J. Quantization of Gauge Theories

Path-integral methods are indispensible when we quantize Yang-Mills theory. We will review some of the relevant formalism now, and, as a warm up, apply it to quantization of QED

In a generic notation in which ϕ represents all the fields of a theory, which are not necessarily scalar fields, the vacuum-to-vacuum persistence amplitude in the presence of a source J can be represented by a functional integral ("= sum over histories")

$$\langle 0+|0-\rangle_J = Z(J) = \int (d\phi)\, e^{iS[\phi]+i\int J\phi}$$

Z is the generating functions for the Green's functions $G^{(n)}$

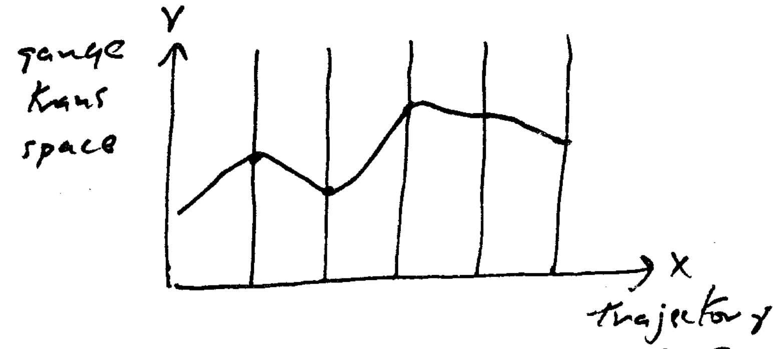
$$G^{(n)} = \frac{(-i)^n}{Z}\frac{\delta^{(n)}}{\delta J \cdots \delta J} Z \Big|_{J=0}$$

(A proof of the sum-over-histories formula, starting from the canonical formalism, can be found in, for example, Itzykson and Zuber, Quantum Field Theory.)

In gauge theories, there is a subtlety. Two histories which differ by only a gauge transformation should be considered the same history. To sum over the infinite volume of the local gauge group would introduce spurious infinities.

This problem is cured by the Fadeev-Popov method; a (functional) δ-function is introduced which specifies a surface of integration in the space of all histories which intersects each gauge-equivalence class at exactly one point.

E.g. suppose I integrated, on a finite dimensional space, a function which depended on $x_1 \ldots x_n$ but not $y_1 \ldots y_m$



then I could write
$$I = \int \prod_a dx_a \, F(\bar{x})$$
$$= \int \prod dx_a \prod dy_b \, F(x) \prod_b \delta(y_b)$$

Or choosing surface of integration defined by
$$G_b(x,y) = 0,$$
$$I = \int \prod dx_a \prod dy_b \, F(x) \prod_b \delta(G_b) \, \det \frac{\partial G_b}{\partial y_c}\Big|_{G=0}$$

Jacobian determinant ensures I does not depend on "gauge-fixing" function G

The infinite-dimensional analog of this statement is the *Fadeev-Popov ansatz* for a gauge theory
$$Z = \int [d\phi] \, e^{iS} \, \delta(G) \left(\det \frac{\delta G}{\delta \omega}\right)_{G=0}$$

δ is a functional δ function, normalized so
$$\int df \, \delta(f) = 1$$
– i.e. a δ at each point in space time. G is chosen so that specifying it at each point uniquely specifies the gauge. Now we have defined the sum over histories in the space where gauge theory dynamics is really defined — trajectories modulo gauge transformations. ($\delta\omega$ is the infinitesimal parameter of an infinitesimal gauge transformation).

(The equivalence of the Fadeev-Popov method to canonical quantization is demonstrated by, e.g. S. Coleman, Erice Summer School 1973, ed. A. Zichichi.)

If we apply the ansatz to QED, we have

$$\psi \to e^{i\delta\omega}\psi \sim (1+i\delta\omega)\psi$$
$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \delta\omega$$

and $\int (dA_\mu)$ is replaced by

$$\int (dA_\mu)\ \delta[F(A)]\ \det\left(\delta F/\delta\omega\right)$$

If we choose a "covariant" gauge, $F = \partial^\mu A_\mu - f$, then $\delta F/\delta\omega = \frac{1}{e}\partial^2$, which is independent of A, so $\det(-)$ can just be absorbed into the (irrelevant) normalization of Z.

So we have

$$= \int (dA_\mu)\ \delta[\partial^\mu A_\mu - f] \underbrace{\int df\ G[f]}_{\text{just another normalization factor}}$$

Integrate over f

$$= \int (dA_\mu)\ G[\partial^\mu A_\mu]$$

so any reasonable functional of $\partial^\mu A_\mu$ will suffice to fix the gauge

We should choose $G[-]$ so that Feynman rule will be simple. The standard choice

$$G[\partial^\mu A_\mu] = \exp\left(\frac{-i}{2\alpha}\int (\partial^\mu A_\mu)^2\right)$$

where α is an arbitrary constant, which we can choose at our convenience

Now let's find the covariant-gauge photon propagator. We have

$$\exp i S_{eff} = \exp i \int \left[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2\alpha} (\partial^\mu A_\mu)^2 + \text{electron term} \right]$$

$$= \exp \int \tfrac{i}{2} \left[A_\mu (\eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A_\nu + \tfrac{1}{\alpha} A_\mu (\partial^\mu \partial^\nu) A_\nu + \cdots \right]$$

So the inverse propagator is

$$(\Delta^{-1})^{\mu\nu} = i(\eta^{\mu\nu} p^2 - p^\mu p^\nu) + \frac{i}{\alpha} p^\mu p^\nu$$

$$\Rightarrow \boxed{\Delta_{\mu\nu} = \frac{-i}{p^2} \left[\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \alpha \frac{p_\mu p_\nu}{p^2} \right]}$$

Note that Δ blows up in the limit $\alpha \to \infty$, in which the gauge fixing term is absent; we can't invert Δ^{-1} if it has no longitudinal part. This is an infinity associated with integrating over the volume of the gauge group

The two conventional choices for α are:

$\alpha = 1$ (Feynman Gauge)

$\alpha = 0$ (Landau Gauge)

of course, no gauge-invariant quantity depends on α

Ward Identities

Let's reexamine a point discussed in Section I. There we claimed that, if a gauge-invariant regularization scheme (like dimensional regularization) is used in QED, the counterterms were guaranteed to be gauge-invariant. But now this is not so obvious, because we needed to introduce the non-gauge-invariant gauge-fixing term. Let's show that it is true, using functional methods.

What we will want to do is show that the one-particle-irreducible (1PI) Feynman graphs (calculated in the regularized theory) have certain symmetry properties. It is very convenient to find a generating functional for 1PI Green's functions

To begin with, we notice that it is easy to find a generating functional for connected Green's functions, using the theorem

$$\text{All Graphs} = \exp[\text{Connected Graphs}]$$

i.e. if $G_r^{(c)}$ $r = 1, 2, \ldots$ are all connected graphs, then

$$\text{All Graphs} = \sum_{n_r} \prod_r [G_r^{(c)}]^{n_r} \frac{1}{n_r!}$$

$$= \exp\left(\sum_r G_r^{(c)}\right)$$

So if we write $Z(J) = \exp i W(J)$, then $W(J)$ is the generating functional for the connected Green's functions

Now let $\Gamma^{(n)}(x_1, \ldots x_n)$ denote the 1PI n-point function, the sum of all 1PI graphs with n external lines. It has the generating functional

$$\Gamma[\bar\phi] = \sum_n \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \Gamma^{(n)}(x_1 \ldots x_n) \, \bar\phi(x_1) \ldots \bar\phi(x_n)$$

i.e.

$$\Gamma^{(n)}(x_1 \ldots x_n) = \frac{\delta^{(n)}}{\delta\bar\phi(x_1) \ldots \delta\bar\phi(x_n)} \Gamma[\bar\phi] \Big|_{\bar\phi = 0}$$

just as

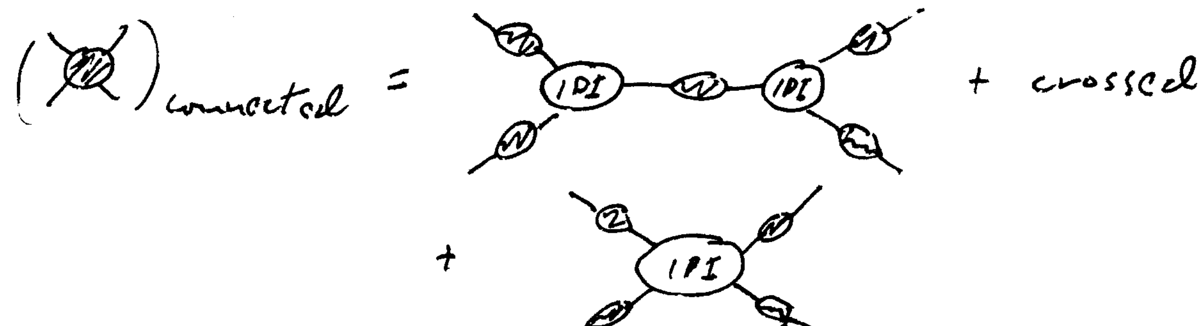
$$G^{(n)}_{\text{connected}}(x_1 \ldots x_n) = \frac{(-i)^{n-1} \delta^{(n)}}{\delta J(x_1) \ldots \delta J(x_n)} W[J] \Big|_{J=0}$$

Actually, we define $\Gamma^{(n)}$ = 1PI Graphs for $n>2$, but for $n=2$, we define

$$\Gamma^{(2)}(p) = \frac{i}{\Delta(p)}$$ – the inverse propagator (Δ is the exact propagator)

Now, we would like to relate $W[J]$, for which we have a functional integral representation, to $\Gamma[\bar{\phi}]$. To do so, note that all connected green's functions can be obtained from the 1PI parts by the following procedure:
Construct all "tree" graphs with the exact two point function as the propagator and the 1PI Green's functions as the vertices

E.g.



All the loop graphs are already included in  or the 1PI parts. So the exact greens functions are the tree approximation to an effective field theory with the (very complicated) effective action

$$S_{eff}[\bar{\phi}] = \Gamma[\bar{\phi}]$$

The tree approximation to a field theory can also be described as the small $\hbar$ (classical) approximation, where $\hbar^{-1}$ is a parameter which multiplies the whole action.

