2. Introduction to Confinement

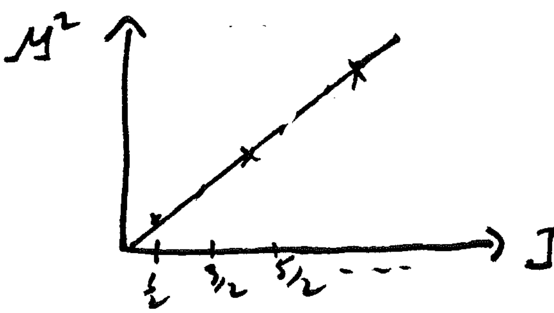
A. The String Picture of Hadrons

We have seen reasons for believing that quarks exist, and that QCD describes their interactions at short distances. But we have not yet even begun to explain, on the basis of QCD, one of the most conspicuous qualitative properties of the strong interactions — quarks appear to be permanently confined to hadrons. (Nonperturbative: $r_H^{-1} \sim \Lambda \sim \mu \exp[\quad]$.)

It is clear what this statement means. Quarks are tagged by the exotic quantum numbers they carry, fractional electric charges. These fractional electric charges are not observed in isolation.

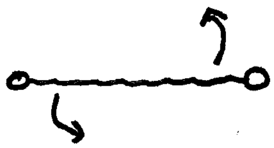
Dare we hope that the detailed dynamics of QCD will provide an explanation for this spectacular qualitative property? An instructive analogy, which will be seen to be extremely appropriate as we proceed, is to superconductivity. The properties of a superconductor are indeed spectacular, and were bewildering when first discovered. But eventually it was found that a detailed microscopic theory could explain these properties.

An important hint about the nature of confinement emerged from the study of orbitally excited hadron resonances. It was found that both meson and baryon resonances could be grouped into families lying on straight lines in a plot of M^2 vs J



(Universal slope for baryons and mesons) one speaks of "linear Regge trajectories." (e.g., Baryon resonances up to $J = 19/2$ have been seen; linearity works best for large J)

The linear trajectories are explained by a childishly simple model of the hadron resonances.



Consider a spinning _relativistic_ string. The statement that the string is relativistic means that all the rest energy of the string is generated by stretching it; i.e., in its rest frame, its mass density per unit length is the tension in the string

$$\rho = T$$

If the ends are massless, then the ends of the string must move at the speed of light. If an element of string spins about its center, the force acting on the tip and due to the string tension is Lorentz transformed to zero.

Exercise 2.1:

Show that, for a spinning relativistic (classical) string, the mass M and angular momentum J are related by

$$M^2 = c\, T J .$$

where 'T' is the string tension and c is a numerical constant. Find c. (T is measured to be ~ 13 tons.)

Inspired by the linearity of the Regge trajectories, attempts were made to devise a theory of hadronic structure in which strings were the fundamental dynamical degrees of freedom.

Eventually the string theory (or "dual model") approach was superseded by the gauge field theory approach. An important step was taken by Nielsen and Olesen, who suggested an interpretation of strings in terms of gauge field theory.

The model considered by Nielsen-Olesen is the abelian Higgs model — A particle physicists version of a superconductor.

It is a $U(1)$ gauge theory with a charged scalar:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu \phi|^2 - \lambda\left(|\phi|^2 - a^2\right)^2, \quad D_\mu = \partial_\mu - ieA_\mu$$

Spontaneous symmetry breaking occurs $\langle 0|\phi|0\rangle \neq 0$ — the vacuum is a coherent superposition of charged particles, a charge "condensate."

First consider the classical theory. The potential is minimized by

$$\phi = a e^{i\sigma}, \quad \text{where } \sigma \text{ is arbitrary (real).}$$

Let us consider classical field configurations at fixed time in two spatial dimensions. The only configurations with finite energy (relative to the vacuum configuration with σ = constant), are those for which

$$|\phi| \to a, \quad \text{as } r \to \infty$$

Therefore, associated with any finite-energy classical field configuration is a mapping from the circle at $r = \infty$ to the circle $e^{i\sigma}$.

It is possible for this mapping to be

topologically interesting. A mapping from a circle to a circle has associated with it a winding number, an integer n, which is unchanged by continuous deformations of the mapping.

Since time evolution is continuous, it follows that n is a conserved quantity ("topological conservation law")

The gradient term in the energy must also be finite for configuration to have finite energy. This means, in particular

$$\int d^2x \, \left|\left(\frac{1}{r}\frac{\partial}{\partial\theta} - ieA_\theta\right)\phi\right|^2 < \infty .$$

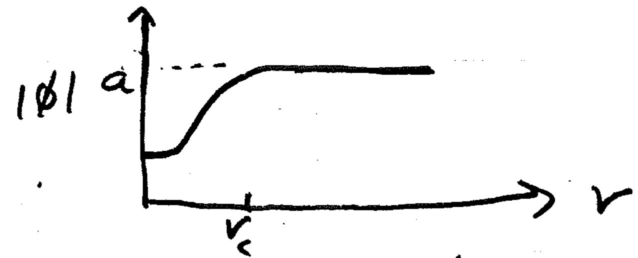
At the circle at infinity $\frac{\partial}{\partial\theta}\phi =$ $a\left(i\frac{d\sigma}{d\theta}\right)e^{i\sigma}$

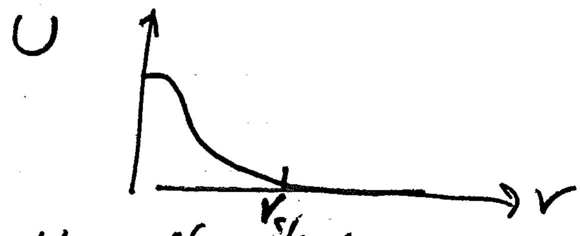
so we require

$$A_\theta \sim \frac{1}{er}\frac{d\sigma}{d\theta} + O\left(\frac{1}{r^2}\right)$$

for a finite energy configuration.
The leading term is a pure gauge, so the magnetic field F_{12} goes to zero at infinity (at least as fast as $\frac{1}{r^3}$).

But if the mapping $\sigma(\theta)$ is topologically nontrivial $(n \neq 0)$ then we know $\phi = 0$ somewhere. Otherwise we could shrink the circle at infinity to a point, with the mapping from circle to circle always well-defined. This would define a continuous deformation of the original mapping to a trivial mapping — impossible for $n \neq 0$. Thus, the configuration must have nonzero energy (relative to the vacuum). We can find a nontrivial soln to the static eqns of motion by minimizing the energy for $n \neq 0$.





The modulus of ϕ is appreciably different from a, and the energy density appreciably different from zero, only within a distance r_c from the center of the "vortex." Inside the vortex there is a magnetic field. The total magnetic flux can be calculated by Stokes theorem

$$\Phi = \int d^2x\, F_{12} = \int_0^{2\pi} r\,d\theta\, A_\theta \Big|_{r=\infty} = \frac{1}{e}\int_0^{2\pi} d\theta \frac{d\sigma}{d\theta} = \frac{2\pi}{e} n$$

– The vortex flux is quantized, in units of $2\pi/e$

The magnetic field inside the vortex stabilizes it against collapse. E.g., if the field is constant inside we would have

$$E_{\text{magnetic}} = \tfrac{1}{2}\pi r_c^2 F_{12}^2 = \tfrac{1}{2}\pi r_c^2 \left[(2\pi/e)/\pi r_c^2\right]^2 = \frac{2\pi}{e^2 r_c^2}$$

which blows up as $r_c \to 0$. The Higgs potential prevents the vortex from spreading out:

$$E_{\text{Higgs}} = \pi r_c^2 \lambda a^4$$

blows up for $r_c \to \infty$. A compromise between these effects determines r_c

(Actually, in general there are two distance scales, r_s for the scalar field, r_m for the magnetic field, and a contribution from the $|D_i\phi|^2$ term. For $r_s < r_m$

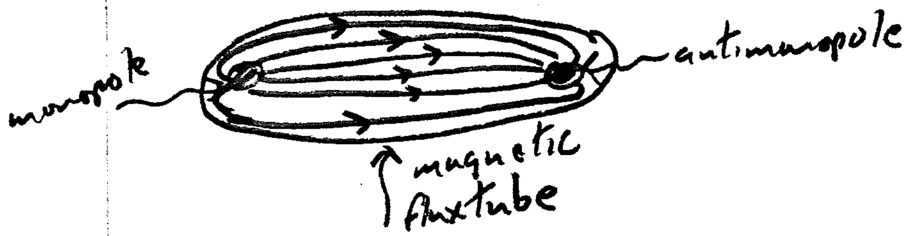
$$E \sim \pi\left(\frac{1}{e^2 r_m^2} + \lambda r_s^2 a^4 + a^2 \ln\frac{r_m}{r_s}\right)$$

we minimize E by choosing $r_m \sim \dfrac{1}{ea}$ $\quad r_s \sim \dfrac{1}{\sqrt{\lambda}\, a}$)

(Note: In the quantum field theory, the strings survive to all orders in the expansion in $\hbar$.)

