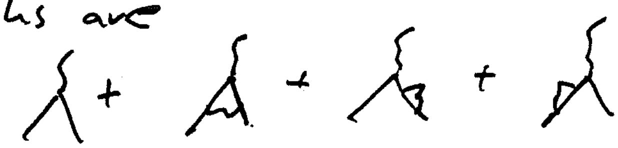
Infrared divergences in QED

Before we consider jets, let's see how the infrared divergences cancel in QED.

Consider scattering of an electron off an external electromagnetic field. To one-loop order, the graphs are



the infrared divergence is in since the electron propagators are (in the denominator)

$$\frac{1}{(k+p)^2-m^2} = \frac{1}{k^2+2p\cdot k},$$

the loop integration behaves like $\int \frac{d^4k}{k^4}$ for small k (logarithmic infrared divergence). Let's calculate the IR divergent part of the graph by giving the photon a fictitious mass μ, and finding the part which diverges as $\mu \to 0$. To keep things simple, we'll calculate only the leading term for

$$|q^2| = |(p-p')^2| \gg m^2 \gg \mu^2$$

Note here

$$(ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}(p')\gamma^\nu \frac{i(\not k+\not p'+m)}{k^2+2k\cdot p'}\gamma^\mu \frac{i(\not k+\not p+m)}{k^2+2k\cdot p}\gamma_\nu u(p)\, \frac{-i}{k^2-\mu^2}$$

we'll drop k in the numerator, since we're only interested in the IR divergent part, and we'll drop m, since we're keeping only the leading term for large $|q^2|$.

$$= (ie)^2(i)\int \frac{d^4k}{(2\pi)^4}\, N\Big/\left[(k^2+2k\cdot p')(k^2+2k\cdot p)(k^2-\mu^2)\right] + \text{IR finite}$$

$$N = \bar{u}(p')\gamma^\nu \not p' \gamma^\mu \not p \gamma_\nu u - 4(p\cdot p')\, \bar{u}(p')\gamma^\mu u(p)$$

To make Feynman parameter integrals easier, combine denominators as follows:

$$\frac{1}{(k^2+2p\cdot k)}\frac{1}{(k^2+2p'\cdot k)} = \int_0^1 dx \frac{1}{[k^2+2p_x\cdot k]^2} \qquad p_x = xp+(1-x)p'$$

$$\frac{1}{k^2-\mu^2}\times \text{above} = \int_0^1 dy\, 2y \int_0^1 dx \frac{1}{[(k+yp_x)^2 - y^2p_x^2 - (1-y)\mu^2]^3}$$

Shift doesn't change numerator,

$$\hat{\Lambda} \simeq -ie^2\int_0^1 2y\,dy\int_0^1 dx \int \frac{d^4k}{(2\pi)^4}[k^2-y^2p_x^2-(1-y)\mu^2]^{-3}(\bar{u}\gamma^\mu u) \times (4p\cdot p')$$

$$= -ie^2\int_0^1 dy\,2y\int_0^1 dx \frac{(i)(-1)\frac{1}{2}}{16\pi^2}\frac{1}{y^2p_x^2+(1-y)\mu^2}(\bar{u}\gamma^\mu u)(4p\cdot p')$$

To take $\mu^2\to 0$ limit of y integral, note that log divergence at $y^2=0$ in

$$\frac{1}{p_x^2}\int_0^1 \frac{dy^2}{y^2+(1-y)(\mu^2/p_x^2)}$$

is cutoff by μ^2. For $\mu^2\to 0$, we can replace $(1-y)$ by 1, and evaluate

$$\sim \frac{1}{p_x^2}\ln\left[\frac{1+\mu^2/p_x^2}{(\mu^2/p_x^2)}\right] + \text{vanishing as } \mu^2\to 0$$

Now we have

$$\hat{\Lambda} \simeq \frac{-e^2}{32\pi^2}\bar{u}(p')\gamma^\mu u(p)(4p\cdot p')\int_0^1 dx \frac{1}{p_x^2}\ln\left(\frac{p_x^2}{\mu^2}\right)$$

$$+ (\text{finite for } \mu^2\to 0)$$

$$p_x^2 = xp+(1-x)p' = [x^2+(1-x)^2]m^2 + x(1-x)(2m^2-q^2)$$
$$= m^2 - x(1-x)q^2$$

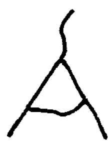
So we need to evaluate

$$\int_0^1 dx \frac{1}{m^2-x(1-x)q^2}\ln\frac{m^2-x(1-x)q^2}{\mu^2} \sim \ln\frac{-q^2}{\mu^2}\int_0^1 dx\frac{1}{m^2-x(1-x)q^2}$$

for $q^2 \lesssim 0$ and $|q^2| \gg m^2$

We could evaluate this integral explicitly, but it is clear that, as $m^2 \to 0$, there are log divergences at $x \to 0$ and $x \to 1$, cut off by m^2. So we have

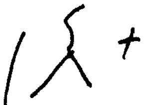
$$\sim \quad \frac{-2}{q^2} \ln {-q^2/m^2} \ln {-q^2/\mu^2} \quad , \text{ or}$$

 $\sim \; (\bar{u}\gamma^\mu u)\left(\frac{-e^2}{8\pi^2} \ln(-q^2/m^2) \ln(-q^2/\mu^2) \right)$

zeroth order result

(The double log, which takes the above form for $q^2 \gg m^2 \gg \mu^2$, would become $\ln^2(-q^2/\mu^2)$ + subleading log if we took $m^2 \to 0$ limit with μ^2 fixed.)

We see that the $O(\alpha)$ interference term in --

$| \;$  $+$  $\; |^2$ is infrared divergent.

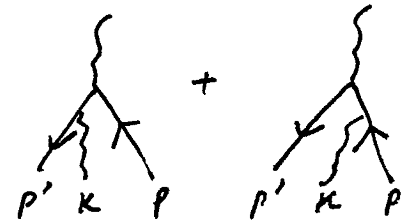
$$\boxed{(d\sigma)_{\text{coulomb}} = (d\sigma_0)_{\text{coulomb}} \left[1 - \frac{\alpha}{\pi} \ln(-q^2/\mu^2) \ln(-q^2/m^2) + \cdots \right]}$$

What is going on? How can coulomb scattering cross section be infinite in $O(\alpha)$? A key hint comes from the realization that the divergence is due to arbitrarily soft virtual photons.

Bremsstrahlung

The problem is, we have asked an unphysical question. In an experiment with nonzero energy resolution, we cannot tell whether the recoiling electron is accompanied by an arbitrarily soft photon; i.e., a real photon emitted during the scattering process.

Amplitude for emission of one photon is



$$= (ie)\,\bar u(p')\,\gamma^\nu \frac{i(\not p' + \not k + m)}{k^2 + 2p'\cdot k}\gamma^\mu\, u(p)\,\epsilon_\nu$$
$$+ ie\,\bar u(p')\,\gamma^\mu \frac{i(\not p - \not k + m)}{k^2 - 2p\cdot k}\gamma^\nu\, u(p)\,\epsilon_\nu$$

To isolate IR sensitive term (very soft real photon), ignore k in numerator

$$= e\,\bar u(p')\gamma^\mu u(p)\left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{p'\cdot\epsilon}{p'\cdot k}\right)$$

therefore

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\; \underbrace{e^2 \sum_\lambda \left(\frac{\epsilon\cdot p}{k\cdot p} - \frac{\epsilon\cdot p'}{k\cdot p'}\right)^2}_{\substack{\text{matrix element squared,}\\ \text{summed over photon}\\ \text{polarizations}}} \underbrace{\frac{d^3k}{(2\pi)^3 2k}}_{\text{photon phase space}}$$

Polarization sum $\quad \sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}$

Ignoring m^2 compared to q^2

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\,\frac{2e^2\, p\cdot p'}{(k\cdot p)(k\cdot p')}\,\frac{d^3k}{(2\pi)^3 2k}$$

Integrating over photon phase space,

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\,\frac{e^2\, p\cdot p'}{(2\pi)^3}\int dk\, k \int d\Omega_{\hat k}\,\frac{1}{(k\cdot p)\, k\cdot p'}$$

The angular integral can be done by the Feynman trick:

$$\int d\Omega_{\hat k}\int_0^1 dx\,\frac{1}{(k\cdot p_x)^2} \qquad p_x = xp + (1-x)p'$$

$$= \int_0^1 dx\; 2\pi\int_{-1}^1 d\cos\theta\;\frac{1}{k^2[\,E_x - |\vec p_x|\cos\theta\,]^2}$$

$$= \frac{2\pi}{k^2}\int_0^1 dx\,\frac{1}{|\vec p_x|}\left[\frac{1}{E_x - |\vec p_x|} - \frac{1}{E_x + |\vec p_x|}\right]$$

$$= \frac{4\pi}{k^2}\int_0^1 dx\,\frac{1}{p_x^2} = \frac{4\pi}{k^2}\int_0^1 \frac{dx}{m^2 - x(1-x)q^2} \cong \frac{8\pi}{-k^2 q^2}\ln\left(-q^2/m^2\right)$$

or

$$(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\, \frac{e^2}{2\pi^2}\left(\int \frac{dK}{K}\right) \ln\left(-q^2/m^2\right)$$

or

$$\boxed{(d\sigma)_{brem} = (d\sigma_0)_{coulomb}\, \frac{\alpha}{\pi} \ln\left(\frac{K_{max}^2}{\mu^2}\right) \ln\left(-q^2/m^2\right)}$$

is the cross section for emission of a bremsstrahlung photon with momentum $|\vec{k}| < K_{max}$ (the photon mass μ has been reinstated here.)

The cross section for soft photon emission is also infrared divergent, but the sum

$$(d\sigma)_{coulomb} + (d\sigma)_{brem} = (d\sigma_0)_{coul}\left[1 - \frac{\alpha}{\pi}\ln\left(-q^2/m^2\right)\ln\left(-q^2/K_{max}^2\right)\right]$$

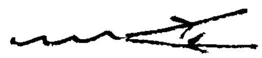
is finite in $O(\alpha)$, for a finite photon resolution K_{max}.

The cancellation of IR divergences due to real and virtual photons works to all orders (S. Weinberg, Phys. Rev. 140, B516 (1965).) The leading (infrared) logs of q^2/K_{max}^2 and q^2/m^2 can be summed to all orders. They exponentiate (D.R. Yennie, S. Frautschi, and H. Suura, Ann. Phys. 13, 379 (1961)):

$$(d\sigma)_{coul} + (d\sigma)_{brem} = (d\sigma_0)_{coul} \exp\left[-\frac{\alpha}{\pi}\ln\left(-q^2/m^2\right)\ln\left(-q^2/K_{max}^2\right)\right]$$

So the cross section does not really blow up as the resolution shrinks to zero

If we wish to take also the limit $m \to 0$, we must exercise more care to obtain finite quantities. Beginning in next order the process

 can occur, $\gamma \to e^+e^-$.

The final state can contain an arbitrary number of soft real e^+e^- pairs. ("Colinear divergence": one massless particle cannot be distinguished from several colinear massless particles.)

Jets in QCD perturbation theory

To discuss jets in QCD perturbation theory we need to define an "infrared safe" variable which describes the shape of the final state, and is "physically sensible" when applied to massless quarks and gluons. Of course, the situation is different than in QED, since hadrons, not quarks and gluons, are observed in the final state. But the effect of hadronization on the energy distribution of the final state (which gives jets a finite width) should be a power correction to the asymptotic QCD prediction. This expectation cannot be rigorously demonstrated, and the nature of the power corrections is less clear than in the case of the total e^+e^- annihilation cross section.

We will consider a variable prefered by the experimentalists, called thrust. For a hadronic final state in e^+e^- annihilation, the thrust is defined by

$$T = \max\left(\frac{\sum_i |\vec{p}_{iL}|}{\sum_i |\vec{p}_i|}\right) ;$$

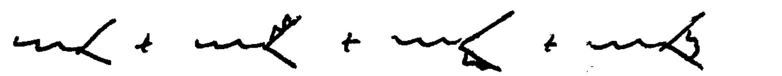
i.e., T is the sum of longitudinal momenta along an axis divided by the total momentum of all particles, where the axis is chosen to maximize this quantity. Thus

$T = 1$ for a perfect jet — all momenta along jet axis

$T = \frac{1}{2}$ for an isotropic event.

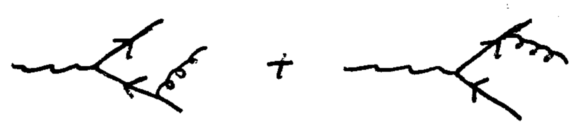
Thrust is an infrared safe jet variable because it is linear in longitudinal momentum, and is therefore unchanged by the collinear fragmentation of gluon into quark-antiquark or quark into quark-gluon.

We wish to calculate the thrust distribution of final-state hadrons to $O(\alpha_s)$. Which means we can ignore



all of which have $q\bar{q}$ in final state, and therefore $T-1=0$. The rule of thumb guarantees that the IR singularities of these graphs cancel the $T\to 1$ IR divergence of the graphs with real final state gluons. In fact, to order α_s the gluon mass is an adequate IR cutoff – there are no collinear $q\bar{q}$ pairs to this order. (Also, the ultraviolet infinities of these graphs cancel; gauge invariance guarantees that the strong interactions do not renormalize the electric charge.)

To calculate thrust distribution to order α_s, we must find $d\sigma(e^+e^-\to q\bar{q}g)$ due to



this is a tedious calculation which I won't do here (See J. Ellis et al., Nucl. Phys. B111, 253 (1976), but be sure to also see the erratum, Nucl. Phys B130, 516 (1977).) Defining

$$x_1 = \frac{E_1}{\frac{1}{2}\sqrt{s}}, \quad x_2 = \frac{E_2}{\frac{1}{2}\sqrt{s}}$$

for the two quarks (so that $x_i = 1$ is the maximum kinematically allowed value),

one finds

$$\frac{d^2\sigma}{dx_1\,dx_2} = (\sigma_0)\,\frac{\alpha_s}{2\pi}\big(\underbrace{C_2(R_f)}_{4/3\text{ for each quark}}\big)\,\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

where $\sigma_0 = \dfrac{4\pi\alpha^2}{3s}\left(N_c \sum_R Q_R^2\right)$ is the total cross section to order α_s^0.