If $\quad S = \frac{1}{\hbar} S$

then $\quad$ propagator $= O(\hbar)$ $\quad$ and graph $= O(\hbar^n)$
$\quad\quad$ vertex $= O(\hbar^{-1})$ $\quad$ where $n = I - V$

The topological theorem $L = I - V + 1$
(where L: no. loops, I: no. internal lines, V: no. vertices)
then tells us

$$n = L - 1$$

The loop expansion is an expansion in powers of $\hbar$,
so tree graphs dominate as $\hbar \to 0$

Thus, the exact Green's functions of the original theory are obtained as the small $\hbar$ limit of the effective theory

$$\int (d\bar\phi)\, e^{i(\Gamma[\bar\phi] + \int J\bar\phi)/\hbar} = \exp[iW[J]/\hbar\,(1 + O(\hbar))]$$

so we can find $W[J]$ in terms of $\Gamma[\bar\phi]$ by the method of stationary phase

$$\boxed{W[J] = \left(\Gamma[\bar\phi] + \int d^4x\, J\bar\phi\right)\Big|_{\text{stationary point}}}$$

and the stationary point is determined by

$$\boxed{\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi} = -J}$$

So $W[J]$ is just a (functional) Legendre transform of $\Gamma[\bar\phi]$, familiar from thermodynamics and mechanics,

E.g. $\quad F = U - TS \quad$ where $\quad \frac{\partial U}{\partial S} = T$

We know how to invert

$$U(S) = F(T) + TS \qquad \frac{\partial F}{\partial T} = -S$$

or

$$\boxed{\Gamma[\bar\phi] = W[J] - \int d^4x\, J\bar\phi \quad \text{where} \quad \frac{\delta W}{\delta J} = \bar\phi}$$

therefore $\bar\phi$ is just $\langle 0|\phi|0\rangle$, the vacuum-expectation value of ϕ, in the presence of the current J

We are finally in a position to discuss the connection between symmetries of the action $S[\phi]$ and symmetries of the effective action $\Gamma[\bar\phi]$

Consider an infinitesimal linear transformation of the fields ϕ in a theory

$$\delta\phi(x) = A(\phi)\,\delta\omega(x)\,, \quad A \text{ is linear in } \phi$$

and suppose the change in the action under this transformation is also linear in ϕ

$$\delta S = \int d^4x\, B(\phi)\,\delta\omega(x)\,, \quad B \text{ is linear in } \phi$$

We will now show that the change in the effective action, $\delta\Gamma$, has the same form as the change in the action δS, provided that the transformation $\delta\phi$ does not change the function space measure $(d\phi)$ (at least in order $\delta\omega$)

To see this, just perform a change of integration variable in

$$e^{iW[J]} = \int (d\phi)\, e^{i[S[\phi] + \int J\phi]}$$

$$= \int (d\phi)\, e^{i(S+\int J\phi)}\left[1 + i\int \delta\omega\,(B(\phi) + JA(\phi))\right]$$

integration measure unchanged

$$\text{or} \quad 0 = \int (d\phi)\, e^{iS+\int J\phi} \int \delta\omega\,(B(\phi) + JA(\phi))$$

But since A, B are linear in ϕ, the $\int d\phi$ integral just give A, B evaluated at $\bar\phi$, the expectation value of ϕ in presence of current J

$$\int \delta\omega\left[B(\bar\phi) + JA(\bar\phi)\right]$$

But, since $J = -\dfrac{\delta\Gamma}{\delta\bar\phi}$ we have

$$\boxed{\delta\Gamma[\bar\phi] = \int \frac{\delta\Gamma}{\delta\bar\phi}\,\delta\bar\phi = \int \delta\omega\, B(\bar\phi) = \delta S[\bar\phi]}$$

(Ward Identity)

which was to be shown; i.e. the change in Γ is the same as the change in S

As a consequence, we see that exact symmetries of the action are exact symmetries of the 1PI graphs (assuming it is possible to regularize w/o explicitly breaking the symmetries).

Now let's apply the theorem to QED with a gauge fixing term

$$S_{eff} = S_{\substack{gauge \\ invariant}} - \frac{1}{2\alpha}\int d^4x \, (\partial^\mu A_\mu)^2$$

An infinitesimal gauge transformation is

$$\delta A_\mu = \frac{1}{e}\partial_\mu \delta\omega$$
$$\delta\psi = i\,\delta\omega\,\psi$$

which doesn't change the integration measure, and

$$\delta S_{eff} = \text{linear in } A_\mu = -\frac{1}{\alpha}\int d^4x \, \partial^\mu A_\mu (\partial^2 \delta\omega)$$

So the theorem applies and we have

$$\Gamma[\bar{A}] = \Gamma[\bar{A}]_{\substack{gauge \\ invariant}} - \frac{1}{2\alpha}\int d^4x \, (\partial_\mu \bar{A}^\mu)^2$$

the non gauge-invariant part of the effective action is the same as the non gauge invariant part of the action. So $\frac{1}{\alpha}(\partial A)^2$ is unrenormalized, and all counterterms are indeed gauge-invariant.

This means that α is renormalized by Z_3^{-1}, but if we do not want to worry about renormalizing it, we can work in Feynman gauge

You may be wondering what the identity

$$\int d^4x\, \delta\omega(x) \frac{\delta\Gamma}{\delta\bar\phi(x)} \frac{\delta\bar\phi(x)}{\delta\omega} = \delta S[\bar\phi],$$

which I called the "Ward identity" has to do with what is normally called the Ward identity. In fact, a lot of information is contained in this equation, obscured, perhaps, by the compact abstract notation. In QED, ϕ represents the fields $A_\mu, \psi, \bar\psi$ and

$$\frac{\delta A_\mu}{\delta\omega} = \frac{1}{e}\partial_\mu, \quad \frac{\delta\psi}{\delta\omega} = i\psi, \quad \frac{\delta\bar\psi}{\delta\omega} = -i\psi,$$

so we have

$$\int d^4x\, \delta\omega(x)\left[\frac{\delta\Gamma}{\delta\psi(x)} i\psi(x) + \frac{\delta\Gamma}{\delta\bar\psi(x)}(-i)\bar\psi(x) - \frac{1}{e}\partial^x_\mu \frac{\delta\Gamma}{\delta A_\mu(x)}\right] = \delta S$$

(I stopped writing bars over the fields, to avoid confusion.) This is true for arbitrary $\delta\omega(x)$, so we have

$$\boxed{e i\left[\frac{\delta\Gamma}{\delta\psi(x)}\psi(x) - \frac{\delta\Gamma}{\delta\bar\psi(x)}\bar\psi(x)\right] = \partial^x_\mu \frac{\delta\Gamma}{\delta A_\mu(x)}}$$

ignoring the δS term, which doesn't affect any Green's functions of interest.

This identity summarizes and infinite number of relations among one-particle-irreducible Green's functions. For example, by equating the coefficients of

$$\int d^4y\, d^4z\, \psi(y)\bar\psi(z)$$

on the two sides of the eqn

$$ie\left[\Gamma^{\psi\bar\psi}_{(y,z)}\delta^4(x-y) - \Gamma^{\psi\bar\psi}_{(y,z)}\delta^4(x-z)\right] = \partial^\mu_x \Gamma^{\psi\bar\psi A}_\mu(y,z,x)$$

and taking the Fourier transform $\int d^4x\, e^{iqx} d^4y\, e^{ipy} d^4z\, e^{ip'z}$ gives

$$ie\left[\tilde\Gamma^{\psi\bar\psi}(p') - \tilde\Gamma^{\psi\bar\psi}(p)\right] = -iq^\mu \tilde\Gamma^{\psi\bar\psi A}_\mu(p,p';q)$$

Since $\tilde{\Gamma}^{\psi\bar\psi} = i S_F^{-1}$, the inverse propagator, this is

$$ie\left(S_F^{-1}(p) - S_F^{-1}(p')\right) = -(p+p')^\mu \tilde{\Gamma}_\mu^{\psi\bar\psi A}(p,p';q),$$

which is the familiar Ward identity, relating the inverse propagator to the divergence of the vertex function. Usually it is derived in a quite different way, using current conservation, and retaining commutator terms generated by the $\frac{\partial}{\partial x^0}$ acting on the θ functions in the T product

K. <u>The Background Field Method in Yang-Mills Theory</u>

Our next task is to calculate the β function (to leading order) in a nonabelian gauge theory.

We have not, and will not, go through the proof of the renormalizability of Yang-Mills theory here. But it turns out that counterterms cannot be absorbed into redefinitions of the parameters in the Lagrangian unless the group generators are rescaled. The bare Lagrangian can contain a term

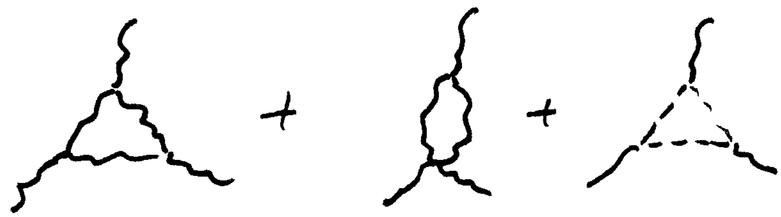
$$\mathcal{L}_B = -\tfrac{1}{2} Z_A \mathrm{tr}(\tilde{F}_{\mu\nu})^2$$

where $\tilde{F}_{\mu\nu} = \frac{-1}{ig}[\tilde{D}_\mu, \tilde{D}_\nu]$ and $\tilde{D}_\mu = \partial_\mu - igZ_gZ_A^{\frac{1}{2}}A$

and $Z_g Z_A^{\frac{1}{2}}$ <u>need not</u> be 1. Therefore, we cannot find Z_g by computing Z_A, as we did in QED.

Instead, we must compute the infinite parts of

to find Z_A (the dotted line represents a ghost), and then compute the vertex function



which is multiplicatively renormalized by $Z_A^{3/2} Z_g$. This determines Z_g (where $g_B = g_R Z_g$), and also the β function.

Actually, we can save some work by noting that the three-gluon vertex is related by Ward identities to the ghost-ghost-gluon vertex, which is easier to calculate. But it is far easier still to work in a special gauge in which explicit gauge-invariance of the counterterms is retained. In this gauge gA is unrenormalized, so we can find Z_g by calculating only Z_A.

The special gauge we will use is called Background Field Gauge. Before introducing it, we will briefly review the procedure for quantization of gauge theories.

We will consider pure Yang-Mills theory with Lagrangian

$$\mathcal{L} = -\tfrac{1}{2} \operatorname{tr} F^{\mu\nu} F_{\mu\nu}$$

where $F^{\mu\nu} = \frac{-1}{ig}[D^\mu, D^\nu]$, $\quad D^\mu = \partial^\mu - igA^\mu$,

and $\quad A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{1}{ig}(\partial_\mu \Omega)\Omega^{-1}$ under a gauge transformation.

