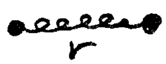
D. Scaling and the Renormalization Group

References on QCD and the renormalization group:
H.D. Politzer, Physics Reports 14, 129 (1974).
D. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973); D9, 980 (1974)
D. Gross, Les Houches Lectures, 1975; in Methods in Field Theory, ed. R. Balian and J. Zinn-Justin, North-Holland (1976).
W. Marciano and H. Pagels, Physics Reports 36, 137 (1978).

Consider QCD without quarks, i.e., pure Yang-Mills theory. We calculated above the interaction energy of two static color sources in this theory, to leading order



$$E(r) = \frac{g^2}{4\pi r} C$$, where C is a group-theoretic constant

The result $E(r) \propto \frac{1}{r}$ could have easily been guessed. Since g^2 is dimensionless, the (classical) theory sets no distance scale. (Actually $g^2 \sim$ Energy × distance)

But scale invariance does not survive in the quantum theory. Why not?

Physically, virtual gluon pairs occur as quantum fluctuations $\longrightarrow$ quantum mechanically, the vacuum is a polarizable medium, and charge screening (or antiscreening) occurs.

$$E(r) = \frac{C g^2(r)}{4\pi r}$$ and $g^2(r) = g^2(\mu r)$, since g^2 is dimensionless. A mass scale μ has appeared.

Mathematically, when quantum corrections are calculated, logarithmic ultraviolet divergences are encountered, and subtractions must be made. We subtract at some Euclidean

momentum $p^2 = -\mu^2$. A mass scale appears. We cannot choose $\mu^2 = 0$, because of infrared singularities associated with gluon poles and cuts.

The Main Idea of the Renormalization Group is: In a massless (classically scale-invariant) theory, we must choose a renormalization scale μ to define renormalized green's functions, but physics is independent of μ.

The "group" transformation is simply a rescaling of μ.

The renormalization group is useful because it allows us to predict how Green's functions behave when all four momenta are scaled by the same constant. The renormalized coupling constant g_μ and the field renormalization constants $Z_i(\mu)$ must become functions of the subtraction point μ, if we demand that physics is independent of μ.

To illustrate how the Main Idea is applied, consider the renormalized Green's functions of a scalar field theory,

$$G_R^{(n)} = \langle 0 | T[\phi_R \cdots \phi_R] | 0 \rangle .$$

Let E represent the "overall" energy scale of the Green's function and x represent all the dimensionless kinematic "angles" - i.e. ratios of invariants. $G_R^{(n)}$ is a function of the coupling constant g_μ and the renormalization scale μ, as well as of the kinematic variables.

Mere dimensional analysis tells us that

$$G_R^{(n)}(E, x, g_\mu, \mu) = E^D f\left(\frac{E}{\mu}, g_\mu, x\right),$$

if $G_R^{(n)}$ has dimension D. But we can learn more about the dependence of $G_R^{(n)}$ on E by exploiting the fact that "physics is really independent of μ!"

The "bare" unrenormalized Green's functions are functions of a bare coupling constant g_0 and of a cutoff Λ which must be introduced to make them finite:

$$G_U^{(n)}(E, x, g_0, \Lambda)$$

The unrenormalized Green's functions are independent of μ, which is introduced only when we specify a way of expressing the renormalized coupling in terms of g_0 and Λ.

$$g_\mu = g_\mu(g_0, \Lambda/\mu)$$

The renormalized Green's functions are related to the unrenormalized Green's functions by

$$G_R^{(n)}(E, x, g_\mu, \mu) = \left[Z(\Lambda/\mu)\right]^{-n/2} G_U^{(n)}(E, x, g_0, \Lambda).$$

Thus, we know that

$$\left[Z(\Lambda/\mu)\right]^{n/2} G_R^{(n)}(E, x, g_\mu, \mu)$$

is independent of μ. Since we can choose μ to be whatever we please, we may as well choose $\mu = E$

$$[Z(\mu)]^{n/2} G_R^{(n)}(E, x, g_\mu, \mu) = [Z(E)]^{n/2} G_R^{(n)}(E, x, g_E, E)$$

or
$$G_R^{(n)}(E, x, g_\mu, \mu) = \left[\frac{Z(E)}{Z(\mu)}\right]^{n/2} G_R^{(n)}(E, x, g_E, E)$$

$$= \left[\frac{Z(E)}{Z(\mu)}\right]^{n/2} E^D f(x, g_E)$$

(The ratio $Z(E)/Z(\mu)$ does not depend on Λ.)

The behavior of $G_R^{(n)}$ as we scale E with x fixed is seen to be determined by the functions $Z(E)$ and g_E. This equation for $G_R^{(n)}(E)$ is called the "integrated form" of the renormalization group equation. We will see that, when g_μ is small, g_μ and $Z(\mu)$ can be determined using perturbation theory.

How can g_μ and $Z(\mu)$, which are dimensionless, depend on μ? The theory does have a mass scale, but it is an arbitrary mass scale,

$$g_{\mu'} = g_{\mu'}\left(\tfrac{\mu'}{\mu}, g_\mu\right).$$

If we differentiate with respect to μ' and set $\mu'=\mu$, we must obtain a dimensionless function of g alone,

$$\mu \frac{d}{d\mu} g_\mu = \beta(g_\mu).$$

Similarly, $$Z(\mu')^{\frac{1}{2}} = Z(\mu)^{\frac{1}{2}} F(\mu'/\mu, g_\mu).$$

Here F is not a function of $Z(\mu)$, since $Z(\mu)$ is fixed only by a renormalization convention for the renormalized fields. Therefore

$$\mu \frac{d}{d\mu} \ln Z(\mu)^{\frac{1}{2}} = \gamma(g_\mu);$$

also a function of g_μ only.

[Incidently, we can now derive the "differential form" of the renormalization group eqn from the μ-independence of G_U:

$$0 = \mu \frac{d}{d\mu}[Z(\mu)]^{n/2} G_R^{(n)}(E, x, g_\mu, \mu) \Rightarrow$$

$$\left[\mu \frac{\partial}{\partial\mu} + \beta(g_\mu)\frac{\partial}{\partial g_\mu} + n\gamma(g_\mu)\right] G_R^{(n)} = 0]$$

If the functions β, γ are known, we can determine the scaling behavior of all Green's functions:

$$\mu \frac{d}{d\mu} g_\mu = \beta(g_\mu) \Rightarrow \frac{d\mu}{\mu} = \frac{dg}{\beta}$$

or
$$\boxed{\ln \frac{\mu}{\mu_0} = \int_{g_0}^{g_\mu} \frac{dg'}{\beta(g')}}$$

And, if g_μ is known, we can integrate

$$\mu \frac{d}{d\mu} Z^{\frac{1}{2}} = \gamma(g_\mu) Z^{\frac{1}{2}}$$ where

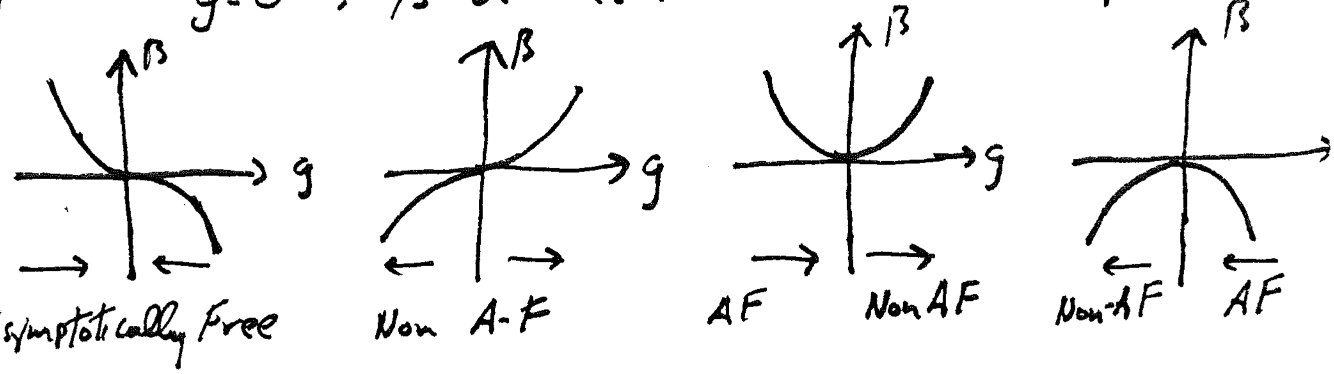
$$d \ln Z^{\frac{1}{2}} = \frac{d\mu}{\mu} \gamma(g_\mu) \Longrightarrow$$

$$\boxed{Z^{\frac{1}{2}}(\mu) = Z^{\frac{1}{2}}(\mu_0) \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(g_{\mu'}) \right]}$$

Thus $$G = E^D f(g_E, x) \exp\left[n \int_{\mu}^{E} \frac{d\mu'}{\mu'} \gamma(g_{\mu'}) \right]$$

We will discuss below how β, γ are calculated in perturbation theory. First let's make the motivation clear by considering different types of possible behavior.

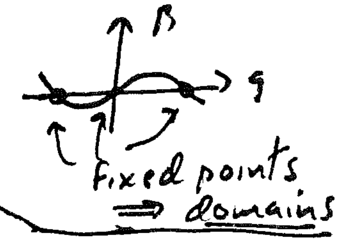
$\beta = 0$ at $g = 0$ (Free field theory stays free for any value of μ) In the vicinity of $g=0$, β behaves in one of four ways



(More complicated types of behavior can occur if there is more than one coupling constant.)

In "asymptotically free" theories, g_μ is drawn toward $g_\mu = 0$ as $\mu \to \infty$.

note:



The Approach to Asymptotia

Suppose $\boxed{\beta(g) \cong -b g^{n+1}}$ for g small $\quad n > 0,\ b > 0$

$$\ln \mu/\mu_0 = \int_{g_0}^{g_\mu} \frac{dg}{\beta} = \frac{1}{nb}\left[\frac{1}{g_\mu^{\,n}} - \frac{1}{g_0^{\,n}}\right]$$

$$\Rightarrow g_\mu^{\,n} = \frac{g_0^{\,n}}{1 + n b g_0^{\,n} \ln \mu/\mu_0}$$

If we absorb the 1 into the argument of the logarithm

$$\boxed{g_\mu^{\,n} = \frac{1}{nb \ln(\mu/\Lambda)}}$$

g_0 has disappeared. There is no dimensionless coupling constant – Just a mass scale Λ (finite; not to be confused with cutoff!)

— Dimensional Transmutation

In perturbation theory, γ is also a power of g. Suppose

$$\gamma(g) \cong c g^m$$

$$G^{(p)} = E^D f(g_E, x) \exp\left[p \int^E \frac{d\mu'}{\mu'} \gamma(g_{\mu'})\right]$$

and

$$\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} c \left[\frac{1}{nb \ln(\mu'/\Lambda)}\right]^{m/n}$$

$$= c(nb)^{-m/n} \frac{1}{1 - m/n} \left[\ln(\mu'/\Lambda)\right]^{1-m/n} \Big|_{\mu_0}^{\mu}$$

Especially relevant in QCD is the case $n=m=2$

$$\frac{1}{1-m/n}\ln(\mu^2/\Lambda)^{1-m/n}\Big|_{\mu_0}^{\mu}$$

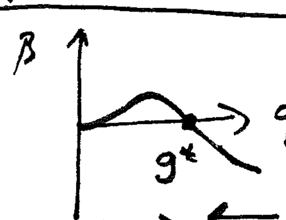
$$\longrightarrow \ln\ln(\mu^2/\Lambda)\Big|_{\mu_0}^{\mu}$$

up to a multiplicative constant,

$$G^{(p)} = E^D f(q_E, x)\left[\ln(E/\Lambda)\right]^{pc/2b}$$

Naive scaling (according to dimensional analysis) spoiled by logarithms

Nontrivial Fixed Point



The β function may have a zero for a nontrivial value of g

As $\mu \to \infty$, g_μ may be attracted to this "nontrivial fixed point"

Note that $\gamma(g^*)$ need not vanish

Exercise 1.4

Suppose $\beta(g) \cong -b(g-g^*)$, for g near g^*

and $\gamma(g) \cong \gamma^* + \gamma'^*(g-g^*)$, for Green's function G.

Find g_μ for $g_\mu - g^*$ small, and the asymptotic behavior of G

Spontaneous Mass Generation

Consider a <u>physical</u> quantity σ which has dimension D and is not a function of any kinematic variables. (E.g., a particle mass, with $D=1$) The statement that σ is <u>physical</u> means that it is not rescaled when we change μ.

Now, σ is a function of only μ and g_μ, and we know by dimensional analysis that

$$\sigma = \mu^D f(g_\mu).$$

We can find f by using the differential form of the RG eqn,

$$0 = \mu \frac{d}{d\mu}\sigma = \mu^D \left[D + \beta(g_\mu) \frac{d}{dg_\mu} \right] f(g_\mu)$$

$$\Rightarrow \; df = -D \frac{dg}{\beta(g)} \; \Rightarrow \; f = C \exp\left[-D \int^{g_\mu} dg'/\beta(g') \right].$$

If we use the perturbative expression for β, $\beta \sim -b g^{n+1}$

$$f = C \exp\left[-D/n b g_\mu^n \right]$$

$$\Rightarrow \; \sigma = C \mu^D \exp\left[-D/n b g_\mu^n \right].$$

We see that if masses are spontaneously generated in a theory which has no mass scale at the classical level, this is a non-perturbative phenomenon; σ has no expansion in powers of the coupling constant. And in a nonasymptotically free theory with $b<0$, we see that σ blows up as $g_\mu \to 0$ if $C \neq 0$, which suggests that C must vanish

Notice that $\quad 1/nbg_\mu^n = \ln(\mu/\Lambda) \Longrightarrow$

$$\sigma = C \Lambda^D,$$

which is obvious, since Λ is the only mass scale the theory has.

<u>Leading Logarithms</u>: The RG eqn sums powers of $\ln(E/\mu)$ to "improve" the perturbation expansion.

Let's consider, for concreteness, the case $n=1$

$$\beta(g) \cong b g^2 + \cdots$$

for which we found

$$g_\mu = g_0 \left[1 - b g_0 \ln \mu/\mu_0\right]^{-1}$$

This can be expanded as a power series in g_0 and $t = \ln \mu/\mu_0$.

Now what if we include terms in β which are higher order in g?

$$\beta(g) = b_0 g^2 + b_1 g^3 + \cdots$$

We can still solve

$$\frac{d}{dt} g(t) = \beta(g)$$

as a double power series in g_0 and t.

$$g(t) = g_0 + \sum_{n=2}^{\infty} \sum_{p=1}^{n-1} C_{n,p}\, g_0^n\, t^p \quad (= g_0 \text{ at } t=0 \Rightarrow C_{n,0} = 0)$$

But the higher order terms in β alter only the "nonleading logs" in each order in g_0

We have

$$\sum_{n,p} C_{n,p}\, g_0^n (p\, t^{p-1}) = b_0 \left[g_0 + \sum C_{n,m} g_0^n t^p\right]^2 + b_1 \left[g_0 + \sum C_{n,m} g_0^n t^p\right]^3 + \cdots$$

Equating coefficients of ---

g_0^2: $C_{2,1} = b_0$

$g_0^3 t$: $2C_{3,2} = 2b_0 C_{2,1}$

$g_0^4 t^2$: $3C_{4,3} = 2b_0 C_{3,2} + b_0 C_{2,1}^2$

etc. Leading Logs

g_0^3: $C_{3,1} = + b_1$

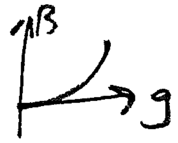
$g_0^4 t$: $2C_{4,2} = 2b_0 (C_{3,1}) + b_1 (3C_{2,1})$

etc.

Next-to-leading logs

Thus, the β function in leading order determines leading logs to arbitrary order in perturbation theory. Next-to-leading order in β determines next-to-leading logs, and so forth.

Landau Singularity



what if $\int^{\infty} \frac{dg}{\beta(g)}$ converges?

E.g. $\beta \sim g^n$, $n>1$, for $g \to \infty$

then $\int_{g_0}^{g_\mu} \frac{dg}{\beta(g)} = \ln \frac{\mu}{\mu_0}$

$\Rightarrow$ For some finite μ $\quad g_\mu \to \infty$

Hence, Green's functions have a singularity at finite energy. This singularity is called a "Landau Ghost". In a sensible theory, there must be no Landau ghosts; β must not grow too rapidly as $g \to \infty$. (Of course, the large g behavior of β cannot be studied in perturbation theory.)

E. Dimensional Regularization and Minimal Subtraction

To proceed with the perturbative calculation of β and γ, we must specify how renormalized couplings and fields are defined.

As an example of a massless field theory, we will consider $\lambda\phi^4$ theory:

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu \phi)^2 - \frac{\lambda}{4!}\phi^4$$

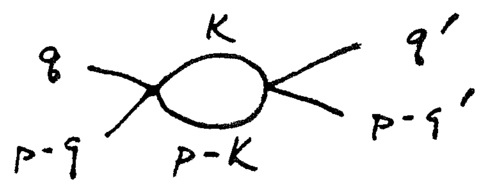
A good reference: Field Theory by P. Ramond

The μ-dependence arises because of infinities encountered in perturbation theory. We must discuss how these infinities are dealt with. The scheme we will use is

- Dimensional Regularization
- Minimal Subtraction

— because this will be convenient when we consider gauge theories, and also will enable us to find a generalized RG eqn which applies to theories with masses.

