Free Energy of a Flux Configuration

Our box is finite in the (Euclidean) time direction, as well as in spatial directions. This means we are studying thermodynamics of a finite box

In $A_0=0$ gauge, the free energy is defined by

$$Z = e^{-\beta F} = \sum_{\alpha} e^{-\beta E_\alpha}$$

where we sum only over states obeying the Gauss' Law constraint – i.e. gauge-invariant states. The matrix element between states $|A_1(\vec{x})\rangle$ and $|A_2(\vec{x})\rangle$ can be represented by a functional integral

$$\langle A_2 | e^{-\beta H} | A_1 \rangle = \int (dA)\, e^{-S_E[A]}$$

where S_E is the Euclidean action, and A obeys the boundary conditions

$$A(t=0) = A_1$$
$$A(t=\beta) = A_2$$

Since we wish to sum over only gauge-invariant states, we introduce the normalized group integration

$\int d\Omega(\vec{x})$ such that

$$\int d\Omega(\vec{x})\, \langle A_2 | U(\Omega) = \int d\Omega\, \langle A_2^{\Omega} |$$

projects out gauge invariant states

Thus we may write

$$Z = \mathrm{tr}\, e^{-\beta H} = \int (d\Omega)(dA(\vec{x})) \langle A^{\Omega} | e^{-\beta H} | A \rangle$$

where $\Omega(\vec{x})$ is a periodic function of spatial $\vec{x}$, and $A(\vec{x})$ is summed over all configurations which are spatially periodic up to gauge transformations.

Thus $Z = \int (dA_i) e^{-S_E[A]}$
periodic up to g.t.

Now, we may consider the contribution to the partition function coming from configurations with definite magnetic and electric flux. We specify the magnetic flux $\vec{m}$ by restricting to gauge field configurations with

$$m_1 = n_{23}, \quad m_2 = n_{31}, \quad m_3 = n_{12}.$$

To specify electric flux we insert the projection operator
$$P(\vec{e}) = \frac{1}{N^3} \sum_{\vec{k}} e^{-2\pi i \vec{k}\cdot\vec{e}/N} \Omega(\vec{k})$$

$\Omega(\vec{k})$ is not a gauge transformation here; it acts on e.g. the configuration at $t=0$, but not at $t=\beta$. It thus introduces a twist in the $4j$ plane with

$$k_j = n_{4j}$$

Thus, the contribution to the partition function due to configurations with electric flux $\vec{e}$ and magnetic flux $\vec{m}$ may be written

$$\boxed{e^{-\beta F(\vec{e}, \vec{m}, \vec{a}, \beta)} = \frac{1}{N^3} \sum_{\vec{k}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}} \int (dA)_{\vec{k},\vec{m}} e^{-S_E[A]}}$$

where $\int(dA)_{\vec{k},\vec{m}}$ is restricted to the topological class $n_{4j} = k_j$, $n_{ij} = \epsilon_{ijk} m_k$, and includes $\int d\Omega(\vec{x})$ — the integral over the gauge transformation relating $A(t=\beta)$ and $A(t=0)$.

We may rewrite this as
$$e^{-\beta F(\vec{e}, \vec{m}, \vec{a}, \beta)} = \frac{1}{N^3} \sum_{\vec{k}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}} W(\vec{k}, \vec{m}, \vec{a}, \beta)$$

where
$$W(\vec{k}, \vec{m}, \vec{a}, \beta) = \int (dA)_{\vec{k},\vec{m}} e^{-S_E[A]}$$

The important feature of W is that, although we have described how to calculate it in a particular gauge ($A_0 = 0$ gauge), in which the (Euclidean) time direction is given a special status, it is really Euclidean invariant. (See page 2.47)

Exercise 2.3

Show that the integers $n_{\mu\nu}$ defined on p 2.33 are unchanged by a (not necessarily periodic) gauge transformation

$$A_\mu(x) \rightarrow \Omega(x) A_\mu(x)$$

in the four-dimensional box.

Duality

$\vec{E}$ and $\vec{B}$ are interchanged by a Lorentz boost. The analog of a boost, which is a symmetry of a Euclidean box, is a 90° O(4) rotation. Consider

$$\left[\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right],$$

which interchanges
$x_1 \leftrightarrow x_2$
$x_3 \leftrightarrow x_4$

By Euclidean invariance of TJ, we have ...

$$\mathcal{W}(k_1, k_2, k_3; m_1, m_2, m_3; a_1, a_2, a_3, \beta)$$
$$= \mathcal{W}(m_1, m_2, k_3; k_1, k_2, m_3; a_2, a_1, \beta, a_3)$$

This implies a duality relation relating electric and magnetic flux:

$$\exp -\beta F(\vec{e}, \vec{m}, \vec{a}, \beta) = \frac{1}{N^3} \sum_{\vec{k}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}}\, \mathcal{W}(\vec{k}, \vec{m}, \vec{a}, \beta)$$
$$= \frac{1}{N^3} \sum_{\vec{k}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}}\, \mathcal{W}(m_1 m_2 k_3; k_1 k_2 m_3; a_2 a_1 \beta a_3)$$

Now use the inverse transform:

$$\mathcal{W}(\vec{k}, \vec{m}, \vec{a}, \beta) = \sum_{\vec{e}} e^{(2\pi i/N)\vec{k}\cdot\vec{e}} \exp -\beta F(\vec{e}, \vec{m}, \vec{a}, \beta)$$

So

$$\exp -\beta F(\vec{e}, \vec{m}, \vec{a}, \beta) = \frac{1}{N^3}\left(\sum_{\vec{k}} e^{-(2\pi i/N)\vec{k}\cdot\vec{e}}\right)$$
$$\left(\sum_{\vec{\ell}} e^{(2\pi i/N)(m_1\ell_1 + m_2\ell_2 + k_3\ell_3)}\right) \exp -a_3 F(\vec{\ell}; k_1 k_2 m_3; a_2 a_1 \beta a_3)$$

The sum over k_3 gives $N\delta_{\ell_3, e_3}$

Thus

$$\boxed{\begin{aligned}&\exp -\beta F(\vec{e}, \vec{m}, \vec{a}, \beta)\\ &= \frac{1}{N^2} \sum_{k_{1,2}} e^{-(2\pi i/N)(k_1 e_1 + k_2 e_2)} \sum_{\ell_{1,2}} e^{(2\pi i/N)(\ell_1 m_1 + \ell_2 m_2)}\\ &\qquad \exp -a_3 F(\ell_1, \ell_2, e_3; k_1, k_2, m_3, a_2, a_1, \beta, a_3)\end{aligned}}$$