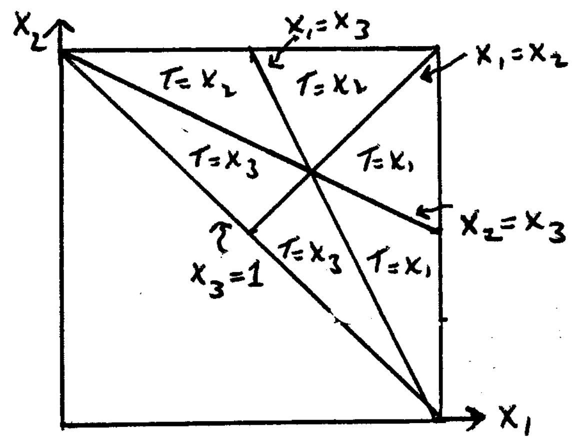
If $x_3 = E_3 / \frac{1}{2}\sqrt{s}$, the scaled gluon energy,

the thrust axis is along the largest of the three momenta, and the thrust is ----

$$T = \max(x_1, x_2, x_3)$$

(Experimentally, we can't distinguish quark, antiquark, gluon jet.)

To find thrust distribution, we sum over phase space where $T = x_1, x_2, x_3$



$x_1 + x_2 + x_3 = 2$

$T = x_1 \qquad 2(1-T) \le x_2 \le T$

$T = x_2 \qquad 2(1-T) \le x_1 \le T$

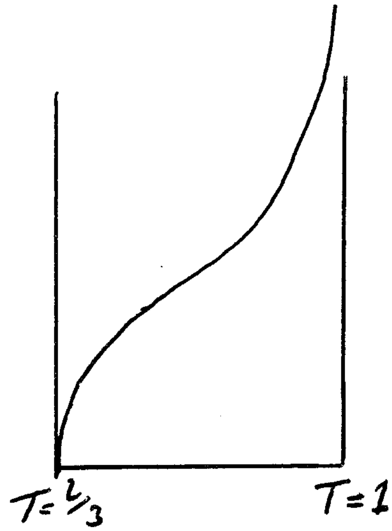
$T = x_3 \qquad 2(1-T) \le x_2 \le T \qquad x_1 = 2 - T - x_2$

$$\frac{d\sigma}{dT} = \sigma_0 \frac{2\alpha_s}{3\pi}\left[2\int_{2(1-T)}^{T} dx \, \frac{T^2 + x^2}{(1-T)(1-x)} + \int_{2(1-T)}^{T} dx \, \frac{(2-T-x)^2 + x^2}{(x+T-1)(1-x)} \right]$$

$$= \sigma_0 \frac{2\alpha_s}{3\pi}\left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)} \right]$$

this has a singularity as $T \to 1$

$$[\quad] \longrightarrow \frac{4}{1-T}\left(\ln(1-T) - \tfrac{3}{4}\right)$$



and looks like this. But if we introduce an IR cutoff the divergence in the three jet differential cross section will cancel the divergence of the two jet cross section, as we saw in QED.

As expected, the mean value of $1-T$ is not IR sensitive:

$$\langle 1-T \rangle = \frac{1}{\sigma_0}\int_{2/3}^{1} dT \frac{d\sigma}{dT}(1-T) \simeq 1.05\, \alpha_s/\pi$$

But a big contribution to $\langle 1-T \rangle$ comes from $T\sim 1$, where nonperturbative corrections are quite important, so it is dangerous to compare this prediction with the existing data.

Since nonperturbative corrections are still too large to allow direct comparison of predicted values of thrust (or other jet variables) with data, one resorts to a cruder approach. Instead of calculating thrust directly from the observed events, a "cluster algorithm" is first applied to fit the event with three jets. Then the "observed" thrust distribution is compared with the data (in the kinematic regime where 3 jets seem to dominate). It fits pretty well, with

$$\alpha_s \sim .17 \pm .01$$ } But quite large systematic error; dependence on "model of hadronization".

at PETRA energies, $\sqrt{s} \sim 35$ GeV.

Order α_s^2 corrections have also been computed, making it possible to extract a value of Λ

$$\Lambda_{\overline{MS}} \sim 100 - 500 \text{ MeV},$$

in agreement with other values of Λ. (For a recent review, G. Kramer, DESY Report 83-068)

The above discussion stressed the angular distribution of three-jet events, since this is a distinctive prediction of QCD which can be tested. More closely analogous to our discussion of Coulomb scattering in QED is the calculation of the 2-jet cross section to order α_s.

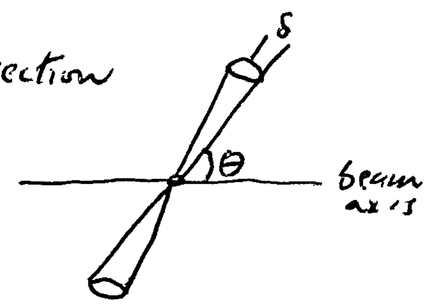
If the quark is taken to be massless, the IR divergence in

 +

is not cancelled by 

for arbitrarily soft gluons. Degenerate with an outgoing quark is a quark accompanied by a hard collinear gluon (i.e. $\vec{p}$ parallel to $\vec{k}$) and we must sum over these final states as well. (The same remark applies to QED with a massless electron)

We can define a two-jet cross section
$d\sigma(s, \epsilon, \theta, \delta)$ for
all but a fraction ϵ of the hadronic energy $\sqrt{s}$ coming out in a pair of back-to-back cones with opening half angle δ.



Jets with these properties are generated in order α_s in perturbation theory in three ways:

i) q and $\bar{q}$; no gluon

ii) $q\bar{q}$ and soft gluon, $k < \epsilon\sqrt{s}$

iii) q and $\bar{q}g$ or $\bar{q}$ and qg, where gluon is hard, and nearly collinear with quark or antiquark ($k > \epsilon\sqrt{s}$, angle $< 2\delta$)

The cross sections for processes (i) (ii) (iii) can be computed, with a gluon mass μ acting as an IR cutoff, (i) and (ii) being similar to the QED calculations we considered.

The results are (G. Sterman and S. Weinberg, Phys. Rev. Lett. 39, 1436 (1977).) ....

$$d\sigma_{(i)} = d\sigma_0 \left[1 + \left(\frac{4}{3}\frac{\alpha_s}{\pi}\right)\left(-2\ln^2\left(\frac{2E}{\mu}\right) + 3\ln\left(\frac{2E}{\mu}\right) + \cdots\right)\right]$$

$$d\sigma_{(ii)} = d\sigma_0 \left(\frac{4}{3}\frac{\alpha_s}{\pi}\right)\left(2\ln^2\left(\frac{4\epsilon E}{\mu}\right) + \cdots\right)$$

$$d\sigma_{(iii)} = d\sigma_0 \left(\frac{4}{3}\frac{\alpha_s}{\pi}\right)\left(-2\ln^2(2\epsilon) - 4\ln(2\epsilon)\ln\left(\frac{2\delta E}{\mu}\right) - 3\ln\left(\frac{2\delta E}{\mu}\right) + \cdots\right)$$

where $2E = \sqrt{s}$; i.e., E is the energy of e^- or e^+

Adding the three contributions to the two-jet cross section together gives

$$d\sigma(E,\epsilon,\theta,\delta) = d\sigma_0 \left[1 + \frac{4\alpha_s}{3\pi}(\quad)\right]$$

where

$$\begin{aligned}(--) &= -2\ln^2\left(\frac{2E}{\mu}\right) + 3\ln\left(\frac{2E}{\mu}\right) + 2\left(\ln\left(\frac{2E}{\mu}\right) + \ln(2\epsilon)\right)^2 \\ &\quad - 2\ln^2(2\epsilon) - 4\ln(2\epsilon)\left(\ln\left(\frac{2E}{\mu}\right) + \ln(\delta)\right) - 3\left(\ln\left(\frac{2E}{\mu}\right) + \ln(\delta)\right) \\ &= -4\ln(2\epsilon)\ln(\delta) - 3\ln(\delta) + \cdots\end{aligned}$$

or

$$\boxed{d\sigma(E,\epsilon,\theta,\delta) = d\sigma_0\left[1 + \frac{4\alpha_s}{3\pi}\left(-4\ln(2\epsilon)\ln(\delta) - 3\ln(\delta)\right)\right]}$$

where $d\sigma_0 = \frac{\alpha^2}{4s}\left(\sum_f 3Q_f^2\right)(1+\cos^2\theta)\, d\Omega$ is the zeroth order cross section for annihilation into $q\bar{q}$ pairs.

The logs of μ cancel as expected. The double logs can be summed as in QED, but perturbation theory breaks down nonetheless for $\alpha_s \ln(1/\delta) \sim 1$. In any case, nonperturbative effects on the width of the jet become important for small enough δ.

But RG-improved perturbation theory does work for large E and δ not too small. Thus, QCD reliably predicts that the e^+e^- annihilation cross section is dominated at large energy by two narrow jets with a $(1+\cos^2\theta)$ angular distribution.

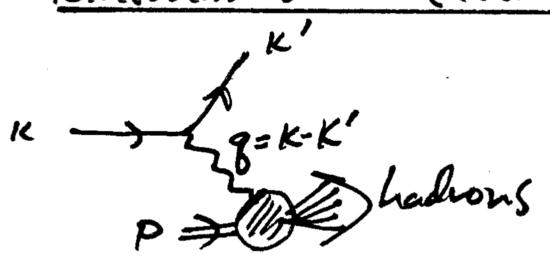
Similarly, higher order calculations will reveal that the order α_s calculation reliably gives the angular distribution of 3-jet events at asymptotically high energy.

A good reference on IR divergences, the KLN theorem and jets is..

T. D. Lee, Particle Physics and Intro to Field Theory, Harwood, 1981.

O. Electroproduction and the Parton Model

Kinematics of electroproduction:



We consider (inelastic) scattering of an electron off a target nucleon which is at rest in the lab, to lowest order in QED

$$\text{Graph} = \frac{ie^2}{q^2}\,\bar{u}(k')\gamma^\mu u(k)\,\langle n|J_\mu(0)|p\rangle$$

– $|n\rangle$ is the hadronic final state

Sum over final electron polarization, average over initial polarization, and sum over final hadron state to obtain inclusive electroproduction cross section:

$$d\sigma = \frac{1}{2m}\,\frac{1}{2E}\left(\frac{e^2}{q^2}\right)^2 \frac{1}{4}\,\mathrm{tr}(\not{k}\gamma^\mu \not{k}'\gamma^\nu)\,\frac{d^3k'}{2E'(2\pi)^3}$$

$$\times \sum_n (2\pi)^4\,\delta^4(p+q-p_n)\,\langle p|J_\nu(0)|\,\rangle\langle n|J_\mu(0)|p\rangle$$

$k=(E,\vec{k})$
$k'=(E',\vec{k}')$
$p=(m,\vec{0})$
$q=k-k'$

$q^2 = -2k\cdot k' = -2EE'(1-\cos\theta) = -4EE'\sin^2\frac{\theta}{2}$

where θ is the electron scat. angle (neglecting the electron mass) writing $d^3k' = E'^2 dE' d\Omega'$, we have

$$\frac{d\sigma}{dE' d\Omega'} = \frac{\alpha^2}{q^4}\frac{E'}{E}\, \ell^{\mu\nu} W_{\mu\nu}$$

where $\ell^{\mu\nu} = 2(k^\mu k'^\nu + k'^\mu k^\nu - \eta^{\mu\nu} k\cdot k')$

$$W_{\mu\nu} = \frac{1}{2m(2\pi)}\int d^4x\, e^{iqx} \langle p| J_\mu(x) J_\nu(0)|p\rangle$$

{ average over proton spin

(by the same reasoning as on p(1.69))

using covariance, current conservation, parity, we may write $W_{\mu\nu}$ in terms of two Lorentz-invariants.

$$W_{\mu\nu} = \left(\frac{q_\mu q_\nu}{q^2} - \eta_{\mu\nu}\right) W_1 + \frac{1}{m^2}\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) W_2$$

(W_1 and W_2 must be functions of the invariants $p\cdot q$, q^2, and $p^2 = m^2$)

We may write the cross section $d\sigma$ in terms of $W_{1,2}$, by contracting the tensors:

$$2\left(\frac{q_\mu q_\nu}{q^2} - \eta_{\mu\nu}\right)(k^\mu k'^\nu + k'^\mu k^\nu - \eta^{\mu\nu} k\cdot k')$$

$$= 4\left[\frac{(q\cdot k)(q\cdot k')}{q^2} - k\cdot k'\right] - 2(k\cdot k' - 4k\cdot k'))$$

and $q = k - k'$, $k^2 = k'^2 = 0$ $\Rightarrow$ $= -4\frac{(k\cdot k')^2}{q^2} + 2k\cdot k' = -2q^2$
$= 8EE'\sin^2\frac{\theta}{2}$

$$2\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right)(2k'^\mu k^\nu - \eta^{\mu\nu} k\cdot k')$$

$$= 4\left[p\cdot k' - \frac{p\cdot q}{q^2} q\cdot k'\right]\left[p\cdot k - \frac{p\cdot q}{q^2} q\cdot k\right] - 2(k\cdot k')\left[p^2 - \frac{p\cdot q}{q^2}\right]$$

$$= 4m^2 \frac{1}{4}(E+E')^2 + q^2 m^2 - m^2(E-E') = 4m^2 EE'\left(1-\sin^2\frac{\theta}{2}\right)$$

or
$$\boxed{\frac{d\sigma}{dE'd\Omega'} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\left[2W_1\sin^2\frac{\theta}{2} + W_2\cos^2\frac{\theta}{2}\right]}$$

$W_{1,2}$ are called structure functions — they can be measured by observing only the final energy and scattering angle of the electron (W_2 is easier.)

"Structure" functions because they reveal, by their dependence on $p\cdot q$ and q^2, the structure of the hadron probed by the photon.

For high energy inelastic scattering ($q^2 \gg m^2$) off an extended object (hadron with size $\sim m^{-1}$) one would expect sensitive dependence on q^2/m^2. E.g. elastic form factors fall off like a power of $(q^2/m^2)^{-1}$. But this behavior is not observed; instead structure functions become independent of q^2 as $q^2 \to \infty$! More precisely define

$$x = -q^2/2p\cdot q \qquad \nu = E-E' = \text{energy transfer} = (p\cdot q)/m$$

One observes that the dimensionless functions

$$m W_1(-q^2/m^2, x) \longrightarrow F_1(x)$$
$$\nu W_2(-q^2/m^2, x) \longrightarrow F_2(x)$$
(scaling functions)

depend only on x for q^2/m^2 large.
This behavior is called (Bjorken) scaling.
Scaling already works for $-q^2 \doteq 2(\text{GeV})^2$.