To see how <u>dimensional regularization</u> works, consider the one-loop graph



$$= \frac{(-i\lambda)^2}{2}\frac{(i)^2}{(2\pi)^4}\int d^4K\left[(K^2+i\epsilon)\left[(K-p)^2+i\epsilon\right]\right]^{-1}$$

using the standard Feynman trick, we have

$$I = \int d^4K\ \left[(K^2+i\epsilon)\left((K-p)^2+i\epsilon\right)\right]^{-1}$$

$$= \int_0^1 dx\int d^4K\ \left[K^2+2p\cdot Kx+p^2x+i\epsilon\right]^{-2}$$

$$= \int_0^1 dx\int d^4K\ \left[K^2+p^2x(1-x)+i\epsilon\right]^{-2}$$

where we have completed the square and shifted K

The $i\epsilon$ tells us how to rotate the contour

$$= i\int_0^1 dx\int d^4K_E\ \left[K_E^2-p^2x(1-x)\right]^{-2}$$

So now we have

$$\text{Graph} = \frac{+i\lambda^2}{2} \frac{1}{(2\pi)^4} I(p^2)$$

$$I(p^2) = \int_0^1 dx \int d^4K \,[K^2 - p^2 x(1-x)]^{-2}$$

The integral $I(p^2)$ is logarithmically divergent. The idea of dimensional regularization is to analytically continue this integral $\int d^4K$ to an integral $\int d^DK$, and regard the UV divergence as a singularity (a pole) that occurs at $D \to 4$

So let us consider how to evaluate

$$I_D(a^2) = \int d^DK \,(K^2+a^2)^{-2}$$

First, we note that for a spherically symmetric integrand

$$\int d^DK = \Omega_{D-1} \int_0^\infty dK\, K^{D-1}$$

where Ω_{D-1} = area of $D-1$-dimensional sphere

to find Ω_{D-1}:

$$\int d^DK\, e^{-K^2} = \pi^{D/2} = \Omega_{D-1} \int_0^\infty (2K\,dK)\, \tfrac{1}{2}(K^2)^{\frac{D}{2}-1} e^{-K^2}$$

$$= \Omega_{D-1}\, \tfrac{1}{2}\,\Gamma\!\left(\tfrac{D}{2}\right)$$

$$\boxed{\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}}$$

Note that this is an analytic function of D which interpolates between the integers.

$$I_D(a^2) = \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty d(K^2)\,(K^2)^{D/2-1}\,(K^2+a^2)^{-2}$$

Now evaluate

$$\int_0^\infty dk^2\,(k^2)^p (k^2+a^2)^{-q}, \quad \text{using Schwinger's trick:}$$

$$= \int_0^\infty ds \int_0^\infty dk^2 (k^2)^p\, s^{q-1} \exp[-s(k^2+a^2)] \,/\, \Gamma(q)$$

$$= \frac{1}{\Gamma(q)} \int_0^\infty ds \quad s^{q-1} e^{-sa^2} s^{-(p+1)}\, \Gamma(p+1)$$

$$= (a^2)^{p-q+1} \frac{\Gamma(p+1)}{\Gamma(q)} \Gamma(q-p-1)$$

For later reference, we assemble the result

$$\boxed{\int d^D k\, (k^2)^r (k^2+a^2)^{-q} = \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})} \frac{\Gamma(r+D/2)}{\Gamma(q)} \Gamma(q-r-\tfrac{D}{2})(a^2)^{r-q+\frac{D}{2}}}$$

And this is what we wanted; an analytic continuation in D of the Feynman integral which is meromorphic (analytic except for poles)

A similar continuation can be performed for <u>any</u> Feynman integral

Applying this to graph =  , we have

$$\text{Graph} = +\frac{i\lambda^2}{2} \frac{1}{(2\pi)^D} \int_0^1 dx\; \pi^{D/2}\, \Gamma(2-\tfrac{D}{2}) \left[-p^2 x(1-x)\right]^{\frac{D}{2}-2}$$

– this has the expected pole at $D=4$

Minimal Subtraction

We have regularized the Feynman integrals. The next step in the renormalization procedure is to introduce counterterms which remove the divergences order by order in perturbation theory

There is an awkward point here. The tree graph $\times = O(\lambda)$ is dimensionless; but the one-loop graph $>\!\!\bigcirc\!\!<$ has dimension $[M]^{D-4}$ – How does this graph generate the one-loop coupling renormalization?

We must remember to dimensionally continue λ along with everything else – A quartic coupling constant is dimensionless only in four dimensions. To keep λ dimensionless, make the replacement

$$\lambda \to \lambda \mu^{4-D}$$, where μ is an arbitrary mass.

Then $\lambda \times \sim (M)^{4-D} \quad \lambda^2 >\!\!\bigcirc\!\!< \sim M^{4-D}$

Now define $\epsilon = 4-D$, and look at the behavior of the graph as $\epsilon \to 0$

$$\text{Graph} = \frac{+i\lambda^2\mu^\epsilon}{32\pi^2}(4\pi)^{\epsilon/2}\,\Gamma\!\left(\frac{\epsilon}{2}\right)\int_0^1 dx\left[\frac{-p^2x(1-x)}{\mu^2}\right]^{-\epsilon/2}$$

and we want to keep only the pieces of this expression which survive as $\epsilon \to 0$

We have $\quad \Gamma(z) = \dfrac{1}{z} - \gamma + O(z) \quad$ for z small

$$\gamma = .57721$$

$$a^\epsilon = e^{\epsilon \ln a} = 1 + \epsilon \ln a + \cdots$$

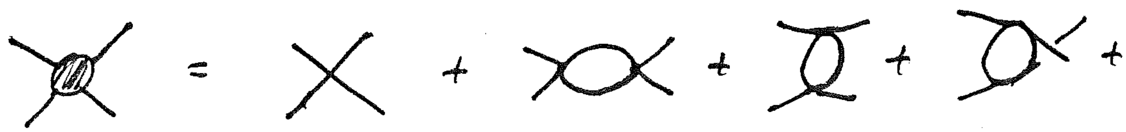
or

$$\text{Graph} = \frac{+i\lambda^2\mu^\epsilon}{32\pi^2}\left(\frac{1}{\epsilon/2} - \gamma\right)\left(1 + \frac{\epsilon}{2}\ln 4\pi\right)\left(1 - \frac{\epsilon}{2}\int_0^1 dx \ln\left[\frac{-p^2 x(1-x)}{\mu^2}\right]\right) + \cdots$$

$$= \frac{+i\lambda^2\mu^\epsilon}{32\pi^2}\left(\frac{1}{\epsilon/2} - \gamma + \ln 4\pi - \int_0^1 dx \ln\left[\frac{-p^2 x(1-x)}{\mu^2}\right]\right)$$

$$= \frac{+i\lambda^2\mu^\epsilon}{16\pi^2}\frac{1}{\epsilon} + \text{finite part}$$

There are actually three graphs which contribute to the four-point function in one-loop order



counter terms + more loops.

But all three have the same infinite part therefore, we have

$$= -i\lambda\mu^\epsilon + \frac{3i\lambda^2\mu^\epsilon}{16\pi^2}\frac{1}{\epsilon} + \text{counter terms} + \text{finite}$$

The counterterms must be chosen to remove the poles at $\epsilon = 0$ order by order. Therefore, the <u>bare</u> coupling constant, to one loop order, must be

$$\lambda_0 = \mu^\epsilon\lambda\left(1 + \frac{3\lambda}{16\pi^2\epsilon}\right) + \text{finite part}$$

$$+ \text{higher order}$$

<u>Minimal Subtraction</u> means the scheme in which there are no finite counter terms. We subtract away the pole in ϵ, but nothing else

(In a modified version of this scheme, one also subtracts the constant $-\gamma + \ln 4\pi$.) Therefore, in minimal subtraction, the bare coupling is

$$\lambda_0 = \mu^\epsilon \lambda \left(1 + \frac{3\lambda}{16\pi^2 \epsilon}\right) + \text{higher order}$$

If we carried out this procedure to higher order, we would find higher order poles in ϵ, and the residues of the poles would be power series in λ, the renormalized coupling.

$$\lambda_0 = \mu^\epsilon \left[\lambda + \frac{a_1(\lambda)}{\epsilon} + \frac{a_2(\lambda)}{\epsilon^2} + \cdots\right]$$

The β-function

Now we can compute the β function in minimal subtraction. When μ changes, we demand that λ changes so that λ_0 remains fixed (for fixed ϵ)

Thus
$$\mu \frac{d}{d\mu}\lambda_0 = 0 = \mu^\epsilon \Big[\epsilon\Big(\lambda + \frac{a_1(\lambda)}{\epsilon} + \cdots\Big) + \beta(\lambda,\epsilon)\Big(1 + \frac{a_1'(\lambda)}{\epsilon} + \cdots\Big)\Big],$$

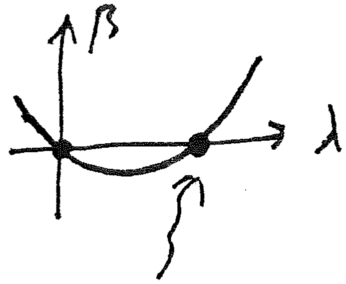
and we find $\beta(\lambda,\epsilon)$ by equating powers of ϵ. Since β is regular at $\epsilon = 0$, we see that

$$\boxed{\beta(\lambda,\epsilon) = -\epsilon\lambda - a_1(\lambda) + \lambda a_1'(\lambda)}$$

β is determined by just the residue of the simple pole. All the other residues may be obtained recursively from this one.
(cf. discussion of leading logs on p 1.18)

For $\lambda\phi^4$ theory, we have $a_1(\lambda) = \frac{3\lambda^2}{16\pi^2}$

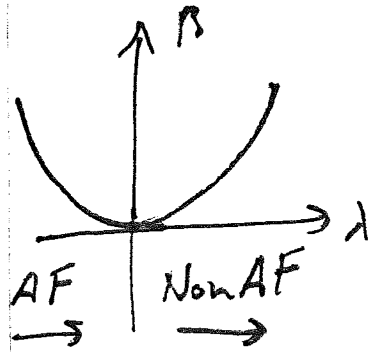
and
$$\boxed{\beta(\lambda) = -\epsilon\lambda + \frac{3\lambda^2}{16\pi^2} + \cdots}$$



For $\epsilon > 0$ $(D<4)$ this theory is asymptotically free and for ϵ small, it has a nontrivial fixed point

this is the Wilson-Fisher fixed point which has proved useful in the study of critical phenomena.

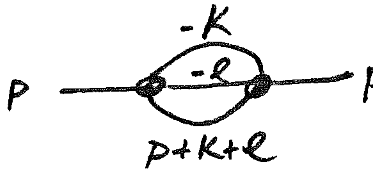
(For $\epsilon > 0$, this theory is superrenormalizable, and superrenormalizable theories are trivially asymptotically free.)



For $\epsilon = 0$ (the case we are really interested in) the theory is AF only if $\lambda < 0$.

The Anomalous Dimension

To find Z, we must compute a two loop graph

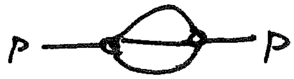


$$= \frac{i\mu^{2\epsilon}\lambda^2}{6(2\pi)^{2D}} \int d^D k\, d^D l \left[k^2 l^2 (p+k+l)^2\right]^{-1}$$

Now, you are to do

Exercise 1.5*

Show



$$= i\frac{\lambda^2}{12(16\pi^2)^2\epsilon} p^2 \quad + \text{finite part}$$

The Lagrangian can be written, in terms of renormalized fields, but including counterterms, as

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}(Z-1)(\partial_\mu \phi)^2 + \text{interactions}.$$

For the counterterm to cancel the infinity in $O(\lambda^2)$, we need

$$+ip^2(Z-1) = -i\frac{\lambda^2}{12(16\pi^2)^2\epsilon} p^2$$

or

$$\boxed{Z = 1 - \frac{\lambda^2}{12(16\pi^2)^2\epsilon}}$$

As before, the finite part of $Z-1$ is chosen to be zero; i.e., we subtract away only the pole in ϵ

The anomalous dimension is defined as

$$\gamma = \mu\frac{\partial}{\partial\mu}\ln Z^{\frac{1}{2}} \Rightarrow Z\gamma(\lambda) = \tfrac{1}{2}\beta(\lambda,\epsilon)\frac{d}{d\lambda}Z$$

and Z can be expressed as

$$Z = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{\epsilon^n}$$

Now, using $\beta = -\epsilon\lambda + \cdots$, we have

$$\gamma\left(1 + \frac{c_1}{\epsilon} + \cdots\right) = \frac{1}{2}(-\epsilon\lambda + \cdots)\left(\frac{c_1'(\lambda)}{\epsilon} + \cdots\right)$$

Since γ is regular at $\epsilon = 0$, we find, equating terms of 0th order

$$\boxed{\gamma = -\frac{1}{2}\lambda c_1'(\lambda)}$$

And $\quad c_1(\lambda) = -\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + \cdots$

$$\Rightarrow \boxed{\gamma = \frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 + \cdots}$$

Mass Renormalization

So far we have considered only massless field theories. Now lets introduce a mass, and consider how the mass is renormalized in minimal subtraction.

We will treat the renormalized mass in a special way. In our renormalization scheme, it is like any other coupling constant, and depends on the scale μ. It is not the physical mass – the position of the pole in the propagator.

Renormalizing this way, we make subtractions which are mass independent, and therefore no mass dependence is introduced by the renormalization procedure. We regard $m(\mu)/\mu$ as another parameter on which Green's functions can depend:

$$G^{(n)}(E, x, g_\mu, m_\mu, \mu) = E^D f\left(\frac{E}{\mu}, g_\mu, \frac{m_\mu}{\mu}, x\right)$$
$$= E^D \left[\frac{Z(E)}{Z(\mu)}\right]^{n/2} f\left(g_E, \frac{m_E}{E}, x\right)$$

Now, <u>if</u>

i) $m_E/E \to 0$ as $E \to \infty$

ii) f has a smooth limit as $m_E/E \to 0$

then we can justify the use of the <u>massless</u> renormalization group eqns to extract asymptotic behavior; i.e., our previous analysis applies.

We will see that it is not hard to verify (i) in an asymptotically free theory, but (ii) involves some mathematical subtleties which we must eventually address. It is believed that these subtleties are "merely" mathematical, however; that any suitably defined <u>physical</u> quantity, something which can in principle be measured, is free of "infrared singularities" as $m \to 0$.

If m_E/E is small but not totally negligible, we can do an improved renormalization group analysis including mass corrections.

To begin, let's see how the renormalized mass is defined to lowest order

$$= \frac{-i\lambda\mu^\epsilon}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-m^2} = \frac{-i\lambda\mu^\epsilon}{2}\int\frac{d^Dk_E}{(2\pi)^D}\frac{1}{k_E^2+m^2}$$

It is convenient to <u>not</u> normal order the interaction

$$= -\frac{i\lambda\mu^\epsilon}{2}\frac{1}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{D/2-1}$$

and, since $\Gamma(1-\frac{D}{2}) = \frac{1}{1-D/2}\Gamma(2-\frac{D}{2}) \sim \frac{-1}{\epsilon/2}$,

we have

$$= \frac{i\lambda}{16\pi^2\epsilon}m^2 + \text{finite part}$$

The mass can be renormalized multiplicatively:

i.e. we remove the $1/\epsilon$ pole by choosing the bare mass to be—

$$m_0^2 = m^2\left(1 + \frac{\lambda}{16\pi^2\epsilon} + \cdots\right)$$

Since mass renormalization is multiplicative (in minimal subtraction), the scaling behavior of the renormalized mass is determined by an anomalous dimension which is a function of only the coupling constant:

$$\gamma_m = \frac{1}{2}\mu\frac{d}{d\mu}\ln m^2$$

If $m_0^2 = m^2\left(1 + \sum_{n=1}^{\infty}\frac{b_n(\lambda)}{\epsilon^n}\right)$, then

$$\mu\frac{d}{d\mu}m_0^2 = 0 = 2\gamma_m m^2\left(1 + \frac{b_1}{\epsilon} + \cdots\right) + m^2\beta(\lambda,\epsilon)\left(\frac{b_1'}{\epsilon} + \cdots\right).$$

Equating terms of zeroth order in ϵ, we find (Recall $\beta \sim -\lambda\epsilon$)

$$\boxed{\gamma_m = \frac{1}{2}\lambda b_1'(\lambda)}$$

and, in the ϕ^4 theory we had

$$b_1 = \frac{\lambda}{16\pi^2} + \cdots$$, so that

$$\boxed{\gamma_m = \frac{\lambda}{32\pi^2} + \cdots}$$

Notice that the multiplicative mass renormalization is a special feature of dimensional regularization. In dimensional regularization, the two point function is a function of $p^2, m^2, \epsilon, \lambda$, and so has the form

$$A(\epsilon,\lambda)m^2 + B(\epsilon,\lambda)p^2$$

where A, B are dimensionless, and represent mass, wave-function renormalization respectively. But if we regularize by introducing an explicit cutoff Λ, the divergence

$$C(\lambda)\Lambda^2$$

can appear, which must be subtracted away

Dimensional regularization/Minimal subtraction is very convenient because it allows us to define a mass parameter $m(\mu)$ which depends on μ in a simple way.

