JOHN PRESKILL
Fall, 1983

Physics 230
Elementary Particle Theory

Topic: Quantum Chromodynamics (QCD) - The Theory of The Strong Interactions

Course Outline:

1) The Renormalization Group and Perturbative QCD.
2) Quark Confinement.
3) Chiral Symmetry and Anomalies.
4) The Limit N_c (number of colors) $\rightarrow \infty$.
5) QCD on The Lattice.

Prerequisite: A course in quantum field theory (e.g., Physics 205).

Text: Lecture Notes

Requirements: Problem Sets

J. PRESKILL

Physics 230

1. Introduction to QCD

A. The Road to QCD

Quantum chromodynamics is a great triumph for quantum field theory. It turned out that no fundamentally new principles were needed to understand the strong interactions. Instead, the old principles –

Causality } i.e., quantum field theory
Unitarity }
Gauge invariance
Renormalizability

were eventually found to have remarkable, qualitatively new consequences: e.g., asymptotic freedom and confinement.

Concepts which paved the way for QCD were…

i) The Quark Model:
Hadron resonances → Eightfold Way → Quark model

$\bar{q}q$ } A classification of resonances –
qqq } No exotics, no isolated quarks.

But are quarks fundamental degrees of freedom?
An affirmative hint came from current algebra

ii) Color:
$SU(3)_c$ symmetry. Quark model "explained" if color singlets are much lighter than nonsinglets

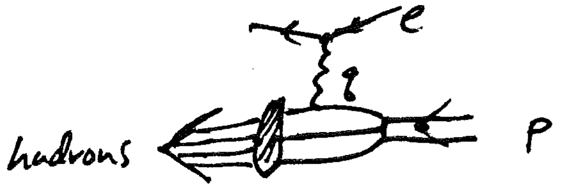
$\bar{q}q \supset \underline{1}$ $\quad \underline{8}, \underline{8}\,\underline{8}, \cdots \not\supset \underline{1}$

$qqq \supset \underline{1}$

Also, statistics problem –
$\Delta^{++} = uuu$ inconsistent with Fermi stat.

Later confirmation that no. of colors is $N_c = 3$
came from: $\Gamma(\pi^0 \to \gamma\gamma) \propto N_c^2$
$\sigma(e^+e^- \to \text{hadrons}) \propto N_c$

iii) Parton Model:



Electroproduction scaling –
$$d\sigma \sim \frac{1}{|q^2|} f(q^2/2p\cdot q)$$

This is behavior expected if masses become irrelevant at large $|q^2|$.
But that can occur only if strong interactions become weak at large $|q^2|$ (short distances) – i.e., if there are no heavy resonances

⇒ we need asymptotic freedom, and the only renormalizable quantum field theories which are asymptotically free are non-abelian gauge theories.

We already need a color degree of freedom, so gauge it!

iv) Etc.:
The above considerations already lead us to an essentially unique theory of the strong interactions: QCD. This theory better be able to explain:

Quark confinement, and linear potential at large distances
(Regge trajectories, quarkonia)
Spontaneous breakdown of chiral symmetry
(light pion)

and, more quantitatively:

Hadron spectrum, cross sections, structure functions and scaling violations, ---

B. Formulating QCD

We need to construct a theory of colored quarks, with local SU(3) symmetry:

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \qquad q(x) \to \Omega(x)\, q(x)$$

$$\Omega(x) \in SU(3)$$

To construct gauge-invariant kinetic term for quark, note

$$\partial_\mu(\Omega q) = (\partial_\mu \Omega) q + \Omega \partial_\mu q$$

So define

$$D_\mu q = \partial_\mu q - i g A_\mu q$$

where $A_\mu(x)$ is hermitian, traceless 3×3 matrix, and

$$A_\mu(x) \to \Omega A_\mu \Omega^{-1} + \frac{1}{ig}(\partial_\mu \Omega)\Omega^{-1}$$

then $D_\mu q \to \Omega D_\mu q$ (transforms covariantly)

and $\bar{q} i \not{D} q$ is gauge-invariant

choose a basis: $A_\mu = A_\mu^a T^a$, $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$
— eight gluon fields

Gluons are also dynamical. To construct kinetic term, note

$$[D_\mu, D_\nu] q = \left\{ -ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2 [A_\mu, A_\nu] \right\} q$$

Define $F_{\mu\nu} = -\frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$

then $F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1}$

$\Rightarrow \mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}$ is gauge-invariant

Let's express this in terms of gluon fields A_μ^a:

$$[T^a, T^b] = i c^{abc} T^c$$

$$\Rightarrow \quad c^{abc} = -2i \, \mathrm{tr}[T^a, T^b] T^c$$

c^{abc} is totally antisymmetric

$$F_{\mu\nu}{}^a = 2 \, \mathrm{tr}(F_{\mu\nu} T^a) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g c^{abc} A_\mu^b A_\nu^c$$

and $-\frac{1}{2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$ — conventionally normalized vector meson kinetic energy

The Lagrangian

Suppose there are N quark *flavors*

$q_n \quad n = 1, 2, \ldots, N$

Most general gauge-invariant, Lorentz invariant, *renormalizable* Lagrangian is

$$\mathcal{L} = -\frac{1}{2} Z \, \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + \bar{q}_L A i \not{D} q_L + \bar{q}_R B i \not{D} q_R$$
$$- \bar{q}_L \mathcal{M} q_R - \bar{q}_R \mathcal{M}^\dagger q_L + \frac{1}{2} Z' \epsilon_{\mu\nu\lambda\sigma} \mathrm{tr} F^{\mu\nu} F^{\lambda\sigma}$$

Here $A, B, \mathcal{M}$ are $N \times N$ matrices acting on flavor indices, A, B are hermitian, and

$$q_{L,R} = \frac{1}{2}(1 \pm \gamma_5) q$$

Now we perform some redefinitions to put $\mathcal{L}$ in a standard form:

i) First, we dispose of the last term:

Exercise 1.1: Show that

$\epsilon_{\mu\nu\lambda\sigma} \mathrm{tr} F^{\mu\nu} F^{\lambda\sigma} = \partial^\mu K_\mu$ and find K_μ.

This term is irrelevant in the classical theory; we may drop it. (But, we will need to reconsider this argument later, when we study the quantum theory.)

ii) $A_\mu \to Z^{\frac{1}{2}} A_\mu$

$g \to Z^{-\frac{1}{2}} g$ — This gets rid of Z

iii) A, B hermitian $\Rightarrow$ they can be diagonalized by $A \to U_L^+ A U_L$, $B \to U_R^+ B U_R$ — $U_{L,R}$ unitary

then rescale q_{Ln} q_{Rn} to get $\bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R = \bar{q} i \not{D} q$

iv) Now we have masses

$$\bar{q}_L M' q_R + q_R M'^+ q_L$$

But M' can be diagonalized by

$$M' \to V_L^+ M' V_R \ , \quad V_{L,R} \text{ unitary.}$$

(without changing kinetic term)

Proof/ $M'^+ M'$ is hermitian, positive.

$$U^+ M'^+ M' U = D^2$$

U unitary

D real, diagonal, positive

If D has no zero eigenvalues —

$$D = D^{-1} U^+ M'^+ M' U$$

Let $U = V_R$

$M' U D^{-1} = V_L$

(and $V_L^+ V_L = D^{-1} U^+ M'^+ M' U D^{-1} = 1$)

<u>Exercise 1.2</u>:

Complete the proof by disposing of the case in which D^2 has zero eigenvalues.

We have

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{tr}\,F^2 + \sum_n \left(\bar{q}_n i \not{D} q_n - m_n \bar{q}_n q_n\right)$$

g and the m_n are the free parameters. (We can remove g from D_μ by rescaling A_μ, but it then reappears in front of F^2)

Symmetries of the Lagrangian

i) $\mathcal{L}$ is invariant under C, P, T

(Recall P: $\bar{A}_L B_R \to \bar{A}_R B_L$, $\bar{A}_L \gamma^\mu B_L \to -\bar{A}_R \gamma^\mu B_R$
C: $\bar{A}_L B_R \to \bar{B}_L A_R$, $\bar{A}_L \gamma^\mu B_L \to -\bar{B}_R \gamma^\mu A_R$)

ii) Flavor symmetry
$q_n \to e^{i\theta_n} q_n \Rightarrow$ u, d, s -- <u>conserved</u>
(charge, baryon number, strangeness)

iii) Chiral Symmetry
QCD <u>interaction</u> (and whole Lagrangian in $m \to 0$ limit) respects chiral symmetry

$U(N)_L \times U(N)_R \qquad q_L \to V_L q_L$
$\qquad\qquad\qquad\qquad\quad q_R \to V_R q_R$

More quarks won't spoil this symmetry

In case N=3, success of current algebra suggests that breakdown occurs:

$SU(3)_L \times SU(3)_R \to SU(3)_V$ — the diagonal SU(3) is the Gell-Mann SU(3)

(π, K, η are light)

(whether this actually occurs in QCD is a difficult dynamical question, to which we shall return)

Note, quark masses transform as $(3, \bar{3}) + (\bar{3}, 3)$ rep of $SU(3)_L \times SU(3)_R$
or 8+1 of $SU(3)_V$

$\Longrightarrow$ Gell-Mann-Okubo formula, among other things.
— isospin

iv) Problem (U(1) Problem):

what about $U(1)_A$? There is no ninth Goldstone boson, and if $U(1)_A$ is not spontaneously broken, spectra should be in $SU(3)_V \times U(1)_A \times \underset{\sim}{P}$ representations, where $\underset{\sim}{P}$ is parity

Axial charge $\underset{\sim}{Q}_A$ does not commute with parity

= If $\underset{\sim}{P}|\psi\rangle = -|\psi\rangle$, then

$$\underset{\sim}{P}\,\underset{\sim}{Q}_A|\psi\rangle = +Q_A\underline{|\psi\rangle}$$

spectrum "parity-doubled"

We will say more about this later.

<u>Note</u>:

These symmetries are <u>not</u> <u>completely</u> destroyed by weak-interaction radiative corrections, even though these violate C, P, CP.

Why?

These corrections generate operators of --

- Dimension ≤ 4 ⇒ Absorbed into our most general $\mathcal{L}$
- Dimension > 4 ⇒ Suppressed by powers of $\frac{1}{M_W}$.

Things would have been different if we had chosen the strong interaction gauge group to be, e.g.

$$SU(3)_L \times SU(3)_R$$

Then parity is conserved if $g_L = g_R$, but not by the most general Lagrangian. Thus, weak radiative corrections will generate parity violation in the strong interaction in $O(\alpha)$.

$\frac{1}{p^2 - M_W^2} \sim \frac{1}{M_W^2}$

dim ≤ 4

dim $\gtrsim 4$

Aside:
The above remark helps to clarify the special status of renormalizable field theories, i.e.

