta} \left(T^3 - \sin^2\theta\, Q \right)$$

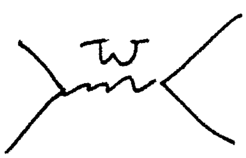
Charged Current

Now let's see how the Fermi interaction is recovered in the low-energy limit. Charged W's have couplings

$$\frac{ig}{\sqrt{2}}(\bar{\nu}\ \bar{e})_L\left(W_\mu^+ T^+ + W_\mu^- T^-\right)\gamma^\mu \begin{pmatrix}\nu\\ e\end{pmatrix}_L$$

$$= \frac{ig}{\sqrt{2}}\left[W_\mu^+ \bar{\nu}_L \gamma^\mu e_L + W_\mu^- \bar{e}_L \gamma^\mu \nu_L\right]$$

So, in tree approximation



$$= \left(\frac{ig}{\sqrt{2}}\right)^2 \frac{-i}{k^2-\mu_W^2}\ \bar{\nu}_L\gamma^\mu e_L\ \bar{e}_L\gamma_\mu\nu_L$$

$$= \frac{-ig^2}{2\mu_W^2} J^{\mu+}_{Lep} J_{\mu Lep} \quad (\text{for } k^2 \ll \mu_W^2)$$

Comparing to $2\sqrt{2}\, G_F J^{\mu+} J_\mu$,

we see that

$$\sqrt{2}\, G_F = \frac{g^2}{4\mu_W^2} = \frac{1}{v^2}$$

So

$$\boxed{v = 2^{-\frac{1}{4}} G_F^{-\frac{1}{2}} = 247\,\text{GeV}}$$

We can express μ_W, μ_Z in terms of v, e, and θ:

$$\boxed{\mu_W = \frac{1}{2}\frac{e}{\sin\theta}v = 37.3\,\text{GeV}/\sin\theta,}$$

$$\boxed{\mu_Z = \frac{1}{2}\frac{e}{\sin\theta\cos\theta}\,v = \mu_W/\cos\theta}$$

Experimentally (more about this later),

$$\sin^2\theta \cong .22$$

$$\Rightarrow \quad \mu_W = 79.5 \text{ GeV}$$
$$\mu_Z = 90.0 \text{ GeV}$$

Actually, these are tree approximation predictions. If one-loop corrections are included, the predicted masses are pushed up about $3\frac{1}{2}$ %, so

$$\mu_W = 82 \text{ GeV}$$
$$\mu_Z = 93 \text{ GeV}$$

which agrees with the measured values.

Neutral Current

This model predicts new weak processes (that had not yet been seen when the model was first proposed) mediated by Z exchange (Neutral weak currents).

We have seen that Z couples to

the charge

$$\frac{e}{\cos\theta\sin\theta}\left(T^3-\sin^2\theta\, Q\right),$$

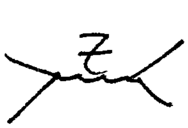
or, in other words, to the current

$$\frac{e}{\cos\theta\sin\theta}\Big[\left(-\tfrac{1}{2}+\sin^2\theta\right)\bar{e}_L\gamma^\mu e_L+\sin^2\theta\,\bar{e}_R\gamma^\mu e_R$$
$$+\tfrac{1}{2}\bar{\nu}_L\gamma^\mu\nu_L\Big]$$

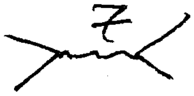
$$=\frac{e}{\sin 2\theta}\,J^\mu_{Neutral}$$

where
$$J^\mu_{Neutral}=\left(2\sin^2\theta-\tfrac{1}{2}\right)\bar{e}\gamma^\mu e+\tfrac{1}{2}\bar{e}\gamma^\mu\gamma_5 e+\bar{\nu}_L\gamma^\mu\nu_L$$

Z exchange generates:


$$\cong\frac{(ie)^2}{\sin^2 2\theta}\,\frac{-i}{-\mu^2}\,J^\mu_{Neut}J_{\mu\,Neut}$$

And $\mu_Z^2=\dfrac{e^2v^2}{\sin^2 2\theta}$, $\quad\dfrac{1}{v^2}=\sqrt{2}\,G_F$, so


$$=-i\sqrt{2}\,G_F\,J^\mu_{Neut}J_{\mu\,Neut}.$$

Neutral current effects are comparable in magnitude to charged current effects. (But harder to see because backgrounds are different.)