Now we can check the naive assertion that the <u>massless</u> RG eqn correctly describes scaling behavior of Green's functions for $E \gg m$. The "running mass" is

$$m_\mu = m_{\mu_0} \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_m(\lambda_{\mu'}) \right]$$

and becomes unimportant in the ultraviolet if $m_\mu/\mu \to 0$ as $\mu \to \infty$. This is a condition on the anomalous dimension $\gamma_m(\lambda)$.

- In a fixed point theory, $\gamma_m \sim \gamma_m^* \Rightarrow$
$$m_\mu \sim m_{\mu_0}\left(\frac{\mu}{\mu_0}\right)^{\gamma_m^*} \Rightarrow \text{ we require } \gamma_m^* < 1$$

- In an asymptotically free theory. $\gamma_m \sim C(\ln\mu)^{-p}$, $p>0$
$$\int d\ln\mu' \frac{C}{(\ln\mu')^p} \sim \frac{C}{1-p}\frac{1}{(\ln\mu')^{p-1}} \qquad (p \neq 1)$$
$$\Rightarrow m_\mu \sim \exp\left[\frac{C}{1-p}(\ln\mu)^{1-p}\right]$$
and
$$\frac{m_\mu}{\mu} \sim \exp\left[\frac{C}{1-p}(\ln\mu)^{1-p} - \underbrace{(\ln\mu)}_{\text{dominates for large } \mu}\right] \longrightarrow 0$$

and, if $p=1$, we have
$$\frac{m_\mu}{\mu} \sim \mu^{-1}(\ln\mu)^C$$

So the leading asymptotic behavior of an asymptotically free theory for $E \gg m$ is the same as in a massless theory.

Scheme Dependence

We have described one mass-independent renormalization scheme. How would our results be changed in a different mass-independent scheme?

A mass-independent redefinition of the coupling takes the form

$$\lambda' = \lambda + A_1\lambda^2 + A_2\lambda^3 + \cdots$$

– the renormalized coupling in the approx is independent of scheme.

$$\beta'(\lambda') = \mu\frac{d}{d\mu}\lambda' = \beta(\lambda)\left(1 + 2A_1\lambda + 3A_2\lambda^2 + \cdots\right)$$

$$= \left(b_0\lambda^2 + b_1\lambda^3 + \cdots\right)\left(1 + 2A_1\lambda\right)$$

$$= b_0\lambda^2 + \left(b_1 + 2A_1b_0\right)\lambda^3 + \cdots$$

Now substitute: $\lambda = \lambda' - A_1\lambda'^2$

$$= b_0\lambda'^2 + \left(b_1 + 2A_1b_0 - 2A_1b_0\right)\lambda'^3 + \cdots$$

$$\beta'(\lambda') = b_0\lambda'^2 + b_1\lambda'^3 + \cdots$$

- The β function is scheme-independent through two-loop order (this result will apply to mass-dependent schemes only if we drop all $O(m/\mu)$ terms in the β function.)

Furthermore:

Exercise 1.6:

Show that a mass-independent redefinition of the coupling leaves invariant --

a) Fixed point: If $\beta(g^*) = 0$ then $\beta'(g'^*) = 0$

b) Slope at a fixed point: $\frac{d}{dg}\beta\Big|_{g^*} = \frac{d}{dg'}\beta'\Big|_{g'^*}$

c) γ at a fixed point: $\gamma(g^*) = \gamma'(g'^*)$

d) First term in expansion of γ: $\gamma(g) \sim cg^2 \Rightarrow \gamma'(g') = cg'^2$

[To show (c) and (d), assume that field renormalization in the new scheme is given by $Z'(g') = Z(g)[1 + O(g^2)]$]

F. The Renormalization Group and Critical Phenomena

References: K.G. Wilson and J.B. Kogut, Physics Rep. 12C, 77 (1974)
D.J. Amit, Field Theory, the Renormalization Group, and Critical Phenomena, McGraw-Hill (1978)

We developed the RG formalism (as did Gell-Mann and Low) as a tool for studying the asymptotic short-distance behavior of a quantum field theory. We now digress briefly to consider the application of the RG to asymptotic long-distance behavior.

The digression can be justified in various ways:

i) We will see that the RG parameters we have calculated in ϕ^4 theory have a genuine physical application. They lead to predictions about critical phenomena which can be, and have been, experimentally tested.

(i) Asymptotic long-distance behavior is a crucial concept if we consider constructing a continuum field theory as a limit of a cutoff theory. It therefore plays an important role in our thinking about nonperturbative quantum field theory

iii) The ideas underlying the notion of renormalization and the renormalization group are clarified by the theory of critical phenomena.

The connection between the calculation we have already done and critical phenomena is established via the equivalence of quantum field theory in D-dimensional spacetime and classical statistical mechanics in D-dimensional space

i.e. $$\mathscr{d}\phi\, e^{\int d^Dx\,\mathcal{L}} \xrightarrow[\text{rotation}]{\text{wick}} d\phi\, e^{-\int d^Dx\,\mathcal{H}}$$

$$\mathcal{L} = \tfrac{1}{2}(\partial_0\phi)^2 - \tfrac{1}{2}(\partial_i\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{\lambda}{24}\phi^4 \Rightarrow \mathcal{H} = \tfrac{1}{2}(\vec\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{\lambda}{24}\phi^4$$

So we can use the RG eqn for a $D=3$ quantum field theory to describe the long-distance behavior of a 3-dim classical statistical-mechanical system.

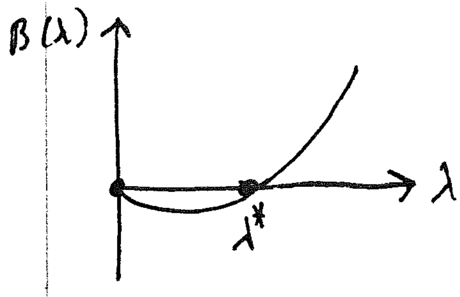
Actually, a 3-dimensional field theory describes the long-distance behavior of a 3-dim quantum-statistical system also, but I won't bother to show that here.

We wish to apply the integrated RG eqn to the Green's functions = correlation functions. We have

$$G_R^{(n)}(E, x, \lambda_\mu, m_\mu, \mu) = E^D\left[\frac{Z(E)}{Z(\mu)}\right]^{n/2} f(x, \lambda_E, \frac{m_E}{E})$$

where we may now identify $E = L^{-1}$ where L is the overall distance scale of the correlation function $G^{(n)}$.

The far infrared $E \to 0$ behavior of $G^{(n)}$ will be controlled by an infrared-stable fixed point of the RG, if there is one.



Suppose such a fixed point exists, at $\lambda = \lambda^*$ and the anomalous dimensions at the fixed point are

$$\gamma = \gamma^* \;;\quad \gamma_m = \gamma_m^*$$

then

$$m^2(\mu) \sim C(\mu^2)^{\gamma_m^*}$$
$$Z^{\frac{1}{2}}(\mu) \sim C' \mu^{\gamma^*}$$

and, as $E \to 0$, the correlation function scales as

$$G^{(n)} \sim E^{D+n\gamma^*} f[x, C(E^2)^{\gamma_m^*-1}]$$

$$\text{or} \quad \boxed{G^{(n)} \sim L^{-D} L^{-n\gamma^*} f(C L^{2(1-\gamma_m^*)})}$$

Masses become irrelevant at high energy, but are very relevant at large distance. Near the fixed point, the mass alone sets the scale of length

$$G^{(n)} \sim L^P f(m_L^2 L^2)$$

The coefficient C is in general a function of temperature, and might have a zero at $T = T_c$. Presumably it is analytic, so

$$C \sim T - T_c \quad \text{for } T \sim T_c$$

and we have

$$G^{(n)} \sim L^{-D} L^{-\frac{n}{2}\eta} f[(T-T_c) L^{\nu^{-1}}]$$

where the "critical exponents" are

$$\boxed{\eta = 2\gamma^* \qquad \nu^{-1} = 2(1-\gamma_m^*)}$$

Here η gives the anomalous scaling behavior of the two point function, and ν determines the behavior of the correlation length ξ as $T \to T_c$

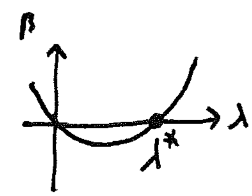
i.e. we have a function of $(T-T_c)^{\nu} L$, so the correlation length must behave like

$$\xi \sim (T - T_c)^{-\nu}$$

Now to calculate the critical exponents ν, η, we must find γ^*, γ_m^*. First we must find an infrared-stable fixed point in $D=3$ dimensional $\lambda\phi^4$ theory.

(ϕ is to be regarded as the scalar order parameter for a phase transition, e.g., a gas-liquid transition or a magnetic transition in a uniaxial magnet -- one without rotational symmetry)

Generally speaking, we cannot find a nontrivial fixed point ($\lambda^* \neq 0$) using perturbation theory, but we will use a trick, due to Wilson and Fisher: In $D = 4-\epsilon$ dimensions, $\epsilon \ll 1$, there is a nontrivial fixed point at $\lambda^* = O(\epsilon)$ which we can study in perturbation theory. We can expand γ^*, γ_m^* in powers of ϵ, and hope that the resulting series converges fairly rapidly for $\epsilon \to 1$



Previously, we calculated

$$\mu \frac{d}{d\mu}\lambda = \beta(\lambda) = -\epsilon\lambda + \frac{3\lambda^2}{16\pi^2} + \cdots$$

$$\mu \frac{d}{d\mu}\ln Z^{\frac{1}{2}} = \gamma(\lambda) = \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2} + \cdots$$

$$\mu \frac{d}{d\mu}\ln m = \gamma_m(\lambda) = \frac{\lambda}{32\pi^2} + \cdots$$

So the fixed point occurs for

$$\beta(\lambda^*) = 0 \Rightarrow \lambda^* = \frac{16\pi^2}{3}\epsilon + O(\epsilon^2)$$

and

$$\gamma^* = \gamma(\lambda^*) = \frac{1}{12}\left(\frac{\epsilon}{3}\right)^2 = \frac{\epsilon^2}{108} + O(\epsilon^3)$$

$$\gamma_m^* = \gamma_m(\lambda^*) = \epsilon/6 + O(\epsilon^2)$$

Thus

$$\boxed{\begin{aligned} \eta &= 2\gamma^* = \epsilon^2/54 + \cdots \\ \nu^{-1} &= 2 - 2\gamma_m^* = 2 - \epsilon/3 + \cdots \end{aligned}}$$

The correlation length ξ diverges like

$$\boxed{\begin{aligned} &\xi^{-1} \sim (T-T_c)^{-\nu} \\ &\nu = \frac{1}{2-\epsilon/3+\cdots} = \frac{1}{2}+\frac{\epsilon}{12}+\cdots \cong .58 \end{aligned}}$$

We have also found that the two-point correlation function at $T=T_c$ scales like

$$\boxed{\begin{aligned} &G^{(2)}(L) \sim L^{-D-\eta} \\ &\eta = \epsilon^2/54+\cdots \simeq .02 \end{aligned}}$$

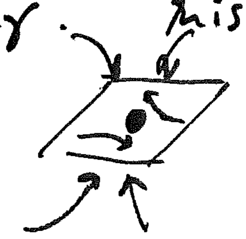
The calculated exponents ϵ have been evaluated to higher order in ϵ, and agree well with observation (for $\epsilon = 1$).

<u>Universality</u>

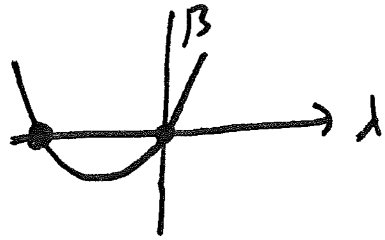
The critical behavior is determined by the fixed point; it does not matter how it is approached. We could also add higher dimension operators to the Lagrangian (e.g. higher derivatives corresponding to next-to-nearest neighbor interactions) the dimensionless couplings

$$\mu^p \lambda, \quad p>0$$

flow toward zero as $\mu \to 0$. (Anomalous dimensions are small, at least for ϵ small.) Thus, all 2nd-order phase transitions with a scalar order parameter should behave the same way. This is called <u>universality</u>.



For $\epsilon < 0$ $(D>4)$, the infrared stable fixed point moves to the origin. Here there are no anomalous dimensions, so



$$\eta = 0$$

and $G^{(n)} \propto f[(T-T_c)L^2]$

$$\Rightarrow \quad \nu = \frac{1}{2}$$

The classical (Landau) theory of 2nd-order phase transitions applies for $D>4$. Four is a critical dimension. (The IR-unstable fixed point can be reached only if m and λ vanish simultaneously, which is unlikely.)

The Continuum Limit of a Cutoff Field Theory

Now lets consider asymptotic infrared behavior from a different point of view. Suppose a theory is cutoff at a short distance scale a, and bare parameters are defined at this distance scale:

$$g(a), \quad a\, m(a).$$

Now, as we make a even smaller, we can redefine $g(a)$ and $m(a)$ so as to leave the physics unchanged at larger distances. The renormalization group functions β and γ_m tell us how to redefine $g(a)$ and $m(a)$. We define a continuum theory by scaling $g(a)$ and $m(a)$ appropriately as $a \to 0$.

The connection with critical phenomena is clear. In a second order phase transition, correlations grow arbitrarily large compared to the short-distance cutoff a;

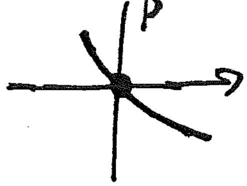
$$\xi >> a.$$

As we near the continuum limit of a cutoff field theory, we want physical distance scales m^{-1} and p^{-1} to become arbitrarily large compared to a:

$$\left.\begin{matrix} m^{-1} \\ p^{-1} \end{matrix}\right\} >> a$$

The continuum limit of a cutoff field theory <u>is</u> a second order phase transition.

We expect to be able to define such a limit if the theory has an ultraviolet-stable fixed point. For example, the gaussian fixed point in $\lambda\phi^4$ theory in $D<4$ dimensions is UV-stable. We can define a nontrivial continuum field theory by letting:

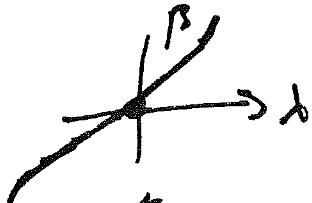


$$\left.\begin{matrix} a^{\epsilon}\lambda_0 = \lambda(a) \to 0 \\ am \to 0 \end{matrix}\right\} \text{ as } a \to 0$$

Why can we get away with tuning only these two parameters? Because there are only two operators which are <u>relevant</u> in the IR limit. An enlightening way of viewing the continuum limit is: Leave a fixed, and consider physics at distances asymptotically large compared to a. At asymptotically large distances, all the higher dimension operators have a negligible effect. If we want physical distance scales to get large, we <u>need</u> to make only λ and m small at distance scale a. No other parameters appear in the description of large distance physics.

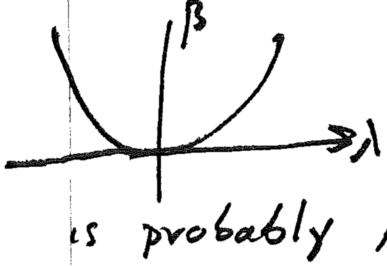
We see what is special about renormalizable theories. Only interactions of renormalizable type (with coupling dimension = mass to a nonnegative power) survive in the infrared limit. Only renormalizable theories occur as the continuum limits of cutoff theories.

If there is no UV-stable fixed point, there may be no sensible continuum limit. To keep large-distance physics unchanged,

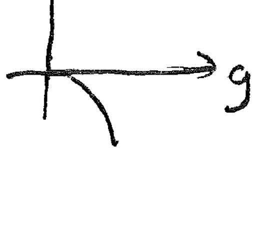


the bare coupling has to get stronger and stronger as we go to short distances; the theory may just cease to make sense at very short distances.

In ϕ^4 theory in $D>4$ dimensions, only the mass is relevant at large distances. The only continuum theory is free field theory.



For $D=4$, the $\lambda\phi^4$ operator is marginal. But because the theory is non-asymptotically free, the only sensible continuum theory is probably free field theory.



But QCD is AF in 4 dim, so it is likely that a continuum theory exists. The bare coupling vanishes. The UV divergences encountered order by order are summed up by the RG!

Renormalization from the RG viewpoint

The usual discussion of order-by-order renormalizability in perturbation theory is based on the superficial degrees of divergence of Feynman graphs

E.g. consider a graph in a scalar field theory with

L = # of loops
I = # of internal lines
E = # of external lines
n_i = # vertices type i
d_i : # derivatives in type i vertex
b_i : # legs in type i vertex

Superficial degree of divergence (in D dimensions) is

$$\Delta = DL - 2I + \sum_i n_i d_i$$

Now use the topological identity $L = I - (\sum_i n_i) + 1$, (for connected graphs)

$$\Delta = (D-2) I + \sum_i n_i (d_i - D) + D ,$$

and the identity $E + 2I = \sum n_i b_i$ (conservation of lines),

$$\boxed{\Delta = D - \left(\frac{D-2}{2}\right) E + \sum_i \left(d_i + \frac{D-2}{2} b_i - D\right) n_i}$$

Thus adding a vertex with

$$d_i + \frac{D-2}{2} b_i \leq D$$

does not increase the degree of divergence; these are the renormalizable vertices.