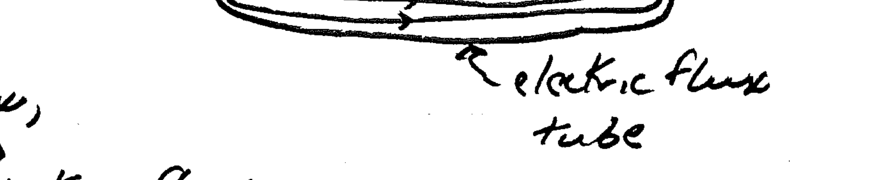
In 3 dimensions, the vortices become <u>strings</u> with constant energy/length. These strings cannot end. But in the string picture of hadrons, the string is of finite length.

But we can end strings, and obtain pieces of string resembling hadrons, if we introduce magnetic monopoles into our superconductor, to soak up the conserved magnetic flux.



Monopoles are <u>confined</u> in a superconductor. A monopole and antimonopole introduced into superconductor will be connected by a <u>flux tube</u> of width r_m and energy per length E; giving rise to a linear potential attracting monopole to antimonopole. They form a "meson"

While this model is wrong in detail, the same physical picture may apply to the confinement of quarks. Quarks are not magnetic monopoles, but carry color electric charges. As a superconductor expels magnetic flux, perhaps the QCD vacuum expels electric flux (electric Meissner effect) perhaps. The strings indicated by the linearity of the Regge trajectories may be color-electric flux tubes



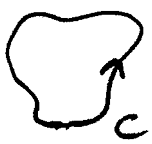
B. A Criterion for Confinement

The flux-tube interpretation of the string picture of hadrons suggests that, to understand confinement, we should study the dynamics of flux tubes in the QCD vacuum. That is, we want to understand what, in QCD, prevents a flux tube from spreading out.

We can consider flux tubes without introducing quarks, just as we were able to discuss vortices in a superconductor without introducing magnetic monopoles. So, for now we will forget about quarks.

We wish to formulate in a precise mathematical way what the property of the QCD vacuum is which corresponds to confinement; i.e., to the stability of flux tubes.

An operator which we will find convenient to study is the Wilson loop. Associated with any closed loop C in spacetime and a sense around the loop is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$$A(C) = \mathrm{tr}\, P \exp\left[ig \oint_C dx^\mu A_\mu\right]$$

The symbol P denotes path ordering. E.g. if the path is parametrized as $x^\mu(s)$ $0 \le s \le 1$, then

$$A(C) = \mathrm{tr}\, S \exp\left[ig \int_0^1 ds \frac{dx^\mu}{ds} A_\mu^a T^a\right]$$

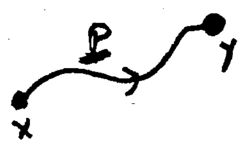
where the exponential is ordered in the same sense as Dyson's formula.

Let's first of all verify that the operator $A(C)$ is gauge-invariant. Consider an infinitesimal element of the loop, for which the contribution to $A(C)$ can be written ...

$$e^{igA_\mu dx^\mu} \sim 1 + igA_\mu dx^\mu$$

This object transforms "bilocally" i.e.

$$\Omega(x+dx)\left[1+igA_\mu dx^\mu\right]\Omega^{-1}(x)$$
$$= 1 + ig\left[\Omega(x)A_\mu(x)\Omega^{-1}(x) + \frac{1}{ig}(\partial_\mu\Omega)\Omega^{-1}\right]dx^\mu + O(dx^2)$$



If we build up a finite open string from many such string bits,

$$U(P; y, x) = \prod_n \exp\left[igA_\mu(z_n)(z^\mu_{n+1} - z^\mu_n)\right]$$
$$= P\exp\left[ig\int_x^y dx^\mu A_\mu\right],$$

This transforms as

$$U(P; y, x) \to \Omega(y)\,U(P; y, x)\,\Omega^{-1}(x)$$

Now, if we close up the loop and take a trace,

$$A(C) \to A(C)$$

– Also $A(C)$ is independent of where the loop starts.

Note that we can also construct a gauge invariant object from an open string operator by placing "quarks" at the ends of the string:

$$\bar{q}(y)\,U(P; y, x)\,q(x)$$

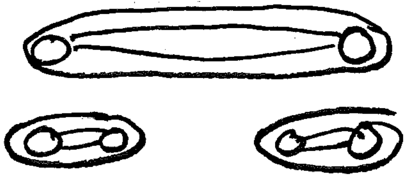
is gauge invariant, if the "quark" is any local object (including a classical source), which transforms under a gauge transformation as the defining representation:

$$q(x) \to \Omega(x)q(x)$$

What does the Wilson loop have to do with confinement and flux tubes? To understand the connection, we must consider several different interpretations of the loop operator.

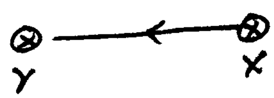
Timelike Loops

To study the energetics of flux tubes, we may consider a theory with gauge fields coupled to heavy quarks, and compute energies of states with a widely separated quark and antiquark. (The quarks must be very heavy, so that it is not energetically favorable for flux tube to "break" by $q\bar{q}$ pair formation.)



A gauge-invariant operator which creates a widely separated quark-antiquark pair is

$$M(y,x) = \bar{q}(y)\, U(P; y, x)\, q(x)$$



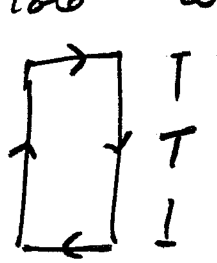
Suppose $y^0 = x^0 = 0$; we choose the path to be a straight line with $t=0$

We can destroy the pair with M^+. Consider, then,

$$W(T,R) = \langle 0 | M^+(T,\vec{0}; T,\vec{x})\; M(0,\vec{0}; 0,\vec{x}) | 0 \rangle$$

$$= \sum_n |\langle n | M(0,\vec{0}; 0,\vec{x}) | 0 \rangle|^2 \, e^{-iE_n T}$$

In Euclidean time, the exponential is $e^{-E_n T}$, and the large time asymptotic behavior is dominated by the least energetic state which couples to M.