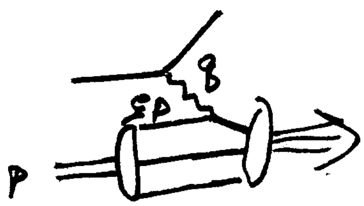
To understand the interpretation of this observation, consider the significance of x:

The invariant mass of the excited hadronic system is greater than or equal to m, the mass of the nucleon

$$M^2 = (p+q)^2 = m^2 + q^2 + 2p\cdot q \geq m^2$$

$$\Rightarrow -q^2 \leq 2p\cdot q \quad \text{or} \quad x \leq 1$$

and $x=1$ corresponds to elastic scattering



Suppose, now, that the virtual photon is actually absorbed by a pointlike constituent of the nucleon — a parton. The scattering is incoherent off the nucleon, but coherent (elastic) off the parton.

If the parton carries a "fraction" of the nucleon 4-momentum ξ (neglecting transverse momentum),

$$(\xi p)^2 = (q+\xi p)^2 \qquad \text{or} \quad q^2 + \xi(2p\cdot q) = 0$$

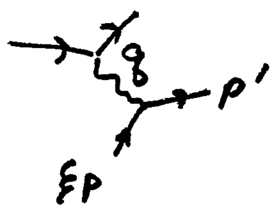
$$\Rightarrow \xi = x$$

Scaling occurs because the scattering is incoherent, i.e., the electron does not care about the large-distance physics which causes the parton to "fragment" into hadrons. This separation of scales can be justified in QCD, because of asymptotic freedom.

Lets now explore the implications of this parton picture a little more fully, before getting into the details of how the picture can be justified in QCD.

We imagine that the nucleon contains several partons, all moving collinearly (no transverse momentum). The main idea of the parton model is that we do not need to worry about the interactions which hold the partons together, so we ignore them. How do the partons contribute to

$$W_{\mu\nu} = \frac{1}{2m}\int \frac{d^4x}{(2\pi)} \quad e^{iqx}\langle p|J_\mu(x)J_\nu(0)|p\rangle ?$$



For a parton carrying fraction ξ of the nucleon momentum p, we have the contribution

$$\hat{W}_{\mu\nu}(\xi) = \frac{1}{2m}\,\frac{1}{2}\sum_{\text{spins}}\int\frac{d^4p'}{(2\pi)^4}\,\delta(p'^2)\,\theta(p'^0)$$
$$\times\ \frac{1}{\xi}\langle\xi p|J_\mu(0)|p'\rangle\langle p'|J_\nu(0)|\xi p\rangle(2\pi)^4\delta^{(4)}(\xi p+q-p').$$

The $\delta(p'^2)\theta(p'^0)$ restricts the sum to on-shell final state partons and the factor ξ^{-1} is stuck in to normalize the matrix element, since $\langle\xi p|\xi p\rangle = \xi\langle p|p\rangle$.

$$= \frac{1}{2m\xi}\,2\left[\xi p_\mu(\xi p+q)_\nu + (\xi p+q)_\mu p_\nu - \eta_{\mu\nu}\,\xi p\cdot(\xi p+q)\right]$$

(assuming parton of unit charge)
$$\times\ \delta[(q+\xi p)^2]\,\theta(\xi p^0+q^0)$$

and $\delta(q^2+2\xi q\cdot p)\,\theta(\xi p^0+q^0) = \frac{1}{2q\cdot p}\delta(\xi - x)$

so we have
$$\hat{W}_{\mu\nu}(\xi) = \frac{\delta(\xi-x)}{2\xi m(p\cdot q)}\left[2\xi^2 p_\mu p_\nu - \xi\eta_{\mu\nu}p\cdot q + \xi(p_\mu q_\nu + q_\mu p_\nu)\right]$$

Now let $f(\xi)d\xi$ be a probability distribution, giving the number of partons carrying momentum fraction ξ (times Q^2, where Q is the parton charge) between ξ and $\xi+d\xi$

i.e. $f(\xi) = \sum_i Q_i^2 [q_i(\xi) + \bar{q}_i(\xi)]$, (1.130)

where q_i is the distribution function for parton of type i, which has charge Q_i.
Then we find $W_{\mu\nu}$ by integrating:

$$W_{\mu\nu} = \int_0^1 d\xi\, f(\xi)\, \hat{W}(\xi)$$

$$\text{or } W_{\mu\nu} = \frac{f(x)}{2m^2\nu}\left[2x\, p_\mu p_\nu - \eta_{\mu\nu}\, p\cdot q + p_\mu q_\nu + p_\nu q_\mu\right]$$

and we can extract

$$W_1 = \text{coef of } (-\eta_{\mu\nu}) \Rightarrow W_1 = \frac{1}{2m} f \Rightarrow \boxed{F_1(x) = \tfrac{1}{2} f(x)}$$

$$W_2 = \text{coeff of } \frac{p_\mu p_\nu}{m^2} \Rightarrow W_2 = \frac{x f}{\nu} \Rightarrow \boxed{F_2(x) = x f(x)}$$

The scaling structure functions measure the parton distribution function. In this "free" parton model they are not independent. We have

$$\boxed{2x F_1(x) = F_2(x)} \quad \text{Callan-Gross relation}$$

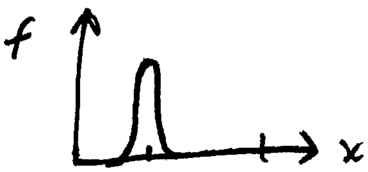
This relation is a consequence of the fact that the partons have spin $\frac{1}{2}$. It has a simple interpretation.
Consider a frame in which the parton "bounces off" the space-like photon, i.e. in which the parton direction reverses.

Since J_z of the parton changes by ± 1, the photon ~~must~~ have helicity ± 1. It cannot be longitudinal. The Callan-Gross relation just expresses the fact that the longitudinal photon contribution to the electroproduction cross section vanishes (in the massless free parton model)

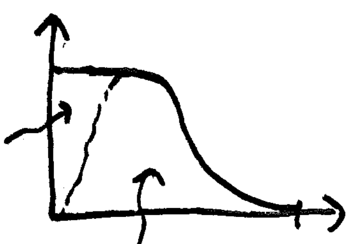
I leave it to you to

show that the Callan-Gross relation is equivalent to $\boxed{R = \frac{\sigma_{\text{Longitudinal}}}{\sigma_{\text{Transverse}}} = 0}$ ... in the scaling limit: $-q^2 \to \infty$, $-q^2/\nu$ fixed

The distribution function $f(x)$ can be extracted from the measured structure functions. If the quarks were really free, we would expect to see δ functions at $1/3$:



But the measured distribution functions look like:



Gluon exchange presumably smears out the "valence" quark contribution, and there is also a small x contribution due to sea quark pairs produced by virtual gluons

Parton Model Sum Rules

We have $2F_1 = \frac{1}{x}F_2 = f(x)$ ← there are different functions for proton and neutron

$$f = \sum Q_i^2 (q^i + \bar{q}^i)$$

E.g. $f_p(x) = \frac{4}{9}(u_p + \bar{u}_p) + \frac{1}{9}(d_p + \bar{d}_p + s_p + \bar{s}_p)$

$f_n(x) = \frac{4}{9}(u_n + \bar{u}_n) + \frac{1}{9}(d_n + \bar{d}_n + s_n + \bar{s}_n)$

We have six functions for each nucleon, but isospin symmetry implies the relations

$$u_p = d_n \equiv u \qquad \bar{u}_p = \bar{d}_n \equiv \bar{u}$$
$$d_p = u_n \equiv d \qquad \bar{d}_p = \bar{u}_n = \bar{d}$$
$$s_p = s_n \equiv s \qquad \bar{s}_p = \bar{s}_n = \bar{s}$$

and, since baryon number of the proton is 1, its charge 1, its strangeness zero, we obtain the three integrated relations

$$\int_0^1 dx\,(u-\bar{u}) = 2$$
$$\int_0^1 dx\,(d-\bar{d}) = 1$$
$$\int_0^1 dx\,(s-\bar{s}) = 0$$

The positivity of the distribution functions imply inequality constraints on the structure functions. For example

$$\frac{f_p}{f_n} = \frac{\frac{4}{9}(u+\bar{u}) + \frac{1}{9}(d+\bar{d}) + \frac{1}{9}(s+\bar{s})}{\frac{4}{9}(d+\bar{d}) + \frac{1}{9}(u+\bar{u}) + \frac{1}{9}(s+\bar{s})}$$

$$\Rightarrow \quad \frac{1}{4} \le \frac{F_{1,2;p}}{F_{1,2;n}} \le 4 \qquad \text{(Nachtman inequality)}$$

Each distribution function has a valence contribution and a sea contribution

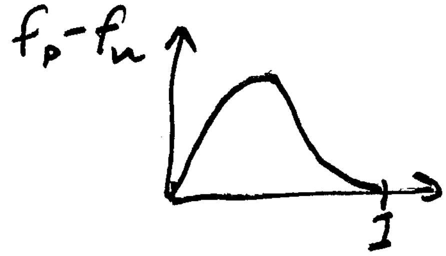
$$q = q_V + q_S$$

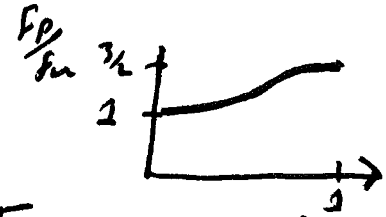
we expect $q_S^i = \bar{q}_S^i$ to be the same for each quark by SU(3) symmetry. That is, if we can neglect quark masses, all three should couple to virtual gluons democratically. We can thus remove the sea by taking the difference

$$f_p(x) - f_n(x) = \left(\frac{4}{9}u_V + \frac{1}{9}d_V\right) - \left(\frac{4}{9}d_V + \frac{1}{9}u_V\right)$$
$$= \frac{1}{3}(u_V - d_V)$$

$$\left(\Rightarrow \int_0^1 dx\,[f_p(x) - f_n(x)] = \frac{1}{3}\right)$$

With the sea subtracted out, one expects to see distribution peaked near 1/3, which *is* seen:





Also, the sea should not contribute near $x=1$. (except for exclusion principle.) and SU(3) symmetry implies $u_v = 2 d_v$] so that

$$f_p/f_n \to 3/2 \quad \text{as } x \to 1 \qquad \text{(which is observed)}$$

For $x \sim 0$ the sea dominates, and one expects

$$f_p/f_n \to 1 \quad \text{as } x \to 0 \qquad \text{(which is observed)}$$

One of the simple parton model predictions does not work well. The total momentum of the proton is the sum of its parts, so

$$1 = \int_0^1 dx\, x \sum_i \left(q^i(x) + \bar{q}^i(x)\right)$$

Note that $f_p + f_n = \frac{5}{9}(u + \bar{u}) + \frac{5}{9}(d + \bar{d}) + \frac{2}{9}(s + \bar{s})$

Since the strange quark sea should make a negligible contribution to the total momentum, we have

$$\int_0^1 (F_{2,p} + F_{2,n})\, dx = \frac{5}{9}$$

But the measured value is .28. This means that nearly half the proton momentum is being carried by neutral constituents — the gluons!

N. QCD Analysis of Electroproduction

To analyze electroproduction scaling, we must determine the asymptotic large-q^2 behavior of

$$W_{\mu\nu} = \frac{1}{2m}\int \frac{d^4x}{2\pi}\, e^{iqx} \langle p | J_\mu(x) J_\nu(0) | p \rangle$$

$$= \quad " \quad " \quad " \quad \langle p | [J_\mu(x) J_\nu(0)] | p \rangle,$$

the last equality following from the same reasoning as on p 1.70 – $\delta^4(q-p+p_n)$ vanishes because $q^0 > 0$ (in lab frame) and $p_n^0 - p > 0$.

What values of x are probed in the large q^2 limit?

In the frame
$$p = (m, 0, 0, 0)$$
$$q = (q^0, 0, 0, q^3)$$

choose variables
$$x_\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \qquad \vec{x}_\perp = (x_1, x_2)$$
$$q_\pm = \frac{1}{\sqrt{2}}(q^0 \pm q^3)$$

$$W_{\mu\nu} = \frac{1}{(2m)(2\pi)} \int dx_- \, e^{iq_+ x_-} \int dx_+ \, e^{iq_- x_+} \int d^2x_\perp \, \langle p | [J_\mu(x), J_\nu(0)] | p \rangle$$

$x_\perp^2 \leq x_+ x_-$ (commutator vanishes for spacelike separation)

In the scaling limit considered in electroproduction, we have $-q^2 \to \infty$, $\dfrac{-q^2}{2p\cdot q}$ fixed

or
$$q_\pm = \frac{1}{\sqrt{2}}\left[\nu \pm \sqrt{\nu^2 - q^2}\right] \Rightarrow$$
$$q_+ \to \sqrt{2}\,\nu \to \infty$$
$$q_- \to \frac{1}{\sqrt{2}}\, q^2/\nu = \sqrt{2}\, m x \ \text{fixed}$$

The asymptotic behavior of $W_{\mu\nu}$ is thus determined by singularities at $x_- \to 0$, x_+ fixed, i.e., for x on the light cone, but not necessarily small. In the old language of the '60s electroproduction explores current commutators <u>on the light cone</u>. Electroproduction scaling was regarded in those days as evidence of "free-field" behavior of such commutators.

Our arguments back on p(1.71) show that

$$\mathrm{Im}\; i\int d^4x\, e^{iqx} \langle p| T J_\mu(x) J_\nu(0) |p\rangle \Big|_{\mathrm{sym}\ \mu\leftrightarrow\nu}$$

$$= \frac{1}{2}\int d^4x\, e^{iqx} \langle p| \; J_\mu(x) J_\nu(0) |p\rangle \Big|_{\mathrm{sym}\ \mu\leftrightarrow\nu}$$

therefore

$$W_{\mu\nu} = \mathrm{Im}\ \frac{1}{m}\, i\int \frac{d^4x}{2\pi}\, e^{iqx}\langle p| T J_\mu(x) J_\nu(0) |p\rangle,$$

this is an object which has an expansion in terms of graphs, and which we know obeys a RG eqn. It is the imaginary part of the forward compton amplitude

The operator product expansion is crucial in the analysis of the asymptotic behavior of $W_{\mu\nu}$. We want to scale q while keeping the protons fixed on mass shell — the scaling behavior as q^2 gets large will be determined by the coefficients of the leading operators in the OPE.