This is the duality relation associated with the above rotation

Flux Tubes

Suppose there are no massless particles. Then, as the box becomes large ($a_i, \beta \to \infty$), $e^{-\beta F}$ is dominated by the state of lowest energy with specified boundary conditions. (Corrections become exponentially small) We renormalize the vacuum energy so that

$$F(\vec{0}, \vec{0}) \longrightarrow 0$$

Then we expect

$$F(\vec{e}, \vec{m}, \vec{a}, \beta) \xrightarrow[\vec{a}, \beta \to \infty]{} 0 \quad (e^{-\beta F} \to 1)$$ if no flux tubes form. "Light flux"

$$F(\vec{e}, \vec{m}, \vec{a}, \beta) \xrightarrow[\vec{a}, \beta \to \infty]{} \infty \quad (e^{-\beta F} \to 0)$$ if there are dynamically stable flux tubes "Heavy Flux"

For simplicity, consider $e_3, m_3 = 0$, since these fluxes appear trivially in the duality relation. The relation then becomes

$$\boxed{e^{-\beta F(\tilde{e}, \tilde{m}, \vec{a}, \beta)} = \frac{1}{N^2} \sum_{\tilde{k}\tilde{e}} e^{(2\pi i/N)(\tilde{e}\cdot\tilde{m} - \tilde{k}\cdot\tilde{e})} e^{-a_3 F(\tilde{e}, \tilde{k}, \cdots}}$$

(We can choose $a_{1,2,3} = \beta$ in what follows). Here $\tilde{e} = (e_1, e_2)$, etc.

We know that

$$1 = e^{-\beta F(\tilde{0}, \tilde{0})} = \frac{1}{N^2} \sum_{\tilde{k}\tilde{e}} e^{-\beta F(\tilde{e}, \tilde{k})}$$

Therefore, if each $e^{-\beta F}$ is either 1 or 0 ("light flux" or heavy flux) there are N^2 light fluxes. Furthermore, if $(\tilde{e}, \tilde{m})$ is a light flux, then

$$1 = e^{-\beta F(\tilde{e}, \tilde{m})} = \frac{1}{N^2} \sum_{\tilde{k}, \tilde{e}} e^{(2\pi i/N)(\tilde{e}\cdot\tilde{m} - \tilde{k}\cdot\tilde{e})} e^{-\beta F(\tilde{e}, \tilde{k})}$$

In the sum, there are N^2 terms, each ≤ 1. Therefore each must be one, and we have

If $(\tilde{e}, \hat{m})$ and $(\tilde{e}', \hat{m}')$ are light fluxes, then
$$\tilde{e}\cdot\hat{m}' - \hat{m}\cdot\tilde{e}' \equiv 0 \pmod N$$

We know more. Since
$$e^{-\beta F(\bar{e},\bar{m})} = \frac{1}{N^3}\sum e^{-(2\pi i/N)\bar{k}\cdot\bar{e}}\, \bar{W}(\bar{k},\bar{m})$$

and each $\bar{W}(\bar{k},\bar{m})$ is nonnegative (it is the integral of a nonnegative functional measure) we have
$$e^{-\beta F(\bar{e},\bar{m})} \le e^{-\beta F(0,\bar{m})}$$

and therefore,

If $(\bar{e},\bar{m})$ is light, then $(0,\bar{m})$ is light

(In fact, if $(\bar{e},\bar{m})$ is light, then $e^{-\beta F(\bar{e},\bar{m})} = e^{-\beta(0,\bar{m})}$, and $\bar{W}(\bar{k},\bar{m})$ must vanish for $\bar{k}\cdot\bar{e} \not\equiv 0 \pmod N$.)

In addition to the duality relations, we can use invariance of the box under 90° spatial rotations to derive,

If $(\bar{e},\bar{m})$ is light (heavy), then $(R\bar{e}, R\bar{m})$ is light (heavy), where R denotes a rotation.

Two solutions to the duality relation which also satisfy rotational invariance and are consistent with the positivity of $\bar{W}$ are

i) $(0,\bar{m})$ is light
$(\bar{e},\bar{m})$ is heavy for $\bar{e}\neq 0$
i.e., electric confinement without magnetic confinement

ii) $(\bar{e},0)$ is light
$(\bar{e},\bar{m})$ is heavy for $\bar{m}\neq 0$
i.e., magnetic confinement without electric confinement

The duality relation cannot distinguish between these possibilities, because it treats magnetic and electric flux symmetrically. But, remarkably, it does tell us that these are the only possibilities (for, e.g. $N=3$).

Exercise 2.4

ⓐ Show that, for $N=3$, the only solutions to the duality relation with $e^{-BF} = 0$ or 1 which satisfy rotational invariance and are consistent with positivity of $\tilde{w}$ are (i) and (ii) above.

ⓑ Show that, for $N=4$, there is a solution satisfying these conditions for which only fluxes with $e_i, m_i \equiv 0 \pmod 2$ are light. In what type of Higgs theory might this solution be realized?

ⓒ Find an exotic solution to the duality relation for $N=6$. How might it be realized in a Higgs theory?

Why do these arguments fail to distinguish between electric and magnetic confinement? All of our topological arguments would apply to a gauge theory with Higgs fields in representations of $SU(N)/Z_N$ (e.g., the adjoint representation). We know that, depending on parameters (in particular on the sign of masses squared of the Higgs fields) this theory can be in a Higgs phase. So we cannot expect the general arguments to exclude magnetic confinement

$$\xleftarrow{\text{Higgs}} \,|\, \xrightarrow{\text{confinement}} \mu^2$$

Massless Phase

Our previous arguments fail if there are massless particles, for we cannot claim that $e^{-\beta F} \to e^{-\beta E_{\text{ground}}}$ exponentially as $\beta, \bar{a} \to 0$. So these arguments do not exclude the possibility of a phase containing massless particles. This, again, should not be a surprise. It is possible for $SU(N)/Z_N$ Higgs fields to drive the symmetry breakdown

$$SU(N) \to U(1)^m \quad ; \quad m \le N-1$$

We expect this $U(1)^m$ (Georgi - Glashow) phase to have massless "photons."