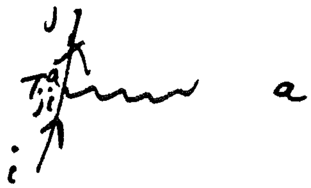
- In general, operators of dimension >4 in the Lagrangian are expected to be suppressed by inverse powers of a large mass scale.

This is one reason why it was sensible to confine our attention to renormalizable theories when we constructed the QCD Lagrangian.

C. Interaction Vertices

What do the interaction terms in the QCD Lagrangian look like?

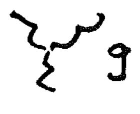
We have: $\bar{q}\, g\, \gamma^\mu A_\mu^a T^a q$



There are eight gluons; six change color, two couple to color, but do not change it.

Since gluons carry color charge, they have self-couplings:

$$\mathrm{tr}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right)^2$$ contains



trilinear coupling

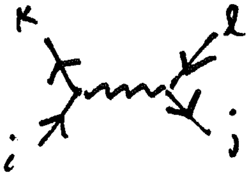
quartic coupling

] A crucial difference between QED, QCD.

One gluon exchange:

In the classical theory, what is the force between static quarks?

This is generated by the tree graph, and differs from Coulomb potential only by a group-theoretic factor



$$E(r) = \frac{g^2}{4\pi r} (T_1{}^a)_{ki} (T_2{}^{a*})_{lj}$$

(sum on a)

Now, write

$$T_1{}^a T_2{}^{a*} = \tfrac{1}{2}\left[(T_1{}^a + T_2{}^{a*})(T_1{}^a + T_2{}^{a*}) - T_1{}^a T_1{}^a - T_2{}^{a*} T_2{}^{a*} \right]$$

$T^a T^a = c\,\mathbb{1}$ acting on an irreducible representation of SU(3)

E.g. consider interaction between quark and antiquark

$3 \times \bar{3} = 1 + 8$ - there is a singlet channel and an octet channel

$$(T^a T^a)_{\text{triplet}} = \left(\tfrac{8}{3}\right)\left(\tfrac{1}{2}\right) = \tfrac{4}{3}$$

$$(T^a T^a)_{\text{octet}} = 3$$

[$8 \to 3+2+2+1$ under SU(2); $\mathrm{tr}\, T^a T^b = c\,\delta^{ab}$; $c = 2 + \frac{1}{2} + \frac{1}{2} = 3$]

Thus

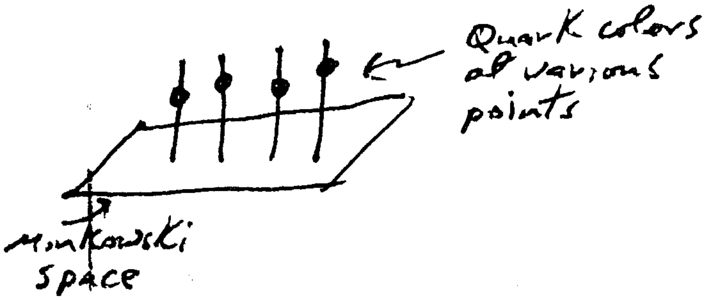
$$E(r) = \frac{g^2}{4\pi r} \begin{cases} -\frac{4}{3} & \text{- singlet} \\ +\frac{1}{6} & \text{- octet} \end{cases}$$

$q\bar{q}$ attract in singlet channel
repel in octet channel

Exercise 1.3: Find interaction energy between two quarks in $\bar{3}$ and 6 channels.

Addendum to IB: Gauge Field as a Connection

The gauge field can be regarded as a "connection" on a "fibre bundle", as was first pointed out by Hermann Weyl. This language means that the gauge field defines a notion of "parallel transport."



Suppose I wish to compare the colors of two quarks at different points in spacetime; e.g., to see if they are the same. In principle, I could do this by transporting one quark to the position of the other, and then comparing, but there is an ambiguity; really, two ambiguities. First, we must have a criterion for deciding that the color of the quark is not changed as it is transported; that is, a notion of "parallel transport." This notion is provided by the gauge field. Second, even for a given gauge field, the outcome of the comparison may depend on the path taken between the positions of the two quarks.

The gauge field is related to parallel transport in the following way: the covariant derivative $D_\mu q$ of a quark field q is the change in q relative to a parallel transported q. That is $\epsilon^\mu D_\mu q$ is the difference between the change in q going from x to $x+\epsilon$, and the corresponding change if q were parallel transported.

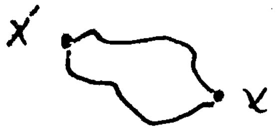
The condition for parallel transport is therefore

$$D_\mu q = 0 \quad \text{or} \quad \partial_\mu q = iA_\mu q$$

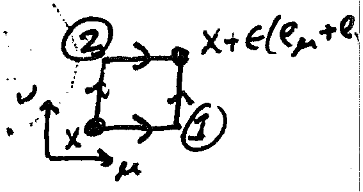
(the coupling constant g has been absorbed in A_μ, for convenience.) Integrating this equation gives

$$q(x') = \left[\underline{P} \exp i\int_x^{x'} A_\mu dx^\mu\right] q(x)$$

where $\underline{P}$ is a "path-ordering" symbol, analogous to the time ordering symbol in Dyson's formula, $U(t', t_0) = [T \exp\int_t^{t'} -iH_I]\, U(t, t_0)$.



Now we know how to compare the colors of quarks at different points, but the comparison is not very interesting, because its outcome is changed by a gauge transformation. A mathematician would say that the comparison, or the matrix $\underline{P} \exp i\int_x^{y} A_\mu dx^\mu$, depends on the *coordinates* of the "bundle", and not on its intrinsic *geometry*. The way to identify geometric properties is to determine how the color of a parallel transported quark depends on the path along which it is transported, or, equivalently, how the color changes when the quark is transported around a closed path.



For example, compare the two paths shown around infinitesimal square. Parallel transport along ①:

$$q \to [\underline{1} + i\epsilon A_\nu(x+\epsilon e_\mu)][\underline{1} + i\epsilon A_\mu(x)]$$

Parallel transport along ②:

$$q \to [I + i\epsilon A_\mu(x+\epsilon e_\nu)][I + i\epsilon A_\nu(x)]q$$

Now, take difference:

$$([I + i\epsilon A_\nu + i\epsilon^2 \partial_\mu A_\nu][I + i\epsilon A_\mu]$$
$$- [I + i\epsilon A_\mu + i\epsilon^2 \partial_\nu A_\mu][I + i\epsilon A_\nu])q$$
$$= i\epsilon^2(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])q$$
$$= \epsilon^2 i F_{\mu\nu} q$$

We conclude that the change in q under parallel transport about area ϵ^2 oriented in $\mu\nu$ plane is

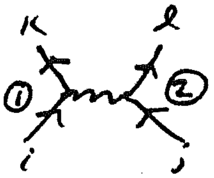
$$\boxed{\delta_{\mu\nu} q = \epsilon^2 i F_{\mu\nu} q}$$

$F_{\mu\nu}$ depends on coordinates in only a trivial way, the orientation of our axes in the color space; its eigenvalues are gauge-invariant. $F_{\mu\nu}$ is obviously antisymmetric — interchanging μ and ν corresponds to traversing the square in the opposite sense. In analogy to Riemannian geometry, $F_{\mu\nu}$ can be called the "curvature" of the "bundle."

The similarity to general relativity is evident. A gravitational field is a connection which defines parallel transport for a gyroscope (or locally inertial frame) in spacetime. The Yang-Mills field is a connection which defines parallel transport for a colored quark in spacetime.

(In electromagnetism, that the gauge field behaves as a connection is dramatically illustrated by the Aharanov-Bohm experiment.)

Addendum to I.C:



The interaction energy of two static color sources in the one-gluon-exchange approx is proportional to the group theory factor

$$(T_1{}^a)_{ki}\,(T_2{}^a)_{lj} \qquad (\text{sum on } a)$$

This interaction is invariant under simultaneous SU(3) color rotations of objects ① and ②. To diagonalize it and find its eigenvalues, we thus go to the basis of irreducible SU(3) representations contained in

$$R_1 \otimes R_2$$

where source ① transforms as the irreducible representatio (IR) R_1 and ② transforms as the IR R_2.

The above remark is exactly analogous to the observation that the $\vec{L}\cdot\vec{S}$ coupling, familiar in atomic physics, is rotationally invariant and can therefore be diagonalized by transformin to the basis of IR's of the rotation group, generated by $\vec{J} = \vec{L} + \vec{S}$.

As in the case of the $\vec{L}\cdot\vec{S}$ coupling, we may use the trick of writing

$$T_1{}^a T_2{}^a = \tfrac{1}{2}\left[(T_1{}^a + T_2{}^a)(T_1{}^a + T_2{}^a) - T_1{}^a T_1{}^a - T_2{}^a T_2{}^a\right],$$

since the $T_1^a + T_2^a$ are the generators of simultaneous color rotations of both objects. If $T_R{}^a$, $a = 1, \ldots, 8$ are the generators of any <u>irreducible</u> representation. Then

$$T_R{}^a T_R{}^a = C(R)\,\mathbb{1}, \quad \text{a multiple of the identity}$$

To see this, note first that $T_R^a T_R^a$ commutes with all the generators:

$$[T^aT^a, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a$$
$$= i\,C^{abc}(T^aT^c + T^cT^a) = 0,$$

because we have chosen the generators so that C^{abc} is totally antisymmetric. Since T^aT^a is hermitian and can be diagonalized, we see that all the generators T^b must preserve the eigenvalues of T^aT^a; if R is irreducible, $T_R^aT_R^a$ must be a multiple of the identity

Acting on the representation $R \subset R_1 \otimes R_2$, we have

$$T_1^aT_2^a = \frac{1}{2}[C(R) - C(R_1) - C(R_2)],$$

so to complete the task of finding the eigenvalues of $T_1^aT_2^a$, we need only compute C (the quadratic Casimir invari for the IR's R, R_1, R_2. To simplify the computation take the trace of both sides of

$$T_R^aT_R^a = C(R)\,\mathbb{I}$$

to obtain (# of generators) $T(R)$ = $C(R)$ (dim of rep)
where $T(R)$ is defined by

$$\mathrm{tr}\, T^aT^b = T(R)\,\delta^{ab}$$

(According to the convention on page 1.3, then, $T(R) = \frac{1}{2}$ for the defining, or triplet, rep of $SU(3)$).)
We now have

$$\boxed{C(R) = \frac{(\text{No. of generators})}{(\text{dim of } R)}\,T(R)}$$

which is useful, because it is easy to calculate $T(R)$ directly.