Origin of Electron Mass

An interesting property of the standard model is that gauge invariance forbids a bare mass for the electron. The reason is that the fields e_L and e_R transform differently under $SU(2)_L \times U(1)_Y$

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} : \; SU(2)_L \text{ doublet} \quad Y = -\tfrac{1}{2}$$

$$e_R : \; SU(2)_R \text{ singlet} \quad Y = -1$$

A mass term has the form $\bar{e}_L e_R + \bar{e}_R e_L$, and evidently would not be invariant under $SU(2)_L \times U(1)_Y$ gauge transformations.

But the electron has a mass. Where does it come from? It, like the mass of the W and Z, must arise as a consequence of spontaneous symmetry breakdown of the gauge symmetry. We should note that a (dimension 4) <u>Yukawa coupling</u> of the electron to the Higgs doublet <u>is</u> allowed by gauge invariance:

$$\phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix} : \; SU(2) \text{ doublet} \quad Y = -\tfrac{1}{2}$$

As discussed on, e.g. page (3.28) ff, the complex conjugate of an $SU(2)$ doublet also transforms as a doublet, and, in fact,

$-i\sigma_2\,\phi^*$ transforms exactly as ϕ does, because

$$\sigma_2\,\sigma^{a*}\sigma_2 = -\sigma^a$$

So $\phi^c = -i\sigma_2\,\phi^* = \begin{pmatrix} -\phi^+ \\ \phi^{0*} \end{pmatrix}$ is an SU(2) doublet with $Y = \frac{1}{2}$

(It is the "charge conjugate" of ϕ)

Apparently, then,

$$\mathcal{L}_{Yuk} = -f_e\,(\bar\nu_L\ \bar e_L) \begin{pmatrix} -\phi^+ \\ \phi^{0*} \end{pmatrix} e_R + h.c.$$

(Yukawa coupling constant)

is gauge-invariant, and should appear in the most general renormalizable Lagrangian with local $SU(2)_L \times U(1)_Y$ gauge symmetry.

In unitary gauge, this coupling becomes

$$\mathcal{L}_{Yuk} = f_e\,\bar e_L\,\frac{-1}{\sqrt{2}}(v+\phi')\,e_R + h.c.$$

with a suitable phase convention for e_R, f_e may be chosen real and positive

So

$$\mathcal{L}_{Yuk} = \frac{-f_e}{\sqrt{2}}\,\bar e e\,(v+\phi')$$

So the electron has acquired a mass

$$m_e = (f_e/\sqrt{2})\,v$$

and has a coupling to the Higgs boson

$$-\frac{m_e}{v}\phi'\bar{e}e.$$

Since $m_e = .511\,\text{MeV}$ $v = 247\,\text{GeV}$, we see

$$f_e = \sqrt{2}\,\frac{m_e}{v} = 2.9\times 10^{-6}$$

There is no explanation within the model for why this dimensionless parameter is so small.

Note that, in the standard model, there is no right handed neutrino (by assumption) and so the neutrino is exactly massless. (That is, any neutrino mass must be attributed to new physics not described by the model)

More Generations

We will now incorporate the muon and its neutrino into the model. Since the weak interactions of μ and ν_μ are similar to those of e and ν_e, we hypothesize that they transform the same way under $SU(2)_L\times U(1)_Y$:

$$\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_L \qquad \mu_R$$

$$Y=-\tfrac{1}{2} \qquad Y=-1$$