Of course, we cannot really evaluate the proton matrix elements — they involve large distance physics which is essentially nonperturbative. But the OPE provides a <u>factorization</u> of the structure function into q^2 dependent pieces which can be calculated in pert. theory, and the q^2-independent pieces involving non-pert physics.

OPE Analysis of Forward Compton Amplitude

We decompose the forward compton amplitude into $T_{1,2}$ defined by

$$i\int d^4x\, e^{iqx} \langle p|\, T J^\mu(x) J^\nu(0)\,|p\rangle\big|_{\substack{\text{spin}\\ \text{ave.}}}$$

$$= \left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \frac{1}{m^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right) T_2$$

So

$$\begin{aligned} \operatorname{Im} T_1 &= 2\pi m W_1 \\ \operatorname{Im} T_2 &= 2\pi m W_2 \end{aligned} \quad \text{and} \quad \left.\begin{aligned} F_1 &= m W_1 = \frac{1}{2\pi} \operatorname{Im} T_1 \\ F_2 &= \nu W_2 = \frac{\nu}{2\pi m} \operatorname{Im} T_2 \end{aligned}\right\} \text{scaling functions}$$

We invoke the operator product expansion of $T J^\mu(x) J^\nu(0)$ to determine the asymptotic large q^2 behavior of $T_{1,2}$. Denoting the ith operator with dimension d, spin n by

$\mathcal{O}_i^{d,n}$, we may write

$$i\int d^4x\, e^{iqx}\, T J^\mu(x) J^\nu(0) \xrightarrow[-q^2\to\infty]{}$$

$$\sum_{d,n,i} \left[\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) C_{1,i}^{d,n}\left(\frac{Q}{\mu}, g_\mu\right) + \frac{1}{m^2}\left(p^\mu - \frac{p\cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p\cdot q}{q^2} q^\nu\right)\left(\frac{2xm}{\nu}\right) C_{2,i}^{d,n}\left(\frac{Q}{\mu}, g_\mu\right)\right]$$

$$(p^2 = m^2)$$

$$\times \left(\frac{2^n q_{\mu_1} \cdots q_{\mu_n}}{(q^2)^{\frac{1}{2}(d+n-2)}} \left(\mathcal{O}_i^{d,n}\right)^{\mu_1 \cdots \mu_n}\right)$$

Here $C_{1,2}$ are dimensionless, $Q \equiv \sqrt{-q^2}$, and the factors $\frac{2xm}{\nu} = -q^2/\nu^2$ and 2^n have been inserted for convenience, to simplify later formulas.

The spin-averaged proton matrix elements of these operators have the the form.

$$\langle p | (\mathcal{O}_i^{d,n})^{\mu_1 \cdots \mu_n} | p \rangle = A_i^{d,n}(p^2)(p^{\mu_1} \cdots p^{\mu_n} - \text{traces})$$

Therefore, the contribution from the operator $\mathcal{O}_i^{d,n}$ is proportional to

$$\left(\frac{2p\cdot q}{q^2}\right)^n \left(\frac{1}{q^2}\right)^{\frac{1}{2}(d-n-2)} \left(1 + O\left(\frac{m^2}{q^2}\right)\right)$$

In the scaling limit $-q^2 \to \infty$, $x = -q^2/2p\cdot q$ fixed, the forward Compton amplitude is dominated by the contribution due to operators with minimal

$$d - n \equiv \text{"}\underline{\text{twist}}\text{"}$$

If we ignore $O(m^2/q^2)$ power corrections, we retain only twist two operators (the leading twist)

$$T_1 = \sum_{n,i} C_{1,i}^n\left(\frac{Q}{\mu}, g_\mu\right) A_i^n(p^2,\mu)\left(\frac{1}{x}\right)^n$$

$$T_2 = \frac{2xm}{\nu} \sum_{n,i} C_{2,i}^n\left(\frac{Q}{\mu}, g_\mu\right) A_i^n(p^2,\mu)\left(\frac{1}{x}\right)^n$$

Only even values of n occur, because $\langle p | T\, J_\mu(x) J_\nu(0) | p\rangle$ is invariant under $\mu \leftrightarrow \nu$, $x^\mu \to -x^\mu$. Since, by parity, it is also symmetric under $\mu \leftrightarrow \nu$, its Fourier transform is an even function of q, and hence of $x = -q^2/2p\cdot q$

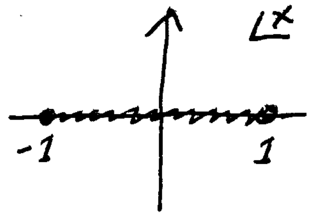
The scaling functions are

$$F_1 = \frac{1}{2\pi} \operatorname{Im} T_1 = \frac{1}{2\pi} \operatorname{Im} \sum_{n,i} C_{1,i}^n\left(\frac{Q}{\mu}, g_\mu\right) A_i^n \left(\frac{1}{x}\right)^n$$

$$F_2 = \frac{\nu}{2\pi m} \operatorname{Im} T_2 = \frac{2}{2\pi} \operatorname{Im} \sum_{n,i} C_{2,i}^n\left(\frac{Q}{\mu}, g_\mu\right) A_i^n \left(\frac{1}{x}\right)^{n-1}$$

(Note that these satisfy the Callan-Gross relation $F_2 = 2xF_1$ if $C_1 = C_2$.)

The OPE becomes an expansion in powers of $\frac{1}{x}$. But we are interested in $F_{1,2}$ for physical values of x, $0 \le x \le 1$, where this expansion is divergent.

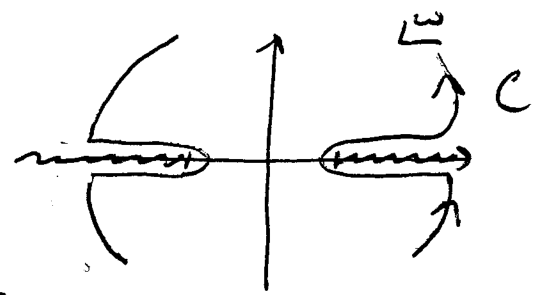


In fact, $T_{1,2}$ have a cut in the complex x plane running from -1 to 1. The structure functions are given by the discontinuity across the cut. It is this discontinuity, then, which can be obtained from experiment.

The coefficients in the expansion of $T_{1,2}$ in powers of $\frac{1}{x}$ are related to the moments of the structure functions. E.g., suppose we have

$$T_1 = \sum_{n=0}^{\infty} B_n \omega^n \quad \text{where } \omega = \frac{1}{x}$$

We know experimentally that $x F_1(x)$ is integrable as $x \to 0$ or $\frac{1}{\omega} \operatorname{Im} T_1(\omega)$ is integrable as $\omega \to \infty$. If we can justify ignoring the contour C at infinity, we can relate the integral of the discontinuity of $\frac{1}{\omega^{m+1}} T_1(\omega)$ along the cut to the coefficient B_m:



$$\frac{1}{2\pi i} \int_C d\omega \, \frac{T_1(\omega)}{\omega^{m+1}} = B_m = \frac{4i}{2\pi i} \int_1^{\infty} d\omega \, \frac{\operatorname{Im} T_1(\omega)}{\omega^{m+1}}$$

$$= 4 \int_0^1 \frac{dx}{x^2} \, x^{m+1} \frac{1}{2\pi} \operatorname{Im} T_1(x) = 4 \int_0^1 dx \, x^{m-1} F_1(x)$$

We can justify ignoring the contour at $\omega = \infty$ only by an argument which goes beyond perturbation theory. In any case, since this contribution is finite for any $m > 1$, it is plausible that it is zero.

The same argument can be applied to $\frac{1}{x} F_2(x)$.

we find, then, for $n>1$

$$\boxed{\begin{aligned} \int_0^1 dx\, x^{n-1} F_1(x,Q) &= \frac{1}{4}\sum_i C^n_{1,i}\left(\tfrac{Q}{\mu}, g_\mu\right) A^n_i(\mu) \\ \int_0^1 dx\, x^{n-2} F_2(x,Q) &= \frac{1}{2}\sum_i C^n_{2,i}\left(\tfrac{Q}{\mu}, g_\mu\right) A^n_i(\mu) \end{aligned}}$$

— The moment sum rules (Christ-Hasslacher-Mueller).

The moments of the structure functions, which in the "naive" parton model would be independent of Q, acquire a Q-dependence which resides in the coefficient functions $C_{1,2}$. We cannot calculate the n-dependence of the moments — that depends on A^n_i, which is essentially nonperturbative. We don't have a prediction for the shape of $F_{1,2}$, only for how the shape changes as a function of Q (that is, we can calculate the logarithmic corrections to naive scaling; not the $O(m^2/Q^2)$ power corrections.)

Will find the Q dependence of $C(\frac{Q}{\mu}, g_\mu)$, and hence of the moments of the structure functions, by using the renormalization-group-improved perturbation theory. We saw how the scaling behavior of C could be determined back on p. 1.57. First we note that the electromagnetic current J_μ is unrenormalized, so $\gamma_J=0$. This can be regarded as a consequence of electromagnetic gauge invariance (eA is unrenormalized), or as a special case of a more general phenomenon — Conserved symmetry currents (gauge or global) are always unrenormalized. This result follows from the Ward identity:

$$\partial^\mu \langle 0|T J_\mu(x)\,\phi(y)|0\rangle = \delta^4(x-y)\,\langle 0|\delta\phi(y)|0\rangle, \quad (\partial^\mu J_\mu = 0)$$

$$\text{where} \quad [J_0(\vec{x},t), \phi(\vec{y},t)] = \delta^3(\vec{x}-\vec{y})\,\delta\phi(\vec{x},t)$$

If ϕ and $\delta\phi$ have the same anomalous dimension, then applying the RG eqn to both sides gives $\gamma_J=0$

The scaling behavior of $C_\mathcal{O}$ is thus determined by the anomalous dimension of $\mathcal{O}$. That is,

$$C_\mathcal{O}(Q,\mu,g_\mu)\,\mathcal{O}(\mu) \quad \text{is} \quad \mu\text{-independent}$$

$$C_\mathcal{O}(Q/\mu,g_\mu)\,\mathcal{O}(\mu) = C(\,1\,,g_Q)\,\mathcal{O}(\mu=Q)$$
$$= C(\,1\,,g_Q)\,Z_\mathcal{O}(Q)\,Z_\mathcal{O}^{-1}(\mu)\,\mathcal{O}(\mu)$$

At asymptotically large Q, we can calculate $C(g_Q)$ as a power series in g_Q, and

$$Z_\mathcal{O}(Q)\,Z_\mathcal{O}^{-1}(\mu) = \exp\left[\int_\mu^Q \frac{d\mu'}{\mu'}\gamma_\mathcal{O}(\mu')\right] = \exp\left[\int_{g_\mu}^{g_Q}\frac{dg}{\beta(g)}\gamma_\mathcal{O}(g)\right]$$

If $\gamma_\mathcal{O}$ and β have expansions

$$\gamma_\mathcal{O} = \gamma_{\mathcal{O},0}\,g^2 + \cdots \qquad = \left(\frac{g^2_Q}{g^2_\mu}\right)^{-\gamma_{\mathcal{O},0}/2b_0}$$
$$\beta = -b_0 g^3 + \cdots$$

or

$$\boxed{C_\mathcal{O}(Q,\mu,g_\mu) = C(g_Q)\left(\frac{g_Q^2}{g_\mu^2}\right)^{-\gamma_{\mathcal{O},0}/2b_0}}$$

So, to find the Q-dependence of the structure functions, we need to calculate the coefficient functions of the leading (twist-two) operators, to leading order, and the anomalous dimensions of these operators, to order g^2.

Coefficient Functions

The relevant twist-two operators are

Quark operators:

$$\mathcal{O}_q^n = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S(\bar{q}\,\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_n} q) - \text{traces}$$

denotes symmetrization

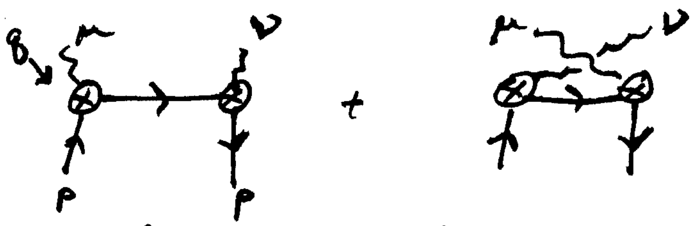
Gluon operators:

$$\mathcal{O}_A^n = \frac{1}{2}\left(\frac{i}{2}\right)^{n-2} S(F^{\nu\mu_1 a}\,\overleftrightarrow{D}^{\mu_2}\cdots\overleftrightarrow{D}^{\mu_{n-1}}F^{\mu_n a}{}_{\nu}) - \text{traces}$$

All the other twist-two operators can be written as total divergences, and therefore make no contribution to the forward matrix element $\langle p|\mathcal{O}|p\rangle$; we may ignore them

We must calculate the coefficient functions of these operators in the OPE, and their anomalous dimensions, to determine the q^2 dependence of the moments of the structure functions

To lowest (zeroth) order in g, only the quark operators occur in the expansion. We find the coefficient functions by calculating quark matrix elements (spin averaged):



$$= i\left(\frac{i}{2}\right)\left[\mathrm{tr}\,\frac{\not p\,\gamma^\nu(\not p+\not q)\gamma^\mu}{(q+p)^2} + \mathrm{tr}\,\frac{\not p\,\gamma^\mu(\not p-\not q)\gamma^\nu}{(q-p)^2}\right]$$

(fermions have been taken to have unit charge)

$$= -\frac{1}{2q^2}\left[\left(1-\frac{1}{x}\right)^{-1}\mathrm{tr}\,(-) + \left(1+\frac{1}{x}\right)^{-1}\mathrm{tr}\,(\quad)\right]$$

$$= \frac{-2}{q^2\left(1-\frac{1}{x^2}\right)}\Big[\left(1+\tfrac{1}{x}\right)\left[p^\nu(p+q)^\mu + p^\mu(p+q)^\nu - \eta^{\mu\nu}p\cdot q\right] + \left(1-\tfrac{1}{x}\right)\left[p^\mu(p-q)^\mu + p^\nu(p-q)^\mu + \eta^{\mu\nu}p\cdot q\right]\Big]$$

$$= -\frac{2}{q^2}\sum_{n=0}^{\infty}\left(\frac{1}{x^2}\right)^n\left[4p^\mu p^\nu - \frac{2}{x}\eta^{\mu\nu}p\cdot q + \frac{2}{x}(p^\mu q^\nu + p^\nu q^\mu)\right]$$

$$\Rightarrow \boxed{T_1 = 2\sum_{n=1}^{\infty}\left(\frac{1}{x^2}\right)^n;\quad T_2 = \frac{-8m^2}{q^2}\sum_{n=0}^{\infty}\left(\frac{1}{x^2}\right)^n = \frac{+4m}{\nu}\sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^{2n+1}}$$

(Note Callan-Gross relation: $2xmT_1 = \nu T_2$)

The quark operators have been defined so that the forward spin-averaged quark matrix elements are

$$\langle p | (\mathcal{O}_q^n)^{\mu_1 \cdots \mu_n} | p \rangle = p^{\mu_1} \cdots p^{\mu_n} - \text{traces}$$

comparing the free-quark calculation of $T_{1,2}$ with the expansion on page (1.137), we have

$$C_{1,q}^{2n} = [2 + O(g_Q^2)]\, Q_f^2$$
$$C_{2,q}^{2n} = [2 + O(g_Q^2)]\, Q_f^2$$

where Q_f is the charge of the quark.