We will use the conventions on p.(1.3-4):

$$A_\mu = A_\mu^a T^a \qquad \operatorname{tr} T^a T^b = \tfrac{1}{2}\delta^{ab}$$

$$[T^a, T^b] = i c^{abc} T^c$$

Functional integrals in a gauge theory are defined by the "Fadeev-Popov" ansatz:

$$Z = \int (dA)\, e^{iS} \det(\delta F^a/\delta\omega^b) \prod_a \delta[F^a(A)]$$

The δ functions "fix" the gauge, and the determinant ensures that Z is independent of the choice of gauge. A popular choice is the covariant gauge

$$F^a = \partial^\mu A_\mu^a - f^a$$

Since Z is independent of f^a, we may write

$$Z = \int (dA)\, e^{iS} \det(\ ) \prod_a \int (df^a)\, G[f^a]\, \delta[\partial^\mu A_\mu^a - f^a]$$

for any functional G, or

$$Z = \int (dA)\, e^{iS} \det(\ )\, G[\partial^\mu A_\mu^a]$$

In particular, we may choose $G = \exp[-\frac{i}{2\alpha}\int d^4x\, (\partial^\mu A_\mu^a)^2]$

$$Z = \int (dA)\, e^{iS} e^{-\frac{i}{2\alpha}\int d^4x (\partial^\mu A_\mu^a)^2} \det(\delta F^a/\delta\omega^b)$$

Now let's take a closer look at the determinant. δF^a is the change in F^a under an infinitesimal gauge transformation parametrized by $\delta\omega$

$$\Omega = 1 + i\delta\omega\ , \qquad A_\mu \to A_\mu + [i\delta\omega, A_\mu] + \frac{1}{g}\partial_\mu \delta\omega$$

or $A_\mu \to A_\mu + \frac{1}{g}[D_\mu \delta\omega]$

Since $F = \partial^\mu A_\mu$, we have $\frac{\delta F}{\delta \omega} = \frac{1}{g}\partial^\mu D_\mu$

(where it is understood that this is D_μ acting on the adjoint rep of the group).

Now the $\det(\partial^\mu D_\mu)$ factor (the $\frac{1}{g}$ may be absorbed into the normalization of the functional integral) may be represented by a an integral over fermion ghost fields transforming as the adjoint representation

$$\det(-\partial^\mu D_\mu) \sim \int (d\bar\eta)(d\eta) \exp\left(i\, \partial^\mu \bar\eta^a D_\mu \eta^a\right)$$

(up to a multiplicative constant). We finally have

$$Z = \int (dA)(d\bar\eta)(d\eta) \exp\left[i\int d^4x\, \mathcal{L}\right]$$

[We need consider only ghost field as internal lines — Ghost loops reproduce the det.

where $$\boxed{\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 + \partial^\mu \bar\eta^a D_\mu \eta^a}$$

from which Feynman rules can be derived in the standard way ($D_\mu \eta^a = \partial_\mu \eta^a + g\, c^{abc} A^b_\mu \eta^c$)

The background field method amounts to a particular way of fixing the gauge, but to motivate it, let us consider quantizing the fluctuations about some arbitrary classical background gauge field. We write the gauge field as

$$A_\mu + Q_\mu$$

where A_μ is the background field and Q_μ is the fluctuation. Green's functions for the Q_μ's are generated by

$$Z[J, A] = \int (dQ)\, e^{iS[A+Q] + i\int J^a_\mu Q^{\mu a} + \text{g.f.} + \text{ghosts}},$$

which is a functional of both A and J.

The action is invariant under the gauge transformation

$$A_\mu + Q_\mu \to A_\mu + Q_\mu + \frac{1}{g}\partial_\mu \delta\omega - i[A_\mu + Q_\mu, \delta\omega]$$

$$\text{or} \quad Q_\mu \to Q_\mu + \frac{1}{g}[D_\mu^{A+Q}, \delta\omega]$$

Here, in an obvious notation, we denote $D_\mu^{A+Q} = \partial_\mu - ig(A_\mu + Q_\mu)$

Now, we specify the background field gauge by choosing the gauge-fixing term to be

$$\mathcal{L}_{g.f} = -\frac{1}{2\alpha}\left([D_\mu^A, Q^\mu]^a\right)^2$$
$$= -\frac{1}{2\alpha}\left(\partial_\mu Q^{\mu a} + g\, C^{abc} A_\mu^b Q^{\mu c}\right)^2$$

This term is invariant under

$$\boxed{\begin{aligned} A &\to \Omega A \Omega^{-1} + \frac{1}{ig}(\partial\Omega)\Omega^{-1} \\ Q &\to \Omega Q \Omega^{-1} \end{aligned}}$$

(Because D_μ^A transforms homogeneously.)

which is also a symmetry of $S[A+Q]$. Thus, while it fixes the gauge for Q, as desired, it respects local gauge transformations on the background field A. This is the crucial property of the background field gauge.

Let's construct the ghost term

$F = D_\mu^A Q^\mu$ (this means D_μ^A acting on adjoint rep.)

$$\frac{\delta Q_\mu}{\delta\omega} = \frac{1}{g} D_\mu^{A+Q} \Rightarrow \frac{\delta F}{\delta\omega} \propto D^{\mu A} D_\mu^{A+Q}$$

Therefore, we write

$$\mathcal{L}_{ghost} = \overline{D^{\mu A}\eta^a}\, D_\mu^{A+Q}\eta^a$$
$$= \left(\partial^\mu \bar\eta^a - g\, C^{abc} A^{\mu b}\bar\eta^c\right)\left(\partial_\mu \eta^a + g\, C^{ade}(A+Q)^{\mu d}\eta^e\right)$$

To summarize

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\alpha}\left(D_\mu^A Q^{\mu a}\right)^2 + \overline{D^{\mu A}\eta^a}\, D_\mu^{A+Q}\eta^a}$$

where $F_{\mu\nu} = \frac{-1}{ig}\left[D_\mu^{A+Q}, D_\nu^{A+Q}\right]$,

and the whole Lagrangian, including gauge fixing and ghost terms, is invariant under

$$A \to \Omega A \Omega^{-1} + \frac{1}{ig}(\partial\Omega)\Omega^{-1} \qquad \text{local g.t. on } A;$$
$$Q \to \Omega Q \Omega^{-1} \qquad \text{homog. g.t. on } Q$$

What do these symmetries imply for the generating functional $Z(J, A)$?

We find

$$Z[J, A] = Z\left[\Omega J \Omega^{-1}, \Omega A \Omega^{-1} + \frac{1}{ig}(\partial\Omega)\Omega^{-1}\right]$$

by the change of variable $Q \to \Omega^{-1} Q \Omega$ in the functional integral.

We may Legendre transform to define a background field effective action

$$Z[J A] = e^{iW(J A)} \qquad \bar{Q} = \delta W/\delta J$$
$$\Gamma[\bar{Q}, A] = W[J, A] - \int d^4x\, J_\mu^a \bar{Q}^{\mu a}$$

The invariance properties of $W(J, A)$ imply

$$\Gamma[\bar{Q}, A] = \Gamma\left[\Omega \bar{Q} \Omega^{-1}, \Omega A \Omega^{-1} + \frac{1}{ig}(\partial\Omega)\Omega^{-1}\right]$$

This is the main result we need, for the calculation of the β function. But it is enlightening to see the connection between $\Gamma[\bar{Q},A]$ and the usual effective action (in a particular gauge).

By the change of variable $Q \to Q - A$, we obtain

$$Z[J,A] = Z[J]_A \exp\left(-i\int d^4x\, J_\mu^a A^{\mu a}\right)$$

where $Z(J)_A$ is the usual generating functional, but evaluated in a gauge

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha}\left(\partial_\mu Q^{\mu a} - \partial_\mu A^{\mu a} + g\, C^{abc} A_\mu^b Q^{\mu c}\right)^2.$$

(Note that $\overline{D^{\mu A}\eta^a}\, D_\mu^Q \eta^a$ is the correct ghost term in this gauge.) We have

$$W(J,A) = W(J)_A - \int d^4x\, J_\mu^a A^{\mu a}$$

$$\bar{Q} = \frac{\delta W(J,A)}{\delta J} = \frac{\delta W(J)_A}{\delta J} - A = Q^c - A,$$

and, when we Legendre transform

$$\Gamma(\bar{Q},A) = W(J)_A - \int d^4x\, J_\mu^a A^{\mu a} - \int d^4x\, J_\mu^a \bar{Q}^{\mu a}$$

or $\boxed{\Gamma(\bar{Q},A) = \Gamma(\bar{Q}+A)_A}$ where $\Gamma(Q^c)$ is the usual effective action, in this peculiar gauge

In particular

$\Gamma(0,A) = \Gamma(A)_A$ — so the background field effective action, with the b.g. field on all external legs, is a generating functional for 1PI graphs, in a particular gauge

When we calculate quantum corrections, we will find that the background field A will have to be multiplicatively renormalized. But, since the effective action is gauge invariant, gA is unrenormalized. To find the β function, we just calculate Z_A

In one loop order, the graphs contributing to Z_A are

where the external lines are A lines and the internal lines are Q lines and ghost lines. The last two graphs vanish in dimensional regularization, so there are only two to calculate.

Exercise 1.11

Show that the β function for pure Yang-Mills theory, to one-loop-order, is

$$\boxed{\beta(g) = -\frac{11}{3} C_2 \frac{g^3}{16\pi^2}}$$

(Politzer; Gross-Wilczek)

where

$$C_{acd} C_{bcd} = \delta_{ab} C_2 \qquad \text{defines } C_2$$

To do the problem, you will need to work out the Feynman rules (in b.g. field gauge) for the vertices

 and

It is probably easiest to do this calculation in Feynman gauge, $\alpha = 1$. (We can't set $\alpha = 0$, because there is a vertex of order $\frac{1}{\alpha}$; α in propagator times $\frac{1}{\alpha}$ in vertex = finite, which we must keep.) Note that it is not necessary to renormalize the Q field, because it appears on internal lines only.

