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3. Introduction to Chiral Symmetry

A. The Pair-Condensate Instability

As was emphasized back in section I.B, the QCD Lagrangian

$$\mathcal{L} = \sum_{a=1}^{n} \Big[\bar{q}_{La} i \not{D} q_{La} + \bar{q}_{Ra} i \not{D} q_{Ra} + m_a \bar{q}_{La} q_{Ra} + m_a^* \bar{q}_{Ra} q_{La} \Big] + \mathcal{L}_{gluon}$$

respects a large group of symmetries in the limit in which all m_a vanish. Only the masses couple together the LH and RH fermions, so if the masses vanish, the Lagrangian is invariant under

$$q_L \to V_L q_L, \quad q_R \to V_R q_R$$

where $V_{L,R}$ are $n \times n$ unitary matrices acting on the n flavor indices. Thus, the symmetry of the Lagrangian is

$$G_{flavor} = U(n)_L \times U(n)_R$$

We will see, as you probably know, that one symmetry does not survive in the quantum theory - the axial $U(1)_A$

$$q_L \to e^{i\theta} q_L, \quad q_R \to e^{-i\theta} q_R,$$

so the actual flavor symmetry of QCD is

$$G_{flavor} = [U(n)_L \times U(n)_R] / U(1)_A$$
$$\sim SU(n)_L \times SU(n)_R \times U(1)_V$$

The question we wish to begin to address is, how is this symmetry realized? Is it manifest, or spontaneously broken to some subgroup, and if so, what subgroup?

Not much will be said here about the consequences of spontaneous breakdown of this chiral symmetry. But you all know (from Phys 234, if not elsewhere) that the approximate (the quark masses are not really zero) $SU(3)_L \times SU(3)_R$ chiral symmetry of the strong interactions is apparently spontaneously broken to $SU(3)_V$

$$q_L \to V q_L\,, \quad q_R \to V q_R\,,$$

and that $\pi^{\pm,0}$ $K^\pm$ $\bar{K}^0, K^0, \eta$ are the eight (pseudo)-Goldstone bosons associated with this symmetry breakdown.

A crucial test that QCD must pass, to be considered the correct theory of the strong interactions is, it must predict the spontaneous breakdown of chiral symmetry

$$SU(n)_L \times SU(n)_R \longrightarrow SU(n)_V\,.$$

(at least for $n = 3$)

Let's begin with a simple plausibility argument. As was the case when we attempted to obtain a heuristic understanding of quark confinement, analogies with the theory of superconductivity are enormously helpful when we attempt to understand the dynamical breaking of chiral symmetry. Superconductivity is a consequence of a dynamical instability toward the formation of a condensate of Cooper pairs. We will argue that chiral symmetry breaking arises in much the same way. (This crucial insight is due to Y. Nambu and G. Jona-Lasinio, Phys. Rev. $\underline{122}$, 345 (1961). Their remarkable paper is still worth reading.)

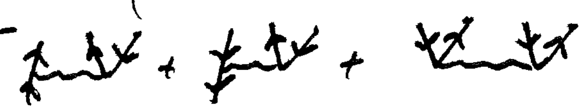
Our plausibility argument is based on a brutal truncation of QCD. We will integrate out gluons, obtaining an effective Hamiltonian for the quarks, but we will retain only terms of order g^2. Then, even though this approximation is justified only for g^2 small, we will ask how the ground state of this Hamiltonian will look for g^2 arbitrarily large.

Schematically, we have

$H_{eff} = H_{Kin} + g^2$ 

and let's split  into two terms, one which preserves both quark number and antiquark number, and one which creates and destroys pairs

$H_{eff} = \Big[H_{Kin} + g^2 \big($  $\big) \Big]$

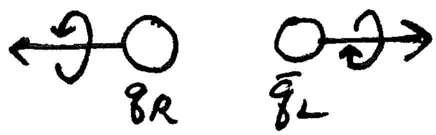
$+ \; g^2 \Big[$  $+ \text{ h.c.} \Big]$

The first bracket contains terms which preserve the number of quark-antiquark pairs; the terms in the second bracket change the number of pairs.

We saw back in Section I.C that the effect of the $O(g^2)$ terms in the first bracket is to generate an interaction which is attractive in one channel and repulsive in another. E.g. for $q\text{-}\bar{q}$ there is an attractive singlet channel and a repulsive octet channel (for SU(3) color). The effect of the terms in the second bracket is to induce matrix elements of Heff between the perturbative vacuum and states containing q-$\bar{q}$ pairs.

As g^2 increases, the energetic cost of producing a pair decreases, in the color-singlet channel, and the importance of the pair-creation term in H_{eff} increases. For large g^2, the vacuum state becomes an approximate eigenstate of the pair-creation term. But this state necessarily contains an indefinite number of $q, \bar{q}$ pairs. It is a "pair-condensate."

The $O(g^2)$ terms in H_{eff} in the first bracket require the pairs to be color singlets. And if we assume that Lorentz invariance is not broken, they must have vacuum quantum nos., vanishing angular momentum and total momentum. The pair looks like this:



q_R moving left pairs with $\bar{q}_L$ moving right (this is the pairing consistent with the conservation laws.) A pair-condensate thus breaks chiral symmetry, for such a pair cannot be invariant under independent LH and RH chiral rotations.

In other words, the operator $\bar{q}_L q_R$, which creates such a pair, has a nonzero vev in the condensate state, $\langle \bar{q}_L q_R \rangle \neq 0$. The most symmetrical form this vev can have is

$$\langle 0| \bar{q}_{La}\, q_{Rb} |0\rangle = \Delta\, \delta_{ab}$$

and, there is no reason to expect the pair condensate to distinguish among flavors.

which preserves the $SU(n)_V$ symmetry and also $U(1)_V$. Thus, this brutally truncated Hamiltonian leads to the expectation that, for large g^2, the vacuum state cannot preserve the full group of chiral symmetries. At best, $SU(n)_V \times U(1)_V$ can survive.

In QCD, the coupling constant (which runs) does become large at large distances. Of course, the approximation of retaining only for large g^2 is very bad. We may nonetheless hope that this naive argument has captured the essential physics.

Remarks

i) The formation of the pair condensate can also be described as the generation of a "constituent mass" for the quark. A massless quark travels at light velocity and has definite helicity (chirality). But a quark in the pair condensate vacuum propagates as a massive "quasiparticle." The vev of $\bar{q}_{La} q_{Rb}$ can be interpreted as a spontaneously generated mass operator which couples RH and LH quarks: $q_L \leftarrow \otimes \leftarrow q_R$.

In terms of the condensate picture (vacuum = overlapping $q\bar{q}$ pairs with size $\sim \Lambda_{QCD}^{-1}$), a q_R comes along and annihilates a $\bar{q}_R$ in a pair, freeing a q_L which continues to propagate, carrying the momentum of the incoming quark. This phenomenon gives rise to an effective quark mass of order Λ_{QCD}, and is presumably the origin of

$$(m_{u,d})_{\text{constituent}} \sim 350 \text{ MeV}$$

which is used in nonrelativistic-quark-model estimates of the hadron spectrum. (The renormalized quark masses appearing in the QCD Lagrangian are much smaller, $m_{u,d} \sim 5, 8$ MeV, as you learned in Physics 2 .)

It is still not very clearly understood why the non-relativistic quark model is so successful. A possible explanation is that chiral symmetry breaking and confinement actually occur at slightly different distance scales. This means that the characteristic size of the pairs in the condensate is smaller than the size of a hadron. Then it may be sensible to describe the inside of a hadron as a pair condensate, so that the hadron can be regarded as a bound state of quarks which carry constituent masses

ii) To argue that a pair condensate should form, we needed to assume g^2 large. This is the generic case; dynamical breaking of chiral symmetry cannot occur unless there are strong interactions. For spontaneous breaking of (exact) chiral symmetry implies the existence of Goldstone bosons, and if there are no elementary scalars, the Goldstone bosons must be bound states. Only a strong interaction can bind zero-mass states. One might argue that this is not obvious, if the constituents are also massless. But saying that the bound state survives as $m \to 0$ means that its size approaches a fixed value independent of m. The constituents, confined to this finite size, have a zero-point energy which must be cancelled by interaction energy.