If we regard the quark fields as classical sources which do not propagate, then $W(T,R)$ is just $\langle A(L) \rangle$ for the rectangular loop shown, in a gauge with $A_0 = 0$. The equality holds in any gauge, because $W(T,R)$ and $A(L)$ are gauge-invariant. (In $A_0 = 0$ gauge, we lose freedom to gauge transform the static source)

We thus have

$$\langle 0| A(C) |0\rangle \xrightarrow[T\to\infty]{} e^{-E(R)T}$$

where $E(R)$ is the energy of $N, \bar{N}$ color sources (of $SU(N)$) separated by distance R

If confinement holds, we expect

$$E(R) \xrightarrow[R\to\infty]{} KR$$

where K is the "string tension" of a flux tube. But, if it is possible to separate quarks, we expect

$$E(R) \xrightarrow[R\to\infty]{} 2\mu \quad , \text{ a constant independent}$$

of R. For large loops then, we expect

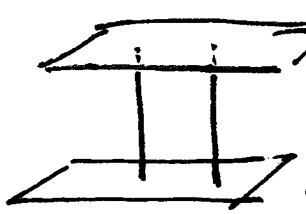
$$\langle 0| A(C) |0\rangle \Big|_{\text{Euclidean}} \longrightarrow e^{-KRT} = e^{-K(\text{Area})} \quad - \text{Confinement}$$

$$\langle 0| A(C) |0\rangle \Big|_{\text{Euclidean}} \longrightarrow e^{-2\mu T} = e^{-\mu(\text{Perimeter})} \quad - \text{No confinement}$$

<u>Spacelike Loops (and the Canonical Formalism)</u>

We'll continue to work in the gauge $A^0=0$. First, let's check that it is always possible to find a gauge transformation such that $A^0=0$:

With each point $(\vec{x},t)$ associate the straight-line path from $(\vec{x},0)$ to $(\vec{x},t)$



Then choose the gauge transformation to be

$$\Omega(\vec{x},t) = U^{-1}(P;\vec{x},t;\vec{x},0)$$

under this transformation U becomes

$$U'(P; \vec{x}, t; \vec{x}, 0) = \Omega(\vec{x}, t)\, U\, \Omega^{-1}(\vec{x}, 0) = \mathbb{1},$$

and the new gauge field can be recovered from U' by differentiation

$$\frac{\partial}{\partial t} U(P; \vec{x}, t; \vec{x}, 0) = i g A_0(\vec{x}, t)\, U = 0$$

The $A_0 = 0$ gauge condition does not remove the freedom to make time-independent gauge transformations $\Omega(\vec{x})$

Before setting $A_0 = 0$, we must derive the eqn. of constraint which is obtained by varying the Lagrange density with respect to A_0. Recall

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} - A_\mu^a J^{\mu a}$$

and $$F_{\mu\nu}^{a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g C^{abc} A_\mu^b A_\nu^c$$

The A_0 eqn of motion is

$$0 = \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i A_0^a} - \frac{\partial \mathcal{L}}{\partial A_0^a} = \partial_i F^{0ia} + F^{0ib} \frac{\partial F_{0i}^{b}}{\partial A_0^a} - J^{0a}$$

$$= \partial_i F^{0ia} + F^{0ib} g C^{bac} A_i^c + J^{0a}$$

or $$\partial_i F^{0ia} + g C^{abc} A_i^b F^{0ic} = -J^{0a}$$

or $$\boxed{[D_i, F^{0i}] = -J^0}$$

This is an eqn of constraint; i.e., involves no time derivatives. It is Gauss's law

The eqns of motion are

$$[D_\mu, F^{i\mu}] = -J^i$$

In the $A_0 = 0$ gauge, we have
$$F^{0i} = \partial^0 A^i,$$
so the term in $\mathcal{L}$ involving time derivatives is simply
$$-\frac{1}{2}\partial^0 A^{ia}\partial_0 A_i^a,$$
and the canonical variable conjugate to A_i is
$$\frac{\partial \mathcal{L}}{\partial \partial_0 A_i^a} = -\partial^0 A^{ia} = \partial_0 A_i^a = E_i^a$$
and $\mathcal{H} = \frac{1}{2}E_i^{a2} + \frac{1}{2}B_i^{a2}$, where $B_i^a = \frac{1}{2}\epsilon_{ijk}F_{jk}^a$

At fixed time, the $A_i^a(\vec{x})$ are a complete set of commuting observables, and they obey with the E_i's the canonical commutation relations
$$[A_i^a(\vec{x}), A_j^b(\vec{y})] = 0 = [E_i^a, E_j^b]$$
$$[A_i^a(\vec{x}), E_j^b(\vec{y})] = i\delta_{ij}\delta^{ab}\delta^3(\vec{x}-\vec{y})$$

Since the $A_i^a(\vec{x})$ are complete, any state may be expanded as a "wave functional" $\psi[A]$ in terms of their eigenstates. In this Hilbert space, E_i^a may be represented by
$$E_i^a(\vec{x}) = -i\frac{\delta}{\delta A_i^a(\vec{x})}.$$

Let's find the generators in this Hilbert space of infinitesimal time-independent gauge transformations. Recall that, for ω infinitesimal,
$$\Omega = 1 + i\omega, \text{ we have } A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{1}{ig}(\partial_\mu \Omega)\Omega^{-1}$$
$$= A_\mu - i[A_\mu, \omega] + \frac{1}{g}\partial_\mu \omega = \frac{1}{g}[D_\mu, \omega] + A_\mu$$
we need, for time-independent $\Omega(\vec{x}) = \mathbb{I} + i\omega(\vec{x}) = \mathbb{1} + i\omega^a T^a$
$$U(\Omega)A_i^a U(\Omega)^{-1} = A_i^a + \frac{1}{g}\partial_i \omega^a + C^{abc}A_i^b\omega^c$$
where $U = \mathbb{1} + i\int d^3x\, \omega^a(\vec{x})\,\Gamma^a(\vec{x})$

or $$i\int d^3x'\, \omega^b(\vec{x}')\,[\Gamma^b(\vec{x}'), A_i^a(\vec{x})] = \frac{1}{g}\left(\partial_i \omega^a(\vec{x}) + g\, C^{abc} A_i^b \omega^c\right)$$

$$\Rightarrow [\Gamma^b(\vec{x}'), A_i^a(\vec{x})] = \frac{1}{ig}\left[\delta^{ab}\left(-\partial_i \delta^3(\vec{x}'-\vec{x})\right) - g\, C^{abc} A_i^c(\vec{x}) \times \delta^3(\vec{x}'-\vec{x})\right]$$

Using the canonical commutation relations, we see that

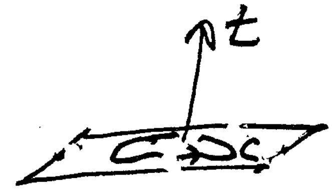
$$\Gamma^b(\vec{x}') = \frac{-1}{g}\left[\partial_i E_i^b + g\, C^{bcd} A_i^c E_i^d\right]$$

or $$\boxed{\Gamma = -\frac{1}{g}[D_i, E_i]}$$

But the Gauss' law constraint (for a vanishing external current) is $\Gamma = 0$; therefore for any state $|\psi\rangle$ satisfying this constraint

$$U(\Omega(\vec{x}))|\psi\rangle = |\psi\rangle$$

Thus, the initial-value problem is well-defined, and the Gauss' law constraint is satisfied, if we restrict our attention to the Hilbert space of states which are invariant under time-independent gauge transformations.



Now we can consider the effect of the operator $A(C)$ acting on a state, where C is a loop at fixed time

$$E_i^a(\vec{x})\, A(C)|\psi\rangle = A(C)\, E_i^a|\psi\rangle + -i\frac{\delta}{\delta A_i^a(\vec{x})} A(C)|\psi\rangle$$

The second term is

$$\mathrm{tr}\, \underline{P}\; g\left[\frac{\delta}{\delta A_i^a(\vec{x})}\left(\oint_C \vec{A}\cdot d\vec{l}\right)\right] \exp\left(ig\oint_C \vec{A}\cdot d\vec{l}\right)|\psi\rangle$$

The $\frac{\delta}{\delta A} \oint A \cdot dx$ factor generates a two dimensional δ function along the loop. Thus, the Wilson loop operator creates a string of electric flux, with infinitesimal width and unit strength (in units of g).