The reason the order-by-order proof of renormalizability is complicated is that we need to consider all the divergent subgraphs of a graph, and verify that e.g. all the divergent subgraphs of a graph which is not superficially divergent are cancelled by counterterms generated in lower orders. The proof thus involves combinatorics associated with the topology of diagrams; it is not completely obvious that the renormalization program has to work to all orders.

But the renormalization group concept makes it easier to understand why renormalization works (See Wilson and Kogut, op. cit., and J. Polchinski, Nuclear Physics B, to appear.) To appreciate this, let's formulate the renormalization problem in the following way:

We define a cutoff field theory by truncating all loop integrations at $k^2 = \Lambda_0^2$, and specifying the bare parameters of the cutoff theory, e.g. m_0^2 and λ_0. Now we allow the cutoff to "float"; i.e. we integrate out all modes with $\Lambda^2 < p^2 < \Lambda_0^2$, and thus generate a theory with cutoff Λ which has all the same physical consequences as the original theory.

This is a type of renormalization group transformation and is expected to generate a very complicated Lagrangian $\mathcal{L}_\Lambda$, with an infinite number of couplings. But we say that the theory is renormalizable if all the parameters in $\mathcal{L}_\Lambda$ can be expressed as functions of a finite number of renormalized parameters (e.g. λ_R and m_R^2) up to corrections which vanish like a power of (Λ/Λ_0).

E.g. if the Lagrangian of the theory with cutoff Λ_0 is

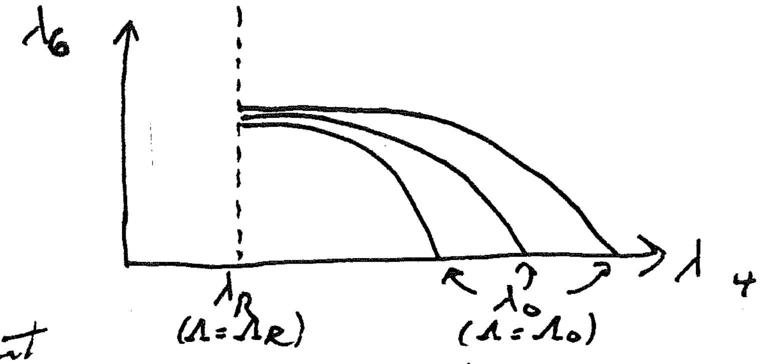
$$\mathcal{L}_{\Lambda_0} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4$$

and the Lagrangian with cutoff Λ_R is (after rescaling the field)

$$\mathcal{L}_{\Lambda_R} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_R^2\phi^2 - \frac{1}{4!}\lambda_R\phi^4 - \frac{1}{6!}\frac{\lambda_6}{\Lambda_R^2}\phi^6 - \cdots$$

then we say the theory is renormalizable if λ_6 and all the other coefficients of higher dimension operators can be expressed as function of m_R^2 and λ_R in the limit $\Lambda_0/\Lambda_R \to \infty$

To keep things simple, consider the projections of renormalization group trajectories onto the λ_4-λ_6 plane. (λ_4 is the coefficient of $\phi^4/4!$) We fix $\lambda_4 = \lambda_R$ at $\Lambda = \Lambda_R$ and consider trajectories with different values of Λ_0 which arrive at $\lambda_4 = \lambda_R$ at $\Lambda = \Lambda_R$; we then generate a function



$$\lambda_6 = \lambda_6(\lambda_R, \Lambda_0/\Lambda_R)$$

Renormalizability in this context means that, as $\Lambda_0 \to \infty$ with Λ_R fixed, $\lambda_6(\Lambda_R)$ converges to a fixed value. In this way, λ_6 becomes a function of λ_R.

The general statement of renormalizability is: The renormalization group trajectories converge to a finite dimensional surface as $\Lambda_R/\Lambda_0 \to 0$. Then only a finite number of renormalized parameters are needed to specify the renormalized theory.

Why should we expect this statement to hold true? Because (naively) all the higher dimension operators should become irrelevant in the infrared, and therefore, that the difference between the values of λ_6 in two trajectories will be suppressed by $(\Lambda_R/\Lambda_0)^2$

Thus, there are two approaches to renormalizability
i) Justify naive power counting for Feynman Graphs
ii) Justify naive notion of irrelevant operator.
And (ii) is really easier to carry out.

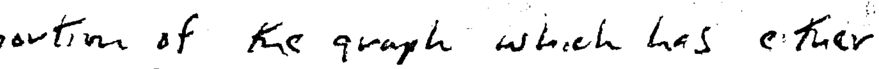
Addendum to 1F: Physics 230

Renormalizability and the Renormalization Group

Continuing the discussion of p. 1.41, let's sketch the proof that $\lambda\phi^4$ theory is renormalizable to all orders of perturbation theory.

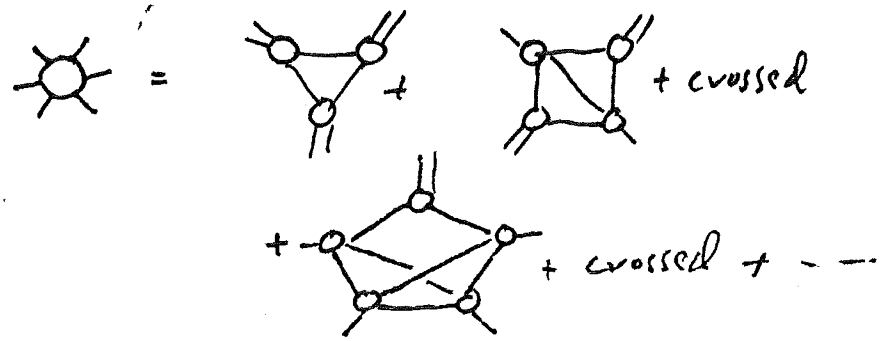
We have already learned from simple power counting that only graphs with 0, 2, or 4 external lines have a nonnegative degree of divergence. Graphs with more than 4 external lines can nontheless be divergent if they contain divergent subgraphs. But it is relatively easy to isolate these divergent parts and associate a skeleton graph with any given graph.

Here is the procedure. Given any connected, one-particle-irreducible graph with more than 4 external lines, draw a box around each connected portion of the graph which has either 2 or 4 lines coming out. A topological theorem (not proved here) says that the boxes will not overlap unless they are completely contained inside an even bigger box. (You will find this statement to be pretty obvious after you have considered a few examples.) By replacing all boxes not contained inside other boxes by or , we associate (unambiguously) with every graph a skeleton graph, in which represents the exact one-particle irreducible 4-point function and internal lines represent the exact propagator:



For example, the one-particle-irreducible 6-point function $\Gamma^{(6)}$ can be expanded ....

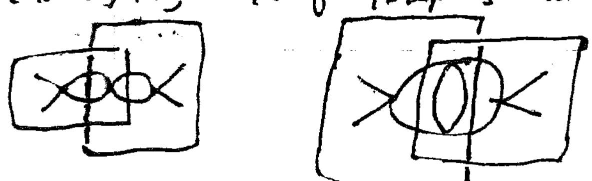
$\Gamma^{(6)} =$



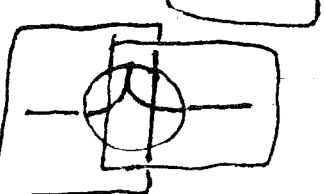
(Any box drawn containing 2 or more vertices of the skeleton graph has 6 or more lines coming out.) Since, to any finite order of perturbation theory, the exact vertex and propagator have an asymptotic momentum dependence differing only by logarithms from the momentum dependence of the lowest order vertex and propagator, all subgraphs of the skeleton expansion converge, by power counting and $\Gamma^{(6)}$ is finite, by Weinberg's theorem (assuming that $\Gamma^{(4)}$ and $\Gamma^{(2)}$ are finite). The same applies to all $\Gamma^{(n)}$, $n > 4$. So the skeleton expansion, and Weinberg's theorem, tell us that all Green's functions are finite in perturbation theory, if we can consistently renormalize $\Gamma^{(4)}$ and $\Gamma^{(2)}$ - i.e., perform coupling, field and mass renormalization.

But renormalizing $\Gamma^{(4)}$ and $\Gamma^{(2)}$ is less trivial; they do not have a skeleton expansion, because the boxes overlap (the "overlapping divergence" problem). For example, loop graphs contributing to $\Gamma^{(4)}$ and $\Gamma^{(2)}$ are ..

$\Gamma^{(4)} \sim$



$\Gamma^{(2)} \sim$



Since the boxes overlap, we cannot necessarily ensure finiteness order by order by working outward from the innermost box.

In the language of dimensional regularization, what we have to fear is a divergence of the form

$$\frac{1}{\epsilon} \ln p^2/\mu^2,$$

which cannot be removed by a local counterterm. In principle, such a divergence could arise in e.g.

 where  represents a counterterm.

The renormalization program fails unless such terms, which can appear in individual graphs, cancel in each order of perturbation theory ("cancellation of overlapping divergences"). Fortunately, the renormalization group formalism is sufficiently powerful to ensure the cancellation of overlapping divergences order-by-order. (See lectures by Callan in Les Houches '75.)

The main point is that a renormalization group eqn. is satisfied by e.g. $\Gamma_R^{(4)}$, as a consequence of the arbitrariness of μ, even for $\epsilon \neq 0$, and even if we have not made the proper subtractions to make $\Gamma_R(\lambda, m, \mu)$ finite. Suppose that we define

$$\lambda_0 = \mu^\epsilon \left(\lambda + \frac{a_1(\lambda)}{\epsilon} + \cdots\right)$$

$$Z = 1 + \frac{c_1(\lambda)}{\epsilon} + \cdots$$

to a given order in λ

Then, even if $\Gamma_R^{(n)} = Z^{-n/2}\Gamma_0^{(n)}$ is not ϵ-independent as $\epsilon \to 0$, it obeys

$$\mu \frac{\partial}{\partial\mu} \Gamma_R^{(n)} = -\left[\beta \frac{\partial}{\partial\lambda} + n\gamma\right]\Gamma_R^{(n)}$$

for $\epsilon \neq 0$. Furthermore β, γ are finite, just as we argued previously, if λ_0 is μ-independent:

$$\beta = -\epsilon\lambda + \beta_2\lambda^2 + \cdots$$

$$\gamma = \gamma_2\lambda^2 + \cdots$$

(We have seen that the μ-independence of λ_0 and $\Gamma^{(n)}_{Bare}$, and hence the finiteness of β, γ, can hold only if the residues of the higher order poles in ϵ in the expressions for λ_0 and Z obey certain recursion relations, but we can choose λ_0 and Z to satisfy these relations, whether or not they render "Γ_R" finite.)

Now, to complete the order-by-order proof of renormalizability, suppose that $\Gamma_R^{(4)}$ is finite (not singular for $\epsilon \to 0$) to order λ^r. The $O(\lambda^{r+1})$ term of the RG eqn is

$$\left[\mu \frac{\partial}{\partial \mu} \Gamma_R^{(4)}\right]_{r+1} = \left[-\left(\beta \frac{\partial}{\partial \lambda} + 4\gamma\right) \Gamma_R^{(4)}\right]_{r+1}$$

(where $[\ ]_{r+1}$ denotes coefficient of λ^{r+1})

$$= \text{finite} + \epsilon(r+1)\left[\Gamma_R^{(4)}\right]_{r+1},$$

since $\Gamma_R^{(4)}$ is finite to order λ^r by hypothesis. So, if we can show that, with β, γ finite to order λ^{r+1}, λ^r respectively and $\Gamma_R^{(4)}$ finite to order λ^r, the only possible divergence in $[\Gamma_R^{(4)}]_{r+1}$ is a <u>simple pole</u> in ϵ, then the RHS is finite, and we conclude that the residue of the pole is μ-independent, a constant which can be removed by an $O(\lambda^{r+1})$ counterterm. But this is more-or-less obvious, because there is only one potentially divergent loop integration, which can only generate a simple pole.

A similar argument, applied to $\Gamma^{(2)}$, completes the proof.

How is the RG eqn. so smart as to know that there can be no uncancelled overlapping divergences? The RG only knows one thing, but it is a big thing; it knows how leading logs in each order are determined by logs appearing in lower order. If a $\frac{1}{\epsilon}\ln\mu$ term really occurred, it would generate a $\ln^2\mu$ term unanticipated in lower order, which the RG eqn forbids.

G. Composite Operators

Before leaving ϕ^4 theory, let's consider how composite operators are renormalized and calculate the anomolous dimension of some composite operators.

Coupling and field renormalization do not suffice to remove all divergences (poles in ϵ) in Green's functions involving composite operators. We must make additional subtractions to define these Green's functions

For example, consider

$\Gamma_\theta^{(n)}$ – the amputated n-point function with one insertion of the operator θ

We renormalize $\Gamma^{(n)}$ (without the insertion) by

$$\Gamma^{(n)}(p_1 \cdots p_n, \lambda, \mu) = Z^{n/2}\, \Gamma^{(n)}_{BARE}(\cdots, \lambda_0)$$

For $\Gamma_\theta^{(n)}$, we need an additional multiplicative renormalization constant Z_θ

$$\Gamma_\theta^{(n)} = Z^{n/2} Z_\theta \left(\Gamma_\theta^{(n)}\right)_{BARE}$$

The anomalous dimension of θ is defined as

$$\gamma_\theta = Z_\theta^{-1}\, \mu \frac{d}{d\mu} Z_\theta$$

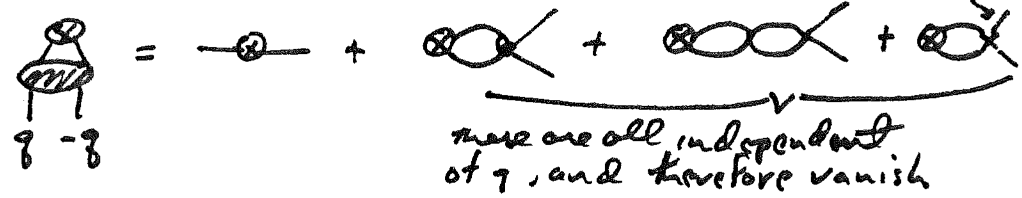
<u>Example:</u>

Consider the operator $\frac{1}{2}(i)^n \phi\, \partial^{\mu_1} \cdots \partial^{\mu_n} \phi = \theta_n$ – traces can invariant tensor – n even)

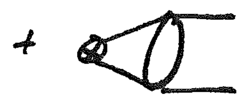
and compute $\Gamma^{(2)}_{\theta_n}$ to $O(\lambda^2)$

in the massless theory, $m^2 = 0$.

counterterm

We have 

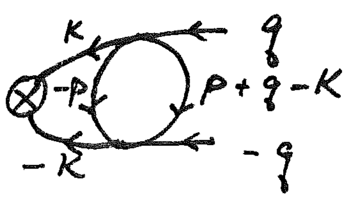
these are all independent of q, and therefore vanish



In zeroth order, we have

 $= q^{\mu_1} \cdots q^{\mu_n} - \text{traces}$

The $O(\lambda^2)$ correction is (in the massless theory)



$$= \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{(k^{\mu_1} \cdots k^{\mu_n} - \text{traces})(i)^2}{(k^2 + i\epsilon)^2}$$

$$\times \lambda^2 \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + i\epsilon)[(p+q-k)^2 + i\epsilon]}$$

We did the p integral back on p (1.23)

$$= \int \frac{d^D k}{(2\pi)^D} (-1) \frac{(k^{\mu_1} \cdots k^{\mu_n} - \text{traces})}{(k^2 + i\epsilon)^2} \frac{i\lambda^2 \mu^{2\epsilon}}{32\pi^2} (4\pi)^{\epsilon/2} \Gamma(\epsilon/2) \int_0^1 dx \left[-x(1-x)(k-q)^2\right]^{-\epsilon/2}$$

To do the k integral, we apply the identity

$$\boxed{\frac{1}{a^p b^q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 dx \frac{(1-x)^{p-1} x^{q-1}}{[(1-x)a + xb]^{p+q}}}$$

obtaining

$$\text{Graph} = \frac{-i\lambda^2 \mu^{2\epsilon}}{32\pi^2} (4\pi)^{\epsilon/2} \Gamma(\epsilon/2) \int_0^1 dx \left[-x(1-x)\right]^{-\epsilon/2} \frac{\Gamma(2+\epsilon/2)}{\Gamma(2)\Gamma(\epsilon/2)}$$

$$\times \int_0^1 dy\, (1-y) y^{\epsilon/2 - 1} \int \frac{d^D k}{(2\pi)^D} \frac{k^{\mu_1} \cdots k^{\mu_n} - \text{traces}}{\left[(k-yq)^2 + y(1-y)q^2 + i\epsilon\right]^{2+\epsilon/2}}$$

when we shift the integral, only the $q^{\mu_1} \cdots q^{\mu_n}$ term in the integrand survives the symmetric integration

$$\text{Graph} = \frac{\lambda^2 \mu^{2\epsilon}}{32\pi^2} (4\pi)^{\epsilon/2} \frac{1}{16\pi^2} \frac{(4\pi)^{\epsilon/2}}{\Gamma(D/2)} (q^{\mu_1} \cdots q^{\mu_n} - \text{traces})$$

$$\Gamma(2+\tfrac{\epsilon}{2}) \int_0^1 dx \, [x(1-x)]^{-\epsilon/2} \int_0^1 dy \, (1-y) \, y^{\epsilon/2 - 1 + n} \underbrace{\int dk^2 (k^2)^{D/2-1} [k^2 - y(1-y) q^2]^{-2-\epsilon/2}}_{\frac{\Gamma(D/2)}{\Gamma(2+\epsilon/2)} \Gamma(\epsilon) \{[-y(1-y)] q^2\}^{-\epsilon}}$$

Using $\int_0^1 dy (1-y) y^{n-1} = \frac{\Gamma(n)}{\Gamma(n+2)} = \frac{1}{n(n+1)}$, we find

$$\boxed{\text{Pole Term} = \frac{1}{2n(n-1)\epsilon} \left(\frac{\lambda}{16\pi^2}\right)^2 (q^{\mu_1} \cdots q^{\mu_n} - \text{traces})}$$

We remove the pole by multiplying by

$$Z Z_{\theta_n} \text{ where } Z = 1 - \frac{1}{12\epsilon} \left(\frac{\lambda}{16\pi^2}\right)^2$$

Thus

$$Z_{\theta_n} = 1 + \frac{1}{12\epsilon} \left(\frac{\lambda}{16\pi^2}\right)^2 \left(1 - \frac{6}{n(n+1)}\right) + \cdots$$

and $\gamma_{\theta_n} = Z_{\theta_n}^{-1} \mu \frac{d}{d\mu} Z_{\theta_n} \Rightarrow Z_{\theta_n} \gamma(\lambda) = -\beta(\lambda, \epsilon) \frac{d}{d\lambda} Z$

If $Z_{\theta_n}^{-1} = 1 + \sum \frac{d_n(\lambda)}{\epsilon^n}$, then $\gamma_{\theta_n} = -\lambda d_1'(\lambda)$.