(Note that the NR quark model does not give a sensible description of the Goldstone boson, the pion, in the chiral limit $m_{u,d} \to 0$.)

iii) In making the statement $\langle 0 | \bar{q}_{La} q_{Rb} | 0 \rangle = \Delta \delta_{ab}$ we are actually adopting a convention. For by doing, e.g., an $SU(n)_R$ chiral rotation we can obtain

$$\langle \mathcal{W} | \bar{q}_{La} q_{Rb} | \mathcal{W} \rangle = \Delta \mathcal{W}_{ab} \quad \text{where } \mathcal{W} \text{ is any } SU(n) \text{ matrix}$$

This is as expected. When spontaneous symmetry breaking occurs, there are are many degenerate, physically equivalent vacua $|\mathcal{W}\rangle$.
Here the "vacuum manifold" is $SU(n)$, which is n^2-1 dimensional. The n^2-1 Goldstone bosons are the "spin waves" associated with space-time dependent rotations of $\mathcal{W}$

Since, in the limit of exact chiral symmetry, all vacua $|\mathcal{W}\rangle$ are equivalent, we may make the standard choice $\mathcal{W} = \mathbb{1}$. But if the quarks have (small) intrinsic masses, which explicitly break chiral symmetry, the states $|\mathcal{W}\rangle$ are no longer equivalent, and we must choose the correct vacuum. (Compare a ferromagnet — In the presence of an external magnetic field, the ground state has the magnetization aligned with the field)

We have added the perturbation

$$H' = m_{ba} \bar{q}_{La} q_{Rb} + \text{h.c.},$$

which, to lowest order in m, shifts the energy of the state $|\mathcal{W}\rangle$ by

$$E(\mathcal{W}) = \langle \mathcal{W} | H' | \mathcal{W} \rangle = m_{ba} \Delta \mathcal{W}_{ab} + \text{c.c.}$$

$$E(\mathcal{W}) = 2\Delta \, \mathrm{Re}\, \mathrm{tr}(m \mathcal{W})$$

To find the true ground state, we must minimize $E(\mathcal{W})$ subject to the constraint $\mathcal{W}$ = unitary matrix of determinant 1.

For example, suppose there are three flavors, and $m = a\mathbb{1}$ (all flavors degenerate in mass), $a > 0$.

The minimum of $E(W)$ depends on the sign of Δ

$$E(W) = 2a\Delta \, \mathrm{Re}\, \mathrm{tr}\, W$$

If $\Delta < 0$, the minimum is clearly $W = \mathbb{1}$. The vacuum lines up with the mass, as we might have expected. But if $\Delta > 0$, ---

Exercise 3.1

Minimize $E(W)$, where $m = a\mathbb{1}$, $a > 0$, and $\Delta > 0$. (W is unitary and has determinant $\mathbb{1}$). Is the solution unique? Will CP be conserved in this theory?

(Although this is not part of the exercise, you might find it interesting to consider how the solution changes when the quark masses m_u, m_d, m_s are not necessarily equal.)

B. The Nambu-Jona-Lasinio Equation

Can we make the discussion of dynamical mass generation more precise? We will attempt to do a variational calculation to test the stability of the perturbative vacuum with $\langle \bar{\psi}_L \psi_R \rangle = 0$.

Writing

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\bar{p}})^{\frac{1}{2}}} \left[b_m^\alpha(\bar{p})\, u^\alpha(\bar{p}) e^{-ip\cdot x} + c_m^{\dagger\alpha}(\bar{p})\, v^\alpha(\bar{p}) e^{ip\cdot x} \right]$$

we will consider trial states $|\psi_m\rangle$ such that

$$b_m^\alpha(\bar{p})\, |\psi_m\rangle = 0$$

$$c_m^\alpha(\bar{p})\, |\psi_m\rangle = 0$$

These are the vacuum states of a free fermion theory with mass m, except that we will allow

m to be position dependent, so that the state $|\Psi_m\rangle$ is a functional of m.

Now let us compute $\langle\Psi_m|H|\Psi_m\rangle = E(m)$, and find m which minimizes $E(m)$. Our set of trial states is not complete, so even if we find that $E(m)$ has a nontrivial minimum at $m \neq 0$, we have not rigorously demonstrated that spontaneous mass generation occurs. But such a nontrivial minimum would be a persuasive indication that spontaneous breakdown of chiral symmetry occurs.

For the fermion field ψ consider

$$\mathcal{L} = \bar\psi i\partial\!\!\!/ \psi - H_I(\bar\psi,\psi)$$

where H_I is a chiral-invariant, but not necessarily local, interaction. To calculate $E(m)$, it is convenient to write

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi + m\bar\psi\psi - H_I(\bar\psi,\psi)$$

If only the first term were present, we would have

$$e^{-iE_0(m)VT} = \det(-\partial\!\!\!/ - im) = \exp \mathrm{tr}\ln(-\partial\!\!\!/ - im)$$

$$\Rightarrow E_0(m)VT = i\,\mathrm{tr}\ln(-\partial\!\!\!/ - im)$$

Adding to this the matrix elements of the other two terms in the state $|\Psi_m\rangle$, we have

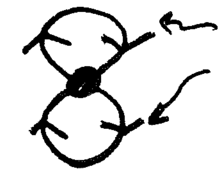
$$VTE(m) = i\,\mathrm{tr}\ln(-\partial\!\!\!/ - im) + \underbrace{\mathrm{tr}\, m(\partial\!\!\!/ + im)^{-1}}_{\text{(remember Fermi }(-1))} + \langle \int \mathcal{H}_I \rangle_m$$

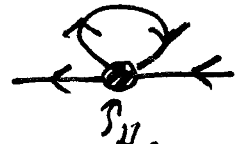
And if we require $E(m)$ to be stationary,

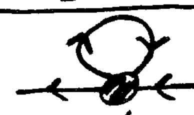
$$0 = \frac{\delta EVT}{\delta m} = (-\partial\!\!\!/ - im)^{-1} + (\partial\!\!\!/ + im)^{-1} - i(\partial\!\!\!/ + im)^{-1} m (\partial\!\!\!/ + im)^{-1}$$
$$+ \frac{\delta}{\delta m}\langle \int \mathcal{H}_I\rangle_m$$

or $\boxed{(\not{\partial}+im)^{-1}(-im)(\not{\partial}+im)^{-1} = -\frac{\delta}{\delta m}\langle \int \mathcal{H}_I \rangle_m}$

Now, suppose, for definiteness, that $\mathcal{H}_I$ is a chiral-invariant 4-fermion interaction, like the one generated by one-gluon-exchange which we considered before.

$\mathcal{H}_I =$  $\Rightarrow \langle \int \mathcal{H}_I \rangle =$ 

and $-\frac{\delta}{\delta m} = -i$  (remember the Fermi (−1))

or  = 

This is the "self-consistent" equation for the spontaneously generated mass, to lowest order in $\mathcal{H}_I$, which was proposed by Nambu and Jona-Lasinio as a criterion for spontaneously broken chiral symmetry. (Our variational computation corresponds to a sort of chiral mean-field theory.)

We can apply this eqn. even if $\mathcal{H}_I$ is nonlocal, e.g. the interaction g^2  generated by gluon exchange. But to demonstrate its use, we will consider a simple model local interaction, following Nambu and Jona-Lasinio.

The model has no flavor. There is a single 4-component fermion, and the Lagrangian is

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi + \tfrac{1}{4} g\left[(\bar{\psi}\psi)^2 + (\bar{\psi} i\gamma_5 \psi)^2\right]$$

This model possesses the symmetry

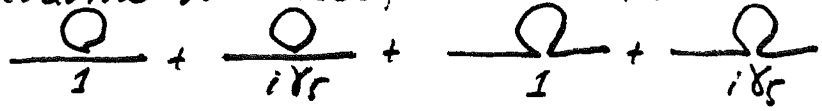
$$G_f = U(1)_V \times U(1)_A$$

The interaction is $U(1)_A$ invariant because $\bar{\psi}\psi$ and $\bar{\psi}i\gamma_5\psi$ transform as an $O(2)$ doublet under

$$\psi \to e^{i\gamma_5\theta}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{i\gamma_5\theta}, \qquad e^{i\gamma_5\theta} = \cos\theta + i\gamma_5 \sin\theta.$$

Now let's solve the Nambu-Jona-Lasinio equation:
(we'll look for a translation invariant solution; m = constant.)

$$-\otimes- \;=\; -im \;=$$



$$= \tfrac{1}{2}(ig)\int\frac{d^4p}{(2\pi)^4}\Big[\underbrace{(-1)}_{\text{fermi}}\,\mathrm{tr}\Big(\frac{i}{\not p - m}\Big) + (-1)\, i\gamma_5\,\mathrm{tr}\Big(i\gamma_5\frac{i}{\not p - m}\Big) + \frac{i}{\not p - m} + i\gamma_5\frac{i}{\not p - m}\, i\gamma_5\Big]$$

or $$m = \frac{g}{32\pi^2}\int p^2\,dp^2\,\frac{4m}{p^2+m^2} \qquad \text{(Euclidean)}$$