By Euclidean invariance, we should be able to apply our Wilson loop confinement criterion to spacelike loops:

$$\langle 0|A(C)|0\rangle \sim e^{-K(\text{Area})} \quad - \text{Confinement}$$
$$\langle 0|A(C)|0\rangle \sim e^{-\mu(\text{Perimeter})} \quad - \text{No confinement}$$

For spacelike loops, the interpretation of this criterion is different. $A(C)$ creates a closed tube of electric flux, and $\langle 0|A(C)|0\rangle$ is the overlap of the state $A(C)|0\rangle$ with the vacuum. The flux tube is not stable, so this overlap does not vanish

But if there is no confinement, the flux tube is not even locally stable – it can simply diffuse away. In this case the flux tube state differs from the vacuum over a volume proportional to the perimeter; it is not surprising that the overlap behaves like $\exp[-\mu(\text{perimeter})]$

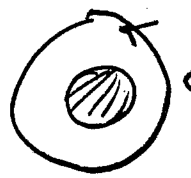
If there is confinement the flux tube cannot spread indefinitely; it is locally stable and must collapse – this state differs significantly from the vacuum over the whole area of the loop, and the overlap behaves like $\exp[-K(\text{Area})]$.

Magnetic Flux and the Wilson Loop

We have seen that the Wilson loop operator creates a closed tube of electric flux. It also, in a sense, measures magnetic flux. For example, in the abelian case

$$\oint_C \bar{A}\cdot d\bar{l} = \int \bar{B}\cdot d\bar{a} = \Phi$$

and $\exp ig\oint_C \vec{A}\cdot d\vec{l} = e^{ig\Phi}$ where Φ is the magnetic flux through the loop C



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Suppose that, in the Abelian Higgs model $F_{\mu\nu}=0$ outside a compact region, so that A_μ is a "pure gauge" in that region

$$A_\mu = \frac{1}{ie}(\partial_\mu \Omega)\Omega^{-1} \quad \text{or} \quad \partial_\mu \Omega = ieA_\mu \Omega$$

which has the solution

$$\Omega(y) = \exp\left[ie\int_x^y A_\mu dx^\mu\right]\Omega(x)$$

(integral independent of path if $F=0$)

The requirement that this gauge transformation is single-valued is

$$e\oint_C A_\mu dx^\mu = 2\pi n$$

– the flux quantization condition we derived earlier.

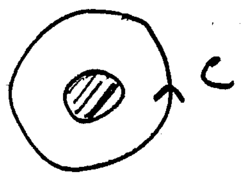
We can determine n by observing A_μ only in the region where $F_{\mu\nu}=0$, and thus find out the total magnetic flux without knowing A_μ in the region where $F_{\mu\nu}$ is nonzero. A gauge transformation in the "unshaded" region which winds around the circle necessarily introduces magnetic flux in the shaded region (E.g. it has a singularity on which it creates a line of magnetic flux.)

An operator which creates a magnetic vortex can be defined whether or not SSB occurs in the model. But the vortex will be stable (unable to spread) only if $\langle\phi\rangle \neq 0$

Analogous magnetic vortices occur in nonabelian gauge theories. The key observation is that the gauge fields really transform as a representation of $SU(N)/Z_N$ where Z_N is the center of $SU(N)$

$$Z_N = \{ e^{2\pi i K/N}, \ K=0,\dots,N-1 \}$$

The gauge group of a pure $SU(N)$ gauge theory is really $SU(N)/Z_N$



If A_μ is a pure gauge outside a compact region in the plane,

$$A_\mu = \frac{1}{ig} (\partial_\mu \Omega)\Omega^{-1} \Rightarrow \Omega(y) = P\left(\exp ig \int_x^y A_\mu dx^\mu\right)\Omega(x)$$

Now single-valuedness of Ω requires

$$P \exp ig \oint A_\mu dx^\mu = \exp[2\pi i K/N] \, \mathbb{1}$$

– i.e. singlevaluedness up to an element of the center of the group. Gauge transformations in the plane with a region excluded have topology classified by Z_N in an $SU(N)$ gauge theory, as such gauge transformations are classified by Z in a $U(1)$ gauge theory.

Thus, the $SU(N)$ theory has Z_N magnetic vortices. Such vortices are stable if $SU(N)$ is completely broken by the Higgs mechanism. The vortex number is conserved mod N.

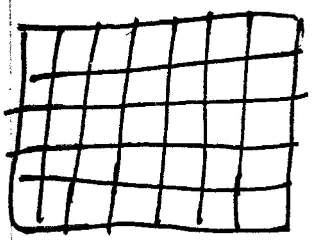
If $SU(N)$ is spontaneously broken, then vortices are heavy, i.e. have a large mass density per unit length. Magnetic vacuum fluctuations (virtual vortex loops) are suppressed. But if $SU(N)$ is not broken, magnetic fluctuations are not so costly.

Confinement can be viewed as a consequence of a magnetically disordered vacuum, i.e. a vacuum in which random magnetic fields fluctuate freely. This statement will gradually become clearer, but to get an idea what it means. we will consider a vastly simplified approximation to an SU(2) gauge theory

A Wilson loop counts the number of magnetic vortices through the loop

$$\langle 0|A(C)|0\rangle \sim \langle (-1)^{n_v} \rangle$$

where n_v is the number of vortices



Suppose that a large Wilson loop can be divided into N squares, such that the probability that a magnetic vortex pierces each square is P, and the squares are completely uncorrelated.

Writing $n_v = Np+m$, the probability that $Np+m$ vortices pierce the loop is

$$P_m = \binom{N}{Np+m} p^{(Np+m)} (1-p)^{N(1-p)-m}$$

Using Stirling approx and $m \ll Np, N(1-p)$, we find

$$P_m \propto \exp\left\{-\frac{m^2}{N}\left[\frac{1}{p(1-p)}\right]\right\}$$

, so m is a Gaussian random variable.

Now

$$\langle 0|A(C)|0\rangle \propto \int_{-\infty}^{\infty} dm\,(-1)^m e^{-am^2} = \int e^{-am^2+i\pi m}\,dm$$

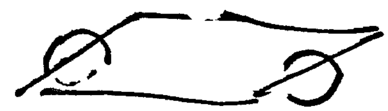
$$= \int e^{-a(m-\frac{i\pi}{2a})^2 - \pi^2/4a} \propto e^{-\pi^2/4a}$$

Thus $\langle 0|A(C)|0\rangle \sim \exp\left[-\frac{p(1-p)}{4}\pi^2 N\right]$

N is the area of the loop, divided by a typical correlation length squared, so this is

$$\langle A \rangle \sim e^{-\text{Area}} \quad \text{behavior}$$
$$\Rightarrow \text{confinement}$$

But, if the magnetic flux is confined to tubes, then we



expect magnetic fluctuations to be suppressed; only short virtual loops of magnetic flux will occur. Thus, we should average over only a number N of cells proportional to the perimeter of the loop:

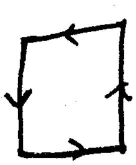
$$\langle A \rangle \sim e^{-\text{Perimeter}} \Rightarrow \text{no confinement}$$

To summarize, if there are no massless particles, and therefore finite correlations, we expect that

Magnetic Disorder ⇒ Electric Confinement
Magnetic Confinement ⇒ Electric Nonconfinement
(Electric Disorder)

at least on the basis of this simple model. We will formulate this "magnetic-electric duality" more precisely later on, and reach more general conclusions.

Exercise 2.2



Calculate to $O(g^2)$ the vacuum expectation value of the Wilson loop operator

$$\langle 0 | A(C) | 0 \rangle = \langle 0 | \mathrm{Tr}\, P \exp ig \oint_C A_\mu dx^\mu | 0 \rangle$$

for a rectangular loop (in Euclidean space) with sides R and T. Assume $T \gg R$ and retain only the term proportional to T/R.

C. Confinement in 2+1 Dimensions

G. 't Hooft, Nucl. Phys. B138, 1 (1978)

We will first consider $SU(N)$ gauge theories in 2+1 dimensions. The topological considerations are similar to those relevant in 3+1 dimensions, but simpler.

In 2 spatial dimensions, the vortices are pointlike objects (solitons). We can define a local field operator which creates a vortex.

Recall that a field configuration with a Z_N vortex

has A_μ = "pure gauge with a twist" outside a vortex.