1.C'3

To calculate $T(R)$, note that we may use any appropriately normalized SU(3) generator. It is convenient to use the generator which in the defining (triplet) representation is

$$T^3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then we can find $T(R) = Tr(T^3)^2$ by decomposing any SU(3) representation R into the direct sum of irreducible representations of the SU(2) generated by T^3, T^1, T^2, and using

$$T(R) = \sum_{\substack{SU(2)\text{ reps} \\ S}} T(S) \; .$$

For example, the 8 rep of SU(3) breaks up under SU(2) into

$$8 = 3 \otimes \bar{3} - 1 \quad \text{of } SU(3)$$

$$\longrightarrow (2+1) \otimes (2+1) - 1 = 3 + 2 + 2 + 1 \quad \text{of } SU(2)$$

For the 3 rep of SU(2), the eigenvalues of T^3 are $1, 0, -1$, so $T(R) = 2$, and we therefore have

$$\boxed{\begin{aligned} T(R=8) &= 2 + \tfrac{1}{2} + \tfrac{1}{2} = 3 \\ \text{and} \quad c(R=8) &= \tfrac{8}{8}\, 3 = 3 \end{aligned}}$$

For the $R=3$ rep of SU(3) we have

$$\boxed{c(R=3) = \tfrac{8}{3} T(R=3) = \tfrac{4}{3}}$$

For $R=6$ of SU(3), we have

$$6 = (3 \times 3)_{\text{symmetric}} \quad \text{of } SU(3)$$

$$\longrightarrow [(2+1) \times (2+1)]_{\text{sym}} = 3 + 2 + 1 \quad \text{of } SU(2)$$

and therefore - - -

$$\boxed{\begin{aligned} T(R=6) &= 2+\tfrac{1}{2} = 5/2 \\ C(R=6) &= \frac{8}{6}\,\frac{5}{2} = \frac{10}{3} \end{aligned}}$$

For $R=10$ of SU(3), we have

$$10 = 6\otimes 3 - 8 \text{ of } SU(3)$$

$$\rightarrow (3+2+1)\otimes(2+1) - (3+2+2+1)$$

$$= 4+3+2+1 \quad \text{of } SU(2)$$

$$\boxed{\begin{aligned} T(R=10) &= 5+2+\tfrac{1}{2} = 15/2 . \\ C(R=10) &= \frac{8}{10}\,\frac{15}{2} = 6 \end{aligned}}$$

Now you know all the Casimirs which are needed in the exercises.

If you would like to know more about the representations of SU(3), two good books are

H. Georgi, "Lie Algebras in Particle Physics"

D. Lichtenberg, "Unitary Symmetry and Elem. Particles"

In particular, you should probably learn how to find the irreducible representations contained in the direct product of two SU(N) reps, but that knowledge is not indispensable for this course.

As an (unofficial) exercise, you may wish to find C(R) for some other representations of SU(N).

Physics 230
Addendum

Renormalization

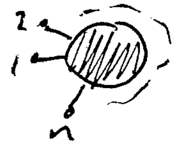
Recall: Scalar field theory

$$G^{(n)}(x_1, x_2, \ldots x_n) \equiv {}_{\underset{\uparrow}{P}}\langle 0 | \overset{\text{Time ordering}}{T} [\phi_H(x_1) \ldots \phi(x_n)] | 0 \rangle_P$$

P = "physical vacuum"

ϕ_H: Heisenberg picture field

$= \boxed{\text{Sum of all Feynman diagrams with } n \text{ external lines}} =$



} Includes disconnected (more than one $\delta^{(4)}(p)$) But "vacuum bubbles" are excluded

Connected: Systematic expansion in number of "loops" L
– or in ħ

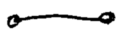
$$\text{Diagram} \sim (\hbar)^{L-1}$$

(# of Loops = # of momentum integrals)

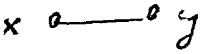
Example:

$$\mathcal{L} = \left(\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 - \tfrac{1}{3!}\lambda \phi^3\right) / \hbar$$

Feynman Rules



$\dfrac{i}{p^2 - m^2 + i\epsilon}$ (mom space)



$$\int \frac{d^4 p}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{p^2 - m^2 + i\epsilon}$$

$= -i\lambda$ (mom space)

Connected ⟹

$$L = \underset{\int d^4 k}{\overset{\uparrow}{I}} - \underset{\delta^4}{\overset{\uparrow}{V}} + \underset{\text{(factor out } \delta^4(p_{\text{total}}))}{\overset{\uparrow}{1}}$$

$\exp(i/\hbar \int \mathcal{L}) \Rightarrow$ contraction ~ ħ
vertex ~ $\hbar^{-1}$

$$\hbar^{(I-V)} = \hbar^{L-1}$$

Or – rescale field: $\phi = \frac{1}{\sqrt{\lambda}}\tilde{\phi}$

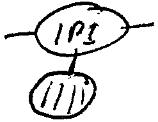
$$\mathcal{L} \to \frac{1}{\hbar \lambda^2} \mathcal{L}'$$

Dim Analy: $m \sim L^{-}$, $\phi \sim M/L$, $\lambda^{-} \sim 1/ML^3$, $\hbar\lambda^2 \sim \frac{1}{L^2}$ $\hbar\lambda^2/m^2$ – dimensionless

Mass (and Field) Renormalization

Define 

, etc.

(subtlety: Tadpole e.g. 

is also included)

– cannot be disconnected by cutting a single internal line

NO 

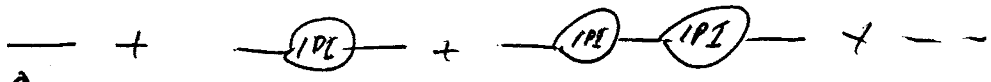
YES 

$$G^{(2)}(x,y) = \langle 0,P|T[\phi_H(x)\phi_H(y)]|0,P\rangle$$

= sum of Feynman diagrams

$G^{(2)}_{(con)}$ ≡ 

connected

= 

$$\frac{i}{p^2-m^2+i\epsilon} \qquad \frac{i}{p^2-m^2+i\epsilon}\left[-i\Pi(p^2)\right]\frac{i}{p^2-m^2+i\epsilon} + \text{—}$$

$\Uparrow$ (definition)

$$= \frac{i}{p^2-m^2-\Pi(p^2)+i\epsilon}$$ Modifies pole in propagator

Note 

$= -i\Pi(p^2)$

↑ Lorentz invariant

How modified?

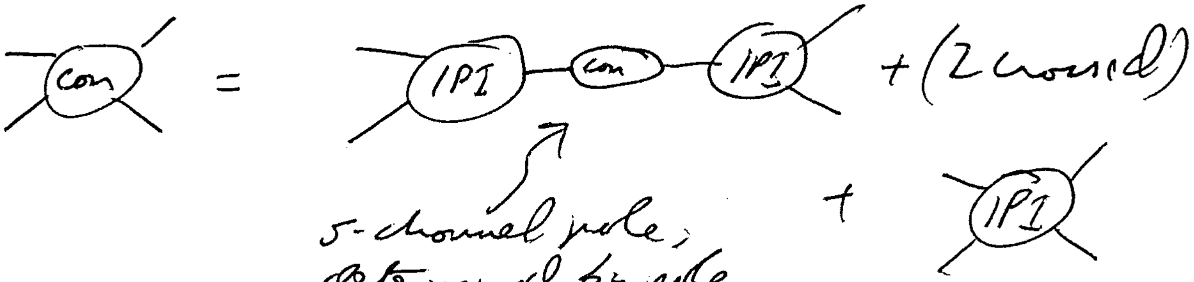
$$\Pi \sim \Pi(m^2) + (p^2-m^2)\,\Pi'(m^2) + \cdots$$

shifts position of pole

rescales residue of pole

Significance of this?

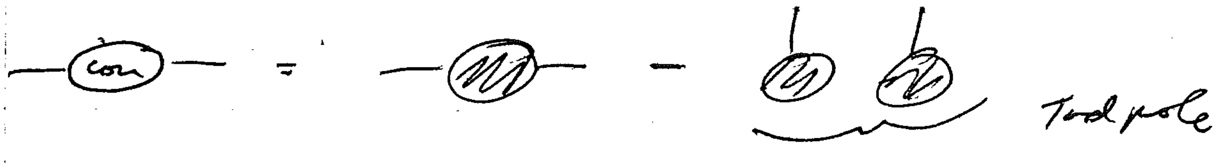
E.g., use Green function to compute scattering amplitude (see later)



s-channel pole; determined by pole of exact propagator

Unitarity ⟹ location of (analytically continued) amplitude is mass of physical particle

See this more directly (work with Green function instead of S matrix):



Tadpole

(Here "vacuum bubbles" = multiplicative constant are excluded, remember)

$$= \langle 0|\, T[\phi(x)\phi(y)]\,|0\rangle$$

$$- \langle 0|\phi(x)|0\rangle \langle 0|\phi(y)|0\rangle$$

Tadpole: $\langle 0|\phi(x)|0\rangle = \langle 0|\phi(0)|0\rangle$
$= \text{constant}$

(Translation invariance of $|0\rangle$)

(Feynman diagram $\sim \delta^4(p)$)

often convenient to "shift" field

$$\phi(x) = \langle 0|\phi(0)|0\rangle + \tilde{\phi}(x)$$

$$\uparrow$$

so $\langle 0|\tilde{\phi}(x)|0\rangle = 0$

So -- consider

$\langle 0|\phi(x)\phi(y)|0\rangle$ (time ordering $x^0 \geq y^0$)

$$= \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle$$

$$= \langle 0|\phi(x)|0\rangle\langle 0|\phi(y)|0\rangle \quad \text{(Tadpole)}$$

$$+ \int\frac{d^3k}{(2\pi)^3 2\omega_k}\langle 0|\phi(x)|k\rangle\langle k|\phi(y)|0\rangle \quad \text{(Rel Norm 1 part states)}$$

$$+ \sum_n{}' \langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle \quad \text{(2 or more particle states)}$$

Trans invariance:

$$\langle 0|\phi(x)|k\rangle = e^{-ik\cdot x}\langle 0|\phi(0)|k\rangle$$

Lorentz invariance:

$$\langle 0|\phi(0)|k\rangle = \langle 0|\phi(0)|\Lambda k\rangle$$

$$\equiv \sqrt{Z} \quad \text{(k-independent constant)}$$

$$= Z \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)}$$

$$+ \sum_n{}' e^{-iP_n(x-y)} |\langle 0|\phi(0)|n\rangle|^2$$

$\nearrow$

same as free-field theory

$$= Z\, \Delta_+(x-y; m^2) + \text{Remainder}$$

$$\text{Remainder} = \int d^4k\, e^{-ik\cdot(x-y)} \sum_n{}' \delta^4(k-P_n) |\langle 0|\phi(0)|n\rangle|^2$$

$$= \int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot(x-y)} \theta(k^0)\, \sigma(k^2)$$