This theory is not renormalizable, so we must impose an explicit (momentum space) cutoff at $p^2 = \Lambda^2$

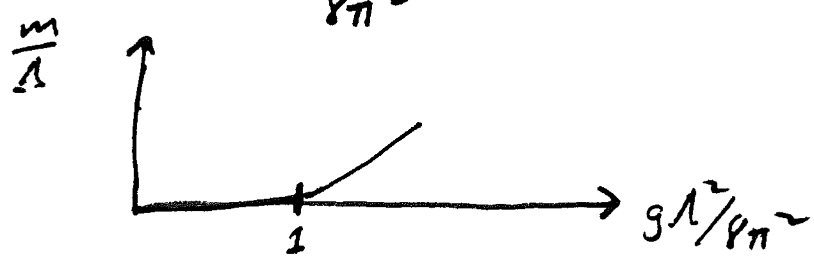
The trivial solution $m=0$ is always allowed; the condition for a nontrivial solution is

$$1 = \frac{g}{8\pi^2}\int_0^{\Lambda^2} dp^2\,\frac{p^2}{p^2+m^2} = \frac{g}{8\pi^2}\int_0^{\Lambda^2} dp^2\Big(1 - \frac{m^2}{p^2+m^2}\Big)$$

or $$\boxed{1 = \frac{g\Lambda^2}{8\pi^2}\Big[1 - \frac{m^2}{\Lambda^2}\ln\Big(\frac{\Lambda^2}{m^2}+1\Big)\Big]}$$

Let us imagine that Λ is fixed and g is allowed to vary. The quantity in brackets ranges from 1 ($m=0$) to 0 ($m=\infty$). So there is no nontrivial solution unless

$$\frac{g\Lambda^2}{8\pi^2} > 1$$



There is a phase transition at a critical value of g. (One can check that the nontrivial solution really has lower energy than the trivial solution.)

This behavior is typical of solutions to the NJL eqn. The nontrivial solution "turns on" only for a large value of the coupling. And this is no surprise, since we know that the interaction must be capable of binding a Goldstone boson. Indeed treating the Bethe-Salpeter eqn to the same order as the self-consistent mass eqn, one should find a massless bound state solution. Perturbation theory can never give an adequate description of a highly relativistic bound state, and the NJL approach can never rigorously establish the existence of a chiral instability, even if it may capture much of the essential physics.

C. Soft Masses and Hard Masses

(see H.D. Politzer, Nucl Phys B117, 397 (1976)).

Is there any way to distinguish (e.g., experimentally) the intrinsic quark masses, the renormalized mass parameter of the QCD Lagrangian, from the spontaneously generated quark mass. One would think so. For if I probe the structure of a hadron at large momentum transfer, $|-q^2|^{1/2}$ small compared to the size of a bound pair in the condensate, the spontaneously generated mass should be irrelevant. That is, the intrinsic mass and spontaneously generated mass should scale differently at short distances; the spontaneously generated quark mass is "softer."

The scaling behavior of a hard intrinsic mass was discussed on page 1.27. It is determined by the anomalous dimension of the mass operator $\bar{q}q$, which, to one loop order in QCD, is generated by the graph

 and has the form

$$\gamma_m = -c g^2 = \mu \frac{d}{d\mu}(\ln m),$$

where c is a number which you can easily compute. If we regard the quark mass m as a coupling constant defined at renormalization scale μ, then

$$m(\mu) = m(\mu_0) \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_m(g_{\mu'}^2)\right]$$

and if $\beta = -b g^3 \Rightarrow g_{\mu'}^2 = \frac{1}{2b \ln(\mu'/\Lambda)}$

we have $\int_{\mu_0}^{\mu} (d\ln\mu') \frac{-c}{2b} \frac{1}{\ln \mu'/\Lambda} = \frac{-c}{2b} \ln\ln(\mu'/\Lambda)\Big|_{\mu_0}^{\mu}$

or $$\boxed{m(\mu) \cong m(\mu_0)\left[\frac{g^2(\mu)}{g^2(\mu_0)}\right]^{c/2b}}$$

that is, the hard mass decays like a power of $(1/\ln\mu)$. (Calculation shows $c > 0$.)

We now want to consider the scaling behavior of the constituent mass, but how shall we define the constituent mass? There is no precise answer.

We will define the mass by considering the quark propagator:

$$\int d^4x\, e^{iqx} \langle 0| T\, \psi(x) \bar{\psi}(0) |0\rangle = \frac{i A}{\not{q} - M} \Bigg|_{q \text{ spacelike}}$$

i.e. M is essentially the ratio of the part of the inverse propagator which commutes with γ_5 to the part which anticommutes.

Unfortunately, this definition is not gauge invariant, so we need to specify a gauge (Landau gauge). So what does this object $M(q^2)$ have to do with a constituent mass?

To answer this question, one must consider how a constituent mass might be measured. One possible way is by looking for quark mass corrections in electroproduction, or (much simpler) e^+e^- annihilation. One argues that it is really $M(q^2)$ which determines the magnitude of these corrections, but it is, unfortunately, difficult to make this argument precise for light quarks.

Let us accept $M(q^2)$ as a definition of constituent mass, at least for large q^2, and consider how it scales, for $-q^2 \to \infty$, in a theory with exact chiral symmetry. The key to the analysis is the operator product expansion. The coefficients in the expansion, derived from perturbation theory, respect chiral symmetry. All the effects of spontaneous breakdown of chiral symmetry may be factored into matrix elements of the local operators.

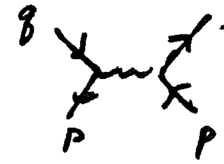
The operator product expansion of the propagator is (cf. p. 1.48)

$$-i\int d^4x\, e^{iqx} \langle 0|T\psi(x)\bar{\psi}(0)|0\rangle = \sum_n C_n(q)\langle 0|O_n|0\rangle_q$$

(means renormalized at scale q)

The leading operators in this expansion are $\mathbb{1}$ and $\bar{\psi}\psi$. In leading order, the coefficient of $\mathbb{1}$ is clearly $1/\not{q} \sim C_{\mathbb{1}}$

To find the leading contribution to (the connected part of) the coefficient of $C_{\bar{\psi}\psi}$, we consider the graph



which contributes to $\langle p|LHS|p\rangle$. It is clear that

$$c_{\bar\psi\psi} \propto \frac{1}{(q^2)} g^2 \quad \text{in lowest order}$$

(You can compute the numerical coefficient, but it is not of interest here; we only want the scaling behavior of M. In fact, the coefficient is gauge-dependent.) If we now, as usual, use the RG to push down the scale at which the matrix elements of the operators are evaluated to $\mu_0 < q$, the leading terms in the expansion become

$$\frac{\not q}{q^2} \exp\left[\int_{\mu_0}^{q} \frac{d\mu'}{\mu'} 2\gamma_\psi\right] + \underset{\substack{\uparrow \\ \text{(numerical coefficient)}}}{C} \frac{g_q^2}{(q^2)^2} \exp\left[\int_{\mu_0}^{q} \frac{d\mu'}{\mu'}(2\gamma_\psi - \gamma_m)\right] \times \langle 0|\bar\psi\psi|0\rangle_{\mu_0}$$

(since γ_m is the anomalous dimension of $\bar\psi\psi$.)

This, by the definition of $M(q^2)$ is

$$\frac{A}{\not q - M} = A\left[\frac{1}{\not q} + \frac{M}{q^2}\right], \text{ and therefore}$$

$$M(q^2) \sim \frac{g^2(q^2)}{q^2} \exp\left[\int_{\mu_0}^{q} -\frac{d\mu'}{\mu'}\gamma_m\right] \langle 0|\bar\psi\psi|0\rangle_{\mu_0}$$

If there is spontaneous breaking of chiral symmetry, then $\langle 0|\bar\psi\psi|0\rangle_{\mu_0}$ is non-vanishing, and our "consituent mass" scales like

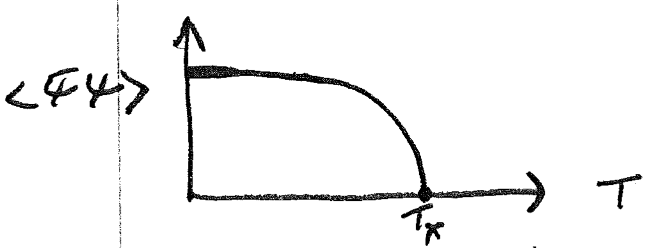
$$\boxed{M(q^2) \sim [g^2(q^2)]^{-4/26+1} \frac{\langle 0|\bar\psi\psi|0\rangle}{q^2}}$$

As expected, this mass is softer (up to logs, by a factor of $1/q^2$) than the intrinsic mass. At short distances, only the intrinsic breaking of chiral symmetry survives.