Could there conceivably be a massless gluon phase in which $SU(N)$ is not spontaneously broken, and there are physical massless gluons? Our arguments have not excluded this, but it is suspected that it cannot exist, because of the severe infrared singularities of a massless gluon theory

Confinement of Quarks and Monopoles

Grand Unified Theories predict that there should exist magnetic monopoles which carry color magnetic flux.

We have seen that, in a phase without massless particles, either color electric flux or color magnetic flux (but not both) is confined to flux tubes.

Apparently, there are no massless hadrons. Therefore, if, as appears to be the case, quarks are exactly confined, color monopoles are not confined; there is no long range color-magnetic interaction

But if, as is suggested by the reported observation of fractional electric charge on matter, quarks are not exactly confined, then color magnetic monopoles are confined.

Either way, the observation of fractional electric charge is inconsistent with the reported observation of a monopole with magnetic charge $m = 1/2e$ (which would be required to carry color magnetic flux)

More specifically:

- Dirac condition $gh + g_c h_c = \frac{n}{2}$

If Quarks exist and $h = \frac{1}{2e}$, monopole must have color magnetic charge.

- Free Quarks $\Rightarrow$ Monopole confinement

Monopole "baryons" have

$$h = \frac{3n}{2e}, \text{ as required}$$

magnetic baryon

Addendum to 2E:

Solutions to the Duality Relations

In a Euclidean box with sides $a_0 = a_1 = a_2 = a_3 = \beta$, the duality relation takes the form:

$$e^{-\beta F(\vec{e},\vec{m})} = \frac{1}{N^2}\sum_{\tilde{k}} e^{-(2\pi i/N)\tilde{k}\cdot\hat{e}} \sum_{\hat{e}} e^{(2\pi i/N)\hat{e}\cdot\tilde{m}} e^{-\beta F(\tilde{e},e_3;\tilde{k},m_3)}$$

where $\hat{e} = (e_1, e_2)$, etc.

Assume there is a mass gap, so that

$$e^{-\beta F(\vec{e},\vec{m})} = \begin{cases} 1, & \text{light flux} \\ 0, & \text{heavy flux} \end{cases}.$$

We also know:

- $(\vec{0},\vec{0})$ is light.
- If $(\vec{e},\vec{m})$ is light, then $(0,\vec{m})$ is light. (positivity of $\widetilde{W}$)
- If $(\vec{e},\vec{m})$ is light, then $(R\vec{e}, R\vec{m})$ is light. (rotational invariance)

Furthermore, from duality relation, we see that

$$e^{-\beta F(\vec{e},\vec{m})} \le e^{-\beta F(0,0,e_3;0,0,m_3)}$$

therefore,

- If $(\vec{e},\vec{m})$ is light, then $(0,0,e_3;0,0,m_3)$ is light.

and...

- there are N^2 light fluxes with given e_3, m_3 (if $(\vec{e},\vec{m})$ is light). And if $(\hat{e}' e_3, \tilde{m}' m_3)$ is also light, then $\boxed{\tilde{e}\cdot\tilde{m}' - \tilde{m}\cdot\tilde{e}' \equiv 0 \pmod N}$

Therefore, by rotational invariance, if $(\vec{e}, \vec{m})$ is light, then

$(e_i, 0, 0\,;\, m_i, 0, 0)$ and $(\vec{0}\,;\, m_i, 0, 0)$ are also light, $(i = 1, 2, 3)$

and we conclude that

$$\boxed{m_i e_j \equiv 0\,, \quad i,j = 1,2,3}$$

And if both $(\vec{e}, \vec{m})$ and $(\vec{e}'\, \vec{m}')$ are light we have

$$\boxed{\begin{matrix} e_i' m_j \equiv 0 \pmod{N} & \\ e_i m_j' \equiv 0 \pmod{N} & \end{matrix} \quad i,j = 1,2,3}$$

We conclude that, if N is prime, the only solutions to duality relation are

(i) $(0, \vec{m})$ is light (electric confinement)

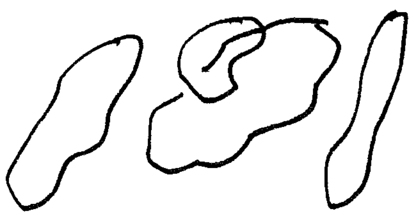
(ii) $(\vec{e}, 0)$ is light (magnetic confinement)

But if N is not prime, there may be other solutions.

F. Deconfinement at Finite Temperature

(see Gross, Pisarski, Yaffe, RMP 53, 43 (1981).)

It is interesting to consider $e^{-\beta F}$ for $a_{1,2,3} \to \infty$ with β fixed, i.e., to study the thermodynamics of the pure gauge theory.



In particular, we expect that, at sufficiently large temperature, large loops of electric flux will appear as thermal fluctuations, even if electric flux is confined to flux tubes at zero temperature. If a "condensate of strings" forms, confinement may be lost. Introducing color sources may not have much effect on the free energy, since adding another flux tube is not very costly.

We might expect a condensate of flux tubes to form because the entropy of a string can overcome its energy density.

As a rough model of a fluctuating string, consider contours on a lattice with lattice spacing a (which we should presumably choose to be of order the width of the physical flux tube) the number of string configurations of length L is of order

$$(2d)^{L/a}$$

- since each link of the lattice has $2d$ directions in which it can turn, in d dimensions

The contribution to the partition function from these string states is

$$Z \sim \sum_L (2d)^{L/a} e^{-\kappa L/T}$$

where κ is the string tension. This blows up (strings condense) if ---

$$KL/T < L/a \ln 2d \text{ or } \boxed{T > \frac{aK}{\ln 2d}}$$

Of course, in an actual gauge theory, we expect $a \sim \Lambda^{-1}$, $K \sim \Lambda^2$.