We have $$\boxed{\gamma_{\theta_n} = -\frac{1}{6} \left(\frac{\lambda}{16\pi^2}\right)^2 \left(1 - \frac{6}{n(n+1)}\right) + \cdots}$$

Now, of course, the way a Green's function with an insertion of Θ scales is determined by an integrated RG eqn. Since

$$\Gamma^{(n)}_{\Theta,R} = Z^{n/2} Z_\Theta \, \Gamma^{(n)}_{\Theta,\mathrm{BARE}}$$

and $\Gamma^{(n)}_{\Theta,\mathrm{BARE}}$ is μ independent, we find, by exactly the same reasoning as before, that

$$\boxed{\Gamma^{(n)}_{\Theta,R}(E, x, g_\mu, \mu) = E^d \left[\frac{Z_\Theta(E)}{Z_\Theta(\mu)}\right]^{-1} \left[\frac{Z(E)^{\frac{1}{2}}}{Z(\mu)^{\frac{1}{2}}}\right]^{-n} f(x, g_E)}$$

where $Z_\Theta(\mu) = \exp\left[\int^\mu \frac{d\mu'}{\mu'} \gamma_\Theta\right]$

Operator Mixing

The renormalization of composite operators is complicated by the tendency of operators to mix under renormalization. In general one has

$$\Gamma^{(n)}_{\Theta_i,R} = Z^{n/2} (Z\;)_{ij} \, \Gamma^{(n)}_{\Theta_j,\mathrm{BARE}} .$$

Green's functions involving insertions of the operator set $\{\Theta_i\}$ are renormalized by multiplication by a matrix.

The matrix $\underset{\sim}{Z}$ is not symmetric. In fact $Z_{ij} = 0$ if $\dim \Theta_i < \dim \Theta_j$

(In minimal subtraction in a massless theory, only operators of the same dimension can mix.)

In some cases, mixing is forbidden

by symmetries. We didn't have to worry about mixing when we renormalized $\phi \partial^{\mu_1} \cdots \partial^{\mu_n} \phi$ - traces, because there were no operators of the same (or lower) dimension with the same quantum nos (except for total derivatives).

To derive the integrated R-G eqn in this more general case, we note

$$\Gamma^{(n)}_{\theta_i, BARE} = Z^{-n/2}_{(\mu)} Z^{-1}_{ij}(\mu) \Gamma^{(n)}_{\theta_j, R} = \mu\text{-independent}$$

$$= Z^{-n/2}(E) Z^{-1}_{ik}(E) E^d f_K(x, g_E)$$

or $$\boxed{\Gamma^{(n)}_{\theta_j, R}(E, x, g_\mu, \mu) = Z_{j\ell}(\mu) Z^{-1}_{\ell K}(E) \left[\frac{Z(E)}{Z(\mu)}\right]^{-n/2} E^d f_K(x, g_E)}$$

we define

$$\boxed{\gamma_{ij} = \left(\mu \frac{d}{d\mu} Z_{iK}\right) Z^{-1}_{Kj}}$$

so we can reconstruct Z by solving the matrix differential eqn:

$$\mu \frac{d}{d\mu} \underset{\sim}{Z} = \underset{\sim}{\gamma} \underset{\sim}{Z}$$

choice of operator basis.

We can solve this by diagonalizing $\underset{\sim}{\gamma}$, but typically we can diagonalize $\gamma(\lambda_\mu)$ only at one value of μ. Exceptional cases are ...

i) At a fixed point $\underset{\sim}{\gamma}(\lambda_\mu) = \underset{\sim}{\gamma}^*$

ii) All entries in γ_{ij} are of the same order in λ_μ

In general, the soln is

$$\underset{\sim}{Z}(\mu) \underset{\sim}{Z}^{-1}(\mu_0) = T\left[\exp \int_0^{\ln \mu/\mu_0} dt\, \underset{\sim}{\gamma}(\lambda_t)\right] \qquad t = \ln\frac{\mu'}{\mu_0}$$

(T meaning largest t at the left.

In general, operators can mix in minimal subtraction if they differ in dimension by an amount of order ϵ; thus, factors of μ^ϵ can appear in off-diagonal entries of $\underset{\sim}{Z}$; i.e. Z can have explicit μ dependence

$$\underset{\sim}{Z} = \underset{\sim}{\mathbb{1}} + \frac{1}{\epsilon}\underset{\sim}{D}_1(\lambda, \mu^\epsilon)$$

and $\underset{\sim}{\gamma}\underset{\sim}{Z} = \mu\frac{d}{d\mu}\underset{\sim}{Z} = \beta(\lambda,\epsilon)\frac{\partial}{\partial\lambda}\underset{\sim}{Z} + \mu\frac{\partial}{\partial\mu}\underset{\sim}{Z}$

Match terms of order 1:

$$\boxed{\underset{\sim}{\gamma} = -\lambda\frac{\partial}{\partial\lambda}\underset{\sim}{D}_1 + \frac{\partial}{\partial\mu^\epsilon}\underset{\sim}{D}_1}$$

Example

To understand how operator mixing arises, let us consider as an example the dimension 4 scalar operators in ϕ^4 theory. They are

$$\mathcal{O}_1 = \frac{1}{4!}\phi^4 \qquad \dim = 2D-4 = D-\epsilon$$
$$\mathcal{O}_2 = -\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi \qquad \dim = D$$
$$\mathcal{O}_3 = -\tfrac{1}{2}\phi\,\partial^2\phi \qquad \dim = D$$

An insertion of operator $\mathcal{O}$ in a graph changes the degree of divergence of the graph by

$$\Delta = (\dim\mathcal{O} - 4) + 4 - n$$

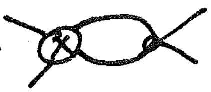
Since these operators are dim 4, we will need to make subtractions to define

$$\Gamma_{\mathcal{O}}^{(0)},\ \Gamma_{\mathcal{O}}^{(2)},\ \Gamma_{\mathcal{O}}^{(4)}$$

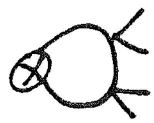
We will work in the $\lambda\phi^4$ theory with vanishing renormalized mass, so $\Gamma_{\mathcal{O}}^{(0)}$, which depends on no external momenta and has dimension 4, must be zero. We need worry only about $\Gamma_{\mathcal{O}}^{(2)}$ and $\Gamma_{\mathcal{O}}^{(4)}$

Let's calculate γ_{ij} to $O(\lambda)$.

First, consider $\Gamma^{(4)}_{\mathcal{O}}$

$\Gamma^{(4)}_{\mathcal{O}_1} =$  $+$  $+$ crossings $+ O(\lambda^2)$

But the divergent contributions to $\Gamma^{(4)}_{\mathcal{O}_{2,3}}$,

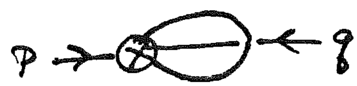
 , etc., are $O(\lambda^2)$ and may be neglected

Thus, in $\Gamma^{(4)}_{\mathcal{O}}$ to order λ, we find only a multiplicative renormalization of $\mathcal{O}_1$

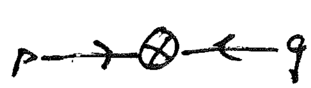
$$\Gamma^{(4)}_{\mathcal{O}_1} = \lambda \frac{A}{\epsilon} + \text{finite} + O(\lambda^2)$$

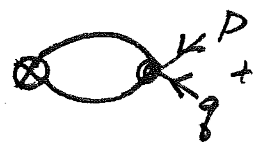
where A is a constant to be computed

Next, consider $\Gamma^{(2)}_{\mathcal{O}}$

$\Gamma^{(2)}_{\mathcal{O}_1} =$ 

$+$ crossed $+\ O(\lambda^2)$

$\Gamma^{(2)}_{\mathcal{O}_{2,3}} =$ 

$+$ 

$+ \quad O(\lambda^2)$

These graphs are all quadratic in momentum, and it is obvious that

$$\Gamma^{(2)}_{\mathcal{O}_1} = \lambda \frac{B\mu^{-\epsilon}}{\epsilon} \tfrac{1}{2}(p^2+q^2) + \text{finite} + O(\lambda^2)$$

$$\Gamma^{(2)}_{\mathcal{O}_2} = \lambda \frac{C}{\epsilon} \tfrac{1}{2}(p+q)^2 + \text{finite} + O(\lambda^2)$$

$$\Gamma^{(2)}_{\mathcal{O}_3} = \lambda \frac{D}{\epsilon} \tfrac{1}{2}(p+q)^2 + \text{finite} + O(\lambda^2)$$

Since, in zeroth order, we have

$$q \rightarrow \underset{\mathcal{O}_2}{\otimes} \leftarrow p \quad = \quad p\cdot q \,,$$

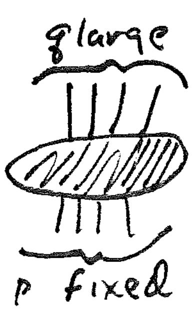
$$q \rightarrow \underset{\mathcal{O}_3}{\otimes} \leftarrow p \quad = \quad \tfrac{1}{2}(p^2+q^2) \,,$$

we see that  cause $\mathcal{O}_1$ to mix with $\mathcal{O}_3$ and  causes $\mathcal{O}_{2,3}$ to mix with each other.

Exercise 1.7*

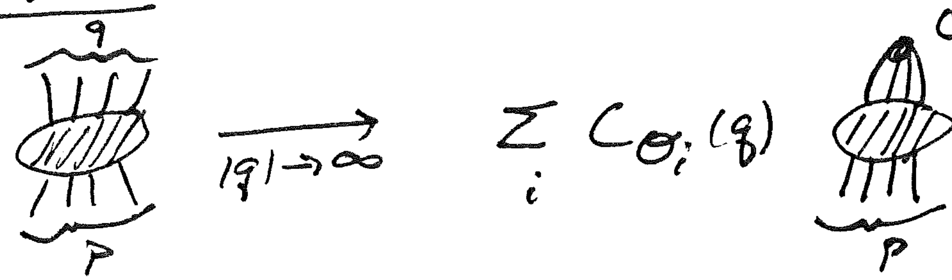
Calculate A, B, C, D, and find Z_{ij} and γ_{ij} to $O(\lambda)$.

H. The Operator Product Expansion



As we have formulated it so far, the renormalization group can be used only to study the scaling properties of Green's functions when all external momenta are scaled by a common factor. Frequently we are interested in the case in which some external momenta get large while the others are held fixed. Can we "factor out" the piece depending on large momenta, and apply the renormalization group to *it*?

Yes, we can! The key is the operator product expansion (OPE):



It says that, asymptotically as $|q|\to\infty$, the Green's function approaches a sum over universal coefficient functions $C_{\mathcal{O}_i}(q)$ times Green's function (with an insertion of the local operator $\mathcal{O}_i$) which are independent of q.

In position space, we write, for operators A and B

$$A(x)\,B(0) \xrightarrow[x\to 0]{} \sum_i C^{AB}_{\mathcal{O}_i}(x)\; \mathcal{O}_i(0),$$

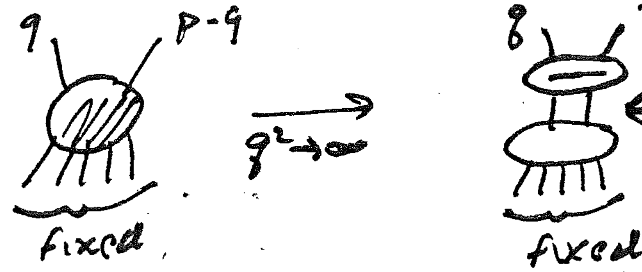
where it is to be understood that this expansion is asymptotic, when inserted inside any Green's function or matrix element. The statement that $C^{AB}_{\mathcal{O}_i}$ is universal means that, while it depends on A, B, and $\mathcal{O}_i$, it does not depend on the Green's function into which $A(x)B(0)$ is inserted.

The basic idea underlying the OPE is that, as $x\to 0$, $A(x)B(0)$ is indistinguishable from a local object, and the local operators give a complete description of all local physics. It can be proved (in momentum space) to arbitrary order in perturbation theory. The proof is difficult, but it is easy to sketch the basic idea.

For example, consider a n+2-point Green's function in $\lambda\phi^4$ theory:

$$G^{(n+2)}_{\text{connected}} = \langle 0 | T\, \tilde{\phi}(q)\, \tilde{\phi}(p-q)\, \tilde{\phi} - \tilde{\phi} | 0 \rangle$$

A naive guess about the $q^2 \to \infty$ asymptotic behavior of this Green's function is the power of q^2 dictated by dimensional analysis (up to logs.) But this guess is wrong, because the large momentum q need not flow through every propagator of the graphs contributing to $p^{(n+2)}$. The leading asymptotic behavior is actually determined by subgraphs which are connected to the rest of the graph by a <u>minimal bridge</u>



no large momentum flows across bridge

( means two particle irreducible in ---- channel)

Now  scales as determined by dimensional analysis. (In this case $\sim q^0 (\log q^2)^P$)

and as $q^2 \to 0$ its leading behavior is independent of the other momentum on its legs. It becomes a coefficient function $C(q^2)$

the rest of the diagram now looks like a contribution to a Green's function $G^{(n)}_{\phi^2}$ with one ϕ^2 insertion.

The subtle part is to show that this picture survives when the $G^{(n)}_{\Theta}$'s are properly renormalized

(Higher dimension operators are generated by non-minimal bridges, or expanding in $1/q^2$)

Proofs of the OPE in perturbation theory can be found in...

W. Zimmerman, Brandeis Summer School 1970, ed. S. Deser et al.
C. Callan, Phys Rev. <u>D5</u>, 3202 (1972);
" , Les Houches Summer School 1975, ed. R. Balian + J. Zinn-Justin

<u>Calculation of Coefficient Functions</u>

If we accept the OPE as correct, then we can calculate coefficient functions in perturbation theory, by taking matrix elements of both sides of—

$$-i\int d^4x\, e^{iq\cdot x} \quad T(A(x)B(0)) = \sum_i C^{AB}_{\mathcal{O}_i}(q)\, \mathcal{O}_i$$

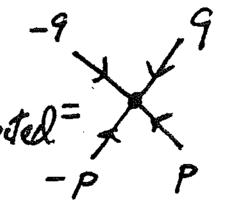
E.g. consider

$$(-i)\int d^4x\, e^{iq\cdot x}\, T(\phi(x)\phi(0)) = C_1(q)\,\mathbb{1} + C_2(q)\,\tfrac{1}{2}\phi^2 + \cdots$$

If all the operators (except $\mathbb{1}$) in the expansion are defined to have vanishing vacuum expectation value (vev) we can isolate the coefficient of $\mathbb{1}$

$$C_1(q) = \langle 0|LHS|0\rangle = \frac{1}{q^2}\left[1 + O(\lambda)\right]$$

We can calculate C_2 by taking the forward matrix element between one-particle states

$\langle p|LHS|p\rangle_{\text{connected}} =$ 

$= (-i)\left(\frac{i}{q^2}\right)^2(-i\lambda) = C_2(q)$

i.e. $\quad C_2(q) = \dfrac{\lambda}{q^4}\left[1 + O(\lambda)\right]$

(We're considering the case of vanishing renormalized mass.)

Exercise 1.8:

Find the coefficient of $\frac{1}{4!}\phi^4$ in this expansion, to order λ^2.