This behavior for $M(q^2)$ can also be derived by solving a RG-improved version of the Nambu-Jona-Lasinio Eqn. (K. Lane, Phys. Rev. D10, 2605 (1974))

D. Chiral Symmetry at Finite Temperature

The pair-condensate ground state of a theory which experiences spontaneous breakdown of chiral symmetry resembles the ground state of a ferromagnet. (Actually, it even more closely resembles an antiferromagnet.) The magnetization of a ferromagnet vanishes at sufficiently high temp. Should we not expect the same behavior for $\langle \bar{q} q \rangle$, the chiral order parameter?



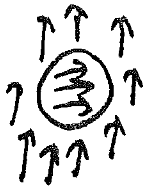
We probably should.

At high temperature, many pions (spin waves) are present, which tend to destroy the long range order by causing $\langle \bar{q} q \rangle = \Delta W$ to wiggle. Also, when the temperature 'T' becomes larger than $M(T)$, the constituent quark mass, there are many thermal $q\bar{q}$ pairs present, which tend to randomize the pair condensate.

What is the critical temperature T_χ? Of course, in QCD $T_\chi \sim \Lambda_{QCD}$ must be correct in order of magnitude. Fluctuations in $W(\vec{x})$ will be controlled by an energy density which will have the form

$$\mathcal{E} = a^{-2}\left[(\vec{\nabla} W) \cdot \vec{\nabla} W^{+} \right] + \begin{array}{l}\text{chiral invariant}\\ \text{terms higher}\\ \text{order in } \vec{\nabla} W\end{array}$$

Here a has the dimensions of length, and is presumably of order the size of a pair in the condensate. The cost of a large (of order one) fluctuation in W in a region of size λ is



$$E \sim \int d^3x \, \frac{a^{-2}}{\lambda^2} \sim a^{-2}\lambda$$

This sort of fluctuation, which destroys order, is suppressed by a Boltzman factor unless $E \lesssim T$ or $T \gtrsim a^{-2}\lambda$

But there is also a short distance cutoff of order a; for $d \leq a$, the notion of a chiral rotation of the vacuum state is meaningless. So large fluctuations which randomize $\overline{W}$ can occur only for

$$T > T_\chi \sim a^{-1}!$$

The chiral transition temp should be of order the size of a pair in the condensate.

What is the order of the transition? We saw in section 1.F that the far infrared behavior is controlled by an infrared stable fixed point of the renormalization group, if one exists. And that the existence of such a fixed point should depend only on the universality class of the system being considered; i.e., its symmetries, spatial dimension, and the nature of the order parameter.

E.g. in the two flavor case, the symmetry-breaking pattern

$$SU(2) \times SU(2) \to SU(2)$$

is isomorphic to

$$O(4) \to O(3),$$

the symmetry-breaking pattern of an $O(4)$ ferromagnet. A straightforward generalization of the analysis we did in sec. 1.F for an Ising ferromagnet to the $O(n)$ case reveals the existence of an IR stable fixed point in $D = 4 - \epsilon$ dimensions. So we expect a 2nd order phase transition in an $O(n)$ ferromagnet in 3 dimensions, and also expect the chiral transition to be 2nd order in the two flavor case.

But what about $n \geq 3$? To determine the order of the chiral transition, we should consider the infrared behavior of a renormalizable field theory (higher dimension operators are presumably irrelevant) with $SU(n) \times SU(n)$ symmetry, and an order parameter transforming as $(n,\bar{n}) + (\bar{n},n)$ under $SU(n)\times SU(n)$ (like the dynamical quark mass $\bar{q}q$). We can write the order parameter

$$U = (\sigma^a + i\pi^a)\lambda^a$$ hermitian $n\times n$ matrix, including $\mathbb{1}$.

– i.e., n^2 complex fields, transforming as

$$U \to V_L^+ U V_R \quad \text{under } SU(n)_L \times SU(n)_R .$$

(this is a reducible representation under $SU(2)\times SU(2)$, since $(2,2)$ is real, but it is irreducible for $n \geq 3$)

The renormalizable Lagrangian with $SU(n)\times SU(n)$ symmetry is

$$\mathcal{L} = \mathrm{tr}\, \partial_\mu U \partial^\mu U^+ - m^2 \mathrm{tr}\, UU^+ - g_1(\mathrm{tr}\, UU^+)^2 - g_2 \mathrm{tr}(UU^+)^2$$

(For $n = 3, 4$, there is also a $\det U$ term, but lets ignore that for now.)

Lets consider the minima of the potential:

$$V(U) = m^2 \mathrm{tr}\, UU^+ + g_1(\mathrm{tr}\, UU^+)^2 + g_2 \mathrm{tr}(UU^+)^2$$

conditions will have to be satisfied by $g_{1,2}$ so that the potential is bounded from below, and the minimum preserves $SU(n)_V$ symmetry

It is convenient to rewrite..

$$V(U) = g_1[\mathrm{tr}(UU^+ - v^2)]^2 + g_2 \mathrm{tr}(UU^+ - v^2)^2$$

(up to an additive constant). If g_1 and g_2 are nonnegative, then V is nonnegative, and its min.

occurs when $V = 0$. So for $v^2 > 0$, the min is

$$U = v\,\mathbb{1} \quad \text{(up to an } SU(n)\times SU(n) \text{ rotation)}$$

which leaves $SU(n)_V$ unbroken.

If $g_1 < 0$, we write

$$V(U) = -g_1 n \,\mathrm{tr}\left[UU^\dagger - \tfrac{1}{n}\,\mathrm{tr}\,UU^\dagger\right]^2$$
$$+(g_2 + g_1 n)\,\mathrm{tr}(UU^\dagger - v^2)^2 \qquad (+\text{ constant})$$

Now we require $g_2 + g_1 n \geq 0$ for the potential to be bounded from below; hence V is again nonnegative, and is minimized (in fact, zero) for

$$U = v\,\mathbb{1} \quad \text{(again)}$$

If $g_2 < 0$, we write

$$V(U) = -g_2\left[(\mathrm{tr}\,UU^\dagger)^2 - \mathrm{tr}(UU^\dagger)^2\right]$$
$$+(g_1 + g_2)\left[\mathrm{tr}(UU^\dagger - v^2)\right]^2 \qquad (+\text{ constant})$$

The first term in brackets is now nonnegative (Schwarz inequality). Boundedness from below requires $g_1 + g_2 \geq 0$ and so minimum occurs for

$$\mathrm{tr}\,UU^\dagger = N v^2$$
$$\mathrm{tr}(UU^\dagger)^2 = (\mathrm{tr}\,UU^\dagger)^2 \implies U = v\sqrt{N}\,\mathrm{diag}(1,0,0,\ldots,0)$$
(up to a $SU(n)\times SU(n)$ rotation)

thus $SU(n-1)\times SU(n-1)\times U(1)$ is unbroken symmetry.

So we should choose

$$g_2 \geq 0$$
$$g_2 + g_1 n \geq 0$$

in order to obtain the desired pattern $SU(n)\times SU(n) \to SU(n)$ (in tree approx.)

Now, in $D = 4-\epsilon$ dimensions, the couplings become $\mu^\epsilon g_{1,2}$. I will not discuss the calculation of the one-loop β-function in detail, since it is easy to guess the result. The eqns

$$\mu \frac{\partial}{\partial \mu} g_{1,2} = \beta_{1,2}(g_1, g_2) = 0$$

are two eqns in two unknowns and have two solutions. One, of course, is free field theory (gaussian fixed point),

$$g_1 = g_2 = 0,$$

and we know this fixed point is not IR-stable for $\epsilon > 0$ $(D < 4)$.