When we go to high temperatures, the free energy becomes dominated by essentially static (time-independent) gauge field configurations. Recall that we may write (p 2.36)

$$e^{-\beta F} = \int (d\Omega(\vec{x})) \int (dA_i(\vec{x},t)) \exp\left[-\int_0^\beta dt \int d^3x \frac{1}{2g^2}\left(\dot{A}_i^{a\,2} + B_i^{a\,2}\right)\right]$$

$$A_i(\vec{x},\beta) = A_i^{\Omega}(\vec{x},0)$$

(g has been absorbed into gaugefield A_i^a)

If we rescale the time $\tau = \beta t$

$$e^{-\beta F} = \int (d\Omega)(dA_i) \exp -\frac{1}{2g^2 T} \int_0^1 d\tau \int d^3x \left(T^2 \dot{A}_i^{a\,2} + B_i^{a\,2}\right)$$

$$(T = \beta^{-1})$$

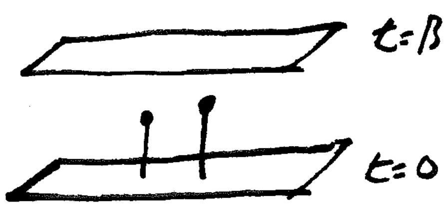
We see that for $T \to \infty$, configurations which are nearly time independent dominate. The integral $d\Omega$ becomes peaked for Ω near the center of the group. Expectation values approach those of a 3 dimensional Euclidean gauge theory with coupling $g^2 T$

Presumably, then, spacelike Wilson loops exhibit area law behavior (for we expect that 2+1 dimensional Yang-Mills theory is confining) But it is not clear that we should interpret this as a signal of confinement. Flux tubes may be locally stable, but at finite temperature, this might not imply confinement of color sources. What criterion should we use for confinement at finite temperature?

To state a criterion, it is convenient to use a different formulation of a gauge theory at finite temperature. The partition function is given by

$$Z = e^{-\beta F} = \int (dA_\mu)_{\substack{\text{periodic} \\ \text{in } \tau \text{ with} \\ \text{period } \beta}} \exp\Big[-\tfrac{1}{4g^2}\int_0^\beta d\tau \int d^3x\, F^a_{\mu\nu}F^{\mu\nu a} \;\; \begin{matrix}+\text{ gauge fixing} \\ +\text{ ghost}\end{matrix}\Big]$$

This differs from our earlier expression by a factor of the volume of the local gauge group, which must be removed by the usual gauge-fixing procedure, but otherwise it is the same. To see this, we try to impose the $A_0 = 0$ gauge condition



We use the procedure described on p (2.10). We associate with each point $(\vec{x}, t)$ a path in the $\hat{t}$ direction beginning at $(\vec{x}, 0)$

$$\text{Define} \quad U(\vec{x}, t) = P\exp i\int_{(\vec{x},0)}^{(\vec{x},t)} A_0\, dt$$

then choose the gauge transformation to be

$$\Omega(\vec{x}, t) = U^{-1}(\vec{x}, t)$$

This transforms A_0 to 0 for all t up to β. But at $t=\beta$ U will typically not be equal to $\mathbb{1}$; it is therefore discontinuous at $t=\beta$ $(t=0)$

By transforming to $A_0 = 0$ gauge, we have destroyed the periodicity in the t direction of the gauge field configuration. Now the remaining integral $\int dA_i(\vec{x}, t)$ is performed over configurations which are periodic up to a gauge transformation:

$$A_i(t=\beta) = A_i^{\Omega(\beta)}(t=0)$$

where $\Omega(\beta, \vec{x})$ must be summed over. That's what we had before. (In effect, we've pushed the region where $A_0 \neq 0$ to an infinitesimally thin constant t slice.)

Now, in this periodic formulation it is easy to state a criterion for confinement. The static potential for colored sources is evidently given by

$$\langle \mathrm{tr}\, U(\vec{x},\beta)\, \mathrm{tr}\, U^{\dagger}(\vec{x}',\beta)\rangle = \exp -\beta V(|\vec{x}-\vec{x}'|,\beta)$$

It is even simpler to consider

$$\langle \mathrm{tr}\, U(\vec{x},\beta)\rangle = e^{-\beta F_{\text{quark}}}$$

which is gauge-invariant and determines the free energy of a single static quark. If quarks are confined, we expect

$\langle \mathrm{tr}\, U(\vec{x},\beta)\rangle = 0$ in the infinite volume limit

The finite-temperature field theory has a global Z_N symmetry — there are gauge transformations which have a twist in the t direction and preserve the periodicity of the gauge field (i.e. are periodic $SU(N)/Z_N$ transformations) Under the Z_N transformation labeled by integer K

$$U(\vec{x},\beta) \longrightarrow e^{2\pi i K/N}\, U(\vec{x},\beta)$$

Thus $\langle \mathrm{tr}\, U\rangle$ is an order parameter for spontaneous breaking of the Z_N symmetry, as well as deconfinement. ($\Rightarrow$ order of transition ~ Potts model?)

We have already argued that, for T large, the ~~functional~~ integral is dominated by configurations with $\Omega(\vec{x}) = U(\vec{x},\beta)$ close to the center of $SU(N)$. This indicates that $\langle \mathrm{tr}\, U\rangle$ is nonzero for T large, and the Z_N symmetry is spontaneously broken. Deconfinement occurs at large temperature. (But note: loss of electric confinement does not imply magnetic confinement)

Order of the Phase Transition

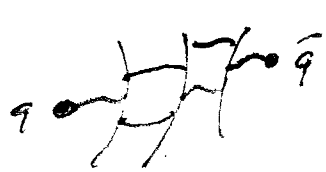
universality suggests that the critical behavior of a gauge theory at the deconfinement transition should be the same as that of a spin system with the same dimensionality and global symmetry — i.e. a Z_N system in 3 dimensions. Thus we expect

$N=2 \Rightarrow$ 2nd order transition (as in 3-d Ising model)
$N=3 \Rightarrow$ 1st order transition (No known fixed point of the renorm. grp. for a Z_3 spin system)

There is a nice physical picture of the difference between $N=2$ and $N=3$. (A. Patel, Caltech Preprint)
For $N=3$

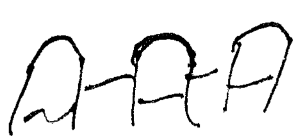
The flux tubes are *directed*, and 3 can meet at a vertex

Thermal fluctuations can produce a percolating network, which fills all space; this can happen at a lower temperature than that at which closed loops of flux start to condense.



Percolation $\Rightarrow$ Deconfinement,
because a quark or antiquark introduced can "hook on" to the existing network, with a finite cost in free energy.

Moreover, the deconfining transition is first order in this picture. To see this, we verify that phases can coexist, separated by a boundary with surface tension. This is true — to make a boundary we must tie up all loose ends of the network, at a cost in free energy proportional to surface area.



For $N \geq 4$, a qualitatively similar picture applies, suggesting transition is still first order (even though, for $N \geq 4$, there *is* a fixed point).