L. Asymptotic Freedom

Incorporation of fermions and scalars:

If there are fermions and scalars coupled to the gauge fields, then the graphs

also contribute to the β function in one loop order. The vertices here are the standard ones, except for the generator T^a. These graphs are the same as those calculated in p 1.58ff and Exercise 1.10, except for the factor

$$T(R), \text{ defined by } \quad \mathrm{tr}\, T^a_R T^b_R = T(R)\,\delta^{ab}$$

where R denotes a representation of the gauge group. From our previous results

$$(\beta(g))_{\text{fermions}} = + \frac{g^3}{12\pi^2} T(R_f) \quad \leftarrow \text{for four-component fermions}$$

$$(\beta(g))_{\text{scalars}} = \frac{g^3}{48\pi^2} T(R_s) \quad \leftarrow \text{for complex scalars}$$

Since $i C_{abc}$ is just $(T^a)_{bc}$ in the adjoint rep, we see that

$$C_2 = T(R_{adj})$$

and the complete expression for the one-loop β-function can be written....

$$\boxed{\beta(g) = \Big[-\frac{11}{3}\,T(R_{adj}) + \frac{4}{3}\,\underbrace{T(R_{ferm})}_{\text{4-component fermion}} + \frac{1}{3}\,\underbrace{T(R_{scal})}_{\text{complex scalar}}\Big]\frac{g^3}{16\pi^2}}$$

For gauge group $G = SU(N)$, we have

$$T(N) = \tfrac{1}{2}$$

$$T(Adj) = N \qquad \text{(By induction: } (Adj)_{N+1} \to (Adj)_N + N + \bar{N} + 1.)$$

For QCD with n_f quarks we have

$$\beta(g) = \left(-11 + \tfrac{2}{3} n_f\right) g^3/16\pi^2 .$$

QCD is asymptotically free only for $n_f \le 16$.

We see that fermions and scalars screen color charge (as they screen electric charge), but that gluons <u>antiscreen</u> color charge. Can we understand why this is so?

The $T(R)$ is simply a sum over all the color charges of the particles. What we would like to understand are the factors

$-11/3$, $4/3$, $\frac{1}{3}$ — i.e., the spin dependence of vacuum polarization.

Heuristic Explanation of Asymptotic Freedom

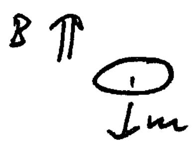
We wish to understand *why* a nonabelian gauge theory is asymptotically free. Why does the vacuum antiscreen color charge; i.e., why is the vacuum a dielectric medium with $\epsilon < 1$?

Since $\mu\epsilon = 1$ in the vacuum, it suffices to explain why $\mu > 1$, i.e. why the vacuum screens color magnetic charge. It is easier to consider the response of the vacuum to a magnetic field than its response to an electric field for several reasons. The vacuum suffers dielectric breakdown (pair production) in an electric field, but not in a magnetic field. In a magnetic field, massless charged particles are confined to orbits with area of order $(eB)^{-1}$, so B provides an infrared cutoff.

Best of all, it is a familiar phenomenon that magnetic materials can either screen or antiscreen magnetic fields.

A magnetic field causes intrinsic magnetic moments to line up with B, giving $\delta\mu > 1$. (Recall $\vec{M} = \frac{\mu - 1}{4\pi\mu}\vec{B}$.)

This is paramagnetism (Pauli).



But a current loop develops a magnetic moment to oppose applied $\vec{B}$ field, giving $\delta\mu < 1$. This is diamagnetism (Landau).

So the possibility of both screening and antiscreening is no more mysterious than the existence of both paramagnetism and diamagnetism!

This remark can be made more quantitative, as we will now see. We will calculate the magnetic susceptibility of the vacuum for a free field theory of particles (either bosons or fermions) of arbitrary spin. This calculation is equivalent to the calculation to one-loop order of the contribution of a charged particle of arbitrary spin to the β function.

How is the β function related to the magnetic susceptibility? Since

$$e_\lambda^2 \cong \frac{e_0^2}{1 - b_0 e_0^2 \ln(\Lambda^2/\lambda^2)} \quad \text{where } \Lambda = \text{cutoff}, \; e_0^2 = \text{bare charge}$$

we identify $\epsilon_\lambda \cong 1 + b_0 e_0^2 \ln\left(\frac{\Lambda^2}{\lambda^2}\right)$ as the "running dielectric constant"

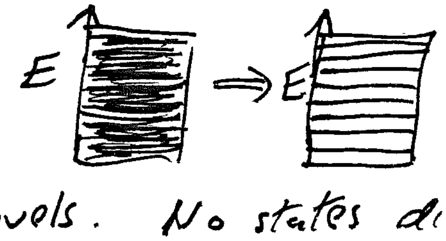
and $\mu_\lambda = \epsilon_\lambda^{-1} \cong 1 - b_0 e_0^2 \ln \frac{\Lambda^2}{\lambda^2}$ is the permeability

and $\boxed{\chi_\lambda = \frac{\mu - 1}{\mu} \cong -b_0 e_0^2 \ln \frac{\Lambda^2}{\lambda^2}}$

And since $dU = -M\,dB = -\chi B\,dB$, $\boxed{U = -\frac{1}{2}\chi B^2}$

We calculate χ by finding the term in the vacuum energy quadratic in B. Then b_0 is the coefficient of $-e_0^2 \ln \Lambda^2$ in the expression for χ.

First, we calculate the diamagnetic term in the vacuum energy, which is independent of the particle spin. Diamagnetism arises because, in the presence of a uniform magnetic field, the continuous energy spectrum of a free particle is replaced by the discrete Landau levels. No states disappear, but the vacuum energy is shifted in order B^2, because of the distortion of the levels.



For a massless charged particle with charge e in a uniform B field with

$$\vec{B} = B\,\hat{e}_z\,, \quad \vec{A} = Bx\,\hat{e}_y\,,$$

we have

$$E^2 = (\vec{p} - e\vec{A})^2 = p_z^2 + p_x^2 + (p_y - eBx)^2$$

(ignoring spin)

p_z, p_x commute with E^2 and are constants of the motion and $p_x^2 + (eB)^2(x-x_0)^2$ has spectrum $(2n+1)\,eB$ so the energy levels are

$$E^2 = p_z^2 + (2n+1)\,eB \qquad \text{(Landau Levels)}$$

the particles move freely in the z-direction, and execute (quantized) circular motion in the x-y plane.

We obtain the vacuum energy density by summing the zero-point energies of all the "vacuum oscillators", which in this case means all Landau levels.

$$U = \sum_n \int \frac{dp_z}{(2\pi)} \left[p_z^2 + (2n+1)eB\right]^{\frac{1}{2}} g_n$$

where g_n is the degeneracy per unit area of the nth Landau level. We quickly obtain g_n by noting that, as $eB \to 0$, U must become

$$U = \int \frac{dp_z}{(2\pi)} \frac{\pi\, dq^2}{(2\pi)^2} \left[p_z^2 + q^2\right]^{\frac{1}{2}},$$

the vacuum energy density in free field theory. One sees that

$$g_n = \frac{eB}{2\pi}$$

or

$$\boxed{U = \int \frac{dp_z}{(2\pi)} \sum_{n=0} \left(\frac{eB}{2\pi}\right)\left[p_z^2 + (2n+1)eB\right]^{\frac{1}{2}}}$$

We need to know how much this expression for U differs from its $B=0$ limit. Use the identity

$$\int_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})\epsilon} dx\, F(x) = \sum_{n=0}^{N} \int_{(n-\frac{1}{2})\epsilon}^{(n+\frac{1}{2})\epsilon} dx\, F(x) = \sum_{n=0}^{N} \int_{(n-\frac{1}{2})\epsilon}^{(n+\frac{1}{2})\epsilon} dx \left[F(n\epsilon) + (x-n\epsilon)F'(n\epsilon) + \tfrac{1}{2}(x-n\epsilon)^2 F''(n\epsilon) + \cdots \right]$$

$$= \sum_{n=0}^{N} \left[\epsilon F(n\epsilon) + \tfrac{1}{24}\epsilon^3 F''(n\epsilon) + \cdots \right]$$

or

$$\sum_{n=0}^{N} \epsilon F(n\epsilon) = \int_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})\epsilon} dx\, F(x) - \tfrac{1}{24}\epsilon^2 \int_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})\epsilon} dx\, F''(\ \ ) + \cdots$$

$$= \int_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})\epsilon} dx\, F(x) - \tfrac{1}{24}\epsilon^2 F'(x)\Big|_{-\frac{1}{2}\epsilon}^{(N+\frac{1}{2})\epsilon} + O(\epsilon^4)$$

Therefore

$$U = U(B=0) + \int \frac{dp_z}{(2\pi)^2}\left(-\frac{1}{24}\right)(eB)^2 \frac{\partial}{\partial}\left[p_z^2 + (2n+1)eB\right]^{-\frac{1}{2}}$$

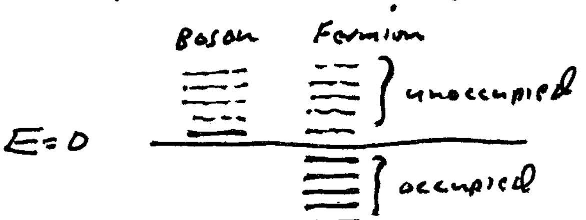
$$= U(B=0) - \left(\frac{1}{24}\right)(eB)^2 \int \frac{dp_z}{(2\pi)^2}\left(-\tfrac{1}{2}\right)\int dq^2 (p_z^2+q^2)^{-3/2}$$

$$= U(B=0) + \frac{1}{24}(eB)^2 \int \frac{d^3p}{(2\pi)^3 |p|^3}$$

$$= U(B=0) + \frac{1}{48\pi^2} e^2 B^2 \int \frac{\frac{1}{2}dp^2}{p^2}$$

$$\Rightarrow \quad \chi_{\text{diamagnetic}} = -\frac{1}{48\pi^2} e^2 \ln \Lambda^2$$

There is a contribution like this for each spin state except that the sign of the vacuum energy is opposite for fermions; i.e. the contribution of each oscillator to the vacuum energy is the gap between the highest occupied hole state and $E=0$, which is less than zero.