— Defined $\sigma(k^2)\theta(k^0) = (2\pi)^3 \sum_n{}' \delta^4(k-\underline{P}_n) |\langle 0|\phi(0)|n\rangle|^2$

(Note: Lorentz-invariant function)

Or --

$$\text{Remainder} = \int d\mu^2\, \sigma(\mu^2) \int \frac{d^4k}{(2\pi)^3} \theta(k^0)\delta(k^2-\mu^2) e^{-ik\cdot(x-y)}$$

$$= \int d\mu^2\, \Delta_+(x-y; \mu^2)$$

We have

$$\langle 0|\phi(x)\phi(y)|0\rangle = Z\,\Delta_+(x-y; m^2)$$
$$+ \int d\mu^2\, \sigma(\mu^2)\, \Delta_+(x-y; \mu^2)$$

Lehmann-Källen spectral representation

- superposition of free field values

combine time orderings

$$\text{—(con)—} = \frac{iZ}{p^2-m^2+i\epsilon} + \int d\mu^2\,\sigma(\mu^2)\,\frac{i}{p^2-\mu^2+i\epsilon}$$

pole term

cut (branch points at thresholds)

So -- • position of pole = physical mass

• residue of pole $Z = |\langle 0|\phi(0)|k\rangle|^2$

— overlap of $\phi|k\rangle$ with physical vacuum

($Z=1$ in free theory)

Also spectral rep for commutator:

$$\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$= Z\,i\Delta(x-y;m^2) + \int d\mu^2\,\sigma(\mu^2)\,i\Delta(x-y;\mu^2)$$

free theory

But — canonical commutator

$$[\phi(x),\dot{\phi}(y)] = i\delta^3(x-y)$$

$$1 = Z + \int d\mu^2\,\sigma(\mu^2)$$

$$= Z\left(1+\int d\mu^2\,\tilde{\sigma}(\mu^2)\right)$$

where $\phi = \sqrt{Z}\,\tilde{\phi}$

$$\tilde{\sigma}(k^2)\,\theta(k^0) = (2\pi)^3 \sum \delta(\ )\,|\langle 0|\tilde{\phi}|n\rangle|^2$$

and $\tilde{\sigma} \geq 0 \Rightarrow \boxed{0 \leq Z \leq 1}$

(coupling of normalized field to many-particle states weakens coupling to one-particle states)

Note: can't find a field with

$$\left.\begin{array}{l} [\phi, \dot{\phi}] = i\delta^3(-) \\ \langle 0|\phi|k\rangle = 1 \end{array}\right]$$

"the interaction picture does not exist"

— Haag's theorem

(for $\tilde{\sigma} > 0$ – i.e., an interacting theory)

MASS RENORM:

What have we learned? Q. corrections modify ("renormalize") mass of particle

$m^2 \sim (\text{Length})^{-2}$ $E_{\text{free}} = \hbar\sqrt{k^2+m^2}$ ← wavenumber

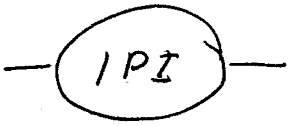
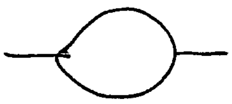
i.e. $\hbar m$ = mass, as $\lambda \to 0$

Denote $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{3!}\lambda\phi^3$

m_0 = "classical mass" = "bare mass"

physical mass m found by solving

$$p^2 - m_0^2 - \Pi(p^2) = 0$$

 =  + higher order

$$m^2 = m_0^2 + O(\lambda^2)$$

Dimensionally $= m_0^2 \, f(\hbar \lambda^2/m_0^2)$

$$f(0) = 1$$

Q. corrections "dress" the particle
Interactions with "vacuum oscillators" change its inertia

<u>Reorganize the expansion</u>

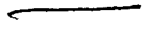
calculate $G^{(n)}(p_1 -- p_n, \lambda, m_0^2)$ ← Not directly measurable

Find m^2 from pole in 

Reexpress $G^{(n)}(p_1 - p_n, \lambda, m^2)$

More <u>convenient</u>: reparametrize

$$m_0^2 = m^2 + \delta m^2$$ ← "mass counterterm"

 $= \dfrac{i}{p^2 - m^2}$ ← physical mass $\qquad \mathcal{L}_{\substack{\text{mass}\\ \text{c.t.}}} = -i\delta m^2 \tfrac{1}{2}\phi^2$

⟹  $= -i(\delta m^2)$

then  $= \dfrac{i}{p^2 - m^2 - \delta m^2} = \dfrac{i}{p^2 - m_0^2}$

Calculate δm^2 order by order, by demanding

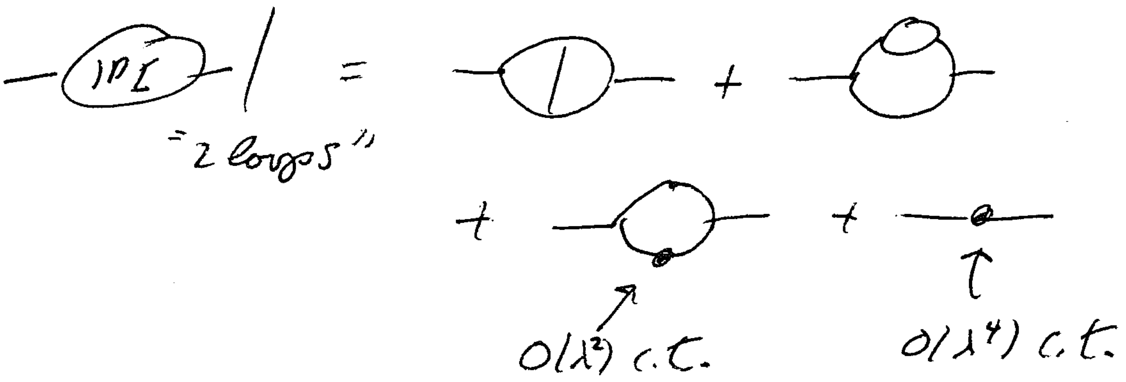
$\Pi(p^2=m^2)=0$ So pole in  stays at m^2

i.e. 

$-i\Pi^{(1\,loop)} + (-i)(\delta m^2)^{(1\,loop)}$

or $(\delta m^2)^{(1\,loop)} = -\Pi^{(1\,loop)}(p^2=m^2)$

Continue to insert counterterms iteratively



- Calculating this way, we don't have to go to the trouble of solving for m^2 and reexpressing answer in terms of m^2
- If we want to know relation of m^2 to m_0^2, we have computed that, too, order-by order, in the form

$$\delta m^2 \equiv m_0^2 - m^2 = m^2 \tilde{f}(\lambda/m^2)$$

(i.e. m_0 in terms of m, instead of m in terms of m_0)

FIELD RENORM

Let's say we have performed mass renorm, so

$$\pi(p^2=m^2)=0$$

Expand $\pi(p^2) = (p^2-m^2)\pi'(m^2) + \frac{1}{2}(p^2-m^2)^2\pi''(m^2) + \cdots$

$$= \frac{i}{p^2-m^2-\pi(p^2)}$$

$$= \frac{i}{p^2-m^2\,(1-\pi'(m^2)) + O(p^2-m^2)^2}$$

(close to position of pole)

$$\sim \frac{iZ}{p^2-m^2+i\epsilon}\text{ , where } \boxed{Z^{-1} = 1-\pi'(m^2)}$$

So — we have calculated how ϕ couples to one-particle states

$$\langle 0|\phi(0)|k\rangle = \sqrt{Z}$$

where $[\phi, \dot{\phi}] = i\delta^{(3)}(-)$

However, it is more convenient to calculate with renormalized fields

$$\phi_R = Z^{-\frac{1}{2}}\phi_B \leftarrow \text{"bare"}$$

so that $\langle 0|\phi_R(0)|k\rangle = 1$

(although $[\phi_R, \dot{\phi}_R] = Z^{-1} i\delta^{(3)}(-)$

More convenient because more directly related to scattering amplitudes (see "LSZ reduction formula", below)

We can reorganize Feynman diagram expansion so that we calculate "renormalized" Green function directly

$$G_R^{(n)}(x_1 - x_n) = \langle 0|T\phi_R(x_1)\cdots\phi_R(x_n)|0\rangle$$
$$= Z^{-n/2}\, G_B^{(n)}(x_1 \cdots x_n)$$

Rewrite:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_B)^2 - \tfrac{1}{2} m_0^2 \phi_B^2 - \tfrac{1}{3!}\lambda \phi_B^3$$

in terms of ϕ_R, m^2

$$\phi_B = Z^{\frac{1}{2}}\phi_R \quad \Longrightarrow$$
$$m_0^2 = m^2 + \delta m^2$$

$$\mathcal{L} = \tfrac{1}{2} Z(\partial_\mu \phi_R)^2 - \tfrac{1}{2}(m^2+\delta m^2) Z \phi_R^2 - \tfrac{1}{3!}\lambda Z^{3/2}\phi_R^3$$

$$= \tfrac{1}{2}(\partial_\mu \phi_R)^2 - \tfrac{1}{2}m^2\phi_R^2 - \tfrac{1}{3!}\tilde{\lambda}\phi_R^3$$
$$+ (Z-1)\left[\tfrac{1}{2}(\partial\phi_R)^2 - \tfrac{1}{2}m^2\phi_R^2\right] - \tfrac{1}{2}(Z\delta m^2)\phi_R^2$$

New Feynman Rules:

$$= -iZ\delta m^2 + i(Z-1)(p^2-m^2)$$

(derivative interaction)

We impose conditions

$$\left\{\begin{array}{l} \Pi(p^2 = m^2) = 0 \\ \Pi'(p^2 = m^2) = 0 \end{array}\right. \Rightarrow \text{determine } \delta m^2 \text{ and } Z \text{ order-by-order}$$

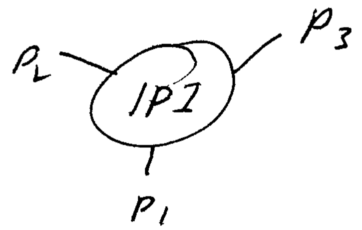
↳ ensures that -- m^2 is physical mass

ϕ_R is renormalized field

COUPLING RENORMALIZATION

- we calculate in terms of m^2, rather than m_0^2, because m^2 can be measured.
- Similarly, convenient to express Green functions, not in terms of λ_0, but in terms of a more directly measurable quantity

Define $-i\Gamma(p_1^2, p_2^2, p_3^2) \equiv$



↗ Lorentz invariant, and $p_1 + p_2 + p_3 = 0$,

so e.g. $(p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = p_3^2 \Rightarrow p_1 \cdot p_2$ not an independent invariant