G. Dynamical Quarks

We found it very convenient to reduce the problem of quark confinement to a question about the stability of electric flux tubes in a pure gauge theory. Now we must consider the effect of introducing quarks as dynamical degrees of freedom. One might worry that introducing dynamical quarks will drastically change the dynamics, and confinement will be lost

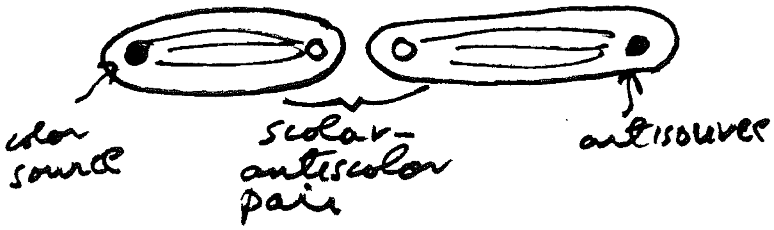
Breaking of the String

Let us first consider a gauge theory coupled to "scalar quarks", i.e. scalar fields transforming as the defining representation of $SU(N)$ - $\phi(\vec{x})$

Now we are unable to define the t'Hooft loop $B(C)$. The discontinuity by an element of Z_N of the gauge transformation $\Omega(\vec{x})$, which occurs on a sheet ending on C, is visible to $\phi(\vec{x})$, although not to the gauge field. The vacuum of this theory simply will not support magnetic vortices, and we have no criterion for magnetic confinement.

The Wilson loop $A(C)$ can be defined and is gauge invariant. How do we expect it to behave. Even if there is exact confinement, we do not expect

$$\langle A(C)\rangle \sim \exp[-\kappa(\text{Area})].$$



Color sources can be screened by the scalar particles. A color electric flux tube will not be stable - It can break if a scalar-antiscalar pair materializes.

So the potential energy of static sources does not rise linearly; we have

$$\langle A(C)\rangle \sim \exp[-\mu(\text{Perimeter})]$$

Since $B(C)$ cannot be defined and $A(C)$ is no help, one begins to wonder how to distinguish the Higgs phase from the confinement phase in this theory.

Higgs Phase vs. Confinement Phase

The criterion we use in practice is — we say that the Higgs phase is characterized by a vev of the scalar field, $\langle\phi\rangle \neq 0$. This criterion is not gauge-invariant, but we usually assume that it is meaningful.

But it is not. A very important difference between global symmetry and local symmetry is this: For local symmetry, there is no local order parameter.

Let's recall what we mean by spontaneous symmetry breaking in the case of a global symmetry. If Ω is an element of a symmetry group acting on an order parameter $\phi(x)$

$$\phi(x) \to \Omega\,\phi(x),$$

then to investigate whether SSB occurs we consider coupling ϕ to a constant external source

$$S(\phi) \to S(\phi) - \int J\phi$$

We say that there is SSB if

$$\lim_{J\to 0} \langle \phi(x)\rangle_J \neq 0$$

Spontaneous symmetry breaking can occur (there are degenerate vacua) because, when the source J is nonzero, the ground state configuration $\phi = v$ differs from the configuration $\phi = \Omega v$ by an infinite amount (if the spatial volume is infinite. SSB occurs only in the infinite volume limit) The configurations are completely decoupled, and the ground state depends on J even as $J \to 0$

(Actually, fluctuations restore uniqueness of the ground state in two or fewer dimensions — the Mermin-Wagner-Coleman theorem.)

Now note the contrast with a local symmetry:

$$\phi(x) \to \Omega(x)\,\phi(x)$$

this symmetry operation can act in a finite volume, and the change in the energy due to a gauge transformation can be finite in the presence of the source. As a result, a unique vacuum is obtained as the source is turned off. <u>Gauge symmetry cannot be spontaneously broken</u>. The vacuum is gauge invariant (Elitzur's theorem)

More precisely,

$$\langle \phi(x)\rangle_J - \langle \Omega(x)\,\phi(x)\rangle_J = \frac{1}{Z}\int (d\phi)\,\phi(x)\, e^{-S(\phi)}\left[e^{\int J\phi} - e^{\int J\Omega^{-1}\phi}\right]$$

$$= \frac{1}{Z}\int (d\phi)\,\phi(x)\, e^{-S(\phi)+\int J\phi}\left[1 - e^{\int J(\Omega^{-1}\phi-\phi)}\right]$$

$$= \left\langle \phi(x)\left[1 - e^{\int J(\Omega^{-1}\phi-\phi)}\right]\right\rangle_J$$

we take Ω to have compact support and J to be small:

$$|\langle\phi(x)\rangle_J - \langle\Omega(x)\phi(x)\rangle_J| \;\approx\; |J\langle \phi(x)\int(\Omega^{-1}\phi-\phi)\rangle_{J=0}|$$

$$\longrightarrow 0 \quad \text{as} \quad J \to 0$$

We conclude $\lim_{J\to 0} \langle \phi(x) \rangle_J = \lim_{J\to 0} \langle \Omega(x)\phi(x) \rangle$

and therefore $\lim_{J\to 0} \langle \phi(x) \rangle_J = 0$

Note: gauge fixing ignored - Applies to a theory with gauge-invariant regularization at short distances, i.e., applies if we take $V\to\infty$ limit before remove short distance regulator

We must use nonlocal gauge-invariant order parameters to classify the phases of a gauge theory, such as $\langle A(C) \rangle$ or $\langle B(C) \rangle$. These fail to distinguish the Higgs and confinement phases of a gauge theory with a scalar field transforming as the defining representation of $SU(N)$! One begins to suspect that there is no distinction, that no genuine phase boundary separates these two "phases." (This suspicion is further reinforced by estimates in lattice gauge theories.)

To check the plausibility of the conjecture that the Higgs "phase" and confinement "phase" are not really separated by a phase boundary, we consider the spectrum of this theory in these two limits. We must argue that the spectra are qualitatively similar, so that one can evolve continuously into the other as parameters change.

For concreteness consider a $SU(2)$ gauge theory with scalar doublet $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$

and also a (4-component) fermion doublet $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

Suppose the scalar potential is

$$V(\phi) = \lambda(\phi^\dagger\phi - v^2)^2$$

our usual expectation is that, if $v^2>0$ (and $g^2 \ll 1$), the theory is in a Higgs phase, while for $v^2<0$, it is in the confinement phase

For $v^2>0$, we usually describe the spectrum of this theory in the following way. We use the gauge freedom to rotate ϕ so that

$$\phi_1 = 0$$
$$\phi_2 = v + \tilde{\phi}.$$

This is the <u>unitary</u> gauge, in which the spectrum contains no unphysical gauge artifacts.