Nonsinglet Structure Functions

Next, we must calculate the anomalous dimensions of the twist-two operators. This calculation is simplest for the isospin nonsinglet operators,

$$\mathcal{O}_{NS}^n = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S(\bar{q}\gamma^{\mu_1}\overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_n}\tau^3 q) - \text{traces},$$

where q is the quark doublet $\binom{u}{d}$ and $\tau^3 = \frac{1}{2}\sigma^3$, because isospin symmetry prevents these operators from mixing with the gluon operators

Experimentally, the effects of these operators are isolated when we measure the differences

$$F_{1,2}^{(n)} - F_{1,2}^{(p)},$$

since the singlet operators drop out when we take difference of proton and neutron matrix elements. In the OPE we have

$$2\left[\frac{4}{9}\mathcal{O}_u^{2n} + \frac{1}{9}\mathcal{O}_d^{2n}\right] = 2\left[\frac{5}{18}(\mathcal{O}_u^{2n} + \mathcal{O}_d^{2n}) + \frac{1}{6}(\mathcal{O}_u^{2n} - \mathcal{O}_d^{2n})\right]$$

$$= \frac{2}{3}\,\mathcal{O}_{i,NS} + \text{singlet}$$

The anomalous dimensions of nonsinglet quark operators are evaluated to $O(g^2)$ by calculating the infinite parts of the graphs.

Exercise 1.13

Calculate the anomalous dimensions of the isospin nonsinglet operators in QCD.

The result is $\gamma^n_{NS} = \gamma^n_{NS,0}\, g^2 + \cdots$, where you are to find $\gamma^n_{NS,0}$. We thus have

$$C^n_{1,2,NS}(Q,\mu) = \tfrac{2}{3}\left(\frac{g_Q^2}{g_\mu^2}\right)^{-\gamma^n_{NS,0}/2b_0}\left(1 + O(g_Q^2)\right)$$

and the moment sum rules take the form

$$M^n_{NS,1} \equiv \int_0^1 dx\, x^{n-1} F_1^{NS}(x,Q) = \tfrac{1}{2} B^n_{NS}\left(\ln Q^2/\Lambda^2\right)^{\gamma^n_{NS,0}/2b_0}$$

$$M^n_{NS,2} = \int_0^1 dx\, x^{n-2} F_2^{NS}(x,Q) = B^n_{NS}\left(\ln Q^2/\Lambda^2\right)^{\gamma^n_{NS,0}/2b_0}$$

$\gamma_0 < 0$, so moments decrease as $Q \to \infty$.

Distributions shift toward small x.

(Note Callan-Gross relation, $F_L \equiv F_2 - 2xF_1 = 0$, survives leading QCD corrections.)

We expect a plot of $\ln M^n$ vs $\ln M^m$ to be a straight line with a known slope. This _is_ seen; the observed slopes are consistent with the QCD predictions.

Attempts have been made to determine Λ_{QCD} by fitting the measured moments of the nonsinglet structure functions. Among the subtleties of this procedure are

i) It is necessary to calculate next-order corrections in α_s, to determine the choice of renorm. scale for which higher order corrections are small (in a given scheme)

ii) The moments depend most sensitively on q^2 at low q^2, so one wants to use data at relatively low q^2. But at low q^2, it is easy to confuse power law scaling violations $\sim m^2/q^2$ with log scaling violations.

iii) The <u>kinematic</u> m^2/q^2 corrections can be removed by taking "Nachtman moments" which project out operators of definite spin for finite q^2. But the intrinsically nonperturbative m^2/q^2 corrections due to twist-4 operators remain.

The result of the analysis, from several independent experiments is

$$\boxed{100\,\text{MeV} < \Lambda_{\overline{MS}} < 400\,\text{MeV}}$$

Singlet Structure Functions

The analysis of the singlet structure functions is complicated by the mixing of quark operators and gluon operators.

In one-loop order, the multiplicative renormalization of the gluon operators arises from the graphs

and mixing of quark and gluon operators is induced by

(It is not completely obvious that we can ignore mixing with the ghost operators, but it is nonetheless true. See, e.g. H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D 12, 3159 (1975).)

Calculating the graphs, one finds a 2×2 anomalous dimension matrix for each spin n

$$\gamma_{ij}^{n} = \gamma_{ij,0}^{n} g^2 + \cdots$$

With mixing, the coefficient functions scale like

$$C^n_{\mathcal{O}_i}(Q,\mu) = \sum_j C^n_{\mathcal{O}_j}(Q,Q)\left[\underset{\sim}{Z}(Q)\,\underset{\sim}{Z}(\mu)^{-1}\right]_{ji}$$

$$= \sum_j C^n_{\mathcal{O}_j}(Q,Q)\, T\!\left(\exp\int_0^{\ln Q/\mu} dt\, \underset{\sim}{\gamma}(g_t)\right)_{ji}$$

But, to order g^2, there are linear combinations of quark and gluon operators which are renormalized multiplicatively. We may write

$$\underset{\sim}{\gamma}_0^n = \sum_k \lambda_k^n \underset{\sim}{P}_k^n$$

where P_k is a (not necessarily orthogonal) projection

$$\sum \underset{\sim}{P}_k = \underset{\sim}{\mathbb{1}}, \quad \underset{\sim}{P}_k \underset{\sim}{P}_j = \delta_{kj}\underset{\sim}{P}_k,$$

and the coefficient functions are

$$\hat{C}^n(\mu) = \hat{C}^n(Q)\exp\left(\int_{g_\mu}^{g_Q} \frac{dg}{-b_0 g}\sum_k \lambda_k \underset{\sim}{P}_k\right)$$

$$= \hat{C}^n(Q)\sum_k\left(\frac{g_Q^2}{g_\mu^2}\right)^{-\lambda_k/2b_0}\underset{\sim}{P}_k$$

where $\hat{C}$ denotes a two component column vector. $C^n(Q)$ vanishes in lowest order for the gluon operators, but both eigenvectors of $\underset{\sim}{\gamma}^n$ have a quark operator component, and we obtain

$$\boxed{\begin{aligned} M^n_{S,2} &\equiv \int_0^1 dx\, x^{n-2} F_2^S(x,Q) \\ &= A^n_{S,1}\left(\ln Q^2/\Lambda^2\right)^{\lambda_1^{(n)}/2b_0} + A^n_{S,2}\left(\ln Q^2/\Lambda^2\right)^{\lambda_2^{(n)}/2b_0}\end{aligned}}$$

Asymptotically, one term will dominate

Energy-Momentum Sum Rule

One singlet twist-two operator has a special status, because its proton matrix element is known — the energy-momentum tensor

$$\Theta_{\mu\nu} = \frac{i}{4}\sum_f \bar{q}_f(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu) q_f + F^{\alpha\,a}_{\mu} F^{a}_{\nu\,\alpha} - \text{traces}$$
$$= 2(\Theta_q^{(n)} + \Theta_A^{(n)}) \quad \text{for } n=2$$

and

$$\int d^3x \langle p|\Theta_{00}(x)|p'\rangle = (2\pi)^3\delta^3(\vec{p}-\vec{p}')\langle p|\Theta_{00}(0)|p'\rangle$$
$$= \langle p|P_0|p'\rangle = 2p_0^2(2\pi)^3\delta^3(\vec{p}-\vec{p}')$$

$$\Rightarrow \langle p|\Theta_{\mu\nu}|p\rangle = 2p_\mu p_\nu$$ (i.e., same for quark as proton)

By calculating, one finds for the twist-two operators of spin 2

$$\underset{\sim}{\gamma}^{(2)} = -\frac{g^2}{6\pi^2}\begin{bmatrix} 2C_2(R) & -T(R) \\ -2C_2(R) & T(R)\end{bmatrix}$$

This annihilates $\begin{pmatrix}1\\1\end{pmatrix}$ (acting to the left), because Θ is unrenormalized (it is conserved). The other eigenvalue is negative so the Θ term dominates asymptotically

The projections are (acting to left)

$$P_\Theta = \left(\frac{1}{2C_2(R)+T(R)}\right)\begin{bmatrix} T(R) & T(R) \\ 2C_2(R) & 2C_2(R)\end{bmatrix}, \quad \begin{bmatrix} 2C_2(R) & -T(R) \\ -2C_2(R) & T(R)\end{bmatrix}\left(\frac{1}{2C_2(R)+T(R)}\right)$$

We therefore have

$$\int_0^1 dx\, F_2^S(x) = \underbrace{\langle Q^2\rangle(1,0)}_{\substack{\text{zeroth order}\\ \text{coefficient}\\ \text{function}}} P_\Theta \begin{pmatrix}\langle\Theta_q^2\rangle\\ \langle\Theta_A^2\rangle\end{pmatrix} = \langle Q^2\rangle \frac{T(R)}{2C_2(R)+T(R)}$$

(i.e. factor of 2 in C cancelled by $\frac{1}{2}$ in moment sum rule) or $$\boxed{\int_0^1 dx\ F_2^S(x) = \langle Q^2\rangle \frac{T(R)}{2C_2(R)+T(R)}}$$

For 3 flavor QCD

$$\begin{array}{l} T(R) = 3/2 \\ C_2(R) = 4/3 \end{array} \Rightarrow \frac{T}{2C_2+T} \quad \frac{3/2}{8/3+3/2} = \frac{9}{25}$$

$$\int_0^1 dx\ F_2^S(x) = \frac{9}{25}\langle Q^2\rangle \qquad \text{(three quarks)}$$

If we compare to our sum rule on p(1.133)

$$\int_0^1 dx\ \tfrac{1}{2}(F_2^n + F_2^p) = \langle Q^2\rangle \times \left(\begin{array}{c}\text{Fraction of momentum}\\ \text{carried by quarks}\end{array}\right)$$

Thus this 9/25 can be interpreted in parton-model language as the fraction r of momentum carried by quarks. At higher q^2 we should include more quarks(?):

$$r = \frac{3n_f}{16+3n_f} \qquad \text{is the general answer}$$

Note that this moment is q^2-independent asymptotically — the area under the curve F_2^S stays fixed in the scaling limit, although its shape changes.

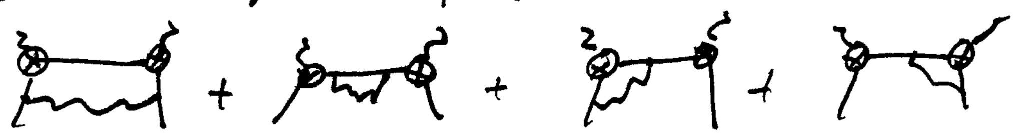
Corrections to σ_L/σ_T

The Callan-Gross relation $\sigma_L = 0$ is satisfied by the lowest-order QCD result there is no contribution to

$W_L \propto \frac{\nu W_2}{2mx} - W_1$ because the coefficient function

$\hat{C}_L^n = \hat{C}_2^n - \hat{C}_1^n$ vanishes to lowest

To find the leading contribution to σ_L, we must calculate coefficient functions to order g^2. If we confine our attention to non-singlet structure functions, then we need only the coefficients of nonsinglet operators, generated by the graphs


+ crossed

But only the first graph contributes to C_L. One finds

$$\frac{C_L^n}{C_2^n} = \frac{\alpha_s}{\pi}\frac{C_2(R)}{n+1} + O(\alpha_s^2)$$

Therefore, the moments of the longitudinal structure functions satisfy

$$\boxed{\frac{M^n_{NS,L}}{M^n_{NS,2}} = \frac{\alpha_s(-q^2)}{\pi}\frac{4}{3(n+1)} + O(\alpha_s^2)}$$

σ_L is difficult to measure, but the experimental situation has recently improved. Now there is a good bound on...

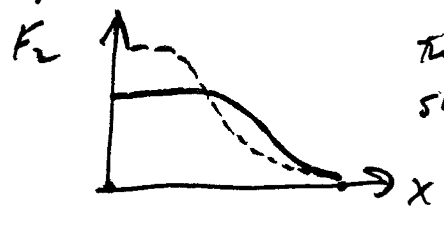
$$R \equiv \frac{\sigma_L}{\sigma_T} < .006 \pm \underset{\substack{\uparrow \\ \text{statistical} \\ \text{error}}}{.012} \pm \underset{\substack{\uparrow \\ \text{systematic} \\ \text{error}}}{.025}$$

for $-q^2 = 38(\mathrm{GeV})^2$

and $.4 \leq x \leq .7$

Parton Model Interpretation of QCD Results

We found $M^n \sim [\ln(-q^2)]^{-\gamma_0^n/2b_0}$ for non-singlet structure functions, and the γ^n are positive for $n \geq 2$



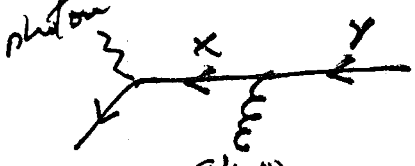
This implies distribution functions shift toward smaller x as $-q^2$ increases. Why does this occur?