Next, lets calculate the coefficient of $\frac{1}{2}(i)^n \phi \partial^{\mu_1} \cdots \partial^{\mu_n}\phi$ - traces.

Actually, we cannot determine the coefficient functions until we specify an operator basis; there are $n/2 + 1$ operators with spin n and dimension $n+2$ (if n is even). It is convenient to use the basis

$$\mathcal{O}_0^n = \tfrac{1}{2}(i)^n \phi\, \partial_+^{\mu_1} \cdots \partial_+^{\mu_n} \phi - \text{traces}$$

$$\mathcal{O}_2 = \tfrac{1}{2}(i)^n S\, \phi\, \partial_-^{\mu_1}\partial_-^{\mu_2}\partial_+^{\mu_3} \cdots \partial_+^{\mu_n}\phi - \text{traces}$$

(where S denotes symmetrization)

$$\vdots$$

$$\mathcal{O}_n^n = \tfrac{1}{2}(i)^n \phi\, \partial_-^{\mu_1} \cdots \partial_-^{\mu_n}\phi - \text{traces},$$

where $$A\partial_\pm^\mu B = A\partial^\mu B \pm (\partial^\mu A)B$$

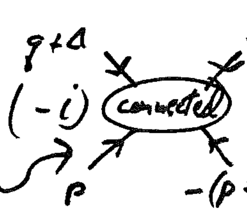
The matrix elements of these operators in $O(\lambda^0)$ are

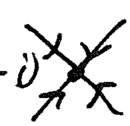
$$p_1 \rightarrow \underset{\mathcal{O}_0^n}{\otimes} \leftarrow p_2 = (p_1+p_2)^{\mu_1} \cdots (p_1+p_2)^{\mu_n} - \text{traces}$$

$$p_1 \rightarrow \underset{\mathcal{O}_2^n}{\otimes} \leftarrow p_2 = S(p_1-p_2)^{\mu_1}(p_1-p_2)^{\mu_2}(p_1+p_2)^{\mu_3} \cdots (p_1+p_2)^{\mu_n} - \text{traces}$$

etc.

We can calculate the coefficients of these operators to $O(\lambda)$ by considering

$(-i)$ 

$= (-i)$  $+ \cdots = (-i)(-i\lambda)\,\dfrac{i}{q^2}\,\dfrac{i}{(q+\Delta)^2}$

w/ these legs amputated

Since $\Delta = -(p_1+p_2)$, we can extract the coefficients of the $\mathcal{O}_j^n$ by ignoring Δ^2 compared to q^2 and $q\cdot\Delta$, and expanding

$(-i)$  $\simeq \frac{\lambda}{q^4}\left(1+\frac{2q\cdot\Delta}{q^2}\right)^{-1} = \frac{\lambda}{q^2}\sum_{n=0}^{\infty}\left(\frac{-2q\cdot\Delta}{q^2}\right)^n.$

This depends on $-\Delta = p_1+p_2$ but not p_1-p_2, so only $\mathcal{O}_0^n$ has a nonzero coefficient. We have

$$C_0^n \langle \mathcal{O}_0^n \rangle = \frac{\lambda}{q^4}\, 2^n \frac{q_{\mu_1}\cdots q_{\mu_n}}{(q^2)^n}(-1)^n \Delta^{\mu_1}\cdots\Delta^{\mu_n};$$

or $$C_0^n = 2^n \lambda \frac{q_{\mu_1}\cdots q_{\mu_n}}{(q^2)^{n+2}}.$$

<u>Exercise 1.9:</u>

Find the coefficients of the operators $\mathcal{O}_j^n$ to order λ^2 in $\frac{\lambda}{3!}\phi^3$ theory. (Or use a more appropriate operator basis, if you wish)

Renormalization Group Eqn for Coefficient Function

In higher order in λ, we must renormalize the operator matrix elements, and the coefficient functions acquire an implicit μ dependence; we have

$$\int d^4x\, T A(x) B(0) e^{iqx} = \sum_i C^{AB}_{\mathcal{O}_i}(q, g_\mu, \mu)\, \mathcal{O}_i(\mu)$$

And a Green's function with an insertion of AB obeys

$$\Gamma^{(n)}_{AB}(q, p, x, g_\mu, \mu) \xrightarrow[q\to\infty]{} \sum_i C^{AB}_{\mathcal{O}_i}(q, g_\mu, \mu)\, \Gamma^{(n)}_{\mathcal{O}_i}(p, x, g_\mu, \mu)$$

q: large momentum scale; p: small momentum scale

We can invoke the OPE and then apply the (integrated) RG eqn, or we can do it the other way around. We'll ignore operator mixing, for simplicity.

$$\Gamma^{(n)}_{AB}(\mu) = \frac{Z_A(\mu)}{Z_A(\mu')}\frac{Z_B(\mu)}{Z_B(\mu')}\left[\frac{Z(\mu)}{Z(\mu')}\right]^{n/2} \underbrace{\Gamma^{(n)}_{AB}(\mu')}_{\simeq \sum_i C^{AB}_{\theta_i}(\mu')\,\Gamma^{(n)}_{\theta_i}(\mu')}$$

and also

$$\Gamma^{(n)}_{AB}(\mu) \simeq \sum_i C^{AB}_{\theta_i}(\mu)\,\Gamma^{(n)}_{\theta_i}(\mu)$$

$$= \sum_i C^{AB}_{\theta_i}(\mu)\,\frac{Z_{\theta_i}(\mu)}{Z_{\theta_i}(\mu')}\left[\frac{Z(\mu)}{Z(\mu')}\right]^{n/2}\Gamma^{(n)}_{\theta_i}(\mu')$$

and this holds for all p, so we must have

$$\boxed{C^{AB}_{\theta_i}(\mu) = \frac{Z^{AB}_{\theta_i}(\mu)}{Z^{AB}_{\theta_i}(\mu')}\,C^{AB}_{\theta_i}(\mu')}$$

where $\quad Z^{AB}_{\theta_i}(\mu) = Z_A(\mu)\,Z_B(\mu)\,Z^{-1}_{\theta_i}(\mu)$

i.e. $\quad Z^{AB}_{\theta_i}(\mu) = \exp\left[\int^{\mu}\frac{d\mu'}{\mu'}\,\gamma^{AB}_{\theta_i}(g_{\mu'})\right]$

$$\gamma^{AB}_{\theta_i} = \gamma_A + \gamma_B - \gamma_{\theta_i}$$

It is easy to generalize the result to the case in which operator mixing occurs. $C^{AB}_{\mathcal{O}_i}$, under a renormalization group transformation, as a triad. There are matrices

$$\left[\underline{Z}_A(\mu)\,\underline{Z}_A^{-1}(\mu')\right]\;\left[\underline{Z}_B(\mu)\,\underline{Z}_B^{-1}(\mu')\right]\;\left[\underline{Z}^{-1}_{\mathcal{O}_i}(\mu)\,\underline{Z}_{\mathcal{O}_i}(\mu')\right]$$

which act on its three indices

To avoid large logs we choose $\mu \sim q$, so

$$C^{AB}_{\mathcal{O}_i}(q, g_\mu, \mu) = q^D\, Z^{AB}_{\mathcal{O}_i}(\mu)\, Z^{AB^{-1}}_{\mathcal{O}_i}(q)\; f(g_q).$$

We can calculate $f(g_q)$ in perturbation theory; then sum logs of q/μ with the RG.

I. The Renormalization Group in QED

We would like to examine the relation between the coupling g_μ defined in minimal subtraction and the physical coupling which is actually measured. We will make this connection in QED. First we calculate the β function in QED.

We need to know how to treat fermions in dimensional regularization. The necessary formulas are

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,\mathbb{1} \quad ; \quad \eta^\mu{}_\nu\,\eta^\nu{}_\mu = D$$

$$\mathrm{tr}(\gamma_\mu \gamma_\nu) = 4\,\eta_{\mu\nu}$$

any smooth function of D which equals 4 for D=4 will do here

Addendum to I H:

LSZ Reduction Formula

In calculating coefficient functions, we used the "LSZ reduction" to evaluate matrix elements of operators. Here, this method, which is central to relativistic quantum field theory, is more fully explained.

References:

Bjorken and Drell, Chapter 16
Itzykson and Zuber, Chapter 5
etc..

The reduction formula establishes the connection between Green's functions and quantities of physical interest, such as S-matrix elements. The basic hypothesis of scattering theory is invoked: At large positive, negative times, all states approach states of noninteracting (free) particles. Thus, as $t \to \pm\infty$, any field $\phi(x)$ can be regarded as a free field $\phi_{\text{out}}^{\text{in}}$ which creates or destroys only one-particle states. Physically, this is because wave packets separate, and their interactions become negligible (this is surely true if the interactions have finite range)

Mathematically, the field ϕ acting on the vacuum creates many eigenstates of H

$$\langle n|\phi(x)|0\rangle = e^{iE_n t}\langle n|\phi(0)|0\rangle$$

But as $t \to \pm\infty$, the rapidly oscillating phase kills all but the one particle states (We choose ϕ so that $\langle 0|\phi|0\rangle = 0$. This argument may fail if there are massless particles, so that there is no gap between one- and many-particle states.

Thus, there is an infrared difficulty which we will need to return to later.) Note that these remarks should apply to any operator which couples to one-particle states and has vanishing vacuum expectation value – it can be either composite or elementary.

As $t \to \mp\infty$, we can choose as a basis of states

$$|k_1, k_2, \ldots\rangle_{in}$$
$$|k_1, k_2, \ldots\rangle_{out}$$

meaning the state which approaches a given state of n free particles as $t \to \mp\infty$. The in basis is different from the out basis if there are interactions (And we must always take superpositions of these plane wave states to make wave packets.)

We may write (in the case of a scalar field)

$$\frac{1}{\sqrt{Z}}\phi(x) \xrightarrow[t\to\mp\infty]{} \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[e^{-ik\cdot x}a_{\substack{in\\out}}(\vec{k}) + e^{ik\cdot x}a^{+}_{\substack{in\\out}}(\vec{k})\right]$$

$(k^0 = \omega_k = \sqrt{\vec{k}^2+m^2})$ (limit in the "weak sense")

where $a_{\substack{in\\out}}$ destroys the relativistically normalized state $|k\rangle_{\substack{in\\out}}$. The factor $\sqrt{Z}$ is included to get the normalization right,

$$[a(\vec{k}), a^{+}(\vec{k}')] = (2\pi)^3 2\omega_k\, \delta^3(\vec{k}-\vec{k}')$$

that is,

$$_{\substack{in\\out}}\langle k|\phi(x)|0\rangle \xrightarrow[t\to\mp\infty]{} \sqrt{Z}e^{ik\cdot x}$$

We can isolate the terms in ϕ with definite frequency by the formulas

$$\frac{-i}{\sqrt{Z}}\int d^3x \begin{bmatrix} e^{-ik\cdot x} \\ e^{+ik\cdot x}\end{bmatrix} \overleftrightarrow{\partial_0}\,\phi(x) \xrightarrow[t\to\mp\infty]{} \begin{bmatrix} a(\vec{k})^+ \\ -a(\vec{k})\end{bmatrix}_{\substack{in \\ out}}$$

and we may write, e.g.

$$a_{out} - a_{in} = \left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right) \frac{i}{Z}\int d^3x\, e^{ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \frac{i}{\sqrt{Z}}\int d^4x\, \partial_0\, e^{ik\cdot x}\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \frac{i}{\sqrt{Z}}\int d^4x\, [e^{ik\cdot x}\partial_0^2\phi(x) - (\partial_0^2 e^{ik\cdot x})\phi(x)]$$

But $\partial_0^2 e^{ik\cdot x} = (\nabla^2 - m^2)e^{ik\cdot x}$, and since, for wave packet states, integration by parts is justified

$$= \frac{i}{\sqrt{Z}}\int d^4x\, e^{ik\cdot x}(\Box + m^2)\,\phi(x)$$

or

$$\boxed{a_{out}(\vec{k}) - a_{in}(\vec{k}) = \frac{-i}{\sqrt{Z}}(k^2 - m^2)\int d^4x\, e^{ik\cdot x}\,\phi(x)}$$

and identical reasoning gives

$$\boxed{a^+_{in} - a^+_{out} = \frac{-i}{\sqrt{Z}}(k^2 - m^2)\int d^4x\, e^{-ik\cdot x}\,\phi(x)}$$

These are the main formulas from which all further results are derived. (True when we take matrix elements and smear in $\vec{k}$ - i.e., really applies to wave packet states)

As an example, consider

$$\int d^4x\, e^{-ik\cdot x}\, {}_{out}\langle\alpha|T(\phi(x)\,\mathcal{O}(y))|\beta\rangle_{in}$$

Because of the time-ordering, a_{out}, a^+_{out} always act to the left, and a_{in}, a^+_{in} always act to the right

$$\int d^4x\, e^{-iK\cdot x}\ {}_{out}\langle\alpha|\, T(\phi(x)\,\mathcal{O}(y))\,|\beta\rangle_{in}$$

$$= \sqrt{Z}\,\frac{i}{K^2-m^2}\Big(\ {}_{out}\langle\alpha|\,\mathcal{O}(y)\,a_{in}^{+}(\vec{K})|\beta\rangle_{in} - {}_{out}\langle\alpha|\,a_{out}^{+}(\vec{K})\,|\beta\rangle_{in}\Big)$$

But we know from the spectral representation that $Z\frac{i}{K^2-m^2}$ is just the propagator $\int d^4x\langle 0|T\phi(x)\phi(0)|0\rangle e^{iK\cdot x}$ as $K^2\to m^2$:

$$\Delta(x) = \langle 0|T\phi(x)\phi(0)|0\rangle$$

$$= \sum_n\left(\Theta(x^0)e^{-iK_n\cdot x} + \Theta(-x^0)e^{iK_n\cdot x}\right)|\langle 0|\phi(0)|n\rangle|^2,$$

summing over intermediate states. We may split the sum into one-particle states and many particle states

$$= \int\frac{d^3K}{(2\pi)^3 2\omega_K}\,Z\left(\Theta(x^0)e^{-iK\cdot x} + \Theta(-x^0)e^{iK\cdot x}\right) + \text{many-particle}$$

$$= Z\int\frac{d^4K}{(2\pi)^4}\,\frac{i\,e^{-iK\cdot x}}{K_0^2-\vec{K}^2-m^2+i\epsilon} + \text{many-particle (non pole terms)}$$

(since the K_0 integral can be done by closing the contour, picking up residue of pole at $K_0 = \pm\,\omega_K$

It is sometimes convenient to define renormalized fields so that $Z=1$. But if we use another convention (e.g. minimal subtraction), we must compute Z, and add a factor of $Z^{\frac{1}{2}}$ when we amputate on-mass-shell propagators, according to the reduction formula.

Example: ⓐ Operator matrix element:

$$[-iZ^{-\frac{1}{2}}(p^2-m^2)][-iZ^{-\frac{1}{2}}(p'^2-m^2)] \langle 0| T(\tilde{\phi}(-p')\tilde{\phi}(p)\mathcal{O})|0\rangle$$

(where $\tilde{\phi}(p) = \int d^4x\, e^{-ip\cdot x}\phi(x)$)

$$= -iZ^{-\frac{1}{2}}(p'^2-m^2) \langle 0| T\tilde{\phi}(-p')\mathcal{O}|\vec{p}\rangle_{in}$$

(since $\langle 0|a^{+}_{out}(\vec{p}') = 0$)

$$= {}_{out}\langle \vec{p}'|\mathcal{O}|\vec{p}\rangle_{in} - \langle 0|\ \mathcal{O}\ a_{in}(\vec{p}')|\vec{p}\rangle$$

$$= {}_{out}\langle \vec{p}'|\mathcal{O}|\vec{p}\rangle_{in} - (2\pi)^3 2E_p\,\delta^3(\vec{p}-\vec{p}')\langle 0|\mathcal{O}|0\rangle$$

$$= {}_{out}\langle \vec{p}'|\mathcal{O}|\vec{p}\rangle_{in} - {}_{in}\langle \vec{p}'|\vec{p}\rangle_{in}\langle 0|\mathcal{O}|0\rangle$$

(we note that positive frequency $(p^0>0) \Rightarrow$ incoming particle; negative frequency $(-p'^0<0) \Rightarrow$ outgoing particle.)