To summarize:

$$\boxed{\chi_{diamagnetic} = -\frac{1}{48\pi^2} e^2 \ln \Lambda^2 \begin{pmatrix} +1, \text{ bosons} \\ -1, \text{ fermions} \end{pmatrix}}$$

Next, we evaluate the paramagnetic contribution to the vacuum energy. This is easier.

$$E^2 = (\bar{p} - e\bar{A})^2 - geB \cdot S_z$$

$$= E^2_{Landau} - geB \cdot S_z$$

if the magnetic moment is $m_z = geS_z$

Expanding E in powers of B gives

$$E = E_{Landau}\left[1 - \frac{geBS_z}{E^2_{Lan}}\right]^{\frac{1}{2}}$$

$$= E_{Landau} - \frac{1}{2}\frac{geBS_z}{E_{Lan}} - \frac{1}{8}\frac{(geS_zB)^2}{E^3_{Lan}}$$

So the term in U of order B^2 is

$$-\frac{1}{2}\chi B^2 = -\frac{1}{8}(gS_z)^2 e^2 B^2 \int \frac{d^3p}{(2\pi)^3 |p|^3}$$

$$\Rightarrow \quad \boxed{\chi_{paramagnetic} = \frac{1}{16\pi^2}(gS_z)^2 e^2 \ln \Lambda^2 \begin{pmatrix} +1, \text{ bosons} \\ -1, \text{ fermion} \end{pmatrix}}$$

(Again there is a sign change for fermions). We expect $g = 2$ for "elementary" particles of any spin (as a consequence of Thomas precession).
So we have now derived that the contribution to the susceptibility of a particle with charge e and spin S_z is

$$\boxed{\chi = \frac{1}{16\pi^2}\left[4S_z^2 - \frac{1}{3}\right] e^2 \ln \Lambda^2 \begin{pmatrix} +1, \text{ boson} \\ -1, \text{ fermion} \end{pmatrix}}$$

And finally, we have derived that, if

$$\beta(g) = \mu \frac{d}{d\mu} g = b_0 g^3 + \cdots,$$

then a massless charged state with charge Q and helicity S_z contributes

$$\boxed{b_0 = -\frac{Q^2}{16\pi^2}\left(4S_z^2 - \frac{1}{3}\right)\begin{pmatrix} +1, \text{boson} \\ -1, \text{fermion}\end{pmatrix}}$$

This reproduces our old result

scalars: $S_z = 0$, 1 helicity state $-$ $b_0 = \frac{Q^2}{16\pi^2}\left(\frac{1}{3}\right)$

fermions: $S_z = \pm\frac{1}{2}$, 2 helicity states $-$ $b_0 = \frac{Q^2}{16\pi^2}\left(\frac{4}{3}\right)$

vectors: $S_z = \pm 1$, 2 helicity states $-$ $b_0 = \frac{Q^2}{16\pi^2}\left(\frac{-22}{3}\right)$

(This counts both particle and antiparticle of charge Q; e.g. complex scalar, 4-component fermion. The contribution due to vectors differs by factor of two from our old result because charges are added up differently here.)

And we have found the generalization to elementary particles of higher spin.

Now we understand asymptotic freedom. It is a consequence of the large magnetic moments of spin-one charged particles, which make the vacuum paramagnetic. Scalar and fermion fluctuations make the vacuum diamagnetic, but for different reasons. The fermions have magnetic moments, but turn out to generate a diamagnetic effect because their zero-point fluctuations carry negative energy.

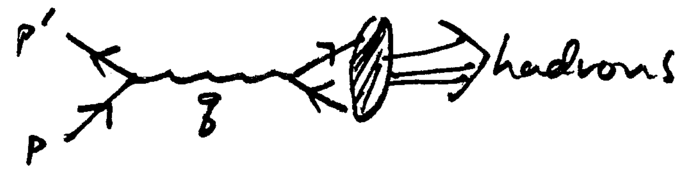
J. PRESKILL
Phys. 230 1.96

M. e+e- Annihilation

As a first example of a strong interaction process, we consider

$$e^+e^- \to \text{hadrons}$$

We would like to calculate the total (inclusive) cross section



Hadron production cannot be treated in P.T. We cannot calculate exclusive cross sections like $e^+e^- \to \pi^+\pi^-$

In position space

$$\int d^4x\, d^4y\; e^{ip'x} e^{-ipy} (ie)^2\, \bar{u}(p')\gamma^\mu u(p)\, D_{\mu\nu}(x-y) \langle n | J_\nu(y) | 0\rangle$$

$$= \left[\int d^4x\, d^4y\; e^{ip'x} e^{-ipy} e^{ip_n y}\, D_{\mu\nu}(x-y)\right] (-e^2)\bar{u}(p')\gamma^\mu u(p) \times \langle n | J_\nu(0)|0\rangle$$

$$(2\pi)^4 \delta^4(q-p_n)\, \frac{(ie^2)}{q^2}\, \bar{u}\gamma^\mu u\; \langle n | J_\mu(0) | 0\rangle$$

Now we sum over final states, and average over initial polarization

$$\frac{1}{4}\sum |A_{fi}|^2 = \frac{1}{4}\underbrace{\left(\frac{e^2}{q^2}\right)^2 \mathrm{tr}(\not{p}'\gamma^\mu \not{p}\gamma^\nu)}_{\left(\frac{e^2}{q^2}\right)^2 p'^\mu p^\nu + p^\mu p'^\nu - \eta^{\mu\nu} p\cdot p'}\; \underbrace{\sum_n (2\pi)^4\delta^4(q-p_n)\langle n | J_\nu(0)|0\rangle\langle 0 | J_\mu(0) | n\rangle}$$

$$\sum_n \int d^4x\, e^{iqx} \langle 0 | J_\mu(x) | n\rangle \langle n | J_\nu(0) | 0\rangle$$

$$= \int d^4x\, e^{iqx} \langle 0 | J_\mu(x)\, J_\nu(0) | 0\rangle$$

$$= \int d^4x\, e^{iqx} \langle 0 | [J_\mu(x), J_\nu(0)] | 0\rangle \qquad \text{(symmetrize } \mu \leftrightarrow \nu)$$

The last equality follows because $\delta^4(q+p_n) = 0$ if $q^0 > 0$ and $p_n^0 > 0$. (And we only care about symmetric part under $\mu \leftrightarrow \nu$)

Since $[J_\mu(x), J_\nu(0)]$ vanishes for spacelike separation, the asymptotic $q^2 \to \infty$ behavior of

$$\int d^4x\, e^{iqx} \langle 0 | [J_\mu(x), J_\nu(0)] | 0 \rangle$$

(and therefore also of $\sigma(e^+e^- \to \text{hadrons})$) is controlled by the short-distance singularities of the current commutator (Schwinger terms = derivatives of δ functions)

We would like to express this object in terms of a time-ordered product, which we can calculate with graphs.

(A, B hermitian)

$$\int d^4x\, e^{iqx} \langle 1 | \, T A(x) B(0) | 2 \rangle$$

$$= \int d^4x\, e^{iqx} \langle 1 | \theta(x^0) A(x) B(0) + \theta(-x^0) B(0) A(x) | 2 \rangle$$

use $\theta(x^0) = \int \frac{ds}{2\pi i} \frac{e^{isx^0}}{s - i\epsilon}$

and sum over intermediate states

$$= \int \frac{ds}{2\pi i} \int d^4x \, \frac{1}{s - i\epsilon} \sum_n \Big[e^{-i(p_n - p_1 - q)x} e^{isx_0} \langle 1 | A(0) | n \rangle \langle n | B(0) | 2 \rangle$$

$$+ e^{-i(p_2 - p_n - q)x} e^{-isx_0} \langle 1 | B(0) | n \rangle \langle n | A(0) | 2 \rangle \Big]$$

$$= -i(2\pi)^3 \int \frac{ds}{s - i\epsilon} \sum_n \Big[\delta^3(\vec{p}_n - \vec{p}_1 - \vec{q})\, \delta(p_n^0 - p_1^0 - q^0 - s) \langle 1 | A | n \rangle \langle n | B | 2 \rangle$$

$$+ \delta^3(\vec{p}_2 - \vec{p}_n - \vec{q})\, \delta(p_2^0 - p_n^0 - q^0 + s) \langle 1 | B | n \rangle \langle n | A | 2 \rangle \Big]$$

$$= -i(2\pi)^3 \sum_n \Big[\delta^3(\vec{p}_n - \vec{p}_1 - \vec{q}) \frac{1}{p_n^0 - p_1^0 - q^0 - i\epsilon} \langle 1 | A | n \rangle \langle n | B | 2 \rangle$$

$$+ \delta^3(\vec{p}_2 - \vec{p}_n - \vec{q}) \frac{1}{p_n^0 - p_2^0 + q^0 - i\epsilon} \langle 1 | B | n \rangle \langle n | A | 2 \rangle$$

Now suppose $|1\rangle = |2\rangle$, and symmetrize in $A \leftrightarrow B$. Then both terms are proportional to

$$\langle 1|A|n\rangle\langle n|B|1\rangle + \langle 1|B|n\rangle\langle n|A|1\rangle,$$

which is real. Using

$$\frac{1}{z-i\epsilon} - \frac{1}{z+i\epsilon} = 2\pi i\,\delta(z),$$ we have

$$\boxed{\begin{aligned} &\mathrm{Im}\; i\int d^4x\, e^{iqx}\left[\langle 1|T(A(x)B(0))|1\rangle + A\leftrightarrow B\right] \\ &= \tfrac{1}{2}(2\pi)^4 \sum_n \left[\delta^4(p_n - p_1 - q) + \delta^4(p_n - p_1 + q)\right]\left[\langle 1|A(0)|n\rangle\langle n|B(0)|1\rangle + A\leftrightarrow B\right] \\ &= \tfrac{1}{2}\int d^4x\, e^{iqx}\left[\langle 1|A(x)B(0) + A(0)B(x)|1\rangle + A\leftrightarrow B\right] \end{aligned}}$$

This is a consequence of unitarity, obviously related to the optical theorem

Since only the symmetric part of $\langle 0|J_\mu(x)J_\nu(0)|0\rangle$ (in $\mu\leftrightarrow\nu$) occurs in $\sum|A_{fi}|^2$, we have derived

$$\sigma_T(e^+e^-\to \text{hadrons}) = \frac{1}{2s}\left(\frac{1}{4}\sum|A_{fi}|^2\right) = \frac{8\pi^2\alpha^2}{s}\,\mathrm{Im}\,\Pi(s)$$

where $i\int d^4x\, e^{iqx}\langle 0|T J_\mu(x)J_\nu(0)|0\rangle = (q_\mu q_\nu - \eta_{\mu\nu}q^2)\,\Pi(q^2)$ (current conserved)

and $s = q^2$

We must find how $\Pi(q^2)$ scales, to determine the asymptotic behavior of the e^+e^- annihilation cross section

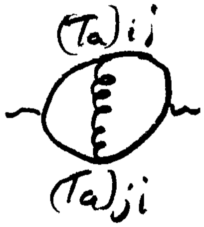
$\mathrm{Im}\,\Pi(s)$ is proportional to a cross-section, and thus is not (multiplicatively) renormalized.

$$\mathrm{Im}\,\Pi\left(\frac{s}{\mu^2}, g_\mu\right) = \mathrm{Im}\,\Pi\left(g_{\sqrt{s}}\right)$$ (neglecting quark masses.)