As $-q^2$ increases, the structure of the nucleon is probed at increasingly short distances. In the naive parton picture, one sees only the pointlike, nearly free partons at very short distances, and scaling is violated only by corrections down by $1/q^2$.

Apparently, in QCD, the interactions do not turn off rapidly enough at short distances for naive scaling to apply. Scaling fails because the partons have structure (As $-q^2 \to \infty$, we are really probing the structure of a quark.)

The point is that a quark can turn into a quark and a gluon if a try to resolve it on a finer scale. Thus, the quark distribution functions depend on q^2

Let $t = \frac{1}{2}\ln(-q^2/\Lambda^2)$. The t dependence of the quark distribution function is determined (to order g_t^2) by an equation of the form



$$\frac{d}{dt}q(x,t) = g_t^2 \int_x^1 dy \int_0^1 dz\, \delta(x - yz)\, P_{qq}(z)\, q(y,t)$$

or

$$\boxed{\frac{dq(x,t)}{dt} = g_t^2 \int_x^1 \frac{dy}{y}\, P_{qq}\left(\frac{x}{y}\right) q(y,t)}$$

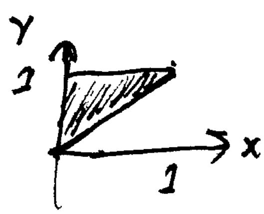
Altarelli-Parisi Eqn.

Here $P_{qq}(z)$ is a kinematic function expressing the variation (with t) of the probability of finding a quark 'inside' a quark carrying momentum fraction z

The function P_{qq} can be computed, but instead we will see how the scaling behavior of the distribution function predicted by this equation relates to what we have already calculated.

Consider a moment of the distribution function

$$M^{n-1} = \int_0^1 dx\, x^n q$$



$$\frac{dM^{n-1}}{dt} = g_t^2 \int_0^1 dx\, x^n \int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) q(y)$$

$$= g_t^2 \int_0^1 dy \int_0^1 d\left(\frac{x}{y}\right) \left(\frac{x}{y}\right)^n P_{qq}\left(\frac{x}{y}\right) y^n q(y)$$

$$= g_t^2 \left[\int_0^1 dz\, z^n P_{qq}(z)\right] M = g_t^2 \underbrace{D^{n-1}}_{\text{moment of Altarelli–Parisi function}} M^{n-1}$$

Thus $\int d\ln M^{n-1} = D^{n-1} \int dt\, g_t^2 = \frac{D^{n-1}}{2b_0} d\ln t$

$$\text{or} \quad \frac{M^{n-1}(t)}{M^{n-1}(t_0)} = \left(\frac{t}{t_0}\right)^{D^{n-1}/2b_0}$$

Thus $\boxed{D^n = -\gamma_{NS}^n}$ the moment of the A-P function P_{qq} is the anomalous dimension of a nonsinglet twist-two operator

In the nonsinglet case, we need to worry about the probability of a gluon containing a quark or a gluon:

$$\frac{dq(x)}{dt} = g_t^2 \int_x^1 \frac{dy}{y}\left[P_{qq}\left(\frac{x}{y}\right)q(y) + P_{Gq}\left(\frac{x}{y}\right)G(y)\right]$$

$$\frac{dG}{dt} = g_t^2 \int_x^1 \frac{dy}{y}\left[P_{qG}\left(\frac{x}{y}\right)q(y) + P_{GG}\left(\frac{x}{y}\right)G(y)\right]$$

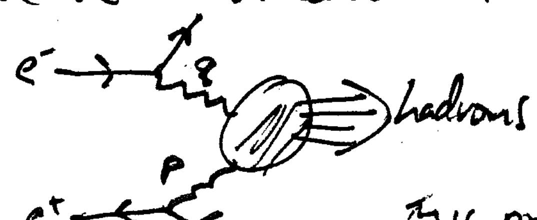
This is the parton model interpretation of operator mixing: Gluons contain quarks, but their quark content is flavor-symmetric.

A reference on the connection between the parton model and the operator analysis is A. Buras, Rev. Mod. Phys. 52, 199 (1980).

Q. Structure Functions of the Photon

(E. Witten, Nucl. Phys. B120, 189 (1977).)

Deep inelastic scattering off a photon target can be studied in e^+e^- collisions

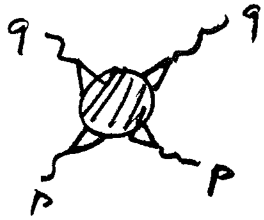


$-q^2$ large
p^2 small

This process is interesting because, at asymptotic $-q^2$, it is possible to calculate the structure functions themselves, not just how the moments scale

(The e is accompanied by a well collimated beam of nearly real photons with dk/k spectrum; we are measuring deep inelastic $e\gamma$ scattering if one electron is scattered at a finite angle, and the other disappears down the beam pipe.)

We need to calculate —



the forward amplitude for scattering of a virtual photon with an on-shell photon (and take the imaginary part)

This differs from electroproduction calculation, because photon matrix elements of quark/gluon operators are of order α_{em}. Hence, to do a consistent calculation to $O(\alpha_{em})$ we must include photon operators

$$\frac{1}{2}\left(\frac{i}{2}\right)^{n-2} F^{\nu\mu_1} \overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_{n-1}} F^{\mu_n}{}_{\nu}$$

Although they have coefficient functions of order α_{em}, photon matrix elements of these operators are order 1.

We cannot calculate photon matrix elements of quark/gluon operators, but we can calculate photon matrix elements of photon operators. It turns out that these dominate for large q^2. That is why it is possible to calculate the structure functions.

We need to integrate the RG eqn to find the coefficient functions

$$\hat{C}^n(q^2/\mu^2, g_\mu, \alpha) = \hat{C}^n_0(g_q, \alpha)\, T\left(\exp\left[\int dt \;\; \underset{\sim}{\gamma}(t)\right]\right)$$

$$= \hat{C}^n_0 \, \underset{\sim}{M} \quad \text{where} \quad \underset{\sim}{M} = \begin{bmatrix} M_H & M_{H\gamma} \\ M_{\gamma H} & M_{\gamma\gamma} \end{bmatrix}$$

need to $O(1)$ need to $O(\alpha)$

not needed in $O(\alpha)$ calculation

can replace by 1

It's $O(\alpha)$ and multiplies

$$\hat{C}_0 = \begin{bmatrix} O(1) \\ O(\alpha) \end{bmatrix} \begin{array}{l} \leftarrow \text{hadron operators} \\ \leftarrow \text{photon operators} \end{array}$$

To the relevant order $\underset{\sim}{\gamma} = \begin{pmatrix} \gamma_H & \gamma_{H\gamma} \\ 0 & 0 \end{pmatrix}$

γ_H is the standard quark-gluon mixing matrix. $\gamma_{H\gamma}$ is generated, in order e^2, by

If we write

$$\gamma_H = g^2 \gamma_0$$
$$\gamma_{H\gamma} = e^2\ \Gamma$$

then, to lowest order in e^2

$$M_{H\gamma} = \int_0^t dt' \left(T \exp \int_{t'}^{t} \frac{dt''}{2b_0 t''} \gamma_0 \right) (\Gamma e^2)$$

Expand γ_0 in terms of projectors onto eigenvectors:

$$\gamma_0 = \sum_K \lambda_K P_K .$$

$$M_{H\gamma} = \int_0^t dt' \Gamma\, e^2 \sum_K \left(\frac{t'}{t} \right)^{-\lambda_K/2b_0} P_K \qquad \text{(as on p 1.145.)}$$

$$M_{H\gamma} = \Gamma e^2 \sum_K \frac{t}{1-\lambda_K/2b_0} P_K$$

This grows like $\ln -q^2$. The factor $(1-\lambda_K/2b_0)^{-1}$ is the renormalization of  by leading-log gluons

The dominant contribution to F_2^γ comes from matrix elements of photon operators. The coefficient function in $M_{H\gamma}$ times the $O(1)$ coefficient of the quark operators. This grows like $\ln -q^2$

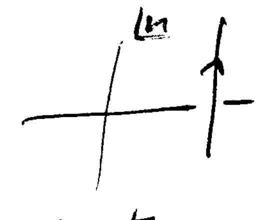
The dominant contributions to F_L^γ also come from photon operators. Because C_L for quark operators is $O(q^2)$, this scales canonically (is independent of q^2)

All the moments of F_2^γ and F_L^γ can be computed. From these F_2^γ and F_L^γ themselves can be recovered.

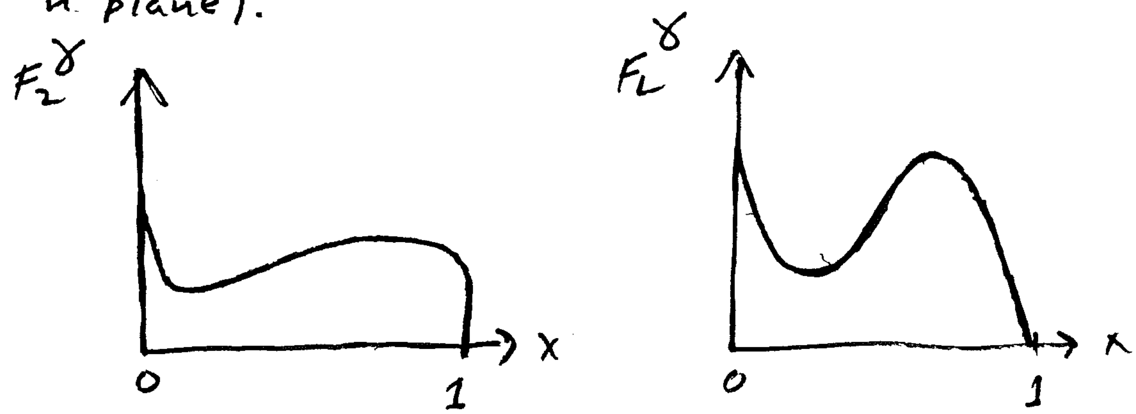
The inversion formula for moments is

$$M^n = \int_0^1 dx\, x^{n-2} F_2(x)$$

$$F_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-(n-1)} M^n$$



(where contour is to right of all singularities in n plane).



Above are sketches of asymptotic forms for $F_{2,L}$. The approach to these curves is not expected to be uniform in x.

Although the data is not that good, since the normalization of F_2 is predicted, one can attempt to find Λ_{QCD} by a fit.

Higher order corrections have been computed and fits give

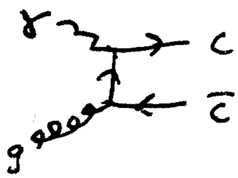
$$100\,\text{MeV} \lesssim \Lambda_{\overline{MS}} \lesssim 300\,\text{MeV}$$

R. Perturbative Miscellany

1. Heavy Quarks

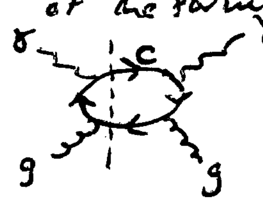
The asymptotic ($Q \to \infty$) form of the contribution to the electroproduction structure functions due to heavy quarks ($M^2 \gg \Lambda^2$) can be calculated in terms of the light quark structure functions (Ref. E. Witten, Nucl. Phys. B104, 445 (1976).)

The leading contribution to heavy quark production in deep-inelastic scattering is "photon-gluon fusion":

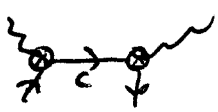


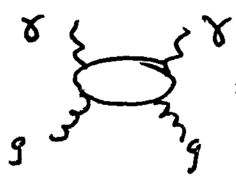
(In parton model language, gluon fragments to heavy sea quarks, which can be struck by photon.)

i.e. there is a contribution to the imaginary part of the γ-g forward amplitude of the form



For $Q \to \infty$, we can factorize this contribution, using the OPE.

 $\Rightarrow C$  Heavy quark twist two operator

and  $\Rightarrow C(Q)$  – Matrix element of heavy quark operator between gluon states

If the quark mass M is very large, the quark loop can be effectively shrunk to a point

 $\Rightarrow A(M)$  – Heavy quark operator matrix elements between light states can be expressed in terms of matrix elements of operators constructed from the light fields, up to $1/M^2$ corrections.

we have the "heavy quark expansion":

$$\langle \text{light} | (\mathcal{O})_{\text{heavy}} | \text{light} \rangle = \sum_i A_i(M, \mu, g_\mu) \langle \text{light} | (\mathcal{O})_{\text{light}, i} | \text{light} \rangle + O\left(\frac{1}{M^2}\right)$$

— sums over operators with same quantum nos as $(\mathcal{O})_{\text{heavy}}$, constructed from light fields

Since the coefficients are universal, i.e. are the same for all matrix elements, an expansion of this kind should apply even for hadron matrix elements, although it can be proved only for quark/gluon matrix elements order by order in perturbation theory.

To demonstrate such an expansion for twist-two operators, note that all matrix elements of these operators except the two-quark and two-gluon matrix elements are convergent by power counting. So graphs contributing to these matrix elements are either finite, and suppressed by $1/M^2$, or contain convergent subgraphs which have already been dealt with in a lower order of P.T.

So we may confine our attention to the two-quark and two-gluon matrix elements of the operators

$$\mathcal{O}^n_c = \tfrac{1}{2}\left(\tfrac{i}{2}\right)^{n-1} S\, \bar{c}\gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \cdots \overleftrightarrow{D}^{\mu_n} c - \text{traces},$$

which are log-divergent by power counting. But any graph contributing to this matrix element becomes superficially convergent when differentiated $n+1$ times wrt external momentum. Then it is either convergent, and down by $1/M^2$, or contains a divergent subgraph which has been dealt with in lower order.