The connected matrix element (the part which is nonvanishing for $p\neq p'$) is associated with the amputated connected Green's function

ⓑ Four particle scattering amplitude:

$$\langle 0| T(\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(-p_3)\tilde{\phi}(-p_4))_{amp}|0\rangle$$

(where amp means with $\sqrt{Z}i/p^2-m^2$ amputated)

$$= \langle 0| T(\tilde{\phi}(-p_3)\tilde{\phi}(-p_4))_{amp}|\vec{p}_1,\vec{p}_2\rangle_{in}$$

$$= {}_{out}\langle \vec{p}_3|\ \tilde{\phi}(-p_4)_{amp}|\vec{p}_1,\vec{p}_2\rangle_{in} - \langle 0|\tilde{\phi}(-p_4)_{amp}\,a_{in}(\vec{p}_3)|\vec{p}_1,\vec{p}_2\rangle_{in}$$

$$= {}_{out}\langle \vec{p}_3,\vec{p}_4|\vec{p}_1,\vec{p}_2\rangle_{in} - {}_{out}\langle \vec{p}_3|a_{in}(\vec{p}_4)|\vec{p}_1,\vec{p}_2\rangle_{in}$$

$$- {}_{out}\langle \vec{p}_4|a_{in}(\vec{p}_3)|\vec{p}_1,\vec{p}_2\rangle_{in} + \langle 0|a_{in}(\vec{p}_3)a_{in}(\vec{p}_4)|\vec{p}_1,\vec{p}_2\rangle_{in}$$

$$= {}_{out}\langle \vec{p}_3\,\vec{p}_4|\vec{p}_1,\vec{p}_2\rangle_{in} - {}_{in}\langle \vec{p}_3\,\vec{p}_4|\vec{p}_1\,\vec{p}_2\rangle_{in}$$

In the last line, we have used the fact that the in and out bases coincide for one particle states (which do not scatter)

The reduction formula gives only the connected scattering amplitude. The disconnected contributions to the Green's function:

are killed by extra powers of $(p^2 - m^2) \to 0$ in the numerator.

Using these formulas we can work out the trace of any string of γ_μ matrices "in D dimensions."

A bit of a problem is the continuation away from $D=4$ of

$$\mathrm{tr}\,\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma = 4i\epsilon_{\mu\nu\lambda\sigma}.$$

Ward identities involving axial currents are violated in $O(\epsilon)$ in the regularized theory, and such corrections, when multiplied by $1/\epsilon$ poles, can give rise to anomalous finite parts. Fortunately, this problem does not arise in QED coupling renormalization, because the electric current is a vector current.

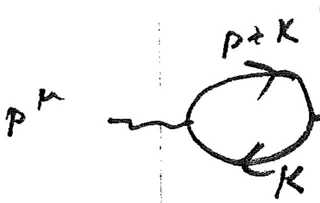
A Ward identity, which continued to hold for $\epsilon \neq 0$, ensures ~~that~~ eA_μ is unrenormalized; i.e.,

$$A_{\mu,B} = A_{\mu,R}\, Z_3^{\frac{1}{2}}$$

$$e_0 = \mu^{\epsilon/2} e\, Z_3^{-\frac{1}{2}}$$, where e is the renormalized coupling.

We can find the β function by computing Z_3. Since the β function in minimal subtraction is mass independent, we can do the calculation for a massless fermion.

We have

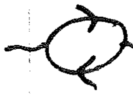


$$= (ie)^2 \mu^\epsilon \underset{\text{fermi}(-1)}{(-1)} \int \frac{d^D K}{(2\pi)^D}\, \mathrm{tr}\left\{ \frac{i(\not p + \not K)}{(p+K)^2 + i\epsilon}\gamma^\mu \frac{i\not K}{K^2+i\epsilon}\gamma^\nu \right\}$$

We apply the identity

$$\mathrm{tr}\,\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu = 4\left[\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}\right],$$

to find...

 $= -4e^2\mu^\epsilon \int \frac{d^Dk}{(2\pi)^D}\left[\frac{(p+k)^\mu k^\nu + k^\mu(p+k)^\nu - \eta^{\mu\nu}(p+k)\cdot k}{((p+k)^2+i\epsilon)\,(k^2+i\epsilon)}\right]$

$$= -4e^2\mu^\epsilon\int_0^1 dx \int \frac{(p+k)^\mu k^\nu + k^\mu(p+k)^\nu - \eta^{\mu\nu}(p+k)\cdot k}{[(k+xp)^2 + x(1-x)p^2+i\epsilon]^2}\,\frac{d^Dk}{(2\pi)^D}$$

Now, shift the k integral, and throw out terms linear in k:

$$= -4e^2\mu^\epsilon\int_0^1 dx\int\frac{d^Dk}{(2\pi)^D}\,\frac{2k^\mu k^\nu - 2x(1-x)p^\mu p^\nu - \eta^{\mu\nu}[k^2 - x(1-x)p^2]}{[k^2+x(1-x)p^2+i\epsilon]^2}$$

By symmetric integration, we replace $k^\mu k^\nu$ by $\frac{1}{D}\eta^{\mu\nu}k^2$, and rotate to Euclidean space, obtaining

$$= -4ie^2\mu^\epsilon\int_0^1 dx\int\frac{[d^Dk]_E}{(2\pi)^D}\,\frac{(1-\frac{2}{D})k^2\eta^{\mu\nu} + x(1-x)[\eta^{\mu\nu}p^2 - 2p^\mu p^\nu]}{[k^2 - x(1-x)p^2]^2}$$

The integral formula on p(1.20) gives

$$\int\frac{[d^Dk]_E}{(2\pi)^D}\,\frac{k^2}{[k^2+a^2]^2} = \frac{1}{(4\pi)^{D/2}}\,\frac{\Gamma(1+\frac{D}{2})}{\Gamma(D/2)}\,\Gamma(1-\tfrac{D}{2})(a^2)^{D/2-1}$$

$$= \frac{1}{(4\pi)^{D/2}}\left(\frac{D}{2}\right)\frac{a^2}{(1-D/2)}\,\Gamma(2-\tfrac{D}{2})(a^2)^{D/2-2}$$

$$\int\frac{[d^Dk]_E}{(2\pi)^D}\,\frac{1}{(k^2+a^2)^2} = \frac{1}{(4\pi)^{D/2}}\,\Gamma(2-\tfrac{D}{2})(a^2)^{D/2-2}$$

and, therefore,

 $= \frac{-ie^2\mu^\epsilon}{4\pi^2}(4\pi)^{\epsilon/2}\,\Gamma(\tfrac{\epsilon}{2})\,2(\eta^{\mu\nu}p^2 - p^\mu p^\nu)$

$$\times\int_0^1 dx\, x(1-x)\,[-x(1-x)p^2]^{-\epsilon/2}$$

Expanding in powers of ϵ

 $= \frac{-ie^2}{2\pi^2}\left(1+\frac{\epsilon}{2}\ln 4\pi\right)\left(\frac{1}{\epsilon/2}-\gamma\right)\left(\eta^{\mu\nu}p^2-p^\mu p^\nu\right)$

$\times \left[\int_0^1 dx\, x(1-x)\left(1-\frac{\epsilon}{2}\ln\left[\frac{x(1-x)p^2}{\mu^2}\right]\right)\right]$

 $\boxed{= \frac{-ie^2}{2\pi^2}\left[\frac{1}{6}\left(\frac{1}{\epsilon/2}-\gamma+\ln 4\pi\right)-\int_0^1 dx\, x(1-x)\ln\left[\frac{x(1-x)p^2}{\mu^2}\right]\right] \times\left(\eta^{\mu\nu}p^2-p^\mu p^\nu\right) + O(\epsilon)}$

$= \frac{-ie^2}{6\pi^2\epsilon}\left(\eta^{\mu\nu}p^2-p^\mu p^\nu\right) +$ higher order

We require  +  = finite

to this order, where the counterterm is

$-(Z_3-1)\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ 

$= -i(Z_3-1)\frac{1}{2}\left(ip_\alpha\eta_{\nu\beta}-ip_\beta\eta_{\nu\alpha}\right)$

$\times\left(-ip_\alpha\eta_{\mu\beta}+ip_\beta\eta_{\mu\alpha}\right)$

$= -i(Z_3-1)\left(p^2\eta_{\mu\nu}-p_\mu p_\nu\right)$

Thus, $(Z_3-1) = \frac{-e^2}{6\pi^2\epsilon} + \cdots$

or $\boxed{Z_3 = 1-\frac{e^2}{6\pi^2\epsilon}+\cdots}$

Now $e_0^2 = \mu^\epsilon e^2 Z_3^{-1} \Rightarrow \boxed{e_0^2 = \mu^\epsilon\left(e^2+\frac{e^4}{6\pi^2\epsilon}+\cdots\right)}$

and by the same reasoning as on p(1.25), we have

$\mu\frac{d}{d\mu}e^2 = -\epsilon e^2 - a_1(e^2) + e^2 a_1'(e^2)$ where

$a_1(e^2) = \frac{e^4}{\cdots} + \cdots$

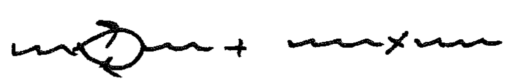
or... $2e\,\mu\frac{d}{d\mu}e = -\epsilon e^2 + e^4/6\pi^2$

$$\Rightarrow \boxed{\beta(e,\epsilon) \equiv \mu\frac{d}{d\mu}e = -\frac{\epsilon}{2}e + \frac{e^3}{12\pi^2} + \cdots}$$

For $\epsilon = 0$, QED is not asymptotically free (Gell-Mann - Low)

Finite Vacuum Polarization

We found $$\Pi(p^2) = \frac{e_\mu^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - x(1-x)p^2}{C\mu^2}\right]$$

where  $\equiv i(g^{\mu\nu}p^2 - p^\mu p^\nu)\Pi(p^2)$,

and the explicit dependence on the renormalized electron mass m has been reinstated. Here

$$C = 4\pi e^{-\gamma},$$

and we can, if we wish, redefine the renormalized coupling e_μ^2 so that C becomes absorbed in μ^2

$\Pi(p^2)$ is said to measure the "vacuum polarization" because it is related to the interaction energy of static charges with separation of order $1/p$; hence, it indicates how the electric charge of the electron is screened by the vacuum.

To understand this recall that the interaction of the photon field with an external charge distribution $J_\mu(x)$ is given by

$$H_I = \int d^4x\; eJ^\mu(x) A_\mu(x)$$

For a stationary charge distribution which persists for time T and then turns off (adiabatically)

$$Z(J) = \langle 0|Te^{-ie\int d^4x\, J^\mu(x) A_\mu(x)}|0\rangle \underset{\text{large } T}{\sim} e^{-iE(J)T}$$

where $E(J)$ is the vacuum energy in the presence of source J

$Z(J)$ is also a generating functional. The propagator is

$$\Delta_{\mu\nu}(x-y) = \frac{1}{(ie)^2}\frac{1}{Z}\frac{\delta^2 Z}{\delta J^\mu(x)\,\delta J^\mu(y)}\Big|_{J=0}$$

or $$\ln Z \sim \ln Z_0 + \tfrac{1}{2}(-ie)^2\int d^4x\, d^4y\, J^\mu(x)\,\Delta_{\mu\nu}(x-y)\,J(y) + \cdots$$

expanding in powers of J.

Therefore
$$E(J)\,T = -\frac{ie^2}{2}\int \frac{d^4p}{(2\pi)^4}\,\tilde{J}^\mu(p)\,\tilde{\Delta}_{\mu\nu}(p)\,\tilde{J}^\nu(-p) + \cdots$$

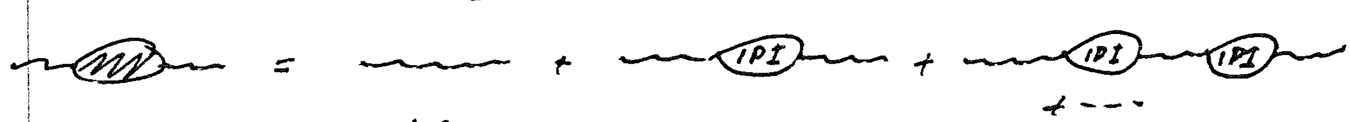
If $J^\mu(x)$ is independent of x^0, then $\tilde{J}^\mu(p) \propto 2\pi\delta(p^0)$ and the $\int dp^0$ gives $(2\pi)^2\delta(0) = 2\pi T$, so

$$E(J) = -\frac{ie^2}{2}\int \frac{d^3p}{(2\pi)^3}\,\tilde{J}^\mu(\vec{p})\,\tilde{\Delta}_{\mu\nu}(\vec{p})\,\tilde{J}^\nu(-\vec{p}) + \cdots$$

and the term quadratic in J is to be interpreted as the sum of two-body interaction energies; that is, the Fourier transform of $-ie^2\tilde{\Delta}_{00}(\vec{p})$ can be interpreted as a static potential.

(This sounds odd because $\tilde{\Delta}_{00}$ is not gauge invariant, but it is actually okay. Gauge transformations change the $p_\mu p_\nu$ part of $\tilde{\Delta}_{\mu\nu}$ but not the $\eta_{\mu\nu}$ part. The static potential is really the coefficient of η_{00}.)

We express the the propagator $\Delta_{\mu\nu}$ in terms of $\Pi(p^2)$ by summing up the 1PI pieces:



or $$\tilde{\Delta}_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2} + \frac{-i\eta_{\mu\alpha}}{p^2}\, i(\eta_{\alpha\beta}p^2 - p_\alpha p_\beta)\,\Pi(p^2)\,(-i)\frac{\eta_{\alpha\nu}}{p^2} + \cdots$$

$$= \frac{(-i)\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)}{p^2\,(1-\Pi(p^2))} - i\,\frac{p_\mu p_\nu}{(p^2)^2}$$

(p^μ p^ν terms are gauge artifacts)

We have found, then, that the potential energy of a pair of static charges is given by

$$V(\vec{r}) = \int \frac{d^3p}{(2\pi)^3}\, \tilde{V}(p)\, e^{i\vec{p}\cdot\vec{r}} \quad \text{and} \quad \hat{V}(p) = \frac{e_\mu^2}{\vec{p}^{\,2}}\left[1 - \Pi(-\vec{p}^{\,2})\right]^{-1},$$

and notice that we can really measure this potential.

i.e., we can measure the photon propagator at spacelike momentum transfer in scattering experiments.

Thus, we have

$$\boxed{\vec{p}^{\,2}\hat{V}(\vec{p}) = \frac{e_\mu^2}{1 - \frac{e_\mu^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 + x(1-x)\vec{p}^{\,2}}{C\mu^2}\right]}}$$

Now, consider the case $\vec{p}^{\,2} \gg m^2$, so we can ignore m^2 in the argument of the ln.

$$\int_0^1 dx\, x(1-x)\left[\ln x(1-x) + \ln \frac{\vec{p}^{\,2}}{C\mu^2}\right] = 2\left(-\tfrac{1}{4}+\tfrac{1}{9}\right) + \tfrac{1}{6}\ln \vec{p}^{\,2}/C\mu^2$$

$$= \tfrac{1}{6}\left(\ln \vec{p}^{\,2}/C\mu^2 - 5/3\right) = \tfrac{1}{6}\ln \vec{p}^{\,2}/C'\mu^2$$

$$\boxed{\Rightarrow \vec{p}^{\,2}\hat{V}(\vec{p}) = \frac{e_\mu^2}{1 - (e_\mu^2/12\pi^2)\ln(\vec{p}^{\,2}/C'\mu^2)} + O\left(\frac{m^2}{p^2}\right)}$$

Comparing with the expression for the running coupling (p. 1.15),

$$e_{\mu'}^2 = \frac{e_\mu^2}{1 - e_\mu^2/12\pi^2 \ln \mu'^2/\mu^2},$$

we see that the <u>effective</u> charge which determines the magnitude of the running coupling constant is

$$e_{\mu'}^2 \quad \text{where } \mu'^2 = \bar{p}^2/C'$$

If the electron were really exactly massless, this would remain true for arbitrarily small p, and

$$V(\bar{r}) \sim = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}\cdot\bar{r}} \frac{12\pi^2}{\bar{p}^2 \ln \Lambda'^2/\bar{p}^2} \sim \frac{3\pi}{2r}\left[\ln(\Lambda' r)\right]^{-1}$$

— for $\Lambda' r$ large

(the coulomb potential modified by a log — an approximate expression for the integral, since the log is slowly varying)

We see $V(\bar{r}) \to 0$ as $r \to \infty$; the electric charge becomes completely (though logarithmically) screened, because QED is <u>infrared free</u>. Here

$$\Lambda'^2 = C'\mu^2 \exp\left[12\pi^2/e_\mu^2\right]$$

is an enormous mass scale.

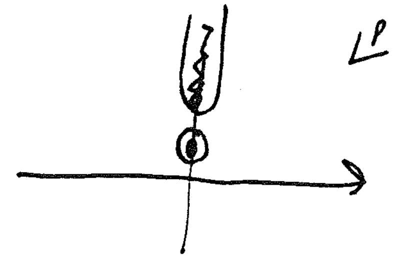
The effective charge increases with decreasing distance because at short distances, the bare charge is screened less; i.e., we are seeing further into the cloud of e^+e^- pairs surrounding the bare charge.