In this gauge, the spectrum consists of

scalar $\tilde{\phi}$
fermions ψ_1, ψ_2
massive gauge bosons W_1, W_2, W_3

(plus possible composite states bound by the (weak) gauge interaction)

For $v^2<0$, we expect that the theory is confining – physical states couple only to gauge-invariant (color singlet) operators. We classify the spectrum by these operators, which can create physical states when acting on the vacuum.

scalar $\phi_i^+\phi_i$
fermions $\phi_i^+\psi_i$
$\epsilon_{ij}\,\phi_i\,\psi_j$

vector bosons $\epsilon_{ij}\,\phi_i D_\mu \phi_j$
$\epsilon_{ij}\,\phi_i^+ D_\mu \phi_j^+$
$\phi_i^+ D_\mu \phi_i$

But these same gauge-invariant operators couple to our states in the Higgs phase. If we impose the unitary gauge condition

$$\phi_i^+\phi_i \longrightarrow v\tilde{\phi} + \cdots$$
$$\phi_i^+\psi_i \longrightarrow v\,\psi_2 + \cdots$$
$$\epsilon_{ij}\,\phi_i\,\psi_j \longrightarrow v\,\psi_1 + \cdots$$

$$\left.\begin{array}{l}\epsilon_{ij}\,\phi_i D_\mu \phi_j \\ \epsilon_{ij}\,\phi_i^+ D_\mu \phi_j^+\end{array}\right\} \to v^2 W_\mu^{1,2} + \cdots$$
$$\phi_i^+ D_\mu \phi_i \to v^2 W_\mu^3 + \cdots$$

we find no reason to believe that the spectrum behaves discontinuously when v^2 changes sign.

One might imagine trying to distinguish the Higgs and confinement phases by a clever trick: couple the scalar and fermion to a $U(1)$ gauge field, through the current

$$J_\mu = \frac{e}{2}\left(\phi^+ \overleftrightarrow{D}_\mu \phi + \bar\psi \gamma_\mu \psi\right)$$

Now the elementary scalar and fermion have charge $e/2$, but the gauge-invariant states have charge ne. Can't we then tell whether the elementary particles are liberated by the Higgs mechanism?

<u>No</u>, because $\langle\phi\rangle = \binom{0}{v}$ breaks $SU(2)\times U(1)$ down to a new $U(1)$ with

$$Q' = Q + T_3$$

under which the elementary fields carry integer charge. The expectations for the charges of the physical states are the same in the Higgs picture and the confinement picture. ($Q' = Q$ for a gauge invariant operator, and v has $Q' = 0$.)

Now contrast the above discussion with the case of a model with a scalar in the triplet representation of $SU(2)$ $\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$

If there are no fermions in the defining rep, then we can define $B(C)$, and also expect $A(C)$ to be a reasonable order parameter for confinement, because color electric Z_2 charge cannot be screened.

Thus, it is possible to distinguish a Higgs phase from a confinement phase. And the distinction survives even when we introduce the fermion doublet $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, as we see by considering the spectra

In the Higgs phase, we expect $SU(2) \to U(1)$ and we have, in unitary gauge

$$\phi_1 = \phi_2 = 0$$
$$\phi_3 = v + \tilde{\phi}$$

Scalar $\tilde{\phi}$ (neutral)
Fermions ψ_1, ψ_2 (charge $\pm\frac{1}{2}$, unconfined)
Massive $W^{\pm}$
Massless W^{0}

But in the confinement phase, we have
Scalars $\phi_i \phi_i$, $\bar{\psi}_i \psi_i$
Massive Vector $\phi_i D_\mu \phi_i$
There are No fermions

Because states with odd fermion number have nontrivial Z_2, and physical states are all trivial under Z_2. Apparently, there is a phase transition.

What is the fundamental difference between these two examples. Clearly it is the existence of a surviving Z_N (Z_2) symmetry which can be used to classify states. When we say that the gauge symmetry is broken, we mean that there exist physical states which transform nontrivially under Z_N. (In the $SU(2)$ model with a Higgs triplet, Z_2 happens to coincide with $(-1)^F$ where F is fermion number) We cannot make sense of this condition, however, if the unitary gauge order parameter transforms nontrivially under Z_N. Then all states have indefinite Z_N quantum numbers

Confinement in QCD:

An $SU(N)$ gauge theory with fermions in the defining representation (N) is like the second example discussed on p. 2.52 ff. There

$$Z_N = \text{quark no. modulo } N,$$

and confinement means that all physical states have the trivial Z_N quantum no.

In the real world $(QCD \times U(1)_{em})$, since quarks carry fractional electric charges, this criterion is equivalent to the requirement that all physical states have integer charge.

Quarks at Finite Temperature

We argued earlier that we expect deconfinement for large T in a pure gauge theory. The order parameter

$$\langle \mathrm{tr}\, U(\vec{x}, \beta) \rangle,$$

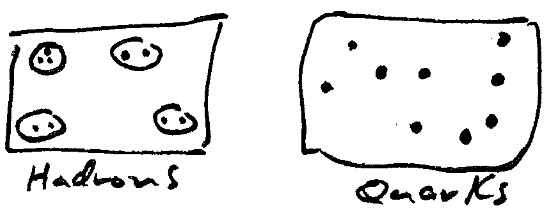
which vanishes at low temperature, is expected to acquire a nonzero value at high temperature. We also noticed that this serves as an order parameter for a global Z_N symmetry (generated by gauge transformations with a twist in the Euclidean time direction), so the question of deconfinement becomes equivalent to the question of the realization of the Z_N symmetry.

If we now introduce dynamical quarks, $\langle \mathrm{tr}\, U \rangle \neq 0$ is no longer an adequate criterion for deconfinement at finite temperature, for we expect an external color source to be screened by a quark, and to have finite free energy.

Also, it appears that the global Z_N is lost, because a gauge transformation with a twist changes the boundary conditions for the quark fields.