To guess what the other fixed point is, note that when $g_2 = 0$, $V(U)$ has a larger symmetry; since

$$\mathrm{tr}\, U U^\dagger = (\sigma^a + i\pi^a)(\sigma^b - i\pi^b)\,\mathrm{tr}\,\lambda^a \lambda^b$$

$$\propto \sum_a (\sigma^a)^2 + (\pi^a)^2,$$

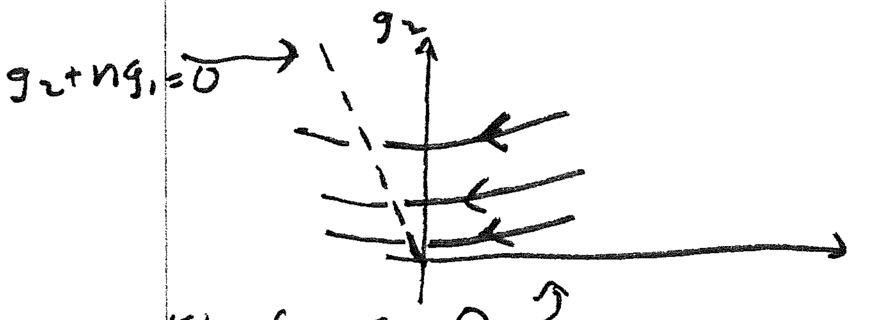
it is an $O(2n^2)$ invariant. Thus, for $g_2 = 0$, the chiral $SU(n) \times SU(n)$ model becomes an $O(2n^2)$ model, which we know has an IR-stable fixed point. So

$$g_1 = O(\epsilon), \quad g_2 = 0$$

is the other fixed point. It is stable along the g_1-axis, but unstable along the g_2-axis. Turning on g_2 is like turning on a perturbation which breaks $O(2n^2)$ symmetry, and prevents the theory from flowing to the fixed point with $O(2n^2)$ symmetry.

Thus, there is no IR stable fixed point in the ϵ-expansion (for $n \geq 3$) and no reason to expect a second-order transition (we need to tune two parameters, m^2 and g_2, to reach the fixed point, but have only one variable — temperature.)

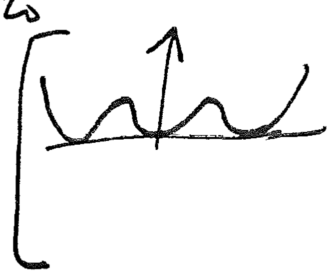
Presumably, this means that the phase transition is first order. In fact, one can understand why it is first-order in detail by studying the renormalization group flows in the g_1, g_2 plane (A.J. Paterson, Nucl. Phys. B190, 188 (1981)).



Flow for $\epsilon = 0$

In the IR, these trajectories are drawn toward the edge of the region in which the tree-approx potential is bounded from below. Near this edge, one loop corrections to the potential become important (S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973); E. Gildener and S. Weinberg, Phys. Rev. D13, 3333 (1976).) and change the shape of the potential to

i.e., in order g_2^2 an interaction $(UU^+)^2 \ln UU^+$ is generated



so that the order parameter jumps discontinuously; i.e., the transition is first order

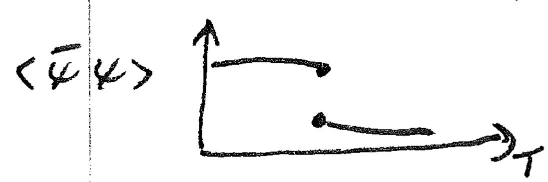
Adding a $\det U$ term to V will not change the result. (In fact, for $n=3$, this is a cubic term which makes the transition 1st-order even in tree approximation, i.e., mean-field theory.)

Note that the key feature which makes a chiral model different than a ferromagnet is the existence of two quartic coupling constants, g_1 and g_2.

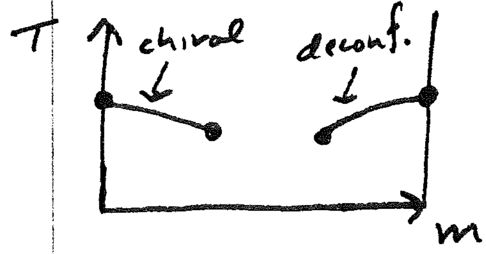
In the $n=2$ case, we expect an actual phase transition only if the intrinsic fermion masses vanish and chiral symmetry is exact.

Intrinsic masses smooth out the transition, so that the order parameter behaves abruptly but analytically (cf. a ferromagnet in an external magnetic field.)

But for $n \geq 3$, the first-order transition should survive even if there are (sufficiently small) intrinsic masses; $\langle\bar\psi\psi\rangle$ changes discontinuously, although it is nonzero in both phases



Thus, we have argued that two first order phase transitions occur in QCD with 3 colors and 3 flavors. But the deconfining transition



is known to occur only if the quark masses are sufficiently large, and the chiral transition is known to occur only if the quark masses are sufficiently small.

In the real world, it is possible that neither phase transition occurs; it is plausible that there is a chiral transition, but no deconfinement transition. (The quarks are light.) It is the s-quark mass which determines whether there is a chiral transition (since there is not one if there are only two light flavors) and the u-quark mass which determines whether there is a deconf. trans. (i.e., how easily the string breaks.)

(Ref: Georgi, *Weak Interactions*)

Chiral Lagrangians and Chiral Pert. Theory

Study properties of soft Goldstone bosons (pions + Kaons). A remarkable amount follows only from symmetry principles. (Current algebra)

Basic idea: Soft pion $\Longleftrightarrow$ slowly varying vacuum orientation

$$\langle \bar{q}_{La} q_{Rb} \rangle_\Sigma = -\Delta \, \Sigma_{ab}$$

Σ_{ab}: SU(n) matrix

Write down "phenomenological Lagrangian" involving Σ, which has the $SU(n)_L \times SU(n)_R$ chiral symmetry, with

$$\Sigma \to V_L^+ \Sigma V_R$$

The invariant $\Sigma^+ \Sigma$ is trivial, so all terms must involve derivatives (Of course, a constant chiral rotation cannot cost any energy). Thus, pions always couple derivatively; coupling $\to 0$ as $p \to 0$.

Identify pions as

$$\Sigma = \exp(2i\pi^a T^a / f_\pi)$$

normalization convention

They transform nonlinearly under chiral symmetry ("Nonlinear realization"). This is necessary unless we introduce auxiliary fields to form complete multiplet. ("Linear sigma model")

There are an ∞ no. of terms we can write down, so how do we calculate? Point is, for "soft" pions, there is a natural expansion parameter, the momentum p. For $p \to 0$, operators with fewest derivatives dominate. ("relevant" in infrared.)

Leading term: two derivatives.
We can construct --

$(\mathrm{tr}\,\Sigma^\dagger \partial_\mu \Sigma)^2 \longrightarrow$ vanishing $\Sigma^\dagger \partial_\mu \Sigma$ is traceless - an $SU(n)$ generator.

$\mathrm{tr}\,\partial^\mu \Sigma^\dagger \partial_\mu \Sigma$

$\mathrm{tr}\,(\Sigma^\dagger \partial_\mu \Sigma)^2$ can be put into previous form by integration by parts.

So there is a unique term that dominates as $p \to 0$

$$\mathcal{L} = \frac{f^2}{4}\mathrm{tr}\,(\partial^\mu\Sigma^\dagger \partial_\mu \Sigma)$$

Expand in terms of pion fields:

$$\Sigma = e^{2iM/f} \qquad M = \pi^a T^a$$

$$= 1 + \frac{2iM}{f} - \frac{2M^2}{f^2} - \frac{4}{3}\frac{iM^3}{f^3} + \cdots$$

$$\mathcal{L} = \frac{f^2}{4}\mathrm{tr}\Big(-\frac{2i}{f}\partial^\mu M - \frac{2}{f^2}\partial^\mu M^2 + \frac{4}{3}\frac{i}{f^3}\partial^\mu M^3 + \cdots\Big)$$
$$\Big(\frac{2i}{f}\partial_\mu M - \frac{2}{f^2}\partial_\mu M^2 - \frac{4i}{3f^3}\partial^\mu M^3 + \cdots\Big)$$

$$= \mathrm{tr}(\partial^\mu M)^2 + \frac{1}{f^2}\mathrm{tr}(\partial^\mu M^2)^2 - \frac{4}{3f^2}\mathrm{tr}\,\partial^\mu M \partial_\mu M^3 + \cdots$$

$$= \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + \cdots$$

where $\mathrm{tr}\,T^a T^b = \frac{1}{2}\delta^{ab}$

Now we see π's are conventionally normalized scalars. And just this lowest term in Σ gives an ∞ no. of pion vertices -- A nonrenormalizable theory. Is nonrenormaliza-bility a problem; does it impair our ability to calculate? No! Not if we are interested in leading $p \to 0$ behavior. We'll need an ∞ no of arbitrary counterterms, but they can all be expressed in terms of Σ and derivatives. And there is only one term with two derivatives, so we get renormalization of f, plus higher-derivative operators whose effects are suppressed as $p \to 0$, when loop graphs are expanded in powers of p.