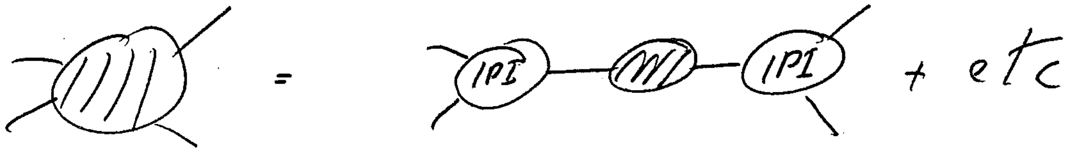
(Rather arbitrary) renormalization condition:

"On shell" $\quad \Gamma(\underline{m^2, m^2, m^2}) = \lambda_R$

Note: momenta must be unphysical (not kinematically allowed for real timelike 4 momenta

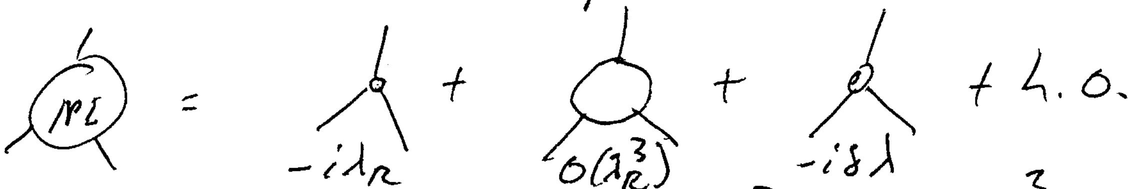
Defined by analytic continuation -- How can we "measure" it?

E.g.



$$\sim \frac{(-i\lambda_R)^2 i}{s-m^2}$$ -- Near the pole

So -- residue of (analytically continued) amplitude at pole in s related to "renormalized coupling"



we write $\mathcal{L}_{int} = -\frac{1}{3}\lambda_0 \phi_B^3 = -\frac{1}{3}\lambda_0 Z^{3/2} \phi_R^3$

$$Z^{3/2}\lambda_0 = \lambda_R + \delta\lambda$$

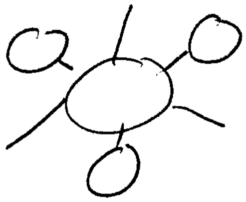
— Cf mass renormalization

the full QM coupling is corrected order by order in $\hbar$ -- we can calculate

$$\lambda_0 = \lambda_R + O(\lambda_R^3)$$
$$= \lambda_R \, g(\lambda^2/m^2)$$
$$g(0)=1$$

For completeness: Two other renorm conditions...

◎ 

$= 0$ Then we don't have to worry about e.g. 

Reflects our freedom to "shift" the field by a constant, so that $\langle \phi_R \rangle = 0$ c.t. q

◎ 

$= 0$

This subtracts vacuum energy

$$\langle 0\,\text{out} | 0\,\text{in} \rangle = \langle 0 | e^{-iHT} | 0 \rangle$$

$$= e^{-iE_{vac}T}$$

$=$  $= \exp\Big[$ 

$\Big]$

 $\sim$ $(2\pi)^4 \delta^4(0)$ $= VT$

(2nd order p.t. ⇒ ground state energy decreases)

●  counterterm subtracts E_{vac} away

(We are free to do this -- unless we couple theory to gravity!)

Summarize:

$= 0$

$= 0$

$= O(p^2 - m^2)^2$

$\Big|\ = -i\lambda_R$

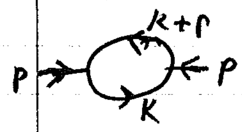
All imposed order-by-order

Infinite Mass Renormalization

Let us now consider, in our model with $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$, the leading contribution to the quantity

$-i\pi(p^2) = -p$ —(1PI)— p $= -p$  p + higher order,

where --



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k+p)^2-m^2+i\epsilon}$$

(symmetry factor)

We can immediately see that this integral is divergent. For large k, it behaves like

$$\int \frac{d^4k}{k^4}.$$

So it will proportional to the log of the ultraviolet cutoff:

 $\sim \ln\Lambda$ + finite part

-- we say that the ultraviolet divergence is "logarithmic"

We can also easily see, without doing an explicit calculation, that the "infinite" part of this diagram (the piece proportional to $\ln\Lambda$) is independent of the external momentum p. To see this <u>differentiate</u> the graph with respect to external momentum:

$\frac{\partial}{\partial p^\mu}$  $= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i2(p+k)_\mu}{((k+p)^2-m^2+i\epsilon)^2}$

Now counting powers of k shows that the integral converges. This shows that if the graph is expanded in powers of p^2 (about any value of p^2) in a Taylor series, only the <u>constant</u> term is divergent (proportional to $\ln \Lambda$).

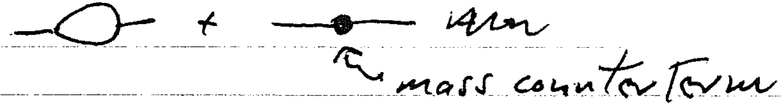
Therefore, without explicitly calculating, we know that

$$\pi(p^2) = \lambda^2 [C \ln \Lambda + \text{"finite part"}] + \text{higher order}$$

where C is a constant, independent of p^2

(The argument of the log should be dimensionless, so the finite part must include a term proportional to $\ln m$. Also, since the integral is dimensionless, C must be a numerical constant, independent of m)

Now suppose we perform mass renormalization, as described previously. That is, we include in $\pi(p^2)$ the contribution from the mass counterterm:

$-i\pi =$ 

The mass counterterm is chosen so that $\pi(p^2)=0$ And thus we see that

$$\delta m^2 = -C\lambda^2 \ln(\Lambda/m) + \text{finite part} + \text{higher order}$$

The infinity (the $\ln \Lambda$ dependence) gets completely absorbed into the relation between the bare and physical masses,

$$m_0^2 = m^2 + \delta m^2 = m^2 - C\lambda^2 \ln(\Lambda/m) + \text{--},$$

and the expression for π as a function of the physical mass m^2 is finite.

(Furthermore, since $\pi'(p^2)$ is finite, we see that the wave function renormalization is finite -- i.e. Z has no dependence on Λ.)

What is the meaning of this logarithmic divergence? It occurs because, for $k^2 \gg p^2, m^2$, the quantum fluctuations have *no* *intrinsic scale*. If λ is small, the quantum corrections are in a sense small, but fluctuations on *all* *scales of length* contribute. For each factor of two, say, in distance scale, the contribution is small if λ is small, but because we get the *same* contribution from each factor of 2 in distance scale, the total contribution from fluctuations on *all* scales is infinite.

Evidently, then, even the low energy ($p^2 \ll \Lambda^2$) physics of our model is sensitive to the fluctuations at arbitrarily small wavelengths This sounds like bad news. Because it may appear that, in order to predict low energy physics, we need to understand physics at arbitrarily short distances. But we can *never* expect *any* model field theory to provide an accurate description of physics at arbitrarily short distances (QED, for example, is not a valid description above 100 GeV, let alone 10^{19} GeV!) Since we can't really *know* what the physics is at very short distances, sensitivity to very short wavelengths appears to mean loss of predictive power.

But the procedure described above indicates that this need not be so! When we perform mass renormalization, we obtain expressions for amplitudes in terms of the *physical* mass, and these expressions are insensitive to the short-wavelength physics. If we wish to relate different measured "low energy" quantities, no dependence on Λ appears. All the sensitivity to short wavelengths can be isolated in the dependence of the physical parameters on the parameters in the "Hamiltonian of the World," e.g., the dependence of the physical mass on the bare mass

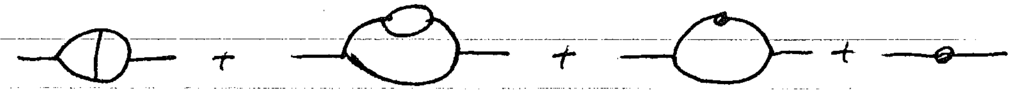
That all of our ignorance about short-wavelength physics can be absorbed into the relation between bare and renormalized parameters (and isolated from the predictions of relations among measurable quantities) is the crucial idea in the theory of renormalization. Indeed, it might even be regarded as the central concept of (relativistic) quantum field theory, because it is only when we grasp this concept that we recognize that we need not be so arrogant as to suppose that we understand physics at the Planck scale, in order to understand physics at 100 GeV.

Higher Orders

The "ultraviolet behavior" of our $\mathcal{H}' = \frac{1}{3!}\lambda\phi^3$ model is actually quite simple, because the sensitivity to short wavelengths becomes milder and milder in higher orders of perturbation theory in λ.

For example, consider the next order contribution to π

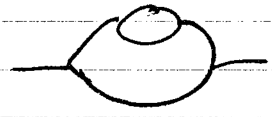
$$-i\pi\Big|_{\text{order }\lambda^4} =$$



The behavior of the loop integrations in  for large loop momenta is evidently

$$\int d^4k_1 d^4k_2 \; \frac{1}{k_1^4} \; \frac{1}{k_2^4} \; \frac{1}{(k_1+k_2)^2},$$

and we can see by counting powers of K that this is convergent for large K. So the only cutoff dependence in π in order λ^4 comes from --



.

But we have already taken care of this with the order λ^2 mass counterterm. There is no new contribution to the infinite part of mass renormalization in order λ^4!

This situation persists in higher orders. Consider a general diagram with

L loops
I internal lines
E external lines
V vertices

The dimensionality of the loop integration (called the "superficial degree of divergence" of the diagram) is

$$D = 4L - 2I$$

But recall the topological identities:

$L = I - V + 1$ (the no. of integrations is the no. of propagators minus V for the momentum-conserving δ-function at each vertex, plus 1, since the overall momentum conserving δ-function can be factored out of the graph.)

$E + 2I = 3V$ (Both sides give the total no. of ends of lines that are absorbed by vertices.)

Therefore, $D = 4(I-V+1) - 2I = 2I - 4V + 4$

$= 3V - E - 4V + 4$

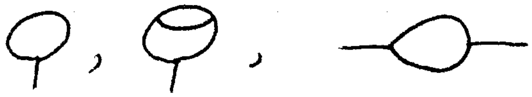
or $D = 4 - E - V$

The number of superficially divergent diagrams is small. For $D \geq 0$, we need

$E = 0, 1, 2, 3$

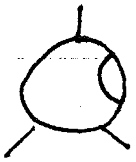
$V \leq 4, 3, 2, 1$

Vacuum diagrams are divergent up to order λ^4, but the only superficially divergent diagrams with external lines are



In particular, 

is (superficially) finite -- thus all the infinities in it come from graphs like



These infinities are removed by an insertion of the order λ^2 mass counterterm.