Thus, we are left without an obvious order parameter for deconfinement in the presence of dynamical quarks. This raises the question: Will there really be a deconfinement phase transition? Or are the high and low temperature "phases" actually not separated by a sharp boundary? (If the quarks are massless, we expect a chiral symmetry transition, but let's not consider that here.) This question is not completely devoid of physical interest. If such a phase transition exists, it presumably occurred in the early universe (during the first 10^{-4} sec), and it can be simulated in high energy heavy ion collisions.

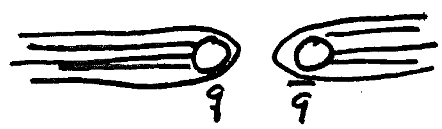
At low temperature, we know what confinement means. There is a physical difference between a hadron gas and a quark gas. But as the temperature increases, the hadron density increases, eventually approaching nuclear density. At that point the hadronic bags can dissolve more or less continuously; i.e., as the temperature increases, a hadron gas could, conceivably, evolve continuously into a quark/gluon gas. (with Debye screening) (Compare the ionization of atomic hydrogen — there is surely a difference between an H gas and an e-p plasma, but there is no phase transition as the H gas heats up and ionizes.)



To argue that there actually is a phase transition, we must find an order parameter which is expected to behave nonanalytically.

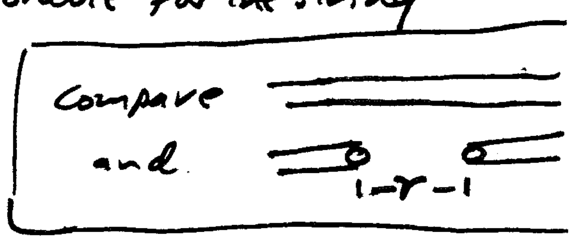
No such order parameter is known, so it seems plausible that a hadron gas (flux tube elec. (color) screening) can evolve continuously into a quark gas (debye electric screening), without a phase transition, if the quarks are sufficiently light.

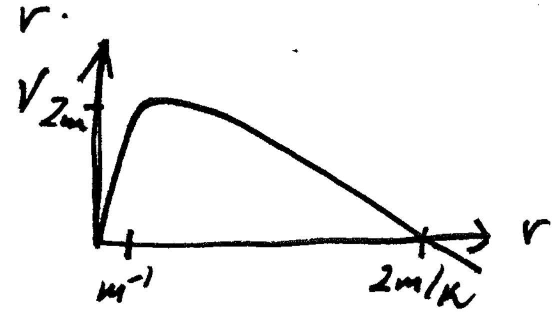
To better appreciate the relevance of the lightness of the quarks, let's try to estimate the probability that a flux tube breaks, by spontaneous creation of a quark-antiquark pair. We'll consider the case of heavy dynamical quarks,



$m \gg \sqrt{K}$, where K is the string tension. In this case, breaking of the string is a highly improbable tunneling event, and it is sensible to try to calculate the rate using semiclassical (e.g., WKB) methods.

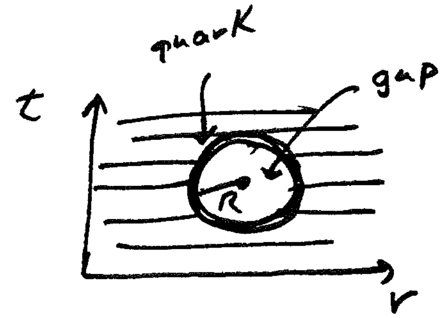
It is energetically favorable for the string to break because the cost $2m$ of creating the pair can be compensated by a savings in string energy $-Kr$ when $q, \bar{q}$ are separated by r.





But the pair must penetrate a barrier of height $2m$ and thickness $\sim 2m/K$.

To estimate the probability of this tunneling event, consider the weighting in the functional integral associated with the "world surface" in two-dimensional Euclidean space of a string in which a gap appears



as a quantum fluctuation
The gap determines a region in space time which is bounded by the world line of a quark (i.e. q-$\bar{q}$ pair).

That is, as a simple model of the string-quark system, we take the action associated with a gap in the string (relative to a string with no gap) to be

$$S = m(\text{Perimeter}) - K(\text{Area})$$

i.e. m = quark mass = action per unit length of gap boundary
K = string tension = $-$action per area of gap.

If we wish to perform a steepest descent approximation to the functional integral

$$\int_{\substack{\text{gap}\\ \text{history}}} e^{-S}$$

, we must first find the stationary point of S. We know that, for fixed perimeter, S is stationary if the gap is a circle, so consider a circle of radius R:

$$S = 2\pi R m - \pi R^2 K$$

the value of R for which S is stationary is

$$\frac{dS}{dR} = 2\pi(m - KR) = 0 \Rightarrow R_0 = m/K$$

and $S_0 = \pi m^2/K$ at the stationary point

Therefore, the vacuum energy E of the string is given by

$$e^{-ET} = e^{-S_0} \int_{\substack{\text{gaussian}\\ \text{fluctuations}\\ \text{about circle, rad } R_0}} e^{-\Delta S}$$

, in the steepest descent (semiclassical) approx.

But now we encounter a problem, because the integral

$$\int dR\, e^{K\pi(R-m/K)^2} \quad \text{diverges.}$$

Actually, this is neither surprising nor a problem. We are attempting to find the energy of an unstable state, which can be defined only by analytic continuation, and which we expect to have an imaginary part that determines the decay rate of the state. We must distort the contour of the R integration so that R runs along the imaginary axis; then the integral is finite and imaginary.

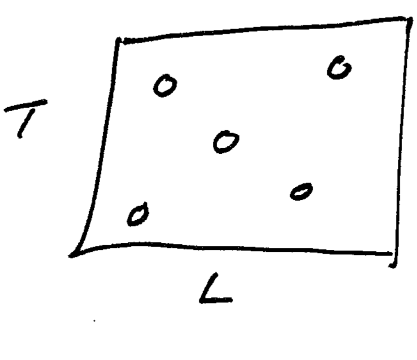
(For a full discussion of this point, see S. Coleman, PR D15, 2929 (1977).)