A Z_N vortex is created by singular gauge transformation which twists the gauge field around a point. There is an $\Omega(\vec{x}, \vec{x}_0)$ which creates a point vortex at $\vec{x}_0$, acting at fixed time.

It is easy to find a corresponding operator in the quantum theory. At a fixed time in $A_0 = 0$ gauge, consider the complete basis $\{ | A_i^a(\vec{x}) \rangle \}$ of eigenstates of A_i. Define the vortex creation operator $\Phi(\vec{x}_0)$ acting on this basis by

$$\Phi(\vec{x}_0) \, | A_i(\vec{x}) \rangle = | A_i^{\Omega(\vec{x}_0)}(\vec{x}) \rangle$$

where A_i^Ω denotes A_i gauge transformed by Ω, and $\Omega(\vec{x}_0)$ is the twisted gauge transformation which creates a vortex at $\vec{x}_0$.

Of course, $\Omega(\vec{x}_0)$ is not really a mere gauge transformation, because of the singularity. If it were, Φ would be trivial acting on the physical states which satisfy the Gauss's law constraint. (But two gauge transformations with the same singularity differ by a nonsingular trans, which does act trivially – so $\Omega(\vec{x}_0)$ is arbitrary, except for the singularity.)

We also know that $\underline{\Phi}(\vec{x}_0)$ preserves the physical subspace of gauge-invariant states, because a gauge transformation cannot change the location of the singularity at $\vec{x}_0$, which is the sole physical content of $\underline{\Phi}(\vec{x}_0)$. ($\underline{\Phi}$ does not create electric flux.)

At fixed time we have

$$0 = [\underline{\Phi}(\vec{x}), \underline{\Phi}(\vec{y})] = [\underline{\Phi}(\vec{x}), \underline{\Phi}^\dagger(\vec{y})] = [\mathcal{O}(\vec{x}), \underline{\Phi}(\vec{y})]$$

for $\vec{x} \neq \vec{y}$, where $\mathcal{O}$ is any gauge invariant local operator and $\underline{\Phi}^\dagger$ induces the inverse gauge transformation. It is in this sense that $\underline{\Phi}(\vec{x})$ may be regarded as a local operator. (That is, the commutators vanish when acting on gauge-invariant states.)

$\underline{\Phi}$ can create an antivortex or annihilate a vortex
$\underline{\Phi}^\dagger$ can create a vortex or annihilate an antivortex

$\underline{\Phi}$ has a Z_N quantum number, because the vortex does. I claim that, if $\underline{\Phi}$ has a vacuum expectation value, and Z_N is spontaneously broken, then quarks will be confined. (Confinement here means Area law confinement, i.e. stable electric flux tubes. In 2+1 dimensions, even the Coulomb potential is logarithmically confining.)

Note the similarity of the above statement to what happens in a superconductor. In a superconductor, a field which creates an electrically charged particle acquires a v.e.v. The vacuum is an electric condensate (contains an indefinite number of electric charges), and, as a result magnetic flux cannot penetrate the vacuum. Here we claim that, if a field which creates a magnetic vortex has a vev, then electric flux cannot penetrate the vacuum

Note also that a vortex condensate, $\langle \Phi \rangle \neq 0$, implies magnetic disorder, as described earlier.

The statement that $\Phi(\vec{x}_0)$ carries a Z_N quantum number means the following: If the magnetic vortices are stabilized (e.g. by the Higgs mechanism), then vortex number is conserved modulo N. An operator U which multiplies a state containing n vortices by $\exp(-in2\pi/N)$ commutes with the Hamiltonian. Since Φ decreases the vortex number by one, we have

$$U \Phi U^{-1} = e^{2\pi i/N} \Phi$$

It is a question of dynamics whether $\langle 0|\Phi|0\rangle \neq 0$. For now we will assume $\langle \Phi \rangle \neq 0$; we'll return to the justification of this assumption later.

If there is a vortex condensate, the topological Z_N symmetry of the Hamiltonian is spontaneously broken

$$\langle 0| U \Phi U^{-1} |0\rangle = e^{2\pi i/N} \langle 0| \Phi |0\rangle$$

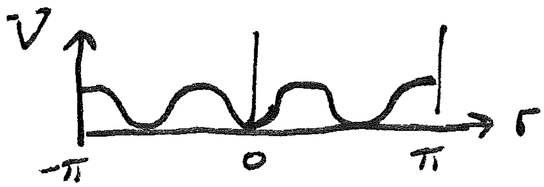
- there are N degenerate vacua

Associated with this spontaneously broken discrete symmetry is a new type of topological object - A domain wall.

There is an effective potential for the vortex field Φ. If $|\langle \Phi \rangle| = a$, we write

$$\Phi = a e^{i\sigma}$$

and the potential as a function of the phase σ, looks like



The energy density has the form

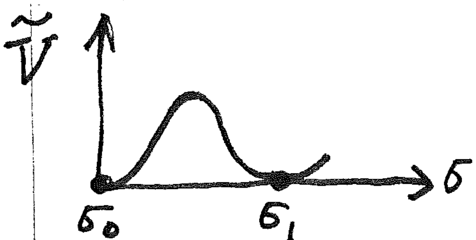
$$U = \tfrac{1}{2}|\nabla \bar{\Phi}|^2 + V(\bar{\Phi}) = \frac{a^2}{2}(\nabla \sigma)^2 + a^6 \tilde{V}(\sigma)$$

where $\tilde{V}$ is dimensionless — a has the dimensions $(\text{mass})^{1/2}$. There may also be terms involving more derivatives of $\bar{\Phi}$; moreover, we cannot really justify freezing the modulus of $\bar{\Phi}$ and considering only σ to be a function of $\bar{x}$. We make these assumptions only to illustrate the character of the domain wall in a simple context.

In 2 spatial dimensions, a "wall" is really like a string. Let's study the wall in one spatial dimension, along a line perpendicular to the wall.

The energy (i.e. energy / length) of a one-dimensional field configuration is

$$E = a^2 \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + a^4 \tilde{V}(\sigma)\right]$$



Finite energy configurations must have $\sigma \to$ min of $\tilde{V}$ as $x \to \pm\infty$ (and $d\sigma/dx \to 0$). If $\tilde{V}$ has more than one minimum, though, σ need not go to the <u>same</u> minimum at $x \to +\infty$ and $x \to -\infty$.

$\lim_{x \to \pm\infty} \sigma = \sigma_\pm$ are <u>topological</u> invariants.
- continuous deformations of $\sigma(x)$ consistent with finiteness of the energy cannot change $\sigma_\pm$

Suppose $\sigma_\pm$ are adjacent minima (σ_0 and σ_1) of the potential $\tilde{V}(\sigma)$. The configuration $\sigma(x)$ which minimizes E is a static soln to the eqn of motion. Since E is stationary, we have

$$0 = \frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 - a^4 \tilde{V}(\sigma)$$ (First integral of Euler-Lagrange eqn.)

Thus $E_{min} = a^6 \int_{-\infty}^{\infty} dx\, 2\tilde{V}(\sigma) = a^6 \int d\sigma\, \frac{dx}{d\sigma}(2\tilde{V})$

$$\text{or } E_{min} = a^4 \int_{\sigma_0}^{\sigma_1} d\sigma\, \sqrt{2\tilde{V}(\sigma)}$$

- this is the energy / length of our "string" (recall a^2 is a mass scale)

The thickness of the wall is determine by solving

$$\frac{d\sigma}{dx} = a^2 \sqrt{2\tilde{V}(\sigma)} \quad \text{or } a\,dx = d\sigma / \sqrt{2\tilde{V}}$$

- it's of order a^{-2}

What determines the energy scale a^2 of the wall (i.e., string)? Recall that a 2+1-dimensional gauge theory is superrenormalizable - g^2 has the dimensions of mass. The coupling constant does not run, and it sets the mass scale. We expect $a^2 = O(g^2)$

We have seen that there are topologically stable strings in this theory with spontaneously broken Z_N symmetry which carry constant energy / length. These are, in 2+1-dimensions, the electric flux tubes which confine quarks.