Thus, the replacement of $(\mathcal{O})_{\text{heavy}}$ by the expansion is valid, up to $O(1/M^2)$ corrections, except for a polynomial in momentum lost by differentiating. The coefficient of this polynomial determines A_i in each order of perturbation theory in g_μ

Operator renormalization generates logs of M/μ; but these can be summed using the renorm. group. Working to first nontrivial (one loop) order, the expansion of the product of two heavy currents,

$$J_H = \bar{c}\gamma_\mu c,$$

can be written (schematically)

$$T(J_H J_4) = C_H(1, g_Q)\, Z(Q) Z^{-1}(M)\, A\, Z'(M) Z'^{-1}(\mu)\, \mathcal{O}(\mu)$$

Here the method of effective field theories is used. The RG eqn is integrated from Q down to M in the "full" field theory, including the heavy quark. Then the heavy quark is integrated out at $\mu \sim M$, and the operators of the full theory are expressed in terms of light-field operators. Finally, the RG eqn is integrated from M down to μ, in the effective theory.

There are three relevant twist-2 operators of spin n

$\mathcal{O}_c^n$, $\mathcal{O}_{S,light}^n = \sum_{i=1}^m \mathcal{O}_{S,i}$ (sum over light flavors)

$\mathcal{O}_A^n$ - gluon operators,

which are conveniently reexpressed as

$\mathcal{O}_{NS}^n = \mathcal{O}_c^n - \frac{1}{m+1}\mathcal{O}_{S,light}^n$ — doesn't mix

$\left.\begin{matrix}\mathcal{O}_{S,light}^n \\ \mathcal{O}_A^n\end{matrix}\right\}$ mix

In lowest order, we have

$$C_H = [\underset{\text{non-singlet}}{1},\ \underset{\text{singlet}}{\tfrac{1}{m}},\ \underset{\text{gluon}}{0}],$$

and, in a leading log calculation, we can integrate out c by replacing $\mathcal{O}_c$ by zero. Thus A is the 2 by 3 matrix

$$A = \begin{bmatrix} -1/m & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

There is also an expansion for the product of (singlet) light quark currents in which the nonsinglet operators do not occur

$$T(J_L J_L) = C_L(1, g_Q)\, Z(Q)\, Z^{-1}(M)\, A\, Z'(M)\, Z'^{-1}(\mu)\, \mathcal{O}(\mu)$$

where $C_L = [0, 1, 0]$.

These eqns can be used to calculate the ratios of contributions due to heavy and light quarks to the structure functions

$$R_n = \frac{\int_0^1 dx\, x^{n-2} F_{2,H}}{\int_0^1 dx\, x^{n-2} F_{2,L,S}}$$

Asymptotically for $Q \gg M \gg \mu$, the nonsinglet operator always decays away (it has a larger anomalous dimension) and we have

$$R_n \to \frac{1}{m}$$

But the approach to this limit is very slow, if $\alpha_s(M)$ is already fairly small.

For $M \gg \mu$, we can ignore the eigenvector of $\underset{\sim}{\gamma}$ with larger anomalous dimension in $Z'(M) Z'^{-1}(\mu) \langle \mathcal{O}(\mu) \rangle$ and predict the R_n's

$$R_n \sim \frac{\frac{1}{m}\left(g_Q^2/g_M^2\right)^{-\lambda_s^n/2b_0} - \frac{1}{m}\left(g_Q^2/g_M^2\right)^{-\gamma_{NS,0}^n/2b_0}}{\left(g_Q^2/g_M^2\right)^{-\lambda_s^n/2b_0}}$$

$$= \frac{1}{m}\left[1 - \left(g_Q^2/g_M^2\right)^{-(\gamma_{NS,0}^n - \lambda_s^n)/2b_0}\right]$$

Numerical values indicate that heavy quark distribution is concentrated at small x (as expected), and R_2 is small (a few percent) for $Q^2 \sim 30\ \text{GeV}^2$ and $M \sim 1.5\ \text{GeV}$ (charmed quark)

2. Semiinclusive Processes (Drell-Yan)

The parton model interpretation of our electroproduction calculation suggests the possibility of extending the QCD analysis beyond processes to which the OPE can be applied.

But when one attempts this, a problem arises — mass singularities. These are logs of $(-q^2/m^2)$ that appear in graphs, where m is a mass of a quark. These spoil perturbation theory for large q^2; we can do a perturbative QCD analysis only if they do not occur

i) For $e^+e^- \to$ hadrons, no mass singularities occur, in total cross section, or jet cross section

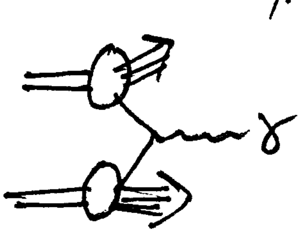
ii) For electroproduction, $\ln q^2/m^2$ terms can be summed and factored out, i.e., we find, after summing leading logs

$$f_1(q^2/\mu^2)\, f_2(m^2/\mu^2)$$

So the q^2 dependence is isolated, and the m^2 dependence gets absorbed into distribution functions, matrix elements.

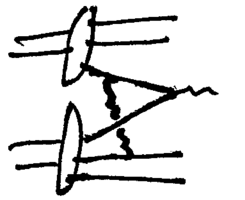
In electroproduction, this factorization is ensured by the OPE, but might other (semi-inclusive) processes factorize in this way?

E.g. Drell-Yan: $pp \to \mu^+\mu^- X$
$p\bar{p} \to \mu^+\mu^- X$



A quark and antiquark annihilate into virtual photon

A graphical analysis appeared to show that the mass singularities do factorize to all orders of perturbation theory (ignoring power corrections).



But recently (about the past 3 years) there has been controversy about this. Some experts claim that soft gluon exchanges with the "spectator" quarks do not really factorize. Others disagree. This is a technical question but a serious one. The applicability of QCD perturbation theory to all hadron-hadron collisions is at stake.

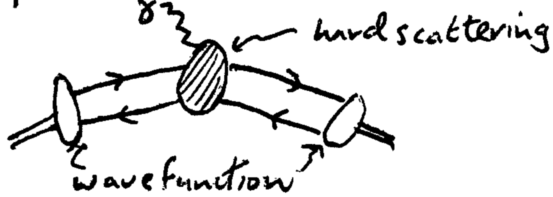
3. Exclusive Processes

Ref: G. P. Lepage and S. J. Brodsky, Phys. Rev. D22, 2157 (1980).
A. H. Mueller, Phys. Rep. 73, 238 (1981).

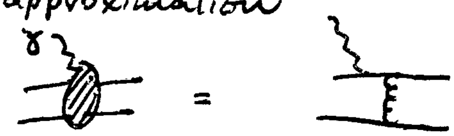
Can QCD perturbation theory be applied to exclusive processes, such as elastic hadron-hadron scattering? As above, the main problem is defining an amplitude which is free of mass singularities, to which the renormalization group can be applied to determine asymptotic behavior. We can attempt to implement the strategy described above of absorbing IR sensitivity into wave functions.

For concreteness, consider the simplest case, the electromagnetic form factor of the pion.

Schematically, the pion form factor is



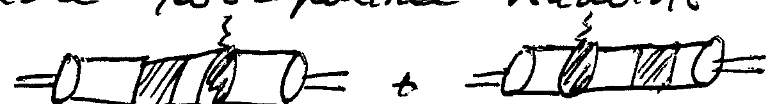
In tree approximation



But are there IR divergences due to soft virtual gluonic corrections?

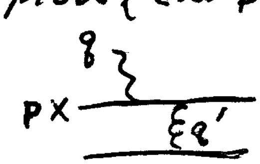
No, the soft gluon divergences are guaranteed to cancel, because the amplitude for emission of a <u>real</u> gluon does. This is because the hadron is a color singlet; a very long wavelength gluon cannot resolve the colored constituents of the pion.

There are <u>collinear</u> divergences, however, which occur when a collinear quark and antiquark exchange a collinear gluon. But these occur only in graphs which are two-particle reducible



So the collinear divergences can be absorbed into the pion wave function.

So we are left with a <u>hard</u> scattering part  in which internal momenta are typically far off shell. But there is a further divergence that occurs at the endpoint of the integration over the longitudinal momentum of the constituents of the pion ("endpoint singularity").



If the quark struck by the photon carries a fraction x of the total pion momentum p, then the momentum which must be transferred to the other quark by the gluon is

$$Q'^2 = (1-x)Q^2$$

In the higher order gluonic corrections to , the argument of g^2_μ is $\mu^2 \sim Q^2(1-x)$, so perturbation theory breaks down for $1-x < \Lambda^2/Q^2$ (the recoil of the pion is not then dominated by short distance physics)

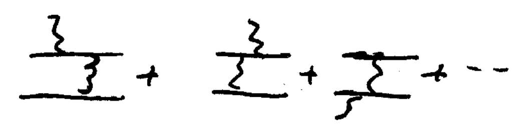
Moreover, soft gluon corrections to the quark-photon vertex are infrared divergent (the other quark acts as a spectator when the first quark is struck) We find double logarithms like

$$-C_2 \frac{\alpha_s}{\pi} \ln^2(Q^2/\Lambda^2) \qquad \text{for quark which goes offshell by of order } \Lambda$$

(cf. p 1.116) But we know that these double logs exponentiate when summed up

$$\sim \exp\Big[-\underbrace{C_2 \frac{\alpha_s(Q)}{\pi\, 2}}_{\text{subleading logs turn } \alpha \text{ into running coupling}} \ln^2(Q^2/\Lambda^2)\Big]$$

Thus, the potentially dangerous endpoint region is actually suppressed by this "Sudakov suppression factor." So the hard scattering amplitude is free of infrared singularities in the relevant kinematic regime, and is susceptible to a RG analysis.

For large Q, the tree graphs  dominate, so

 $$\sim \frac{\alpha_s(Q)}{Q}$$

the $\frac{1}{Q}$ we can obtain from dimensional analysis. An amplitude with n external lines has dimension

$$A \sim \frac{1}{M^{n-4}}$$

for relativistically normalized states (with δ^4 factored out)

(For the graphs shown, this includes a factor of Q^2 from the dirac spinors)

There are also logarithms of Q generated by integrating over the transverse momenta of the quarks in the pion up to $k_\perp \sim Q$. These logs can be analyzed using the OPE and the RG. The light cone ($x = -q^2/2p\cdot q = 1$) limit of

$$\int d^4x\, e^{iqx} \langle 0 | T\, \psi(x)\, \psi(0) | \pi, p \rangle$$

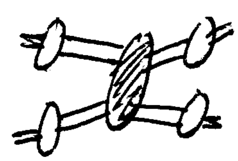
is determined by expanding in twist-two operators.
Since  $\sim q_\mu F_\pi(Q^2)$, we find

$$F_\pi(Q^2) \sim \frac{\alpha_s(Q)}{Q^2} \left| \sum_n \xi_n (\ln Q^2/\Lambda^2)^{\gamma^n_{NS,0}} \right|^2 \left[1 + O(\alpha_s(Q)) \right]$$

$$\sim \frac{\alpha_s(Q)}{Q^2} (\ln Q^2/\Lambda^2)^{2\gamma^{(2)}_{NS,0}}$$

(only isospin nonsinglet operators couple to pion.)

Fixed-Angle Scattering



The hard scattering part of a hadron-hadron scattering process has an amplitude which scales like

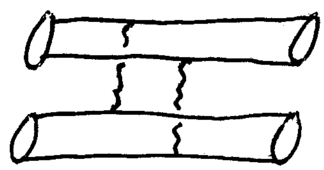
$$A \sim \left(\frac{1}{Q}\right)^{n-4}$$

where n is the total number of constituents in initial and final states. For fixed-angle scattering with $s \to \infty$ and s/t fixed, the cross section scales naively like

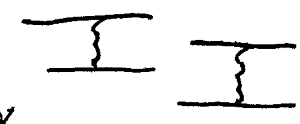
$$\frac{d\sigma}{dt} \sim \frac{1}{s^2}|A|^2 \sim \frac{1}{s^{n-2}}$$

or $\frac{d\sigma}{dt} \sim$ $1/s^6$ π-π scattering
$1/s^{10}$ p-p scattering

But there is a new type of singularity which arises in hadron-hadron scattering (the Landshoff "pinch" singularity) which threatens to invalidate the scaling prediction. E.g. there is a contribution to $\pi\pi$ elastic scattering (the "multiple scattering" contribution) of the form



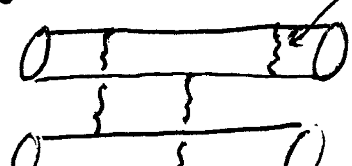
Here the "hard scattering" part is disconnected



and therefore scales differently (enhanced by power of s) relative to connected amplitude

Of course, this power counting also applies if one of the tree gluons in the connected tree graph gets soft compared to s.

Again, Sudakov double logs arise to suppress the nonscaling kinematic region; this time the pinch region:



this gluon is not scaling

If, as $s \to \infty$, the "offshellness" of a quark line is held fixed at Δ, we obtain Sudakov double logs

$$\sim \ln^2(s/\Delta^2)$$

Summing these gives the Sudakov suppression factor

$$\sim \exp\left[-2C_F \frac{\alpha_s(\theta)}{\pi} \ln^2(s/\Delta^2)\right]$$

Does this factor suffice to make the pinch contribution smaller than the naive scaling contribution?

The answer is unknown. It may be that the asymptotic behavior of elastic hadron-hadron scattering is not dominated by a short distance contribution which scales, but rather by a pinch contribution, which does not.

The power law scaling predictions work pretty well experimentally for pp, Kp, πp scattering. There is no experimental evidence for logarithmic scaling violations, or for a pinch contribution, at present energies.

4. <u>Strong Interaction Corrections to Effective Weak Hamiltonian</u>

Covered in Physics 234.

4. Strong Interaction Corrections to Effective Weak Hamiltonian

A long-standing mystery in the theory of weak interactions – the $\Delta I = \frac{1}{2}$ rule

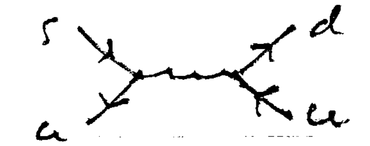
E.g. $$\frac{\Gamma(K_S^0 \to \pi^+\pi^-)}{\Gamma(K^+ \to \pi^+\pi^0)} \sim 450$$ $\pi^+\pi^-$ both $I=2$ and $I=0$

$\pi^+\pi^0$ $I=2$

$\Delta I = \frac{1}{2}$ process in enhanced (Octet enhancement)

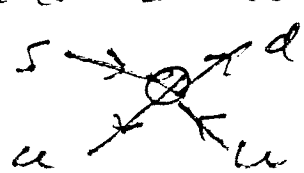
Wilson – this is a <u>dynamical</u> enhancement due to the strong interactions.