Thus, this theory has no infinite coupling (or field) renormalization. The only infinite renormalizations are vacuum energy renormalization in order λ^2 and λ^4, mass renormalization in order λ^2, and the renormalization of  in order λ and λ^3.

(More about this "tadpole" later.)

More general:

coupling $\sim (\text{mass})^{\text{positive}} \Longrightarrow$ "superrenormalizable"

Finite no. of "primitive divergences", so infinite renormalizations cease to be necessary after some finite order

(infinite counterterm = polynomial in coupling constant,

More interesting — "renormalizable"

coupling = dimensionless

E.g. consider $\int \mathcal{L}_I = - \int d^d x \, \lambda \phi^n$

(n = integer)

$\int d^d x \, (\partial \phi)^2 =$ dimensionless $\Longrightarrow$

$$\phi \sim (\text{mass})^{(d-2)/2}$$

$\lambda \, (\text{mass})^{-d + \frac{n}{2}(d-2)} =$ dimensionless $\Longrightarrow$

$$\lambda \sim (\text{mass})^{d - \frac{n}{2}(d-2)}$$

$$D = d - \left(\frac{d-2}{2}\right) E \quad - \left[d - n\left(\frac{d-2}{2}\right) \right] \quad V$$

(Dimensional or topological analysis ~ p. 1.41)

$-E = 2 \Rightarrow$ dimension $(\text{mass})^2$

$E = n \Rightarrow$ dimension (λ)

We see from this argument that the reason for the mild ultraviolet behavior of this theory is that the coupling constant λ has the dimensions of mass to a <u>positive</u> power. If we add a loop to a Feynman diagram, keeping the number of external lines fixed, we must add two vertices (since $V = 2L + E - 2$). Dimensional analysis thus shows that adding a loop decreases the dimension of the Feynman <u>integral</u>, and correspondingly reduces its degree of divergence.

In general, in a theory in which the coupling constant has <u>positive</u> dimension, there are a finite number of "primitive" divergences, and hence infinite renormalizations cease to be necessary after some finite order of perturbation theory. Such theories are said to be "<u>superrenormalizable</u>".

More interesting are the <u>renormalizable</u> theories, in which the coupling constant is dimensionless. In renormalizable theories, the primitive degree of divergence of a graph with a fixed number of external lines remains the same to all orders of perturbation theory, and thus the need for infinite renormalization persists to all orders.

Examples of renormalizable theories are scalar field theories with the interactions...

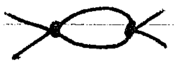
$$\mathcal{H}' = \frac{1}{3!}\phi^3 \text{ -- in 6 dimensions } (\dim\phi = 2)$$

$$\mathcal{H}' = \frac{1}{4!}\phi^4 \text{ -- in 4 dimensions } (\dim\phi = 1)$$

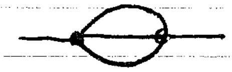
$$\mathcal{H}' = \frac{1}{6!}\phi^6 \text{ -- in 3 dimensions } (\dim\phi = \tfrac{1}{2})$$

(Any polynomial interaction in ϕ is superrenormalizable in 2 dimensions, since ϕ is dimensionless.)

For example, in the four-dimensional theory with $\mathcal{H}' = \frac{\lambda}{4!}\phi^4$, the graph

 $\sim \lambda^2 \ln \Lambda + ---$

is logarithmically divergent, and hence generates (logarithmically) infinite coupling renormalization in order λ^2. The graph

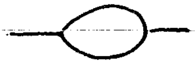
 is quadratically divergent:

$$\sim \lambda^2 (\Lambda^2 + (p^2 - m^2) \ln \Lambda + --)$$

It generates (quadratically) divergent mass renormalization, and (logarithmically) infinite field renormalization.

Dimensional analysis shows that coupling and field renormalization are <u>always</u> logarithmic, and mass renormalization is always quadratic in a renormalizable (scalar) field theory. E.g. consider the 6 dimensional theory with $\mathcal{H}' = \frac{\lambda}{3!}\phi^3$.

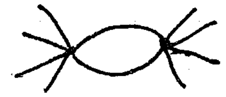
 is logarithmically divergent

 is quadratically divergent

A crucial feature of renormalizable and superrenormalizable theories is that primitive divergences occur only in graphs with some small number of external lines. This feature allows us to isolate the effects of short-wavelength quantum fluctuations in a few incalculable "renormalized parameters." This nice feature is not shared by theories in which the coupling constant has negative dimension, the so-called "*nonrenormalizable*" theories. In these theories, the ultraviolet convergence properties of Feynman diagrams get worse *and worse* in each order of perturbation theory.

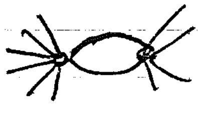
For example, consider a 4-dimensional theory with

$$\mathcal{H}' = \frac{\lambda}{6!}\phi^6$$

We find that the diagram ... 
is logarithmically divergent. To remove this sensitivity to short-wavelengths, we need a ϕ^8 counterterm (since the graph has 8 external lines) Our interaction has become

$$\mathcal{H}' = \frac{\lambda_6}{6!}\phi^6 + \frac{\lambda_8}{8!}\phi^8$$

— where now there are two free parameters λ_6 and λ_8. But we find that



is now divergent.

To absorb *this* cutoff dependence into the relation between a bare and renormalized coupling, we must introduce a ϕ^{10} coupling. It is evident that we can absorb all the sensitivity to short-wavelength physics into our renormalized parameters only at the cost of introducing an infinite number of parameters. Thus, the theory has lost its predictive power. In nonrenormalizable theories, sensitivity to short wavelengths *does* destroy our ability to predict low energy physics.

So, in *renormalizable* and *superrenormalizable* theories, all the sensitivity to short-wavelength physics can be absorbed into a few parameters, which we can take to be the free parameters of the theory. And dimensional analysis determines what these parameters are. (In a renormalizable theory, they correspond to all the local terms in fields and derivatives of fields that have dimension less than or equal to D, in D spacetime dimensions.) In nonrenormalizable theories this procedure fails.

But, why are we so lucky that the world happens to be well-described by a renormalizable field theory (like QED)? Or did it *have* to be this way?

In fact, there is a very good general reason to believe that low energy physics can be described to excellent accuracy by a renormalizable field theory. To understand this, let us take the idea of a cutoff very seriously. That is, we will accept the idea that our scalar field theory is the mere low energy phenomenology of some more fundamental underlying field theory. The description of physics in terms of the Lagrangian

$$\mathcal{L}(\phi(x), \partial_\mu \phi(x), \text{—})$$

is only approximate, and becomes inappropriate at wavelengths much smaller than Λ^{-1}, where Λ is the cutoff. This could happen for various reasons. Perhaps there are new elementary particles, with masses of order Λ. Or perhaps our scalar particles are not really elementary, but have a size of order Λ^{-1}; so that physics at shorter distances must be described in terms of their constituents.

When we adopt this point of view, there is no reason to expect the "phenomenological" Lagrangian to be particularly simple. It ought to be local, — Poincaré-invariant, and should respect whatever other exact symmetries the theory has, but it need not be a polynomial in ϕ and derivatives, and could depend on derivatives higher than the first. Such a Lagrangian, in a four-dimensional theory of a real scalar field, when expanded in powers of ϕ, might have the form

$$\mathcal{L}_\Lambda = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_4\phi^4 - \frac{1}{6!}\tilde{\lambda}_6\phi^6 - \frac{1}{4}\tilde{h}(\partial_\mu\phi\,\partial^\mu\phi)\phi^2 + \cdots$$

(assuming a $\phi \to -\phi$ symmetry).
This is the "bare" Lagrangian of a theory with cutoff Λ; it is very complicated.

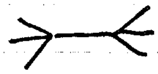
But now we come to a crucial point: the coefficients of the operators in $\mathcal{L}$ that are of dimension > 4 have dimensions of mass to a negative power. Since these coefficients are determined by physics at mass scale Λ and above, dimensional analysis would indicate that, for example,

$$\tilde{\lambda}_6 = \lambda_6\Lambda^{-2},$$

where λ_6 is a dimensionless number expected to be of order one (it has no reason to be very large)

Now, imagine that we compute Feynman graphs using $\mathcal{L}_\Lambda$, with the understanding that all loop integrations are cut off at $k^2 \sim \Lambda^2$, and all external momenta p obey $p^2 \ll \Lambda^2$

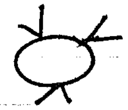
We can see, just as a consequence of dimensional analysis, that the operators of dimension greater than 4 in $\mathcal{L}_\Lambda$ give a small contribution to the graphs, a contribution suppressed by powers of p^2/Λ^2. For example, consider graphs with 6 external lines:



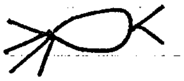
$$\sim \lambda_4^2/p^2$$

$$\sim \lambda_6/\Lambda^2 \quad \text{-- suppressed by } p^2/\Lambda^2$$

Similarly:



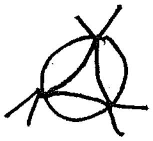
$$\sim \lambda_4^3\int\frac{d^4K}{(K^2+p^2)^3} \sim \lambda_4^3/p^2 \quad \text{(ignoring masses)}$$



$$\sim \lambda_4\frac{\lambda_6}{\Lambda^2}\int^\Lambda\frac{d^4K}{(K^2+p^2)^2} \sim \frac{\lambda_4\lambda_6}{\Lambda^2}\ln\Lambda^2/p^2$$

-- suppressed by p^2/Λ^2 up to a log

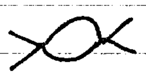
Adding more λ_6 interactions makes graphs more divergent, but the extra powers of Λ^2 from the loop integrations are compensated by extra powers of $1/\Lambda^2$ from the coupling. E.g.

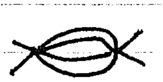


$$\sim \left(\frac{\lambda_6}{\Lambda^2}\right)^3\left(\int^\Lambda d^4K\right)^4\frac{1}{(K^2)^6} \sim \frac{\lambda_6^3}{\Lambda^2}$$

The dimension greater than four couplings are said to be "irrelevant" in the infrared. Their effects are suppressed by powers of p^2/Λ^2 relative to those of the renormalizable couplings.

There is an exception to this observation though. Namely, consider Feynman graphs with four or fewer external lines:

 $\sim \lambda_4^2 \int^\Lambda \frac{d^4k}{(k^2+p^2)^2} \sim \lambda_4^2 \ln(\Lambda^2/p^2)$

 $\sim \left(\frac{\lambda_6}{\Lambda^2}\right)^2 \left(\int^\Lambda d^4k\right)^3 \frac{1}{(k^2)^4} \sim \lambda_6^2$

-- same order, up to a logarithm

But *this* effect of the "irrelevant" couplings, which is *not* suppressed by powers of p^2/Λ^2, can simply be absorbed into the definition of the renormalized coupling $(\lambda_4)_R$.