To show this, we show that the Wilson loop operator $A(C)$ creates a closed domain wall along C. We first observe that, for a curve C in a plane of constant time

$$A(C)\,\Phi(\vec{x}_0) = \Phi(\vec{x}_0)\,A(C)\,e^{2\pi i n/N}$$

where n is the number of times C wraps (counter-clockwise) around the point. This follows from our observation (p 2.16) that the argument of $A(C)$ counts vortices mod N contained inside C. Equivalently, the multivalued singular gauge transformation associated with $\Phi(\vec{x}_0)$ does not leave $A(C)$ invariant but multiplies it by an element of Z_N:

$$P \exp ig\int_x^x dx^\mu_\mu A_\mu$$

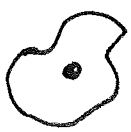
$$\to e^{-2\pi i n/N}\,\Omega(\bar{x})\left[P \exp ig\int(\ )\right]\Omega(\bar{x})$$

Now, what happens if $A(C)$ acts on the vacuum state with $\langle \Phi \rangle = a$? If we expand in terms of eigenstates of the vortex field $|\Phi(\bar{x})\rangle$

$$\underset{\sim}{\Phi}(\bar{x}_0)\,|\Phi(\bar{x})\rangle = \Phi(\bar{x}_0)\,|\Phi(\bar{x})\rangle$$

↖ operator ↗ eigenvalue

then $\underset{\sim}{\Phi}(\bar{x}_0)\,\underset{\sim}{A}(C)\,|\Phi(\bar{x})\rangle = e^{-2\pi i n/N}\,\Phi(\bar{x}_0)\,\underset{\sim}{A}(C)\,|\Phi(\bar{x})\rangle$



C $\underset{\sim}{A}(C)$ changes the eigenvalue of $\Phi(\bar{x}_0)$ by $e^{-2\pi i/N}$ if C winds once around $\bar{x}_0$

In this representation, we see that $\underset{\sim}{A}(C)^\dagger\,\underset{\sim}{A}(C) = \mathbb{1}$ and we may write

$$A(C)^\dagger\,\Phi(\bar{x}_0)\,A(C) = e^{-2\pi i n/N}\,\Phi(\bar{x}_0)$$

and therefore

$$\langle 0|A(C)^\dagger\,\Phi(\bar{x}_0)\,A(C)|0\rangle = e^{-2\pi i n/N}\langle 0|\Phi(\bar{x}_0)|0\rangle$$

For a simple closed loop, $A(C)$ causes $\langle\Phi\rangle$ to jump to another discrete vacuum value at C. It creates a domain wall along C

If we now introduce quarks, the topologically

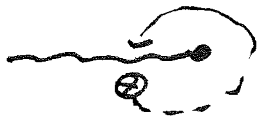
stable strings end on quarks, which are therefore confined.

Why do domain walls end on quarks? It is because quarks transform nontrivially under Z_N, the center of $SU(N)$. Consider

$$\bar{q}(\bar{y})\,P\exp\left[ig\int_{\bar{x}}^{\bar{y}} d\bar{x}'\cdot\bar{A}\right] q(\bar{x})\ \ \Phi(\bar{x}_0)$$

– a gauge invariant operator which creates a (anti)vortex and a quark antiquark pair

If I consider $q(\vec{x})\,\Phi(\vec{x}_0)$, and allow the vortex field to wind around the quark field, it picks up the phase $\exp(2\pi i/N)$, because the quark transforms nontrivially under Z_N. Dynamics determines whether it is energetically favorable for this change in phase to collapse to a string with constant energy per unit length. (We've seen that it is, if there is a vortex condensate.)



If we consider the gauge invariant operator, which attaches a string to the quark, $\Phi(\vec{x}_0)$ changes discontinuously when it crosses the string.

Since there are N degenerate vacua, N strings can meet at a junction. The result can be a baryon:

Higgs vs. Confinement:

Why should we expect a vortex condensate? Imagine introducing elementary scalar fields in $SU(N)/Z_N$ reps, which, if they have vevs, completely break the gauge symmetry, and stabilize the vortices. If μ^2 is the scalar mass, we expect the vortex mass M_V^2 to be given by

$$M_V^2 \propto -\mu^2\,, \quad \text{for } \mu^2 < 0$$

It is plausible that this proportionality holds also for $\mu^2 > 0$, so vortices condense when gauge symmetry is unbroken:

$$\text{Higgs} \;\big|\; \xrightarrow{\text{Confinement}} \mu^2/g^4$$

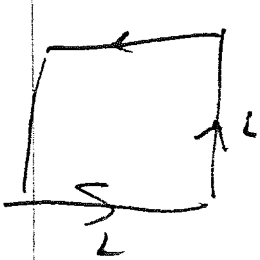
$\uparrow$ Transition could be first order

Is pure gauge theory more like theory with positive or negative mass-squared?

Let's pause to note why we do not expect magnetic disorder, and area law confinement, in quantum electrodynamics in (2+1)-dimensions.

Electrodynamics without electrons is just free-field theory, and we know the vacuum state; it is an assembly of uncoupled oscillators, each in its ground state. The (Euclidean) action can be written (in $A_0=0$ gauge) as

$$S=\frac{1}{2}(\dot{A}^2+B^2)d^3x$$



Magnetic fluctuations which give the dominant contribution to a Wilson loop with side L have wavelength $\sim B$

$$A\sim BL \Rightarrow \dot{A}\sim B \quad \text{and} \quad S\sim B^2L^3$$

Since large fluctuations are suppressed by e^{-S}, we expect

$$\langle B^2\rangle^{\frac{1}{2}}\sim L^{-3/2} \quad \text{and} \quad \langle \Phi\rangle \sim L^{\frac{1}{2}}$$

where Φ is the B-flux through loop.

Reasoning as on page 2.17, we expect

$$\langle e^{i\Phi}\rangle \sim e^{-L} \quad \text{— Perimeter Law}$$

These correlated fluctuations give rise to perimeter law behavior, but they are suppressed in the nonabelian case because there is a term in B of order A^2:

$$B\sim A^2\sim L^{-1} \text{ and } S\sim L.$$

2-dim Yang-Mills theory, as opposed to QED, has a mass gap and area law confinement because magnetic fluctuations are uncorrelated at large distances. The difference is $(2\pi/g)N$ units of flux is equivalent to no flux in YM theory, so A_μ can be zero outside a fluctuation region. But in QED, a large number of magnetic vortices already present prevents the accumulation of more. (R.P. Feynman Nucl. Phys B188, 479(1981))

D. Confinement in 3+1 Dimensions

There are clearly features of our discussion of confinement in 2+1 dimensions which do not readily generalize to the 3+1-dimensional case:

- In 2+1 dimensions, a string is associated with spontaneous breaking of a global Z_N symmetry — a topologically stable string which can confine quarks. But in 3+1 dimensions, the topological structure associated with spontaneously broken Z_N symmetry is a wall, which would confine strings.

- In 2+1 dimensions, we were able to define local operators which created vortices at fixed time. The analogous construction in 3+1 dimensions will yield operators which create closed loops of magnetic flux (at fixed time). Since these closed loops are not topologically stable, there will be no topological global Z_N symmetry in the 3+1-dimensional case.

Nevertheless, topological considerations similar to those applied in 2+1 dimensions can be used in the discussion of confinement in 3+1 dimensions, and will improve our understanding of the stability of flux tubes. To make the 3+1-dimensional generalization more natural, we will first reformulate slightly our discussion of the 2+1-dimensional case.