To what extent can we calculate the QCD corrections and verify Wilson's conjecture? The "effective Hamiltonian" for $\Delta S = 1$ processes, to lowest order in κ, is generated by the graph

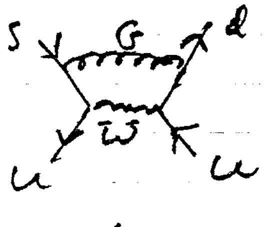
$-iH =$  $$= \frac{-ie^2}{\cos^2\theta_W M_W^2} \sin\theta_c \cos\theta_c$$

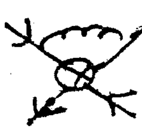
$$\times \left[\bar{s}_L \gamma^\mu u_L \, \bar{u}_L \gamma_\mu d_L - \underset{+h.c.}{u \leftrightarrow c}\right]$$

(At low momentum transfer, we may ignore the K^2 in the propagator) This has $I = \frac{1}{2}$ and $I = \frac{3}{2}$ pieces

For the purpose of calculating matrix elements at low momentum transfer this is the same thing as a pointlike operator  with coefficient $O\left(\frac{1}{M_W^2}\right)$

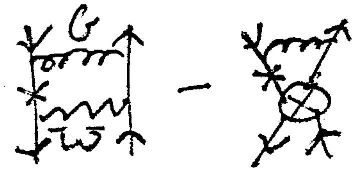
Now, what happens if we allow the strong interactions to dress up this process. In one-loop order we have e.g.



May we regard this, to leading order in $\frac{1}{M_W^2}$, as merely a strong-interaction correction to the matrix element of the operator:  ??

That is, can we find the piece of the graph  which goes like $1/M_W^2$ by ignoring the momentum dependence in the W-propagator?

No, this approximation is not valid because contracting the W propagator makes the graph (logarithmically) UV divergent. However, if we differentiate wrt an external momentum, or a quark mass, then

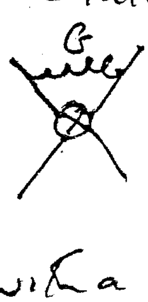


$$= O\left(\frac{1}{M_W^4}\right)$$

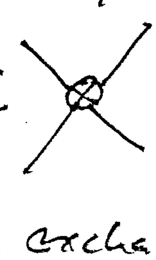
– Because the graph remains convergent when W propagator is replaced by $1/M_W^2$

Therefore, if we are interested only in $O(1/M_W^2)$ contributions

$=$



$+ \; C$



Dressing the graph with a gluon exchange has two effects: there is a "soft-gluon" contribution which renormalizes the matrix element, but there is also a "hard-gluon" contribution (from gluons with $k^2 \sim M_W^2$) to the coefficient of the operator. The coefficient is independent of external masses and momenta.

This factorization of the effects of virtual gluons persists to all orders. We can't use perturbation theory to calculate the matrix elements of the operators appearing in the effective weak Hamiltonian, but we can use the RG to calculate the coefficients in a leading log approximation. We will have

$$H^{|\Delta S|=1} = \frac{e^2}{\cos^2\theta_W M_W^2} \sin\theta_c \cos\theta_c$$

$$\times \left[A_{\frac{1}{2}}\left(\frac{M_W}{\mu}, g_\mu\right) \mathcal{O}_{\frac{1}{2}} + A_{3/2}\left(\frac{M_W}{\mu}, g_\mu\right) \mathcal{O}_{3/2} \right]$$

(where it is understood that matrix elements are to be computed to all orders in the strong interactions, but to zeroth order in the weak interactions)

The coefficient functions depend on μ (we use a mass independent scheme, so that they are independent of masses), and so do the matrix elements, in such a way that the product is independent of μ. In general, there may be mixing, so that

$$\langle H \rangle \propto \sum_i A_i M_i \quad \text{where} \quad M_i = \langle \mathcal{O}_i \rangle$$

$$\vec{M}_{bare} = \underline{Z}(\mu)\vec{M}(\mu) = \underline{Z}(\mu_0)\vec{M}(\mu_0)$$

$$\text{or} \quad \vec{M}(\mu) = \underline{Z}^{-1}(\mu)\,\underline{Z}(\mu_0)\,\vec{M}(\mu_0)$$

$$\mu\frac{d}{d\mu}\underline{Z} = \underline{Z}\,\underline{\gamma} \Rightarrow \underline{Z}^{-1}(\mu)\,\underline{Z}(\mu_0) = T\exp\left[-\int_0^{\ln \mu/\mu_0} dt\,\underline{\gamma}(t)\right]$$

thus $\langle H \rangle \propto \sum_{ij} A_i\left(\frac{M_W}{\mu}, g_\mu\right)\left[T\exp -\int_0^{\mu/\mu_0} dt\, \underset{\sim}{\gamma}(t)\right]_{ij} M_j(\mu_0, g_0)$

If we choose $\mu = M_W$

$$\boxed{\langle H \rangle \propto \sum_{i} A_i(1, g_{M_W})\left[T\exp -\int_0^{M_W/\mu_0} dt\, \underset{\sim}{\gamma}(t)\right]_{ij} M_j(\mu_0, g_0)}$$

To do a leading order QCD calculation, we take the boundary value $A_i(1,0)$ from the free quark model (lowest order weak interaction, zeroth order in g_{QCD}), and calculate γ to one-loop order. Sliding μ_0 down takes hard gluon effects from matrix element to coefficient, where they can be calculated. But we must not slide μ_0 down too far, or perturbation theory will break down.

This calculation has been done (Gaillard and Lee; Altarelli and Maiani). One finds an enhancement

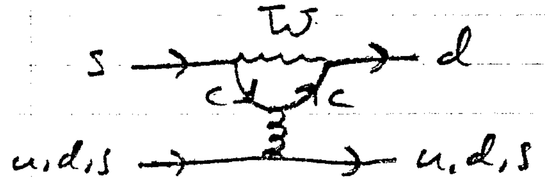
$$\frac{A_{1/2}}{A_{3/2}} \sim 3$$

This is going in the right direction to explain the $\Delta I = \frac{1}{2}$ rule, but is too small in magnitude.

Can a "large-distance" enhancement of the matrix elements of $I = \frac{1}{2}$ operators make up for this shortfall of the "short-distance" enhancement of the coefficient?

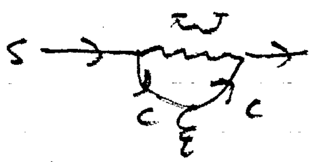
But this might not be the whole story. To obtain our weak Hamiltonian, we should integrate out heavy quarks, as well as the W. E.g., integrating out the c quark produces another $\Delta S = 1$, $\Delta I = \frac{1}{2}$ operator

Penguin diagram:



At low momentum transfer, there is a $1/k^2$ from the gluon propagator, so this does not appear at first to induce a local term in H_{eff}.

But it is! The graph

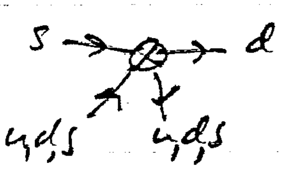


has a divergent piece which is a nondiagonal contribution to wave function renormalization (it gets cancelled by the corresponding graph with a u-quark loop). But there is a remaining finite part

$$\sim \frac{k^2}{M_W^2} \ln\left(\frac{M_W^2}{m_c^2}\right)$$

(Differentiate twice wrt $k \rightarrow$ )

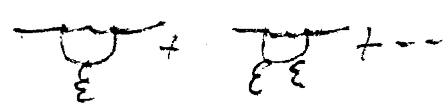
The k^2 cancels the propagator, generating the local interaction



The result persists to all orders — the QCD corrections to the Penguin diagram can be factored into matrix element times coefficient function.

To see this more clearly, note that the operator

$$g\, \bar{s}_L \gamma^\nu T^a d_L\, D^\mu F^a_{\mu\nu}$$

is generated by 

and $D^\mu F^a_{\mu\nu} = J^a_\nu = g(\bar{u}\gamma_\nu T^a u +$ d term + s term$)$

These Penguin operators are pure $\Delta I = \frac{1}{2}$. Could they provide the dominant contribution to $\Delta I = \frac{1}{2}$ processes? Their coefficient functions are small, but it has been suggested that their matrix elements are greatly enhanced. (Vainshtein, Zakharov and Shifman)

The operators generated by Penguin operators have a $(V-A)\times V$ structure; in particular, they have a $(V-A)\times(V+A)$ component. There is a (very unconvincing!) argument suggesting that the $(V-A)\times(V+A)$ $\Delta I = \frac{1}{2}$ operators have much larger matrix elements than the $(V-A)\times(V-A)$ $\Delta I = \frac{1}{2}$ operators. The argument is based on the (crude!) approximation of saturating the matrix element with the intermediate vacuum state:

E.g.

Non Penguin: $\mathcal{M}_{NP} = \langle \pi^+ | \bar{s}_L \gamma^\mu u_L \, \bar{u}_L \gamma_\mu d_L | K^+ \rangle$

$$\sim \langle \pi^+ | \bar{u}_L \gamma_\mu d_L | 0 \rangle \langle 0 | \bar{s}_L \gamma^\mu u_L | K^+ \rangle$$

$$= \frac{f_\pi}{2} \frac{f_K}{2} p_\mu p^\mu = \frac{f_\pi}{2} \frac{f_K}{2} m_K^2$$

Penguin $= \mathcal{M}_P = \langle \pi^+ | \bar{s}_L \gamma^\mu T^a d_L \, \bar{u}_R \gamma_\mu T^a u_R | K^+ \rangle$

$$T^a_{ij} T^a_{kl} \langle \pi^+ | -2 \bar{s}_{Li} u_{Rl} \, \bar{u}_{Rk} d_{Lj} | K^+ \rangle$$

(by a Fierz transformation)

$$\sim -2 T^a_{ij} T^a_{kl} \langle \pi^+ | \bar{u}_{Rk} d_{Lj} | 0 \rangle \langle 0 | \bar{s}_{Li} u_{Rl} | K^+ \rangle$$

$$= -2 T^a_{ij} T^a_{kl} \left(\tfrac{1}{3} \delta_{kj}\right)\left(\tfrac{1}{3} \delta_{il}\right) \langle \pi^+ | \bar{u}_R d_L | 0 \rangle \langle 0 | \bar{s}_L u_R | K^+ \rangle$$

(because only color singlet operators have nonvan. matrix elements.)

Since $\partial^\mu \bar{u}\gamma_\mu\gamma_5 d = i(m_u+m_d)\bar{u}\gamma_5 d$
$\partial^\mu \bar{s}\gamma_\mu\gamma_5 = i(m_s+m_d)\bar{s}\gamma_5 u$

and $\langle 0|\partial^\mu(\bar{s}\gamma_\mu\gamma_5 u)|K^+\rangle = i f_K m_K^2$
etc

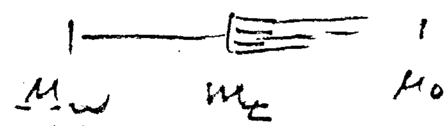
so $$\mathcal{M}_P = -\frac{8}{9}\,\frac{f_K m_K^2}{2(m_s+m_d)}\,\frac{f_\pi m_\pi^2}{2(m_u+m_d)}$$

Finally $$\left|\frac{\mathcal{M}_P}{\mathcal{M}_{NP}}\right| = \frac{8}{9}\,\frac{m_\pi^2}{(m_s+m_d)(m_u+m_d)} \sim 10$$

Take $m_u+m_d \sim 10\,\text{MeV}$
$m_s+m_d \sim 150\,\text{MeV}$

To find coefficients of Penguin operators, we must systematically integrate out heavy quarks. The boundary values (i.e. free quark value) of the coefficients of the Penguin operators vanish, but are generated in the leading log calculation by operator mixing

We want to use RG to sum up leading logs of m_t the heavy quark mass, as we summed up leading logs of M_W. To do this, we must integrate down from M_W to μ_0 in steps, stopping at each quark threshold. Below threshold, we keep only operators involving quark fields lighter than threshold, and use the β_{MS} with heavy quark integrated out



that is, leading log evaluation of some effective operator in 5 quark theory (top quark integrated out) takes the form

$$A_\ell(M_W, m_t, \mu_0, g_0)$$

$$= A_i^0 \exp\Big[-\int_{m_t}^{M_W} \underbrace{\frac{d\mu'}{\mu'}\gamma(g_{\mu'})}_{\text{evaluated in 6 quark theory}}\Big]_{ij} \underbrace{B_{jk}}_{\text{express 6 quark theory operators in terms of 5 quark theory operators @ boundary.}} \exp\Big[-\int_{\mu_0}^{m_t} \underbrace{\frac{d\mu'}{\mu'}\gamma'(g_{\mu'})}_{\text{evaluated in 5 quark theory}}\Big]_{k\ell}$$

or, for some multiplicatively renormalized operator:

$$\boxed{A_i \sim \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{c/2b_0} \left[\frac{\alpha_s(m_t^2)}{\alpha_s'(\mu_0^2)}\right]^{c'/2b_0'} A_i^0}$$

α_s', b_0' are evaluated in 5 quark model

(This procedure is justified in greater detail by E. Witten, Nucl. Phys B122, 109 (1977).)

Coefficients of Penguins and Nonpenguins have been calculated to leading log in M_W, m_t, m_b, m_c (Gilman-Wise). The enhancement of the coefficient of the nonpenguin $\Delta I = \frac{1}{2}$ operator over nonpenguin $\Delta I = 3/2$ is slightly improved over calculation including only ln M_W corrections

The penguins have coefficients smaller by about an order of magnitude. The enhancement of the penguin matrix elements must be large to account for $\Delta I = \frac{1}{2}$ rule

But if penguin operators do dominate, this has an interesting consequence. The coefficient of the most important penguin operator has a much larger phase than non penguins, giving rise to (detectable!) CP violation in $K_L \to \pi\pi$ amplitude (as well as in $K^0 \bar{K}^0$ mass matrix). This means CP violation is not purely superweak and

$$\epsilon' \propto \mathrm{Arg}\,(A_{\text{penguin}})$$

one expects

$$\epsilon'/\epsilon \sim .002 - .006$$

(depending on KM angles) This could be measured soon. But ϵ'/ϵ would be much smaller if non-penguins actually dominate

In addition to uncertainty in KM angles, there is uncertainty concerning the reliability of integrating out c-quark.

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