What we actually have are an ∞ no. of relations satisfied by scattering amplitudes of pions, all generated in the soft limit by a single term of the chiral Lagrangian.

E.g. consider the case of $SU(2) \times SU(2)$:

$$M^2 = \pi^a \pi^b \tfrac{1}{2}\{T^a, T^b\} = \tfrac{1}{4}\pi^a \pi^a$$

In general, we have

$$\mathcal{L}_{int} = \frac{1}{f^2} \mathrm{tr}\Big[\big((\partial^\mu M) M + M(\partial^\mu M)\big)^2 - \tfrac{4}{3} \partial^\mu M (2\partial_\mu M M^2 + M \partial_\mu M M)\Big]$$

$$= \frac{1}{f^2} \mathrm{tr}\Big[2(\partial^\mu M \partial_\mu M) M^2 + 2(\partial^\mu M) M (\partial_\mu M) M - \tfrac{8}{3}(\partial^\mu M)^2 M^2 - \tfrac{4}{3}(\partial^\mu M) M (\partial_\mu M) M\Big)$$

$$= \frac{2}{3f^2} \mathrm{tr}\Big[(\partial^\mu M) M (\partial_\mu M) M - (\partial^\mu M)(\partial_\mu M) M^2\Big]$$

More direct:

$$\mathcal{L}_{int} = \frac{1}{8f^2}(\partial^\mu \pi^a \pi^a)(\partial_\mu \pi^b \pi^b) - \frac{1}{6f^2}\partial^\mu \pi^a \partial_\mu(\pi^a \pi^b \pi^b) + \cdots$$

$$= \frac{1}{2f^2}(\pi^a \partial_\mu \pi^a)(\pi^b \partial^\mu \pi^b) - \frac{1}{3f^2}(\pi^a \partial_\mu \pi^a)(\pi^b \partial^\mu \pi^b) - \frac{1}{6f^2}(\partial^\mu \pi^a \partial_\mu \pi^a)(\pi^b \pi^b) + \cdots$$

or

$$\boxed{\mathcal{L}_{int} = \frac{1}{6f^2}\left[(\vec{\pi}\cdot\partial^\mu\vec{\pi})^2 - \vec{\pi}^2(\partial^\mu\vec{\pi})^2\right] + \cdots}$$

With this effective Lagrangian, we can derive prediction for soft elastic 2π scattering amplitude.

$$= \frac{i}{6f^2}\Big[2\delta^{ab}\delta^{cd}(-1)(p_a\cdot p_c + p_a\cdot p_d + p_b\cdot p_c + p_b\cdot p_d) - 4\delta^{ab}\delta^{cd}(-1)(p_a\cdot p_b + p_d\cdot p_c) + 2 \text{ other channels}\Big]$$

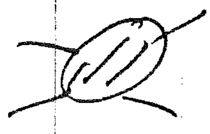
$p_a\cdot p_b = s/2 = p_d\cdot p_c$ (Massless pions)

$p_a\cdot p_c = -t/2 = p_b\cdot p_d$

$p_a\cdot p_d = -u/2 = p_b\cdot p_c$

$$\Rightarrow \; = \frac{i}{6f^2}\left[\delta^{ab}\delta^{cd}(4s - 2(t+u)) + 2 \text{ other channels}\right]$$

and $s + t + u = 0$

$\Rightarrow$  $= \frac{i}{f^2}(\delta^{ab}\delta^{cd}s + \delta^{ac}\delta^{bd}t + \delta^{ad}\delta^{bc}u)$

This is Weinberg's (1966) formula, in the $m_\pi \to 0$ limit. What is remarkable is that this form is completely determined by the chiral symmetry.

The amplitude just computed involves an undetermined parameter f. You may have guessed $f = f_\pi$, measured in e.g. weak decay amplitudes. To show this, we must construct the left-handed current, or axial current, from our chiral Lagrangian, and then find f_π from the definition:

$$\langle 0 | J^\mu_{5a} | \pi_b \rangle = i\delta_{ab} f_\pi p^\mu, \quad f_\pi = 93\,\mathrm{MeV}$$

We find the properly normalized current by the Noether procedure. In general if $\mathcal{L}$ is invariant under linear operator

$$\delta\phi = \epsilon \underset{\text{generic notation}}{A}\phi \quad , \quad \delta\mathcal{L} = 0$$

where ϵ = constant, consider local transformation

$$\delta\phi = \epsilon(x) A\phi(x).$$

Then (see page 3.37 of notes), $\mathcal{L}$ changes by

$$\delta\mathcal{L} = \partial_\mu\epsilon J^\mu,$$

where J^μ is conserved Noether current i.e.

$$\delta\mathcal{L} = \partial_\mu\epsilon\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} A\phi\right) + \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} A\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial\phi} A\phi\right]\epsilon}_{\text{vanishes, because } \delta\mathcal{L}=0 \text{ for } \epsilon = \text{constant}}$$

To find an $SU(n)_L$ current, consider

$$\delta\Sigma = -i\epsilon_a T^a \Sigma \Rightarrow$$

$$\delta\mathcal{L} = f^2/4\, \mathrm{tr}\left[\partial^\mu(\Sigma^\dagger(+i\epsilon_a T^a))\partial_\mu\Sigma + \partial^\mu\Sigma^\dagger \partial_\mu(-i\epsilon_a T^a\Sigma)\right]$$

$$= if^2/4\; \partial_\mu\epsilon_a\, \mathrm{tr}(\Sigma^\dagger T^a \partial^\mu\Sigma - \partial^\mu\Sigma^\dagger T^a\Sigma)$$

$$+ (\text{no deriv of } \epsilon)$$

we have found:

$$J_L^{\mu a} = if^2/4\; \mathrm{tr}[\Sigma^\dagger T^a\partial^\mu\Sigma - \partial^\mu\Sigma^\dagger T^a\Sigma]$$

Expanding $\Sigma = e^{2iM/f}$, we have

$$J_L^{\mu a} = if^2/4\; \mathrm{tr}\Big[\left(1-\tfrac{2iM}{f}\right)T^a\left(\tfrac{2i}{f}\partial^\mu M - \tfrac{2}{f^2}\partial^\mu M^2\right) - \left(-\tfrac{2i}{f}\partial^\mu M - \tfrac{2}{f^2}\partial^\mu M^2\right)T^a\left(1+\tfrac{2iM}{f}\right)\Big]$$

$$+ \cdots$$

$$= -f\, \mathrm{tr}\, T^a\partial^\mu M$$

$$+ i\,\mathrm{tr}(M T^a \partial^\mu M - \partial^\mu M T^a M)$$

$$+ \cdots$$

$$= -f\,\mathrm{tr}\, T^a\partial^\mu M - i\,\mathrm{tr}\, T^a[M, \partial^\mu M] + \cdots$$

(To get RH current, just change sign of all terms odd in π -- the axial pieces.)

Now $J_L^{\mu a} = -\frac{1}{2} f\partial^\mu\pi^a + \cdots$

$$\Rightarrow J_{L5}^{\mu a} = J_L^{\mu a} - J_R^{\mu a} = -f\;\partial^\mu\pi^a + \cdots$$

Finally, we have

$$\langle 0 | J_{5a}^{\mu} | \pi b \rangle = i \delta_{ab} f p^{\mu}$$

— so we have verified $f = f_\pi$

But the expansion of J_L in powers of π has other interesting consequences. It allows us to relate weak interaction amplitudes with different nos. of soft pions.