The semiclassical evaluation of the functional integral thus gives

$$e^{-ET} \simeq 1 + i\,C(LT)\,e^{-S_0}$$

where C is determined by integrating over all the positive frequency modes of oscillation of the circular gap, and the volume factor LT arises because the gap can appear anywhere in Euclidean spacetime. (The 1 comes from the history with no gap at all) The RHS is evidently the expansion to linear order of

$$\mathrm{Exp}\left(i\,C(LT)\,e^{-S_0}\right).$$



In fact, we can obtain the full exponential by summing up contributions to the functional integral due to an arbitrary number of tunneling events (i.e. gaps)

$$e^{-ET} \simeq \sum_n \left(iCe^{-S_0}TL\right)^n \frac{1}{n!}$$ because gaps are identical

$$= \mathrm{Exp}\left[iCLTe^{-S_0}\right]$$

We have, finally

$$\mathrm{Im}\,E \simeq CLe^{-S_0}$$

and the decay rate is $\Gamma = 2\,\mathrm{Im}\,E$, or

$$\Gamma/L \sim 2Ce^{-S_0} = 2Ce^{-\pi m^2/\kappa}$$

– there is a probability of tunneling per unit time and length, since tunneling can occur anywhere on the string.

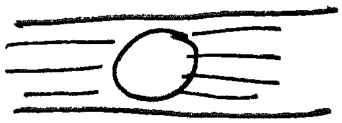
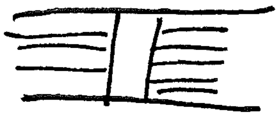
While we have not calculated the prefactor C, it is interesting to see how easy it is to find the correct numerical factor in the exponent; i.e. π.

The estimate $\Gamma/L \sim e^{-\pi m^2/\kappa}$

is the quantum mechanical tunneling rate at zero temperature. At finite temperature, thermal fluctuations can also cause the string to break. We might guess that, since the probability of making a q-$\bar{q}$ pair is suppressed by a Boltzman factor, the rate at which the string breaks due to thermal fluctuations at inverse temperature β is

$$(\Gamma/L)_{\text{thermal}} \sim e^{-2m\beta}$$

We can see that this estimate is correct by doing a semiclassical approximation to the functional integral at temperature β^{-1}

 $\beta \sim m/K$  $\beta \ll m/K$

For $\beta^{-1} \gg R_0 = m/K$, the gap of radius R_0 cannot "fit" in periodically identified imaginary time with period β. So at high temperature we consider configurations which are constant in imaginary time, and the (Euclidean) action is

$$S = \beta(2m - K|R|).$$

This has no stationary point, but we expand around $R_0 = 0$, so

$$S_0 = 2m\beta.$$

i.e., thermal fluctuations don't <u>penetrate</u> the barrier; they climb over it.

So the rate estimate

$(\Gamma/L)_{\text{quantum}} \sim e^{-\pi m^2/K}$ applies at low temp., $\beta \gg m/K$;

while $(\Gamma/L)_{\text{thermal}} \sim e^{-2m\beta}$ applies at high temp., $\beta \ll m/K$.

Consider now the implications for the deconfining transition in the presence of dynamical quarks. The "pure gauge theory limit" is $m = \infty$, since for infinite mass quarks, breaking of the string can be ignored. For $m < \infty$, the Z_N global symmetry discussed on p 2.48 is explicitly broken; quarks, transforming as the defining representation of $SU(N)$, satisfy boundary conditions which are not left-invariant by the $SU(N)/Z_N$ gauge transformations considered there.

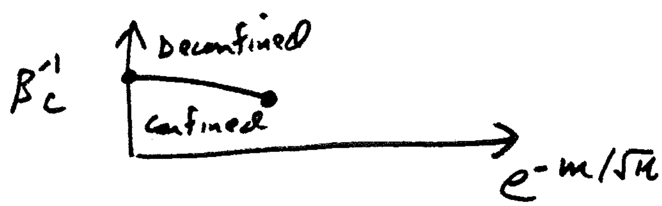
i) $N = 2$

Neglecting dynamical quarks, we argued that large closed loops of electric flux would "condense" at a temperature $\beta^{-1} \sim \sqrt{K}$. In this temp. range, the thermal estimate of the "tunneling" probability applies.

No matter how large m is, breaking of the string will prevent arbitrarily large loops from forming. It seems reasonable to guess, then, that the 2nd-order phase transition will disappear for $m < \infty$. This is consistent with standard wisdom; 2nd-order phase transitions are typically unstable under small perturbations. (E.g., Ising magnet in external field.)

ii) $N \geq 3$

For $m = \infty$, we argued that the deconfining transition, driven by percolation of a string network, was first order. For m large, the time scale $\sim e^{2m\beta}$ for a link in the network to break is large compared to the characteristic time scale for fluctuations of the network. So a phase transition can still be expected. The order parameter $\langle \mathrm{tr}\, U \rangle$, while nonzero in both phases, is discontinuous at some critical temp depending on m.



Since no symmetry distinguishes the two phases (flux tube screening and Debye screening), it is possible for the transition line to terminate, as for a gas-liquid transition, at some critical value of m. That is, when m is small enough, strings break so easily that the percolating network ceases to exist, and the phase transition disappears

Exercises

2.1)  Show that, for a classical spinning relativistic string (with massless ends) the mass M and angular momentum J are related by

$$M^2 = c\,T\,J \qquad \text{(Linear "Regge trajectory")}$$

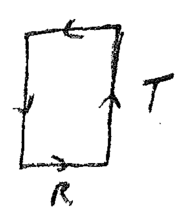
where T is the string tension. Find the numerical coefficient c.

2.2) Show that the energy per unit length of the Nielsen-Olesen string has the form

$$T = \frac{\text{Energy}}{\text{Length}} = a^2 f\left(\frac{\lambda}{e^2}\right),$$

where f is some function of λ/e^2. (Notation as on page 2.3 of notes.) Hint: You do not need to do any calculations, just a clever rescaling of the parameters in the Lagrangian.

2.3) Calculate to $O(g^2)$ the vacuum expectation value of the Wilson loop operator for a rectangular loop (in Euclidean Space) with sides R and T. Assume $T \gg R$, and retain only the term proportional to T/R.



Physics 230
Exercises

2.4) Show that the integers $n_{\mu\nu}$ defined on p. 2.33 are unchanged by a (not necessarily periodic) gauge transformation.

2.5) Show that for $N = p_1 p_2$, where p_1, p_2 are prime, there is an "exotic solution" to the duality relation (neither electric confinement nor magnetic confinement).

2.6) Find an interpretation for the "exotic solution" of Ex(2.5) in terms of the Higgs mechanism. That is, propose a symmetry breaking pattern (and Higgs fields driving the breakdown) which would give rise to these light and heavy fluxes.