We may simply calculate it in powers of g, and then replace g by the running coupling.

This is a nontrivial two loop calculation ---

but we can do it the way the QCD pioneers did it - by looking up the corresponding QED calculation (Jost and Luttinger - 1950). The only new feature is the color indices, contracted like



$$(T^a)_{ij}(T^a)_{ji}$$

- i.e. for each quark color we have the factor $C_2(R)$, defined by

$$(T_R^a)_{ij}(T_R^a)_{jk} = C_2(R)\,\delta_{ik}$$

or $$C_2(R) = \frac{\dim(Adj)}{\dim(R)}\,T(R)$$ (E.g. $(N^2-1)/2N$ for fund. rep of SU(N)))

The result of plagiarizing the QED calculation is

$$\mathrm{Im}\,\Pi = A\left(\sum_R (\dim R)\,Q_R^2\left[1+\frac{3g^2}{16\pi^2}C_2(R)\right]\right.$$

where A is a numerical constant which drops out when we take the ratio ---

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_R (\dim R) Q_R^2 \left[1 + \frac{3g^2}{16\pi^2} C_2(R)\right]$$

If $G = SU(3)$ and all fermions are quarks,

$$\boxed{R(s) = \left(\sum_f 3Q_f^2\right)\left[1 + \frac{\alpha_s(s)}{\pi} + \cdots\right]}$$

where $\alpha_s = g^2/4\pi$

the prefactor, including 5 quarks, should be

$$3\left[3\left(\tfrac{1}{3}\right)^2 + 2\left(\tfrac{2}{3}\right)^2\right] = \frac{11}{3} = 3.67$$

Experimentally (with hadronic decays of τ lepton subtracted out), one finds that R is very flat at large s, and

$$R = 3.93 \pm 0.10 \quad \text{at } s = (34\,\text{GeV})^2$$

this result suggests $\alpha_s \sim 0.2$

It works; but notice that what we are doing is slightly subtle. We have really calculated the cross section for e^+e^- into quarks and gluons. It works because e^+e^- annihilation is really a short distance process

- In the OPE language, we have calculated the coefficient of the operator $\mathbb{1}$, the operator which has a vacuum expectation value in perturbation theory. "soft" nonperturbative physics can produce vacuum expectation values for higher dimension operators like $F^a_{\mu\nu}F^{\mu\nu a}$, but these effects are suppressed by powers of $1/q^2$. (This remark is the basis of an elaborate phenomenology, to be discussed further below.)

- In the language of the "parton model", quarks and gluons are produced with probabilities determined by perturbation theory, due to interactions with a distance scale much less than the confinement scale. Subsequently, quarks and gluons "fragment" into hadrons in a complicated way, determining the details of the final state. The "soft" physics controls fragmentation, but cannot change the probability that something <u>hard</u> happened in the first place.

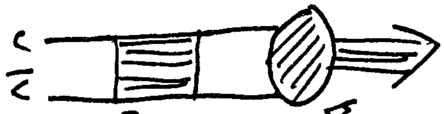
<u>Quarkonium</u>

We have implicitly ignored quark masses so far, which is sensible for $\sqrt{s} \gg m_{quark}$. But masses are quite relevant in the vicinity of a <u>heavy quark threshold</u>,

$$s \sim 4m_q^2$$

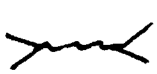
We expect perturbation theory to break down near threshold because there are small kinematic momenta, and large logs occur. In fact, this breakdown occurs even in QED, since e.g. $\mu^+\mu^-$ pair near threshold separates slowly, and many photons are exchanged, building up coulombic bound states. But in QCD the situation is worse, because soft gluon exchange means large effective coupling. (For very large quark mass the color-coulombic bound state size $r \sim 1/\alpha_c m_q$ becomes significantly less than the confinement scale, and quark thresholds are like QED thresholds.) But for charm, bottom threshold, this is not so.

E.g. Charm Threshold



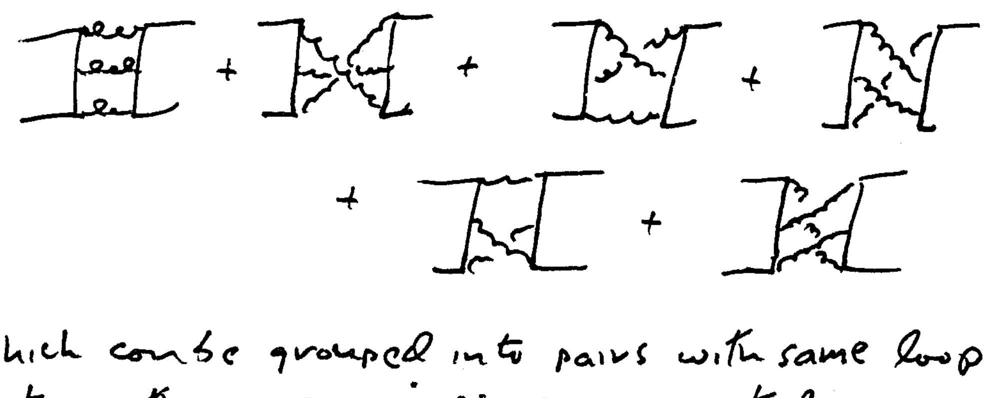
The decay of quarkonium can be treated perturbatively; it involves short distance physics at scale m^{-1} rather than long distance physics at scale $(\alpha m)^{-1}$, because quarks must be close together to annihilate. E.g.

$\Gamma(1^{--} \to \text{hadrons}) = |\psi(0)|^2 \,|$  $|^2$, $\quad 1^{--} = J/\psi$ = "orthocharmonium"

$\Gamma(1^{--} \to \mu^+\mu^-) = |\psi(0)|^2 \,|$  $|^2$,

and the (uncertain) wave function at the origin drops out if we take the ratio. By pirating Schwinger's QED calculation of positronium decay we can predict the leptonic branching fraction of the J/ψ. The only thing we need to do is get the group theory (color) factor right

There are really six graphs. Take Imaginary part of ...



which can be grouped into pairs with same loop integration, but with color indices permuted

$$\text{E.g.: } (\mathrm{tr}\,\lambda^a\lambda^b\lambda^c)(\underbrace{\mathrm{tr}\,\lambda^a\lambda^b\lambda^c + \mathrm{tr}\,\lambda^a\lambda^c\lambda^b}_{= \mathrm{tr}\,\lambda^a\{\lambda^b\lambda^c\}\text{ - totally symmetric}})$$

The trace is taken because J/ψ is a color singlet state, and we can project out singlet by taking the trace

i.e. $\frac{1}{\sqrt{3}}\mathrm{tr}\,(\;)$ projects color singlet; $\frac{1}{\sqrt{3}}(\bar{c}_1c_1 + \bar{c}_2c_2 + \bar{c}_3c_3)$ is normalized singlet

So we have

$$\Gamma(1^{--}\to \text{hadrons}) = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right) \mathrm{tr}\,\lambda^a\{\lambda^b,\lambda^c\} \quad \left(\tfrac{1}{2}\right)\mathrm{tr}\,\lambda^a\{\lambda^b,\lambda^c\}$$

$$\times(\text{Schwinger's positronium result w/ } \alpha\to\alpha_s)$$

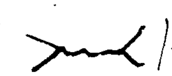
The group theory factor is

$$\frac{1}{48}\, d_{abc}\, d_{abc} \quad \text{where } \{\lambda_a,\lambda_b\} = d_{abc}\lambda_c$$

One can look up tables of d's in books, or work them out, and find

$$d_{abc}\,d_{abc} = \frac{40}{3}, \text{ so the factor is } 5/18$$

The Schwinger result is $\Gamma = \frac{16}{9}(\pi^2-9)\frac{\alpha^3}{m^2}|\psi(0)|^2$

while $\left|\right.$  $\left.\right|^2 = 4\underset{\text{the charge of the quark}}{Q^2}\alpha^2 m^2$

Finally we have

$$\frac{\Gamma(1^{--}\to\mu^+\mu^-)}{\Gamma(1^{--}\to \text{hadrons})} = \frac{81\pi Q^2\alpha^2}{10(\pi^2-9)\alpha_s^3} \simeq .07 \text{ experimentally}$$

Again we conclude

$$\alpha_s \sim 0.2$$

Note that the total width is remarkably narrow, about 60 KeV

Determination of Λ_{QCD}

The next order corrections to the decay amplitude have been calculated, and used to extract one of the more precise determinations of α_s.

In QCD, the β function is

$$\beta(g) = -g\left[\beta_0(g^2/16\pi^2) + \beta_1(g^2/16\pi^2)^2 + \cdots\right]$$

where

$$\beta_0 = 11 - 2/3\, n_f$$
$$\beta_1 = 102 - 38/3\, n_f$$
$$\beta_2 = 2857/2 - 5033/9\, n_f + 325/54\, n_f^2$$

where n_f is the number of quark flavors

Recall that the integral of $\mu \frac{d}{d\mu} g = \beta(g)$ to one loop order is

$$\frac{g_\mu^2}{16\pi^2} = \frac{1}{\beta_0 \ln \mu^2/\Lambda^2}$$

The constant of integration Λ is defined by

$$\ln \mu/\Lambda = \int^{g_\mu} \frac{dg}{\beta(g)}$$

If we keep the two loop term.

$$\ln \mu^2/\Lambda^2 = \int^{g_\mu^2} \frac{dg^2}{g\beta(g)} = \int^{g_\mu^2/16\pi^2} \frac{dx}{-\beta_0 x^2[1+\beta_1/\beta_0 x]}$$

We make an error no worse than the neglect of higher order terms by rewriting $(1+\beta_1/\beta_0 x)^{-1} \sim 1 - \beta_1/\beta_0 x$

$$\Rightarrow \ln \mu^2/\Lambda^2 = \frac{1}{\beta_0 x} + \frac{\beta_1}{\beta_0^2} \ln x \Big|^{g_\mu^2/16\pi^2}$$

or

$$\boxed{\ln \mu^2/\Lambda^2 = \frac{1}{\beta_0 (g_\mu^2/16\pi^2)} - \frac{\beta_1}{\beta_0^2} \ln \frac{16\pi^2}{g_\mu^2}}$$

This is a transcendental eqn for g_μ. Solving by successive approximations....

$$\boxed{\frac{g_\mu^2}{16\pi^2} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} - \frac{\beta_1}{\beta_0^3} \frac{\ln \ln \mu^2/\Lambda^2}{(\ln \mu^2/\Lambda^2)^2} + \cdots}$$

Λ is a useful means of parametrizing the running coupling constant. Strictly speaking, we should do a two loop calculation to determine Λ. The reason is that we want to know how μ should be chosen so that higher order corrections are best under control

The value of α_s, and hence of Λ, is dependent on how subtractions are made; i.e., on the renormalization scheme. It is meaningless to quote a value of Λ without specifying the scheme.