For example, consider charged Kaon decay. Chiral perturbation theory allows us to relate amplitudes of e.g. the semileptonic decays

$$K \to \mu\nu,\ \mu\nu\pi,\ \ldots$$
$$e\nu,\ e\nu\pi,\ e\nu\pi\pi,\ \ldots$$

For a current-current weak interaction

$$\mathcal{L} \propto (J_\mu^+)_{\text{Hadron}} (J^{\mu -})_{\text{Lepton}} + \text{h.c.},$$

these amplitudes have the form

$$\langle \text{final st. had.} | (J_\mu^+)_{\text{Had}} | K^- \rangle \langle \mu\nu | (J^{\mu -})_{\text{Lep}} | 0 \rangle .$$

The leptonic part is always the same, so the ratio of different amplitudes is ratio of hadronic matrix elements

The left-handed weak current responsible for K^- decay is

$$\bar{u}\gamma^\mu(1+\gamma_5)s$$

The associated $SU(3)$ generator is

$$T = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while the matrix $\mathcal{M}$ can be written

$$\mathcal{M} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \eta/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{2/3}\,\eta \end{bmatrix}$$

Thus

$$\bar{u}\gamma^\mu(1+\gamma_5)s \longrightarrow$$

$$\sqrt{2} f \partial^\mu K^- - i\frac{1}{\sqrt{2}}\left(K^-\partial^\mu\pi^0 - \pi^0\partial^\mu K^-\right)$$

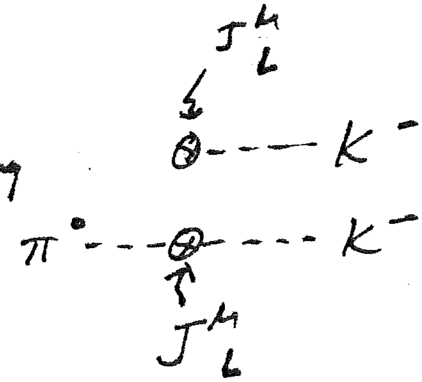
$+$ (other terms not involving $K^- \to \pi^0$)

$+$ (cubic and higher order)

We have amplitudes

$K^- \to \mu^-\nu$ from corresponding

$K^- \to \pi^0\mu^-\nu$ graph



We can also calculate amplitude for emission of another π, $K^- \to \mu^- \nu \left[\begin{matrix}\pi^0\pi^0\\ \pi^+\pi^-\end{matrix}\right.$ but it gets more complicated. We need to include meson interactions as well as current matrix elements:

Exercise:

Calculate

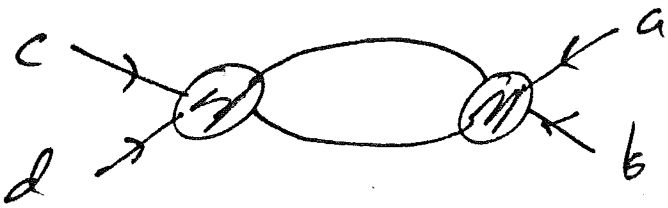
$$\langle \pi^0 \pi^0 | J^\mu(p) | K^- \rangle$$

in the soft limit, using the chiral Lagrangian.

Much more can be done with this chiral perturbation theory. In particular, we can introduce baryons and find relations satisfied by their weak decay amplitudes. I won't go into any of this here. For more, see Georgi, Weak Interactions.

I do want to briefly discuss two further issues concerning chiral perturbation theory. First, what are effects of loop graphs, and second, how do we introduce explicit symmetry breaking (quark masses)?

To discuss loop effects, we return to the example of elastic π-π scattering for purposes of illustration. The one loop diagram that contributes is



+ crossed graphs

where the blob is the four-pion vertex we used earlier in our tree calculation. This graph has a logarithmic ultraviolet divergence, and we need to make a subtraction. We know without doing any explicit computation that the answer will have the form:

$$\frac{1}{f^2}\frac{1}{(4\pi f)^2}\left(\text{Polynomial }^{\text{Quartic}}_{\text{Momentum}}\text{ in}\right)\ln\left(\frac{(\text{mom})^2}{\mu^2}\right)$$

The point is that we have a f^{-2} at each vertex, so momentum must enter quartically by dimensional analysis. (The cutoff dependence is only logarithmic) Where did $\ln \Lambda^2$, the log dependence on cutoff go? Since it multiplies a quartic polynomial in momentum, it can be absorbed into renormalization of a coupling in the chiral Lagrangian that has four derivatives.

changing the renormalization point μ is equivalent to a logarithmic change in the renormalized coupling.

Now, several remarks:

i) Note that the loop correction is of order $(p/f)^4 \ln(p^2)$, so it really is systematically small in our expansion in powers of p, compared to the tree graph, of order $(p/f)^2$.

ii) The contribution to the amplitude that is a polynomial of order $(p/f)^4$ cannot be computed in terms of p and f alone. It involves a new coupling, which is arbitrary, associated with an operator involving more derivatives.

iii) However, in the soft limit $p \to 0$, the leading correction to the $O(p/f)^2$ calculation is not a quartic polynomial, but a quartic polynomial times a logarithm of momentum. This term cannot be generated by a tree graph. It is related (by unitarity) to the order $(p/f)^2$ result, and is generated only by the loop graph. Thus this leading correction actually is computable.

iv) The polynomial piece of the loop graph is suppressed relative to the tree graph by $(p/4\pi f)^2$. This suggests that $p/4\pi f$ is the natural expansion parameter of chiral perturbation theory, which helps explain why soft pion theorems work well, even though f_π is comparable to m_π.

Finally, let's discuss how the effects of finite quark masses are incorporated into the chiral Lagrangian. We have already noted that quark masses will break the degeneracy of the chiral vacuum states, and give mass to the Goldstone bosons. We would like to relate the quark masses to the masses of the Goldstone bosons

The quark masses are a perturbation of the QCD Lagrangian of the form

$$\mathcal{L}_m = -M_{ba}\,\bar{q}_{La} q_{Rb} + h.c.$$

In a chiral vacuum labeled by Σ,

$$\langle \bar{q}_{La} q_{Rb} \rangle_\Sigma = -\Delta \Sigma_{ab},$$

this becomes

$$\mathcal{L}_m = \Delta\left(\mathrm{tr}\, M\Sigma + h.c.\right).$$

Here Δ is a dynamically determined quantity of dimension $(\text{mass})^3$. We can choose a basis in which M is diagonal. In the 3 flavor case, it is

$$\mathcal{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}$$

If we expand to quadratic order

$$\Sigma = 1 + \frac{2i}{f}(\pi\cdot T) - \frac{2}{f^2}(\pi\cdot T)^2 + \cdots$$

$$\Rightarrow \quad 2\Delta \,\mathrm{Re}\, \mathrm{tr}\, \Sigma\mathcal{M} = -\frac{4\Delta}{f^2}\,\mathrm{tr}\left[\mathcal{M}(\pi\cdot T)^2\right]$$

Since $\quad \pi\cdot T = \frac{1}{\sqrt{2}}\begin{bmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & \overline{K^0} & -\sqrt{\tfrac{2}{3}}\,\eta \end{bmatrix}$

we obtain

$$\mathcal{L}_m = -\frac{2\Delta}{f^2}\Big[(m_u+m_d)\left(\frac{(\pi^0)^2}{2} + \pi^+\pi^-\right) + (m_u+m_s)(K^+K^-) + (m_d+m_s)(K^0\overline{K^0}) + \left(\tfrac{1}{3}m_u + \tfrac{1}{3}m_d + \tfrac{4}{3}m_s\right)\left(\tfrac{1}{2}\eta^2\right)\Big]$$

Although Δ/f^2 is not known, we have 3 mass ratios of mesons -- enough to determine ratios of quark masses.

First note that, if we assume $m_u = m_d$ (exact isospin symmetry) we have a <u>relation</u>:

$$3m_\eta^2 + m_\pi^2 = 4m_K^2$$

This is the famous Gell-Mann-Okubo relation. It was originally derived assuming that exact SU(3) sym. was broken by a "medium strong" interaction transforming as an SU(3) octet, conserving charge and isospin. We now understand that this interaction is a quark mass. Chiral perturbation theory also tells us that it is the $(\text{masses})^2$ that satisfy the relation, not masses, a point unclear to GMO.

To extract quark mass ratios accurately, we should take account of SU(3) breaking by electromagnetic effects (which split e.g. $\pi^\pm$ from π^0 by 5 MeV). The electric charge is the $SU(3)_V$ generator:

$$Q = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}.$$

This commutes with a chiral $[SU(2) \times U(1)]_{L,R}$, so $\pi_0, \eta, K_0, \bar{K}_0$ would be exactly massless even in the presence of electrodynamics effects, if the quarks were exactly massless. Furthermore (π^+, K^+) and (π^-, K^-) are doublets under an exact $SU(2)_V$ symmetry preserved by Q, so π, K get equal em contributions to their masses.