Thus, up to an accuracy of order p^2/Λ^2, our very complicated bare theory can be replaced by a renormalized theory, with just one renormalized coupling and a renormalized mass as free parameters. The relation between the bare Lagrangian and the renormalized Lagrangian is extremely complicated. But if we are interested in low energy physics at $p^2 << \Lambda^2$, we may compute to good accuracy using a simple *renormalizable* field theory (the *most* general renormalizable theory of one scalar field, with $\phi \to -\phi$ symmetry)

The shift in viewpoint that leads to the conclusion that nonrenormalizable theories are not without predictive power after all comes about when we take the idea of a *physical* short distance cutoff seriously, for *we* then recognize that the coupling constants should be functions of the cutoff, and scale roughly as indicated by dimensional analysis. The essential physics is that a "decoupling" of long-wavelength physics from complicated details of short-wavelength physics should occur no matter how complicated the short-wavelength physics is.

This new viewpoint is very useful in condensed matter physics as well as relativistic quantum field theory. It indicates that the long-wavelength behavior of any system should have a simple description, however complicated the microscopic physics. This idea is the key to the modern theory of second order phase transitions, for instance, as developed by Ken Wilson. (The idea is called "Universality".)

And it is, perhaps, THE MOST IMPORTANT IDEA IN PARTICLE PHYSICS. Because it enables us to understand why renormalizable quantum field theories work as a description of Nature, even though we may have no idea how things really are at extremely short distances. (And it explains, for example, why we can use QED at energies well below 100 GeV, even though to describe things accurately at energies of order 100 GeV, we need the Weinberg-Salam model.)

You might have noticed one implication of the above analysis. We ought to regard the mass term in the bare Lagrangian as just another coupling, and write

$$m_0^2 = \Lambda^2 \lambda_2$$

The same logic as above indicates that λ_2 is order one. And we therefore expect that the bare mass of our scalar is of order Λ^2

This is right, of course: $-\!\!\!-(\text{1PI})\!-\!\!\!-$ will be quadratically divergent, by dimensional analysis. We need a mass counterterm of order Λ^2.

But, in the phenomenological Lagrangian point of view, we are given a bare theory at scale Λ, determined by some underlying microscopic physics, and this bare theory determines the physical mass in turn. Why should it turn out that

$$(m^2)_{\text{physical}} << \Lambda^2 \quad ?$$

This seems to require an incredible conspiracy among all the bare couplings of the theory. Putting it differently, it requires that the bare mass be "fine tuned" to an accuracy of $m^2_{\text{physical}}/\Lambda^2$, since

$$m^2_{\text{physical}} = m_0^2 + O(\Lambda^2)$$

is small by virtue of a cancellation between two quantities of order Λ^2.

(This is a genuine problem. One says (beginning with Ken Wilson) that

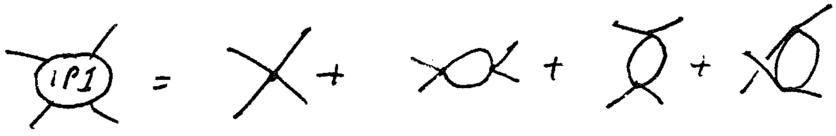
"Elementary scalars are *unnatural*."

A scalar particle wants to acquire a mass of order the cutoff. Things are different with fermions and vector mesons, because their masses can be forbidden by symmetries. There is, in fact, a symmetry called "supersymmetry" that can require that a scalar be massless, and our best reason for believing that supersymmetry has something to do with low energy ($\lesssim$ 100 TeV) physics is that an elementary ("Higgs") scalar seems to be required in the

Weinberg-Salam model. In any case, the need for an elementary scalar in the model encourages the hope that there is new physics --a new cutoff-- not far above 1 TeV. So-- maybe the SSC will not be a waste of money.)

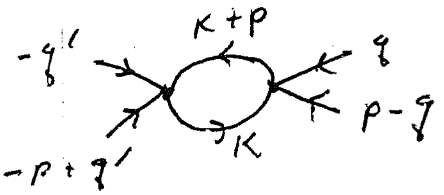
Infinite Coupling Renormalization

Consider $\mathcal{L}' = \frac{1}{4!}\phi^4$, $d=4$ dimensions (renormalizable)



+ counterterm + higher order

E.g.



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k+p)^2-m^2+i\epsilon}$$

(symmetry factor)

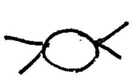
"Feynman trick"
$$\int_0^1 dx\,[ax+b(1-x)]^{-2}$$
$$= \left(\frac{-1}{a-b}\right)\frac{1}{ax+b(1-x)}\Bigg|_0^1$$
$$= \left(\frac{-1}{a-b}\right)\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}$$

So..

 $$= \frac{1}{2}\lambda^2\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2+x2k\cdot p+xp^2-m^2+i\epsilon]^2}$$

And $[\ ] = (k+xp)^2 + x(1-x)p^2 - m^2 + i\epsilon$

We can shift k integration:

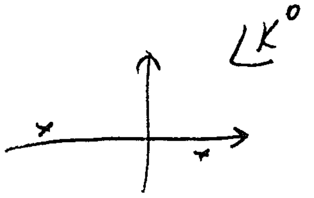
 $= \frac{1}{2}\lambda^2\int_0^1 dx\, I(a)$ $\qquad a = m^2 - x(1-x)p^2$

$$I(a) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-a+i\epsilon]^2}$$

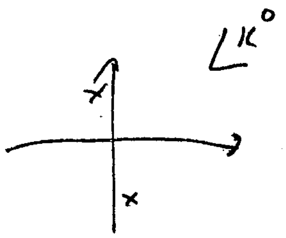
"Wick rotate" the k^0 integral

Singularities at $(k^0)^2 = \vec{k}^2 + a - i\epsilon$

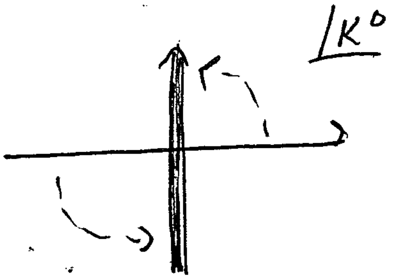
Either



or



Either way, we can rotate k^0 contour



(No contribution from $|k^0| \to \infty$)

So $k^0 \to i k_E^0$

$$I(a) = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[-k_E^2 - a + i\epsilon]^2}$$

↑ Euclidean metric $k_E^2 = k_E^{0\,2} + \vec{k}_E^{\,2}$

- Drop E subscript
- No longer need $i\epsilon$? We <u>do</u> need it, still, should a become negative.

Rotational Invariance: $d^Dk = \Omega_{D-1} k^{D-1} dk$

↑ volume of sphere

$\int d^Dk\, e^{-k^2} = \pi^{D/2}$

(product of Gaussian integrals)

$$= \Omega_{D-1} \int_0^\infty dk\, k^{D-1} e^{-k^2} = \frac{1}{2}\Omega_{D-1} \int_0^\infty dk^2 (k^2)^{\frac{D}{2}-1} e^{-k^2}$$

$$= \frac{1}{2}\Omega_{D-1} \Gamma(\tfrac{D}{2})$$

$$\Longrightarrow \quad \Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}$$

Integrate by parts $\Gamma(z+1) = z\Gamma(z)$

$\Gamma(1) = 1 \Rightarrow \Gamma(2) = 1$ etc

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\quad \Gamma(\frac{3}{2}) = \sqrt{\pi}/2$

$\Longrightarrow \Omega_1 = 2\pi,\ \Omega_2 = 4\pi,\ \Omega_3 = 2\pi^2$, etc

$$I(a) = i\,\frac{2\pi^2}{16\pi^4}\int_0^\infty dK\,K^3\,\frac{1}{[K^2+a-i\epsilon]^2}$$

$$= \frac{i}{16\pi^2}\int_0^{\infty} dK^2\,K^2\,[K^2+a-i\epsilon]^{-2}$$

Need to cut off log divergence

Rotationally invariant (i.e. Lorentz invariant) cutoff: $0 \le K^2 \le \Lambda^2$

Now, integral is elementary

$$\int_0^{\Lambda^2} dz\,\frac{z}{(z+b)^2} = \int_0^{\Lambda^2} dz\left[\frac{1}{z+b} - \frac{b}{(z+b)^2}\right]$$

$$= \left[\ln(z+b) + \left(\frac{b}{z+b}\right)\right]\Bigg|_0^{\Lambda^2}$$

$$= \ln\left(\frac{\Lambda^2+b}{b}\right) + \frac{b}{\Lambda^2+b} - 1$$

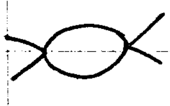
$$= \ln\left(\Lambda^2/b\right) - 1 + O\left(b/\Lambda^2\right)$$

So -- $$I(a) = \int \frac{d^4K}{(2\pi)^4\,[K^2-a+i\epsilon]^2}$$

$$= \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{a-i\epsilon}\right) - 1\right]$$

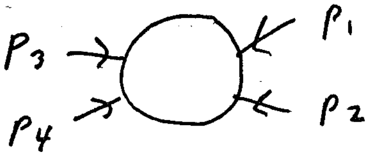
(ignoring terms that $\to 0$ as $\Lambda \to \infty$)

And so --



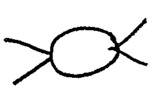
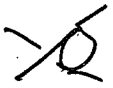
$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{m^2-x(1-x)p^2-i\epsilon}\right) - 1\right]$$

Here...



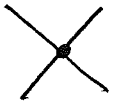
$s = (p_1+p_2)^2$
$t = (p_1+p_3)^2$
$u = (p_1+p_4)^2$

So $p^2 = s$. And ...