Now we consider the one-loop quantum corrections to the Coulomb potential for $p^2 \ll m^2$; i.e., at distances much greater than the electron compton wavelength. We have

$$V(\bar{r}) = e_\mu^2 \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}\cdot\bar{r}} \frac{1}{\bar{p}^2}\left[1 - \pi(-\bar{p}^2)\right]^{-1}$$

$$= \frac{e^2}{(2\pi)^2}\int_0^\infty dp \frac{1}{1-\pi(-p^2)} \frac{e^{ipr} - e^{-ipr}}{ipr} = \frac{e^2}{2\pi r}\int_{-\infty}^{\infty} \frac{dp}{2\pi i p} \frac{1}{1-\pi(-p^2)} e^{ipr}$$

It is convenient to calculate this as



$$V(r) = \lim_{\mu \to 0} \frac{e^2}{2\pi r} \int_{-\infty}^{\infty} \frac{dp}{2\pi i} \frac{p}{p^2+\mu^2} \frac{e^{ipr}}{1-\Pi(-p^2)}$$

We can do the integral by distorting the contour into the upper half-plane; we encounter a pole at $p = i\mu$, and a cut in Π along the imaginary p axis. We obtain

$$V(r) = \frac{e^2}{4\pi r} \frac{1}{1-\Pi(-p^2=0)} + \frac{e^2}{2\pi r} \int_{cut} \frac{i\,dp'}{2\pi i} \frac{e^{-p'r}}{ip'} \, \mathrm{Disc}\left(\frac{1}{1-\Pi}\right)$$

$(p' = ip)$

The first term is

$$\frac{1}{4\pi r} \frac{e_\mu^2}{1 - e_\mu^2/12\pi^2 \ln(m^2/c\mu^2)}$$

The effective charge appearing in the coulomb potential is $e^2_{\mu'}$ where

$$\mu'^2 = m^2/c$$

We see that the physical coupling stops running for $r \gg m^{-1}$. The effect of virtual e^+e^- pairs is small over distances much larger than the electron compton wavelength.

We can also calculate the correction term in $V(r)$

Recall

$$\Pi(p^2) = \frac{e^2}{2\pi^2} \int_0^1 dx \, x(1-x) \ln\left[\frac{m^2 + x(1-x)p^2}{c\mu^2}\right]$$

The discontinuity in Π occurs for p^2 such that $m^2 + x(1-x)p^2 < 0$ for some x in $[0,1]$; i.e. for $-p^2 > 4m^2$

or $p'^2 > 4m^2$

So the cut begins at $p' = 2m$, and we see that the correction is suppressed by e^{-2mr}, which quickly becomes negligible for $mr \gg 1$.

You might enjoy verifying that the correction term, up to order (e^4) is

$$\frac{e^4}{12\pi^2 r}\int_{2m}^{\infty}\frac{dp'}{2\pi}\,\frac{e^{-p'r}}{p'}\left(1-\frac{4m^2}{p'^2}\right)^{\frac{1}{2}}\left(1+\frac{2m^2}{p'^2}\right)$$

$$\sim \frac{1}{4\sqrt{\pi}\,r}\left(\frac{e^2}{4\pi}\right)^2\frac{e^{-2mr}}{(mr)^{3/2}}\left[1+O\left(\frac{1}{mr}\right)\right]$$

(see p (1.69))

We see that the charge actually measured at large distances is $\frac{1}{4\pi}e^2_{\mu'=m/\sqrt{c}} \sim \frac{1}{137}$

However, the coupling constant of minimal subtraction continues to decrease for $\mu < m$

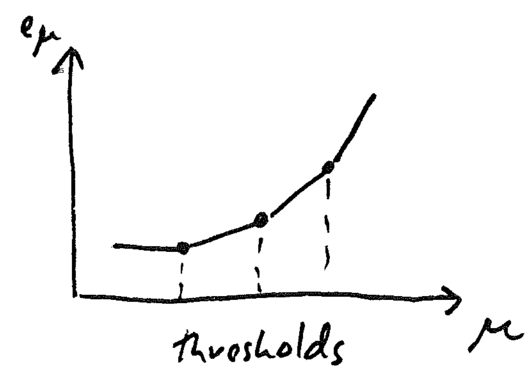
$$e_\mu^2/4\pi = \frac{(1/137)}{1+\left(\frac{1}{(137)3\pi}\right)\ln\left(m^2/c\mu^2\right)}$$

The electron has actually "decoupled" from low-energy physics, but the minimal subtraction scheme does not know this, and includes the electron contribution to the β function even though it is irrelevant.

The situation is even more awkward if there are more heavy particles, with still higher masses. In our mass independent scheme, we have

$$\beta(e) = \left(\sum_i Q_i^2\right)\frac{e^3}{12\pi^2}+\cdots$$

-- all charged particles (Q_i is the charge) contribute, no matter how heavy. If there is a superheavy particle with $M \sim 10^{16}$ GeV, a $\ln M^2/\mu^2$ appears in the relation between e_μ and the physical coupling!



It is convenient to work in an <u>effective field theory</u> with heavy degrees of freedom integrated out; then one includes only <u>light</u> fermions in the calculation of the β function

$$\beta(e) = \sum_{\text{light } i} Q_i^3 \frac{e^3}{12\pi^2}$$

If the charged fermions have masses m_i, the light fermions in the effective theory at scale μ are those with $m_i < \mu$, so the effective theory changes at each <u>threshold</u> where $\mu = m_i$. The definition of the renormalized coupling thus changes at each threshold, and we link the effective theories together by demanding that e_μ is continuous. If we are interested in a process at energy E, we choose $\mu \sim E$. Since all heavy fermions are integrated out, no large logs appear.

The alternative is to use a mass-dependent scheme, with thresholds appearing explicitly in $\beta(g, m/\mu)$.

<u>Exercise 1.10</u>

Calculate the β function to one-loop order for (massless) scalar electrodynamics, with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D^\mu \phi)^\dagger D_\mu \phi \quad , \quad D_\mu = \partial_\mu - ieA_\mu$$

ϕ is a <u>complex</u> scalar field

Addendum:

Here are the details of the calculation described at the top of p. (1.67).

We wish to evaluate $\boxed{I = \frac{e^2}{2\pi r}\int_{2m}^{\infty}\frac{dp'}{2\pi i}\,\frac{e^{-p'r}}{p'}\,\mathrm{Disc}\left[\frac{1}{1-\pi(p'^2)}\right]}$

where

$$\pi(p'^2) = \frac{e^2}{2\pi^2}\int_0^1 x(1-x)\ln\left[\frac{m^2 - x(1-x)p'^2}{c\mu^2}\right]$$

Since $\mathrm{Disc}\,\frac{1}{1-\pi} = \mathrm{Disc}\,\pi + O(\pi^2)$

we have $I = \frac{e^2}{2\pi r}\int_{2m}^{\infty}\frac{dp'}{2\pi i}\,\frac{e^{-p'r}}{p'}\,\mathrm{Disc}\,\pi(p'^2) + O(e^6)$

The log has a discontinuity $2\pi i$ when its argument is negative or $x^2 - x + m^2/p'^2 < 0$ which is true for $x_- < x < x_+$ where $(x_\pm - \frac{1}{2})^2 = \frac{1}{4} - m^2/p'^2$ or $x_\pm = \frac{1}{2} \pm (\frac{1}{4} - m^2/p'^2)^{\frac{1}{2}}$

Therefore $\mathrm{Disc}\,\pi = \frac{e^2}{2\pi^2}\,2\pi i\int_{x_-}^{x_+} dx\, x(1-x) = \frac{e^2}{2\pi^2}\,2\pi i\int_{y_-}^{y_+}(\frac{1}{4} - y^2)\,dy$

$$y_\pm = \left(\tfrac{1}{4} - m^2/p'^2\right)^{\frac{1}{2}}$$

and $\int dy\,(\frac{1}{4} - y^2) = \frac{1}{2}y_+ - \frac{2}{3}y_+^3 = \frac{1}{2}y_+(1 - \frac{4}{3}y_+^2)$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{m^2}{p'^2}\right)^{\frac{1}{2}}\left(1 - \frac{1}{3} + \frac{4}{3}\frac{m^2}{p'^2}\right) = \frac{1}{6}\left(1 - \frac{4m^2}{p'^2}\right)^{\frac{1}{2}}\left(1 + \frac{2m^2}{p'^2}\right)$$

$$\Rightarrow \quad \mathrm{Disc}\,\pi = \frac{ie^2}{6\pi}\left(1 - \frac{4m^2}{p'^2}\right)^{\frac{1}{2}}\left(1 + \frac{2m^2}{p'^2}\right)$$

and $\boxed{I = \frac{e^4}{24\pi^3 r}\int_{2m}^{\infty}\frac{dp'}{p'}\,e^{-p'r}\left(1 - \frac{4m^2}{p'^2}\right)^{\frac{1}{2}}\left(1 + \frac{2m^2}{p'^2}\right)}$

We can approximately evaluate I for $r \gg m^{-1}$, since the integral is dominated by $p' \sim 2m$

$$I = \frac{e^4}{24\pi^3}\int_{2m}^{\infty}\frac{dp'}{p'^2}\,e^{-p'r}\,(p'-2m)^{\frac{1}{2}}(p'+2m)^{\frac{1}{2}}\left(1 + \frac{2m^2}{p'^2}\right)$$

$$\sim \frac{e^4}{24\pi^3 r}e^{-2mr}\left(\frac{3}{2}\right)\frac{(4m)^{\frac{1}{2}}}{4m^2}\int_0^{\infty}dp\,p^{\frac{1}{2}}e^{-pr}\left[1 + O\left(\frac{1}{mr}\right)\right]$$

$$\sim \frac{e^4}{32\pi^3 r}\,e^{-2mr}\frac{1}{(mr)^{3/2}}\,\Gamma\left(\tfrac{3}{2}\right) \qquad \text{and } \Gamma(3/2) = \sqrt{\pi}/2$$

$$\Rightarrow \boxed{I = \left(\frac{e^2}{4\pi}\right)^2\frac{e^{-2mr}}{4\sqrt{\pi}\,r\,(mr)^{3/2}}\left[1 + O\left(e^2, \frac{1}{mr}\right)\right]}$$

Addendum to IJ:
The Path Integral

We can demonstrate the equivalence of canonical quantization and functional integration. As an illustrative example consider quantum mechanics of a single degree of freedom. (Reference: Fadeev's lectures in Les Houches '75 volume.)

Canonical variables, in Heisenberg picture:

$$q(t) = e^{iHt} q(0) e^{-iHt}$$
$$p(t) \quad e^{iHt} p(0) e^{-iHt}$$

At each time t, q, p have complete basis of eigenvectors

$$|q,t\rangle, \quad |p,t\rangle.$$

We have

$$\langle q,t|p,t\rangle = \frac{1}{\sqrt{2\pi}} e^{ipq} \quad (\text{since } p = \tfrac{1}{i}\tfrac{\partial}{\partial q})$$

Also

$$\langle p_1 t_1 | q_2 t_2\rangle = \langle p_1 t_1 | e^{iH(t_2 - t_1)} | q_1 t_1\rangle$$
$$= e^{-iH(p_1,q_1)(t_1 - t_2)} \frac{1}{\sqrt{2\pi}} e^{-ip_1 q_1}$$

(if we order H so that p's are on left, q's on right.)

Now we will derive a formula for $\langle q_f t_f | q_i t_i\rangle$, by dividing the time interval (t_f, t_i) into $(n+1)$ equal fragments of width $\epsilon = |t_f - t_i|/(n+1)$.

Inserting sums over complete sets of intermediate states

$$\langle q_f, t_f | q_i, t_i\rangle$$
$$= \int dp_0\, dq_1\, dp_1 \cdots \langle q_f t_f | p_0 t_f\rangle \langle p_0 t_f | q_1 t_1\rangle \langle q_1 t_1 | p_1 t_1\rangle \cdots \langle p_n t_n | q_i t_i\rangle$$

Using the above formulas, we have

$$\langle q_f t_f | q_i t_i \rangle = \int \left(\frac{dp_0}{2\pi} \frac{dq_1 dp_1}{2\pi} \frac{dq_2 dp_2}{2\pi} \cdots \right) \left[e^{iq_f p_0} e^{-\epsilon i H(p_0 q_1)} e^{-ip_0 q_1} \right.$$
$$\left. e^{ip_1 q_1} e^{-\epsilon i H(p_1 q_2)} e^{-ip_1 q_2} \cdots \right]$$

$$= \int \left(\frac{dp_0}{2\pi} \frac{dq_1 dp_1}{2\pi} \cdots \right) \exp i\epsilon \left[p_0 \dot{q}_1 - H(p_0, q_1) + p_1 \dot{q}_2 - H(p_1, q_2) + \cdots \right]$$

where we write

$$(q_{j+1} - q_j) = \epsilon \dot{q}_j,$$

regarding the q_j's as approximation to a differentiable trajectory. As $\epsilon \to 0$, the argument of the exponential becomes a Riemann integral associated with the trajectory $q(t), p(t)$

$$\boxed{\langle q_f t_f | q_i t_i \rangle = \int (dq)(dp)\, e^{i S_H[q,p]}}$$

where $S_H = \int dt \, (p\dot{q} - H(p,q))$ is the action in Hamiltonian form; the measure $(dq)(dp)$ is defined by the above limiting procedure (the extra $\int dp_0$ is needed for dimensional consistency) and the boundary condition

$$q(t_i) = q_i, \quad q(t_f) = q_f$$

is understood.

If the Hamiltonian is quadratic in momentum, we can do the (dp) integral explicitly: If

$$H = \frac{1}{2m} p^2 + V(q),$$

then use

$$\int \frac{dp}{2\pi} e^{i\epsilon(p\dot{q} - \frac{1}{2m}p^2)} = \sqrt{\frac{m}{2\pi i \epsilon}} \, e^{i\frac{1}{2}m\dot{q}^2},$$

and obtain

$$\langle q_f t_f | q_i t_i \rangle = \frac{1}{\sqrt{2\pi i \epsilon/m}} \int \prod_i \frac{dq_i}{\sqrt{2\pi i\epsilon/m}} e^{i\epsilon[\frac{1}{2}m\dot{q}_i^2 - V(q_i)]}$$

or $\boxed{\langle q_f t_f | q_i t_i \rangle = \int (dq)\, e^{i S_L[q]}}$ – Feynman's Formula

where $S_L = \int dt\, [\frac{1}{2}m\dot{q}^2 - V(q)]$ is the action in Lagrangian form.

This formula is important for three reasons:

i) It provides insight into the nature of the classical limit: $e^{iS/\hbar}$ gives destructive interference except along classical trajectory, where action is stationary

ii) It can be generalized to systems with more degrees of freedom, including field theory:

$$\langle \phi(\vec{x}, t_f) | \phi(\vec{x}, t_i) \rangle_J = \int (d\phi)\, e^{i(S[\phi] + \int J\phi)}$$

By taking $T = |t_f - t_i|$, we pick out vacuum expectation value, up to normalization vector

$$\langle 0|0\rangle_J = Z[J] = \int d\phi\, e^{i(S[\phi] + \int J\phi)}$$

iii) It reduces quantum mechanics to quadrature, and provides basis for numerical solutions to nonperturbative problems.

Canonical Quantization of Electrodynamics

(Reference: S. Coleman's lectures, Erice '73)

The Fadeev-Popov method defines a theory which is independent of the choice of gauge, but is it otherwise a sensible theory? In particular, is it unitary and covariant? If we can justify the formal argument for gauge-independence in the renormalized theory, then it will suffice to show that the theory is covariant in one gauge and unitary in another.

To demonstrate unitarity, we must show that there is a gauge in which the Fadeev-Popov method agrees with canonical quantization. In the canonically quantized theory, time evolution is governed by a Hermitian Hamiltonian, and is guaranteed to conserve probability.

We will make the gauge choice

$$A_3 = 0 \qquad \text{(axial gauge)}$$

(which suffices to fix the gauge uniquely if $A \to 0$ at spatial infinity, which is good enough for the purpose of defining perturbation theory.) We wish to show that, in this gauge, canonical quantization and the Fadeev-Popov procedure give the same result.

What we will actually do is write the action in the so-called "first-order form" (since it is linear in time derivatives)

$$S_{1st\,order} = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right]$$

Here $F_{\mu\nu}$ is a constrained variable (no time derivatives of it appear) which satisfies the eqn. of motion

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and, substituting this in, we obtain the usual action. So $S_{first\text{-}order}$ defines the same classical theory as the usual variational principle

What we will show is, that if we set $A_3 = 0$, then

$$Z = N \int \prod_{\mu\nu} dF_{\mu\nu}\, dA_0\, dA_1\, dA_2\, e^{iS_{first\text{-}order}}$$

agrees with both the function integral for Z obtained from the Fadeev-Popov method, and that obtained from canonical quantization in $A_3 = 0$ gauge (up to normalization)

The basic idea is that, if a variable is a constrained variable (no time derivatives of it occur) and it appears quadratically in the action, with the quadratic term having a constant coefficient (independent of other variables), then eliminating it by solving the constraint is equivalent to integrating over it. That is,

$$\int dy\; e^{-\frac{1}{2}ay^2 + by} = N e^{b^2/2a}$$

– just what we obtain by solving the "eqn of motion" $ay - b = 0$ and substituting back in

Therefore, if we integrate out F_{ij} $i,j = 1,2,3$, F_{03} and A_0, we obtain

$$Z = N' \int dF_{01}\, dF_{02}\, dA_1\, dA_2\, e^{iS_H}$$

where S_H is the action in Hamiltonian form. This is exactly what we obtain from canonical quantization in this gauge ($A_3 = 0$)

If, on the other hand, we integrate out $F_{\mu\nu}$, we have

$$Z = N'' \int dA_0\, dA_1\, dA_2\, e^{iS_2}$$

where S_2 is the action in the usual 2nd-order form, or

$$Z = N'' \int dA_\mu\, e^{iS_2[A]}\, \delta[A_3]$$

But this is precisely of the Faddeev-Popov form (under a gauge trans, $\delta A_3 = \frac{1}{e}\partial_\mu \delta\omega$, and $\det \frac{1}{e}\partial_\mu$ is absorbable into the constant N'')

We have thus succeeded in demonstrating the claimed equivalence.