The most popular scheme in perturbative QCD calculations is called the $\overline{MS}$ scheme. It is related to minimal subtraction by

$$g^2_{\overline{MS}} = g^2_{MS} + 2\beta_0(\ln 4\pi - \gamma)\, g^4_{MS} / 16\pi^2 + \cdots$$

(The finite constant $\ln 4\pi - \gamma$ gets absorbed in the renormalized coupling.) i.e. $g^2_{\overline{MS}} \sim \frac{1}{2\beta_0 \ln(\mu/\Lambda c)}$ $\quad c = 4\pi e^{-\gamma}$

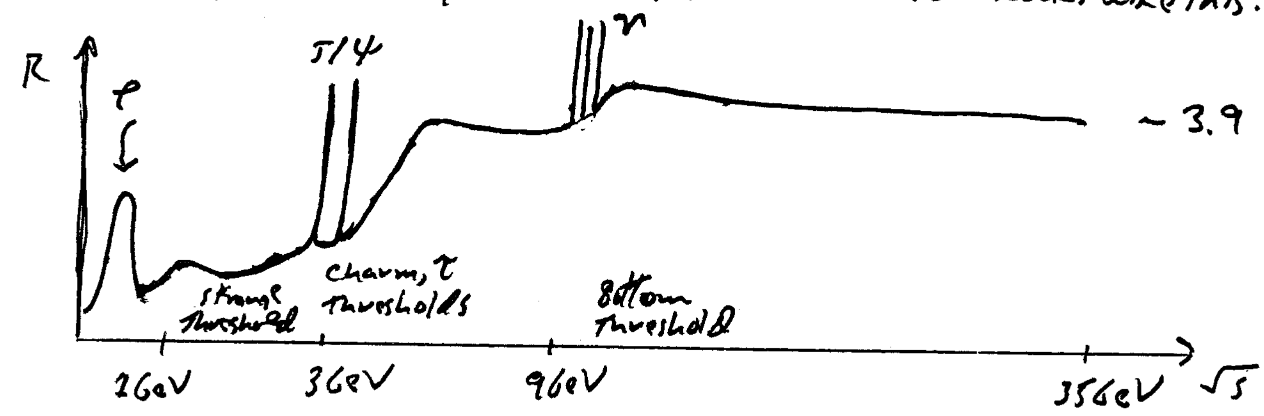
The value of $\Lambda_{\overline{MS}}$ obtained from Υ decay is

$$\Lambda_{\overline{MS}} \sim 100 \text{ MeV}$$

(Lepage and Mackenzie Phys. Rev. Lett. 47, 1244 (1981))

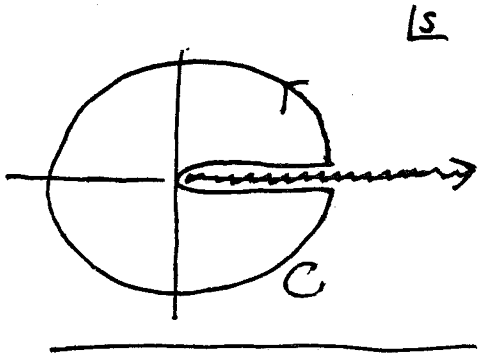
Dispersion Relations for R

A rough sketch of R as a function of $\sqrt{s}$ looks like this:



R is not flat in the resonance and threshold regions, but the QCD predictions can still be applied in these regions to the "smeared" R, i.e. the average value of R over a range large enough to smooth out thresholds and resonances

Another way of saying this is, we can reliably calculate the integrated value of R up to a value of $\sqrt{s}$.



The point is simply that $\Pi(s)$, which has a discontinuity on the positive real axis, obeys the dispersion relation

$$\frac{1}{\pi}\int_0^{s_0} ds\ \mathrm{Im}\,\Pi(s) = \frac{-1}{2\pi i}\int_C ds\ \Pi(s)$$

where C is a circle of radius s_0.

If $\mathrm{Im}\,\Pi(s)$ is not varying rapidly near $s=s_0$, then we can use perturbation theory to calculate the integral over C (perturbation theory only breaks down if there are resonances or threshold singularities, provided α_s is small). So

$$R = R_0\left(1+\frac{\alpha_s}{\pi}+\cdots\right)$$ (Actually should include threshold factor in R_0)

can be regarded as a prediction for the smeared cross section. This prediction works pretty well even if applied at $s \gtrsim (1\,\mathrm{GeV})^2$. (Calculate R_0 in free field theory with massive quarks)

Attempts have been made to use dispersion relations for R to learn something about QCD at relatively low energies (especially by Shifman et al. Nucl Phys B147, 385, 448, 519 (1979).) The starting point is the operator product expansion for the product of two electromagnetic currents

$$i\int d^4x\, e^{iqx}\, T(J_\mu(x)J_\nu(0))$$
$$= (q_\mu q_\nu - \eta_{\mu\nu} q^2)\left[\frac{3Q^2}{12\pi^2}\left(1+\frac{\alpha_s}{\pi}\right)\ln{-q^2/\mu^2} + \sum C_n(q)\,\mathcal{O}_n\right]$$

The leading operators (other than $\mathbb{1}$) in the expansion which have vacuum matrix elements are---

$$\frac{2m}{(q^2)^2}\bar{q}q + \frac{g^2}{48\pi^2(q^2)^2}F^a_{\mu\nu}F^{\mu\nu a} + \cdots$$

The leading nonperturbative effects contributing to $e^+e^- \to$ hadrons are effects which contribute to matrix elements in the vacuum of these operators. Thus, the nonperturbative corrections are suppressed by $\frac{1}{s^2}$ at large s

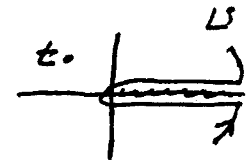
We can attempt to extract the power corrections from the data by the following method:

$$R = \frac{1}{(6\pi)} \operatorname{Im} \Pi(s), \quad \text{and}$$

$$\Pi(s) = \frac{Q^2}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi} + \dots\right) \ln -s/\mu^2 + \sum_n \frac{c_n}{(s)^n}$$

and $\Pi(s)$ obeys a dispersion relation

$$\frac{d\Pi}{dt} = \frac{1}{\pi}\int_0^\infty ds\, \frac{\operatorname{Im}\Pi(s)}{(s-t)^2}.$$



Differentiating again, we find

$$t^n \frac{d^n}{dt^n}\Pi = \frac{n!\, t^n}{\pi}\int ds\, \frac{\operatorname{Im}\Pi(s)}{(s-t)^{n+1}}$$

Now take the limit

$$\lim_{\substack{n\to\infty \\ t=-ns_0}} \frac{(-1)^n}{(n-1)!}\, t^n \frac{d^n}{dt^n}\Pi(t) = \lim_{n\to\infty} \frac{n}{\pi n s_0}\int ds\, \frac{\operatorname{Im}\Pi(s)}{\left(1+\frac{s}{ns_0}\right)^{n+1}}$$

$$= \frac{1}{\pi s_0}\int ds\, \operatorname{Im}\Pi(s)\, e^{-s/s_0}$$

Also, $\displaystyle \lim_{\substack{n\to\infty \\ t=-ns_0}} \frac{(-1)^n}{(n-1)!}\, t^n \frac{d^n}{dt^n}\frac{1}{t^K} = \lim_{\substack{n\to\infty \\ t=-ns_0}} \frac{\Gamma(K+n)}{\Gamma(n)\Gamma(K)}\frac{1}{t^K}$

and Stirling $\Rightarrow$ $\displaystyle \lim_{n\to\infty} \frac{\Gamma(K+n)}{\Gamma(n)\Gamma(K)\, n^K} = \frac{1}{\Gamma(K)}$

so this becomes $\displaystyle \frac{1}{(K-1)!}\frac{(-1)^K}{s_0^K}$

So if we have $\boxed{\frac{d}{ds}\Pi = +\frac{C_0}{s} - \sum_n \frac{nC_n}{s^{n+1}}}$,

Then

$$\boxed{\frac{1}{\pi s_0}\int ds\, \mathrm{Im}\,\Pi(s) e^{-s/s_0} = C_0 + \sum_n \frac{(-1)^n C_n}{(n-1)!\, s_0^n}}$$

This is the basic result, known as a "QCD sum rule". It relates the power corrections to Π to the measured (smeared) value of $R = (\mathrm{Im}\,\Pi)/(6\pi)^{-1}$.

We get further sum rules by differentiating. E.g., taking $\frac{d}{ds_0} s_0(\ )$ of both sides gives

$$\boxed{\frac{1}{\pi s_0^2}\int ds\, \mathrm{Im}\,\Pi(s) e^{-s/s_0} = C_0 + \sum \frac{(-1)^{n+1} C_n}{(n-2)!\, s_0^n}}$$

Here $C_0 = \frac{Q^2}{4\pi^2}\left(1 + \frac{\alpha_s}{\pi} + \cdots\right)$, for one quark flavor

The C_n can be determined by phenomenological fits, and one finds that the power corrections to the QCD perturbation theory predictions for (smeared) R are surprisingly small. If one takes the bold step of applying the sum rules to the region $\sqrt{s} < 1\,\mathrm{GeV}$, where R is dominated by the ρ resonance, some properties of the ρ can be predicted.

Since $R = (6\pi)\,\mathrm{Im}\,\Pi$, the sum rules can be written

$$\int ds\, e^{-s/s_0} R(s) = \left(\tfrac{3}{2}Q^2\right) s_0 \quad + \text{ pert and nonpert corrections}$$

$$\int ds\, s\, e^{-s/s_0} R(s) = \left(\tfrac{3}{2}Q^2\right) s_0^2 + \cdots$$

We can derive analogous sum rules for the isospin 1 currents

$$J_\mu^{I=1} = \tfrac{1}{2}(\bar u \gamma_\mu u - \bar d \gamma_\mu d),$$ to which ρ couples

and the $I=1$ component of R, $R^{I=1}$.