Reformulation of Confinement in 2+1 dimensions

To express our analysis of confinement in 2+1 dimensions into a language more readily generalized to 3+1 dimensions, we turn attention away from the local operator $\Phi(\vec{x})$ in favor of the bilocal object

$$B(\vec{x},\vec{y}) = \Phi^{+}(\vec{x})\,\Phi(\vec{y})$$

B serves as well as Φ as an order parameter for the realization of the gauge symmetry. In a Higgs phase (stable vortex), we expect

$$\langle 0|B(\vec{x},\vec{y})|0\rangle \xrightarrow[|\vec{x}-\vec{y}|\to\infty]{} \exp\left[-M|x-y|\right] \qquad (\text{Higgs})$$

where M: vortex mass

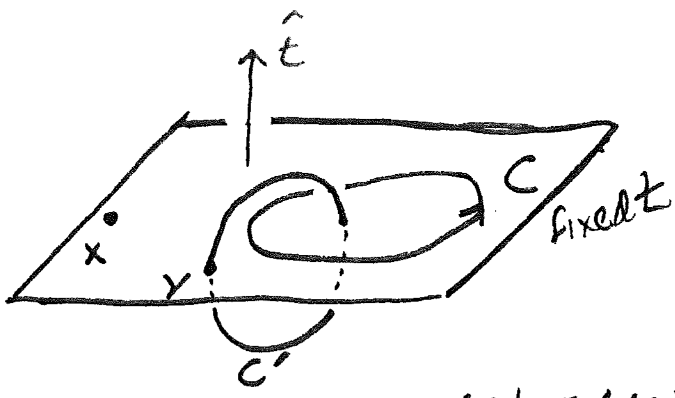
In a confinement phase (vortex condensate), we expect

$$\langle 0|B(\vec{x},\vec{y})|0\rangle \xrightarrow[|\vec{x}-\vec{y}|\to\infty]{} |\langle\Phi\rangle|^2 = \text{constant} \neq 0 \quad (\text{Confinement})$$

It follows from our earlier discussion that the B's commute at fixed time and noncoincident points, when acting on gauge-invariant states. Furthermore, if C is a curve in a plane of fixed time, we have

$$A(C)\,B(\vec{x},\vec{y}) = B(\vec{x},\vec{y})\,A(C)\,e^{2\pi i n/N}$$

where n is the number of times C wraps around y (counterclockwise) minus the number of times it wraps around x — i.e., the number of times C passes between x and y



But now, if we allow x and y to wander out of a plane of fixed time, we see that

$$\langle 0| T A(C)\, B(x,y) |0\rangle = f(\theta)$$

must be multivalued, because the number of times C passes between x and y is no longer unambiguously defined. E.g. suppose we keep x and C fixed, but allow y to traverse a path C' in three-dimensional space which winds once around C, as shown

Because of the time ordering, the value of $f(\theta)$ (θ labels the path C') jumps when C' crosses the plane containing C; we have

$$f(\theta + 2\pi) = f(\theta)\, e^{2\pi i/N},$$

as a consequence of the equal-time commutation relation. The same conclusion applies if we work in Euclidean space and drop the time ordering.

We may choose C to be very large, so that C' and C are always very far apart; then, our naive expectation would be that clustering holds:

$$f(\theta) \sim \langle 0| B(x,y) |0\rangle \langle 0| A(C) |0\rangle$$

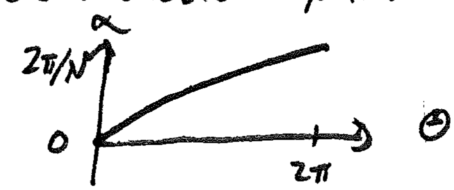
But this cluster property is inconsistent with the multivaluedness of $f(\theta)$, for $\langle 0| B |0\rangle$ is clearly single-valued!

Therefore, the cluster property must be modified:

$$f(\theta) \sim \langle 0|B(x,y)|0\rangle \langle 0|A(C)|0\rangle\, e^{i\alpha(\theta)}$$

where $e^{i\alpha(\theta)}$ has the required multivaluedness.
What can cause clustering to fail?

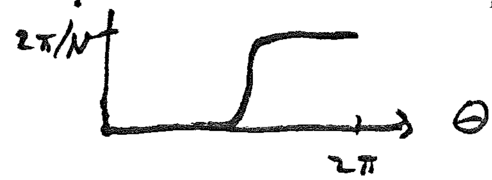
- There may be massless particles; i.e., long-range correlations.



In this case, $\alpha(\theta)$ can change smoothly as θ varies from 0 to 2π

(the discontinuous behavior we noted before corresponding to jumping from one branch of $\alpha(\theta)$ to another; but on each branch $\alpha(\theta)$ might be smooth.)

- If there are no massless particles, then $\alpha(\theta)$ can only jump suddenly (with a width of the order of a typical compton wavelength divided by the radius of C').



Either:

1) Jump occurs when y crosses surface bounded by C. There is a physically observable sheet ending on C – a cross section of this sheet is an electric flux tube. This observable sheet has a finite action density, leading to

$$\langle 0|A(C)|0\rangle \sim \exp[-K(\text{Area})]$$

– confinement phase.

2) Jump occurs when a string running from x to y crosses C. Physical information resides only on the curve C, and not the sheet bounded by C, leading to $\langle 0|A(C)|0\rangle \sim \exp[-\mu(\text{perimeter})]$

– Higgs phase (Note: non-Higgs $\Rightarrow$ Confinement)

Generalization to 3+1 dimensions

The above analysis can now be carried out for the 3+1 dimensional case. At a fixed time we define operators $B(C)$ where C is a closed curve in a (hyper) plane of fixed time:

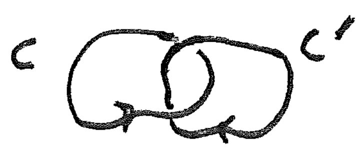
$$\underset{\sim}{B}(C)\ |A_i(\bar{x})\rangle = |A_i^{\Omega(C)}(\bar{x})\rangle$$

i.e. $\underset{\sim}{B}$ acting on the basis $|A_i\rangle$ is a gauge transformation with a singularity along C, and a "twist" around C. $\underset{\sim}{B}(C)$ produces a closed magnetic vortex line. As before, the action of $\underset{\sim}{B}(C)$ on gauge-invariant states is determined by only the singularity of $\Omega(C)$ – other properties of $\Omega(C)$ are superfluous

Furthermore, it is clear that

$$[B(C), B(C')] = 0,$$

acting on gauge-invariant states, if C, C' are curves at the same fixed time with no points in common.



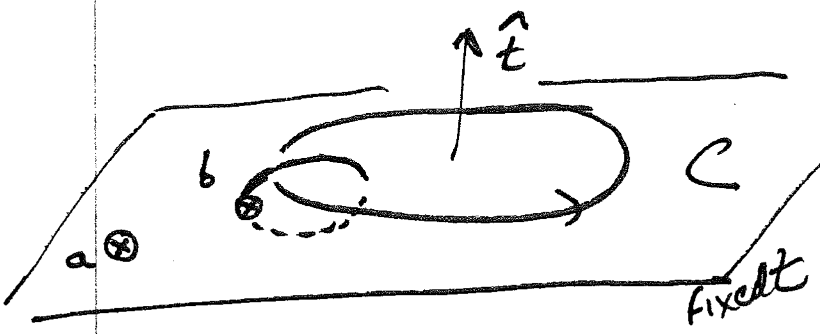
Also, by the same reasoning we used before (p 2.23) we have the important equal-time commutation relation

$$A(C)\,B(C') = B(C')\,A(C)\, e^{2\pi i n/N}$$

where n is the no. of times C' winds around C (in some appropriately prescribed sense). The singular "gauge transformation" about B changes $A(C)$ by some element of Z_N.

This commutation relation is crucial, because it leads in 3+1 dimensions as in 2+1 dimensions, to the conclusion that the cluster property must fail for large interwoven loops.