 +  + 

$$= \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\left(\frac{\Lambda^2}{m^2-x(1-x)s-i\epsilon}\right) - 1 + (s\to t) + (s\to u)\right]$$

We fix the counterterm

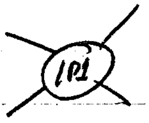
 $= -i\delta\lambda$ by imposing suitable renormalization condition

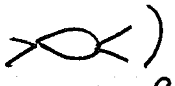
Note
$p_1+p_2+p_3+p_4 = 0 \Rightarrow \quad 2\sum_{i<j} p_i\cdot p_j = -\sum_i p_i^2$

$$2(s+t+u) = (p_1+p_2)^2 + (p_3+p_4)^2 + (p_1+p_3)^2 + (p_2+p_4)^2 + (p_1+p_4)^2 + (p_2+p_3)^2$$

$$= 2\sum_{i<j} p_i\cdot p_j + 3\sum p_i^2 \Rightarrow \boxed{s+t+u = p_1^2+p_2^2+p_3^2+p_4^2}$$

Natural "symmetric point" at which to subtract
$s = t = u = \frac{4}{3}m^2$

i.e.  $\Big|_{s=t=u=\frac{4}{3}m^2} = -i\lambda$ — Defines renormalized coupling

So $-i\delta\lambda = -3\,\Big($  $\Big)_{p^2=\frac{4}{3}m^2}$

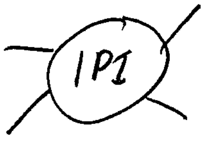
$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\frac{\Lambda^2}{m^2\left(1-\frac{4}{3}x(1-x)\right)} - 1\right]$$

+ higher order

$$= \frac{3\lambda^2}{32\pi^2}\left(\ln \Lambda^2/m^2 + \text{finite}\right) + \text{higher order}$$

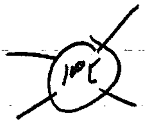
pure number: $-\int_0^1 dx \ln\left[1-\frac{4}{3}x(1-x)\right] - 1$

After renormalization:

 $= -i\lambda - \dfrac{i\lambda^2}{32\pi^2}\displaystyle\int_0^1 dx\left[\ln\left[\frac{m^2 - x(1-x)s - i\epsilon}{m^2\left(1-\frac{4}{3}x(1-x)\right)}\right] + (s\to t) + (s\to u)\right]$

— a <u>finite</u> expression

Note: In <u>massless</u> theory $(m^2=0)$, subtracting at the symmetric point does not work. Need to choose an <u>arbitrary</u> subtraction mass -- e.g.

 $\Big|_{s=t=u=-\mu^2} = -i\lambda$

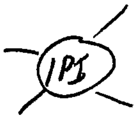
(choose spacelike momenta)

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\int_0^1 dx\left[\ln\frac{\Lambda^2}{\mu^2} - \ln x(1-x) - 1\right]$$

$$= \frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2}{\mu^2} + 1\right)$$

$\left(\int_0^1 dx \ln x = (x\ln x - x)\Big|_0^1 = -1\right.$

and



$$= -i\lambda -$$

$$-\frac{i\lambda^2}{32\pi^2}\left[\ln\left(\frac{-s-i\epsilon}{\mu^2}\right) + \ln\left(\frac{-t-i\epsilon}{\mu^2}\right) + \ln\left(\frac{-u-i\epsilon}{\mu^2}\right)\right]$$

(— quantum mechanics spoils classical scale invariance — —) why?

The renormalized coupling implicitly depends on μ — and we can choose μ to be whatever we like

Λ is a scale. can eliminate it only by introducing μ.

Gell-Mann + Low's Key idea:

- Since μ is arbitrary, we can choose it in any convenient way
- And, indeed, we are free to subtract at an arbitrary point, even in the massive theory

Note: expand around $p^2 = -\mu^2$ in field renormalization also $\Longrightarrow$

$$Z = Z(\Lambda^2/\mu^2)$$

Write the Lagrangian two ways

(NOTE: UNIVERSALITY $\Rightarrow$ this is sufficient)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_B)^2 - \frac{1}{2} m_0^2 \phi_B^2 - \frac{1}{4!}\lambda_0 \phi_B^4$$

— in terms of bare mass, coupling, field

$$\phi_B = \sqrt{Z}\,\phi_R$$

$$= \frac{1}{2} Z\, \partial_\mu \phi_R^2 - \frac{1}{2} m_0^2 Z \phi_R^2 - \frac{1}{4!}\lambda_0 Z^2 \phi_R^4$$

$$= \frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{1}{2} m^2 \phi_R^2 - \frac{1}{4!}\lambda \phi_R^4 + \mathcal{L}_{c.t.}$$

— in terms of renorm mass, coupling, field

$$\mathcal{L}_{c.t} = (Z-1)(\partial_\mu \phi_R)^2 - \frac{1}{2}\delta m^2 \phi_R^2 - \frac{1}{4!}\delta\lambda\, \phi_R^4$$

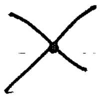
So
$$\boxed{\begin{aligned} Z m_0^2 &= m^2 + \delta m^2 \\ Z^2 \lambda_0 &= \lambda + \delta\lambda \end{aligned}}$$

↑ important — don't forget the Z's

This is a different convention than on page 10 — more natural if we don't subtract on shell

c.t. Feynman Rule

$i(Z-1)p^2 - i\delta m^2$



$-i\delta\lambda$

Renormalization is the procedure of writing

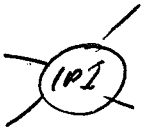
$$\mathcal{L}_{bare} = \underbrace{\mathcal{L}_{renorm} + \mathcal{L}_{counterterms}}$$

↑ this defines the theory

this division depends on <u>conventions</u>

Renormalization prescription or conditions

Renormalization is a convention-dependent reparametrization of the theory, performed (in part) so that calculated quantities make no reference to the cutoff

E.g. — we used 

$$\Big|_{s=t=u=-\mu^2} = -i\lambda$$

Found
$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left(\ln\left(\Lambda^2/\mu^2\right) + 1\right)$$

(in massless theory) + higher order

We also have

$$Z = 1 + O(\lambda^2)$$

So
$$\lambda_0 = \lambda + \frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right) + O(\lambda^3)$$

λ_0 ← convention-independent bare parameter

λ ← renormalized parameter — must depend implicitly on μ

(I've rescaled μ slightly, to absorb finite part into log)

E.g.
$$\lambda_\mu + \frac{3\lambda_\mu^2}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2} = \lambda_0 = \lambda_{\mu'} + \frac{3\lambda_{\mu'}^2}{32\pi^2}\ln\Lambda^2/\mu'^2$$

The relation between the two renormalized couplings λ_μ and $\lambda_{\mu'}$ must not involve Λ

can solve iteratively (implicitly regarding $\lambda_\mu \ln(\Lambda^2/\mu^2)$ as a small quantity)

$$\lambda_{\mu'} = \lambda_\mu + \frac{3\lambda_\mu^2}{32\pi^2}(\ln\Lambda^2 - \ln\mu^2)$$

$$- \frac{3\lambda_{\mu'}^2}{32\pi^2}(\ln\Lambda^2 - \ln\mu'^2) + \cdots$$

subst in lowest order $\lambda_{\mu'} = \lambda_\mu$

$$= \lambda_\mu + \frac{3\lambda_\mu^2}{32\pi^2}\ln\frac{\mu'^2}{\mu^2} + \text{higher order}$$

Important: Λ does drop out (dependence is higher order: so higher order c.t. must cancel that remaining Λ-dependence

$$\beta(\lambda) = \mu'\frac{d}{d\mu'}\lambda_{\mu'}\Big|_{\mu=\mu'} = \frac{3\lambda_\mu^2}{16\pi^2} + \text{h.o.}$$

More traditional derivation:

$$\mu\frac{d}{d\mu}\lambda_0 = 0 = \mu\frac{d}{d\mu}\left(\lambda + \frac{3\lambda^2}{32\pi^2}\ln\Lambda^2/\mu^2 + \text{h.o.}\right)$$

$$= \beta(\lambda)\left[1 + \frac{3\lambda}{16\pi^2}\ln\Lambda^2/\mu^2\right] - \frac{3\lambda^2}{16\pi^2} + \text{h.o.}$$

$$\Rightarrow \beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$$

Now -- let's integrate $\mu \frac{d}{d\mu}\lambda = \beta(\lambda)$

Consider $\beta(\lambda) = b\lambda^{n+1} + \cdots$

$$\mu \frac{d}{d\mu}\lambda = b\lambda^{n+1} + \cdots \Rightarrow b \ln \mu'/\mu = \int_{\lambda_\mu}^{\lambda_{\mu'}} \frac{d\lambda}{\lambda^{n+1}}$$

$$= \frac{-1}{n\lambda^n}\Big|_{\lambda_\mu}^{\lambda_{\mu'}} = -\frac{1}{n}\left[\frac{1}{\lambda_{\mu'}^n} - \frac{1}{\lambda_\mu^n}\right]$$

$$\frac{1}{\lambda_{\mu'}^n} = \frac{1}{\lambda_\mu^n} - nb \ln \mu'/\mu$$

$$\Rightarrow \boxed{\lambda_{\mu'}^n = \frac{\lambda_\mu^n}{1 - nb\lambda_\mu^n \ln \mu'/\mu}}$$

Why be so silly as to differentiate and reintegrate?

E.g. $$\lambda_\mu' = \frac{\lambda_\mu}{1 - \frac{3\lambda_\mu}{16\pi^2}\ln \mu'/\mu}$$

agrees with what we started with, when expanded, up to order λ^2. Do higher order terms have any significance?

Yes! Summation of leading logs

Suppose $\beta(\lambda) = b_0\lambda^2 + b_1\lambda^3 + \cdots$

Solve $\frac{d}{dt}\lambda = \beta(\lambda)$ where $t = \ln \mu/\mu_0$ by double power series expansion

$$\lambda(t) = \lambda_0 + \lambda_0^2 (C_{2,1} t) \qquad (\lambda(0) = \lambda_0)$$
$$+ \lambda_0^3 (C_{3,2} t^2 + C_{3,1} t) \quad \ldots$$
$$+ \cdots$$

(Note: λ_0 is not bare coupling — it is λ_{μ_0})

$$\frac{d}{dt}\lambda = \lambda_0^2 (C_{2,1}) + \lambda_0^3 (2C_{3,2} t + C_{3,1}) + \cdots$$
$$= b_0 (\lambda_0 + \lambda_0^2 (C_{2,1} t) \quad + \cdots)^2$$
$$+ b_1 (\lambda_0 + \lambda_0^2 (C_{2,1} t) \quad + \cdots)^3$$
$$+ \cdots$$

Equate powers of $\lambda_0^2 \Rightarrow C_{2,1} = b_0$

b_0 determines leading logs

$$\begin{cases} \lambda_0^3 t \Rightarrow & 2C_{3,2} = 2b_0 C_{2,1} \Rightarrow C_{3,2} = b_0^2 \\ \lambda_0^4 t^2 \Rightarrow & 3C_{4,3} = b_0 C_{2,1}^2 + 2b_0 C_{3,2} \\ & \Rightarrow C_{4,3} = b_0^3 \end{cases}$$

b_1 determines next-to leading logs

$$\begin{cases} \lambda_0^3 \Rightarrow & C_{3,1} = b_1 \\ \lambda_0^4 t \Rightarrow & 2C_{4,2} = b_0 (2C_{3,1}) + b_1 3(C_{2,1}) \\ & = 2b_0 b_1 + 3b_0 b_1 \\ & \Rightarrow C_{4,2} = \tfrac{5}{2} b_0 b_1 \end{cases}$$

---

our formula for λ_μ sums up the leading logs in each order in λ_μ