The ρ has $I=1$ and couples to the $I=1$ part of the electromagnetic current

$$J^\mu_{em} = \tfrac{2}{3}\bar{u}\gamma^\mu u - \tfrac{1}{3}\bar{d}\gamma^\mu d = \tfrac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d) + \tfrac{1}{6}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d)$$

For the isospin one current, we have

$$\Sigma Q^2 = \tfrac{1}{4}+\tfrac{1}{4} = \tfrac{1}{2}$$

and the sum rule becomes

$$\int ds\, e^{-s/s_0} R^{I=1}_{(s)} \simeq \tfrac{3}{4} s_0$$

We can define the coupling G_ρ of the ρ to the isospin current by

$$\langle 0| J^a_\mu |\rho^b\rangle = \epsilon_\mu \delta^{ab} G_\rho \qquad \text{(a, b are isospin indices)}$$

Approximating the ρ by a narrow resonance, the contribution it makes to $\Pi_{\mu\nu} = i\int d^4x\, e^{ikx}\langle 0|T J^3_\mu(x) J^3_\nu(0)|0\rangle$ is

$$\Pi_{\mu\nu} = i\ \otimes\text{-----}\otimes = i\, G_\rho^2\, \frac{-i(\eta_{\mu\nu} - k_\mu k_\nu/\mu^2)}{k^2-\mu^2+i\epsilon} + \text{real}$$

and $\operatorname{Im}\Pi = -\dfrac{G_\rho^2}{\mu^2}\operatorname{Im}\dfrac{1}{k^2-\mu^2+i\epsilon} = \pi\dfrac{G_\rho^2}{\mu^2}\delta(k^2-\mu^2)$

or $R = 6\pi^2(G_\rho^2/\mu^2)\,\delta(k^2-\mu^2)$.

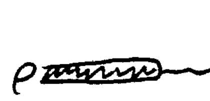
Our sum rule becomes

$$\frac{G_\rho^2}{\mu^2}\, e^{-\mu/s_0} = \frac{1}{8\pi^2} s_0$$

It is sensible to saturate with the ρ only for $s_0 \sim \mu$, so choose $s_0 = \mu$ and obtain

$$\boxed{\frac{G_\rho^2}{\mu_\rho^4} = \frac{e}{8\pi^2} = \frac{2.718}{8\pi^2}}$$

This is a prediction for the leptonic width of the ρ, which can be computed in terms of G_ρ.



$$= e^2 G_\rho \epsilon_\mu \frac{-i}{\mu_\rho^2} \bar{u}\gamma^\mu u$$

$$\langle |A|^2 \rangle = \frac{1}{3}\sum_r e_\mu^{(r)} e_\nu^{(r)} e^4 G_\rho^2 \, k \, \not{p}\gamma^\mu \not{p}'\gamma^\nu$$

$$= \frac{1}{3} e^4 \frac{G_\rho^2}{\mu_\rho^4} \left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}\right) 4\left(p^\mu p'^\nu + p^\nu p'^\mu - \eta^{\mu\nu} p\cdot p'\right)$$

$p\cdot p' = k\cdot p = k\cdot p' = \frac{\mu^2}{2}$ in CM frame $\Rightarrow$ $\langle |A|^2 \rangle = \frac{4}{3} e^4 \frac{G_\rho^2}{\mu_\rho^2}$

$$\Gamma(\rho \to e^+e^-) = \frac{1}{16\pi\mu_\rho}\langle |A|^2\rangle = \frac{1}{3}\frac{e^4}{4\pi}\frac{G_\rho^2}{\mu_\rho^4}\mu_\rho$$

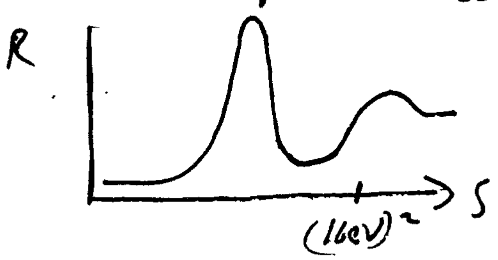
or $$\boxed{\frac{\Gamma(\rho\to e^+e^-)}{\mu_\rho} = \frac{1}{3}\alpha^2\left(\frac{2.718}{2\pi}\right) \sim 7.7\times 10^{-6}}$$

While, experimentally, we have

$$\left.\frac{\Gamma(\rho\to e^+e^-)}{\mu_\rho}\right|^{exp} = 8.6\times 10^{-6}$$

The agreement is remarkably good, but somewhat fortuitous, since the choice $s_0 = \mu_\rho$ in the sum rule was a bit arbitrary.

Why does it work? The key is that the ρ completely dominates the $I=1$ contribution to R up to $\sqrt{s}\sim 1\,\text{GeV}$, where the QCD prediction for smeared R starts to work well. Moreover, the "continuum" contribution is suppressed up to $\sqrt{s}\sim 1\,\text{GeV}$. So integrating up to about $1\,\text{GeV}$, dominating with the ρ, and comparing with the zeroth order QCD prediction is sensible, if power corrections and α_s are small at $\sqrt{s}\sim 1\,\text{GeV}$.



Masses and couplings of other resonances can also be predicted this way, because it seems that every channel is dominated by a resonance, with a continuum contribution turning on at about 1 GeV, where QCD perturbation theory already works okay.

But I do not understand why all channels are dominated by a low lying resonance in this way; we know it just from the data. So we do not really understand from first principles why these sum rule predictions work well.

Exercise 1.12

Assuming that the ϕ is an $s\bar{s}$ bound state, use the QCD sum rule to predict $\Gamma(\phi \to e^+e^-)$. Compare with experiment (write the electromagnetic current as $J_{em} = -\frac{1}{3}\bar{s}\gamma^\mu s + \cdots$, where ϕ does not couple to the remainder.)

N. Infrared Divergences and Jets

We originally formulated the renormalization group as a means of finding the asymptotic behavior at high energies of a massless field theory, and later noted that a massive field theory would behave like a massless one for $E \gg m$, provided the anomalous dimension γ_m of the mass operator is not too big. E.g., if we want to consider the asymptotic behavior of a cross section (which has no anomalous dimension),

$$\sigma(E, x, g_\mu, \mu, m_\mu) = E^{-2} f\left(\frac{E}{\mu}, x, g_\mu, \frac{m_\mu}{\mu}\right)$$
$$= E^{-2} f\left(x, g_E, \frac{m_E}{E}\right)$$
$$\Rightarrow E^{-2} f(x, g_E, 0), \text{ asymptotically.}$$

But this expression for the limiting behavior makes sense only if f is *not* singular as $m \to 0$. That is, it must be free of "infrared" or "mass" singularities.

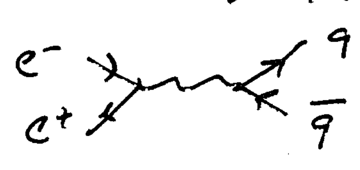
Traditionally, the RG analysis is applied to Green's Functions at large spacelike momentum (in fact, "nonexceptional" momenta, such that no partial sum is zero), since then the external momenta are guaranteed to provide an infrared cutoff on all Feynman integrals; masses are truly irrelevant as $E \to \infty$.

But for *physical* amplitudes, or cross sections, loop integration may be peaked (and divergent) at small k, because propagators blow up there.

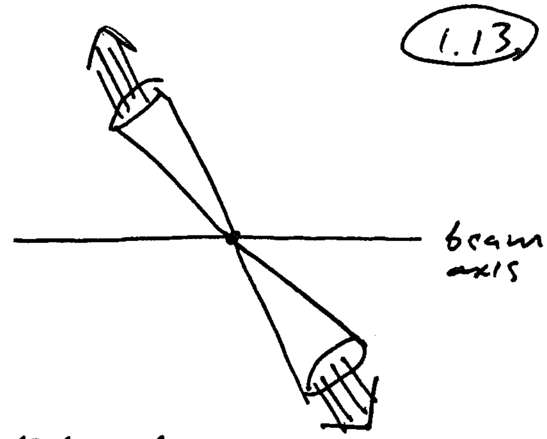
Fortunately, the following "rule of thumb" seems to hold: Any quantity which can, *in principle*, be measured, is free of infrared singularities. As long as we are careful to ask the right questions, the $m \to 0$ limit will be smooth.

This crucial interpretation of the IR divergences of QED was first proposed by Bloch and Nordsieck in 1937, and later extended by Kinoshita and Lee and Nauenberg.

The rule of thumb guarantees that the total e^+e^- annihilation cross section is infrared finite, and susceptible to a RG analysis. But now we would like to make predictions about details of the final state, using QCD perturbation theory. Specifically, it was predicted even before QCD was invented that if the process ---



dominates annihilation into hadrons at high energy,



then one expects to see two hadron "jets" in the final state, each "led" by a quark (or antiquark) produced in the collision. Fairly dramatic jets are observed at PETRA and PEP. The free field theory ("naive parton model," without QCD corrections) prediction for the angular distribution of the jets relative to the beam axis

$$d\sigma/d\Omega \sim (1+\cos^2\theta),$$

is seen to be roughly satisfied, indicating that partons have spin $\frac{1}{2}$, as expected.

Jets should become even better defined at higher energy (e.g. LEP). Non perturbative effects (soft exchanges) produce an intrinsic transverse momentum of the hadrons relative to the jet axis, $\langle p_T\rangle \sim$ 100's of MeV. At energy $\sqrt{s}$, the typical hadron momentum is

$$\langle p\rangle \sim \frac{\sqrt{s}}{\langle n\rangle} \quad \text{where } n = \text{hadron multiplicity.}$$

So the typical opening half-angle of a jet is

$$\langle \delta\rangle \sim \frac{\langle p_T\rangle}{\langle p\rangle} \sim \frac{\langle n\rangle\langle p_T\rangle}{\sqrt{s}}.$$

At PETRA, $\sqrt{s} \sim 40$ GeV, $\langle n\rangle \sim 20$, $\langle p_T\rangle \sim 500$ MeV and

$$\langle \delta\rangle \sim 15^\circ\text{-}20^\circ.$$

Since $\langle n\rangle\langle p_T\rangle$ is known to increase slowly with $\sqrt{s}$, δ will continue to shrink at higher energy. (Very impressive jets have been seen in recent $p\bar{p}$ collider exps.)

Can we use QCD perturbation theory to go beyond the naive parton model? We must define a "jet variable" which measures the "jettiness" of an event, calculate it in QCD, and compare with experiment.