To reach this conclusion, we consider $\langle 0 | T\, A(C)\, B(C') | 0 \rangle$ for contours which are not necessarily in a plane of fixed time. In four-dimensional (Euclidean) space, the no of times one closed curve winds around another is not well-defined — As a result this Green's function must be multivalued



The drawing indicates a hyperplane which contains the curve C but contains only the two points a and b of the curve C'

Suppose we distort C' so that the point b winds around C as shown. The point b crosses the hyperplane of fixed t containing C twice, corresponding to values 0 and 1 for the integer n, the number of times C' winds through C

Because of the Time-ordering, we conclude as before that the Green's function is multivalued, it has values differing by the factor $e^{2\pi i/N}$ as b approaches the plane of C from above and below. Therefore, clustering fails, even if we choose both C and C' to be large, and always distant from each other

$$\langle 0 | A(C) B(C') | 0 \rangle_{\text{Euclidean}} \sim \langle A(C) \rangle \langle B(C') \rangle e^{i\alpha(\theta)}$$

If there are no massless particles, the jump by $2\pi/N$ of $\alpha(\theta)$ must be sudden.

There must be a physically observable sheet linking either C, or C', or both. We have either electric confinement

$$\langle 0|A(C)|0\rangle \sim \exp[-K(\text{Area})]$$

or magnetic confinement

$$\langle 0|B(C)|0\rangle \sim \exp[-K(\text{Area})]$$

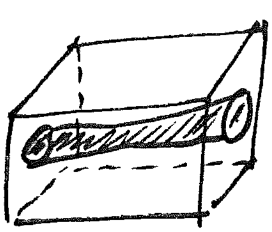
or both. This conclusion is consistent with our heuristic discussion back on p. (2.18), although we have not (yet) excluded the possibility that both magnetic and electric confinement apply

E. Electric and Magnetic Flux in a Box

(G. 't Hooft Nucl. Phys. B153, 141 (1979).)

A striking feature of the above analysis is the symmetrical treatment of $A(C)$ and $B(C)$, of electric flux and magnetic flux. This "electric-magnetic duality" has not yet been fully exploited.

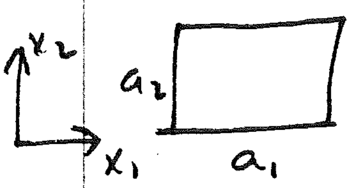
To study further the energetics of magnetic and electric flux tubes, we will consider gauge fields defined in a finite box.



By specifying boundary conditions satisfied by the gauge field, we can require that the box contains magnetic or electric flux. By calculating the ground state energy for specified boundary conditions, we can determine whether the flux is confined to tubes.

Twisted Gauge Fields

We will impose periodic boundary conditions on the gauge fields, but the only requirement that makes physical sense is periodicity up to a gauge transformation. E.g., consider a plane with x_3, x_4 fixed



we have

$$A_\mu(a_1, x_2) = \Omega_1(x_2)\, A_\mu(0, x_2)$$
$$A_\mu(x_1, a_2) = \Omega_2(x_1)\, A_\mu(x_1, 0)$$

Here ΩA_μ is shorthand for $\Omega A_\mu \Omega^{-1} + \frac{1}{ig}(\partial_\mu \Omega)\Omega^{-1}$
Can $\Omega_{1,2}$ simply be gauge transformed away? (by a non-periodic gauge trans)
Not necessarily.

Note the consistency condition:

$$A_\mu(a_1, a_2) = \Omega_1(a_2)\, A_\mu(0, a_2) = \Omega_1(a_2)\, \Omega_2(0)\, A_\mu(0,0)$$
$$= \Omega_2(a_1)\, A_\mu(a_1, 0) = \Omega_2(a_1)\, \Omega_1(0)\, A_\mu(0,0)$$

Thus $\Omega_1(a_2)\Omega_2(0)$ and $\Omega_2(a_1)\Omega_1(0)$ must be equivalent $SU(N)/Z_N$ gauge transformations, or

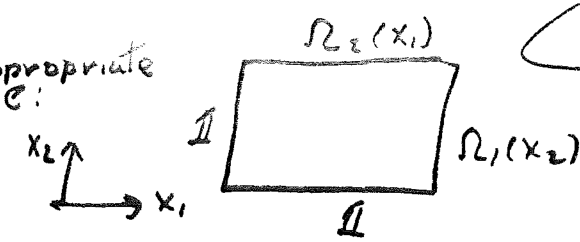
$$\Omega_1(a_2)\,\Omega_2(0)\,\Omega_1(0)^{-1}\,\Omega_2(a_1)^{-1} = \exp(2\pi i n_{12}/N)$$

— an element of the center

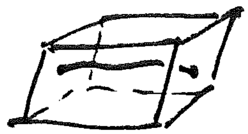
A gauge transformation cannot change n_{12}. Nor can any continuous deformation of the gauge field. It is a topological invariant. If n_{12} is nontrivial, we cannot gauge transform away both Ω_1 and Ω_2; we say that the gauge field is "twisted"

There are altogether $\binom{4}{2} = 6$ integers $n_{\mu\nu}$ which label a gauge field configuration in a 4 dimensional box.

In appropriate gauge:



Magnetic Flux



The integers n_{12}, n_{31}, n_{23} are related to the magnetic flux in the box. E.g. consider the magnetic vortex creation operator $B(C)$ for a contour which wraps once around the hypertorus (periodically identified box) in the 3-direction $B(C)$ generates a twisted gauge transformation in the 1-2 plane. Hence it is sensible to adopt the integers

$$m_i = \tfrac{1}{2} \epsilon_{ijk} n_{jk},$$

defined mod N, as a definition of the magnetic flux in the box.

Electric Flux

The electric flux of a gauge field configuration in the box can be defined in terms of the response of the configuration to a time-indendent gauge transformation.

In the $A_0=0$ gauge, all physical states are required to be invariant under infinitesimal gauge transformations. But there are also pseudo-periodic gauge transformations which cannot be built out of infinitesimals:

$$\Omega(a_1, x_2, x_3) = \Omega(0, x_2, x_3)\, \exp(2\pi i k_1/N)$$
$$\Omega(x_1, a_2, x_3) = \Omega(x_1, 0, x_3)\, \exp(2\pi i k_2/N)$$
$$\Omega(x_1, x_2, a_3) = \Omega(x_1, x_2, 0)\, \exp(2\pi i k_3/N)$$

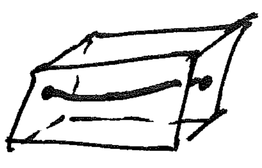
Two such gauge transformations labeled by the same integers $k_{1,2,3}$ have the same action on physical states, for they differ by a "small" gauge transformation which leaves such states inv.

These gauge transformations leave the integers m_i unchanged and commute with the Hamiltonian. We may choose the physical states to be eigenstates under this $(Z_N)^3$ group. The representations are obviously

$$\Omega(\bar{k})\,|\Psi\rangle = \exp\left[\frac{2\pi i}{N}\,\bar{e}\cdot\bar{k}\right]|\Psi\rangle$$

-the integers e_i, defined modulo N, are taken as the definition of the electric flux of the state $|\Psi\rangle$

To see that this definition is sensible, note the effect of $\Omega(\bar{k})$ on the Wilson loop operator $A(C)$, for a curve C which winds around the hypertorus in the 3 direction.



Clearly $A(C)\Omega(\bar{k}) = \Omega(\bar{k})A(C)e^{-2\pi i k_3/N}$
Therefore, $A(C)$ acting on $|\Psi\rangle$ increases e_3 by one unit, which is consistent with our earlier observation that $A(C)$ creates a line of electric flux.

To project out states with a definite electric flux $\bar{e}$ we may define a projection operator

$$P(\bar{e}) = \frac{1}{N^3}\sum_{\bar{k}} e^{-2\pi i\,\bar{k}\cdot\bar{e}/N}\,\Omega(\bar{k})$$

since
$$\Omega(\bar{k}')P(\bar{e}) = \frac{1}{N^3}\sum_{\bar{k}} e^{-2\pi i\,\bar{k}\cdot\bar{e}/N}\,\Omega(\bar{k}+\bar{k}')$$

$$= e^{2\pi i\,\bar{k}'\cdot\bar{e}/N}P(\bar{e})$$

and
$$P(\bar{e})\,P(\bar{e}') = \delta_{\bar{e},\bar{e}'}\,P(\bar{e})$$