Including both em corrections and explicit quark masses, the pseudo-Goldstone bosons have masses---

$$m^2_{\pi^\pm} = C(m_u+m_d) + m^2_{em}$$
$$m^2_{K^\pm} = C(m_u+m_s) + m^2_{em}$$
$$m^2_{K^0} = C(m_d+m_s)$$
$$m^2_{\pi^0} = C(m_u+m_d)$$
$$m^2_\eta = C(\tfrac{1}{3}m_u+\tfrac{1}{3}m_d+\tfrac{4}{3}m_s)$$

There are 5 known masses, in terms of $Cm_{u,d,s}$ and Δm^2_{em}, so all can be determined, and there is one relation. E.g. use

$$m^2_{K^\pm} - m^2_{\pi^\pm} = C(m_s-m_d)$$

and derive

$$Cm_s = \tfrac{1}{2}(m^2_{K^\pm} + m^2_{K^0} - m^2_{\pi^\pm}) = .2359\ \text{GeV}^2$$
$$Cm_d = \tfrac{1}{2}(m^2_{K^0} - m^2_{K^\pm} + m^2_{\pi^\pm}) = .0117\ \text{GeV}^2$$
$$Cm_u = \tfrac{1}{2}(2m^2_{\pi^0} - m^2_{\pi^\pm} + m^2_{K^\pm} - m^2_{K^0}) = .0065\ \text{GeV}^2$$

plus the relation --

$$m^2_\eta = \tfrac{1}{3}(m^2_{\pi^0} + 2m^2_{K^\pm} + 2m^2_{K^0} - 2m^2_{\pi^\pm})$$

or $(549)^2 = (566)^2$ -- reasonable agreement

We don't know C, but we have determined the *ratios* of quark masses:

$$\frac{m_s}{m_d} = 20.2 \qquad \frac{m_d}{m_u} = 1.80$$

Of course, the higher order corrections in chiral perturbation theory could easily modify these ratios by 10-20%

Can we go further and determine the masses themselves? Not very reliably. A common guess is $m_s \sim 100$-150 GeV, based on hadron mass splittings in SU(3) multiplets; then $m_d \sim 5$-7 MeV, $m_u \sim 3$-4 MeV. Note that isospin is a remarkably good symmetry not because m_d/m_u is close to one, but because m_u and m_d are small compared to the QCD scale Λ (and therefore, e.g., the proton mass.)

Chiral Anomalies

Physics 230

Anomaly means symmetry of a classical system fails to survive upon quantization of the system. Anomalies can occur in field theory because of the need for *regularization*. It may be impossible to regulate infinities and at the same time preserve all classical symmetries. (We have already seen this happen in the case of classical scale invariance.)

The simplest example of a chiral anomaly occurs for a massless fermion in (1+1)-dimensions coupled to an abelian gauge field. For a *free* massless fermion, the Lagrange density is

$$\mathcal{L} = \bar{\psi}\, i \not{\partial} \psi = i\psi^\dagger \gamma^0 (\gamma^0 \partial_0 + \gamma^1 \partial_1)\psi$$
$$= i\psi^\dagger \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\gamma_5\right)\psi$$

where "γ_5" $= \gamma^0\gamma^1$, $\gamma_5^2 = I$, gamma matrices are 2×2. We define one-component fermions by

$$\gamma_5 \psi_{R,L} = \pm \psi_{R,L}.$$

Then

$$\mathcal{L} = i\psi_R^\dagger\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\psi_R + i\psi_L^\dagger\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\psi_L,$$

and the general solution to the Dirac equation is

$$\psi_R(x,t) = \psi_R(x-t) \quad \text{-- Right mover}$$
$$\psi_L(x,t) = \psi_L(x+t) \quad \text{-- Left mover}$$

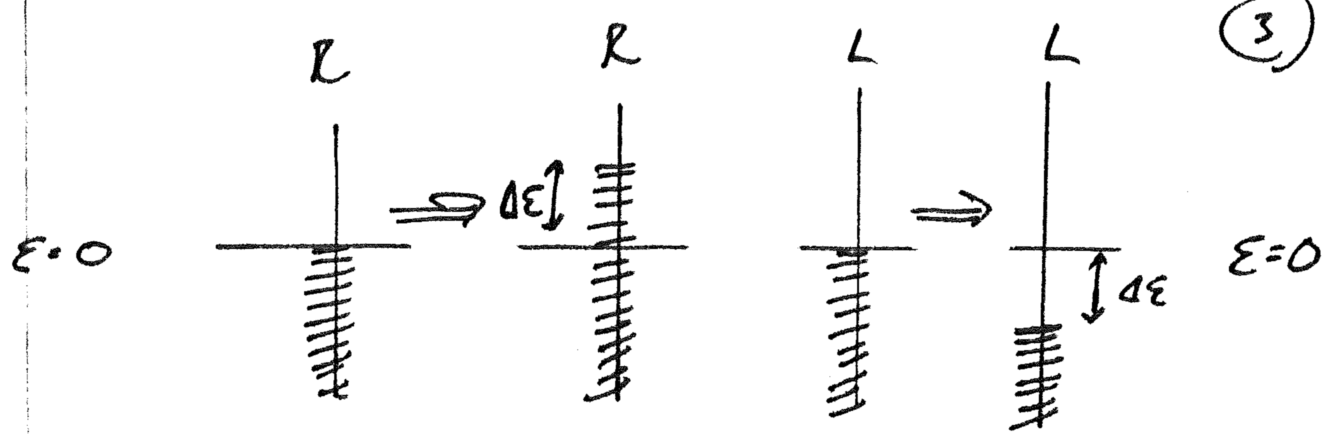
We now see what it means for a fermion to be "chiral" in 1+1 dimensions. Unlike in 3+1 dimensions, chirality makes no reference to helicity; there is no spin in one spatial dimension. Rather, a fermion is said to be right-handed if it propagates only to the right. This definition makes sense only for massless fermions, propagating at velocity c -- direction of propagation is same in all Lorentz frames. And it makes sense only in one spatial dimension, in which no smooth rotational symmetry (only a discrete "parity symmetry" interchanges right and left movers.

In the classical theory, R fermion number and L fermion no are independent constants of the motion; this remains true if the fermions are coupled to an abelian gauge field,

$$\mathcal{L} = \bar{\psi} i \not{D} \psi \qquad D_\mu = \partial_\mu - ieA_\mu$$

But consider now the quantum mechanics of a Dirac fermion (R and L one-component fermions) in an external electric field. If an electric field E pointing right acts for time t, all right-movers gain and left-movers lose energy

$$\Delta E = eEt$$



If the initial state before the Electric field turns on is the Dirac vacuum, with all negative energy levels filled, then after time t, the RH "Fermi level" has increased by ΔE and the LH Fermi level has decreased by ΔE. Thus, R fermions are created, and so are L antifermions (holes). Since the one-dimensional density of states is $dp/2\pi$, the number density per unit length of R and L fermions are

$$\rho_R = \frac{eE}{2\pi} t \qquad \rho_L = -\frac{eE}{2\pi} t$$

Thus, the "vector" fermion number is preserved

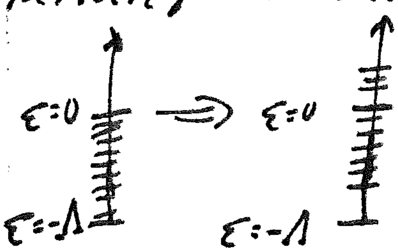
$$\dot{\rho}_V = \dot{\rho}_R + \dot{\rho}_L = 0.$$

But the "axial" fermion number, also conserved at the classical level, changes, according to

$$\dot{\rho}_A = \dot{\rho}_R - \dot{\rho}_L = \frac{eE}{\pi} \qquad \text{-- Axial Anomaly}$$

When an electric field is applied to the vacuum, it suffers dielectric breakdown, and fermion antifermion pairs are produced. The fermions produced are right movers and the antifermions are left movers, so the difference between R no. and L no. increases.

I said that lurking behind the anomaly are the infinities of field theory. Where are the infinities in this discussion? The only infinity is the depth of the Dirac sea.



If we regularize by cutting off the bottom of the sea at $\varepsilon=-\Lambda$, then R fermions are not produced. That is, as many fermions pop out of the sea as holes open up in the sea (the total number of states is finite), so no net R no. is generated. The anomaly occurs only because the number of modes filling the sea is infinite, so a finite number of fermions can check out without any vacancies opening up. (The Dirac sea is a "Hilbert Hotel".)

Does an analogous breakdown of axial charge conservation occur in 3+1 dimensions? The above discussion seems peculiarly one-dimensional, relying as it does on the notion of R mover and L mover.

But the same discussion applies in 3+1 dimensions to massless fermions in a magnetic field. We've seen (page 1.92) that massless fermions in a constant magnetic field occupy Landau levels labeled by integer n, with energy

$$H^2 = p_z^2 + (2n+1)eB - eB\,2S_z$$

(for $\vec{B} = B\hat{z}$ in the $\hat{z}$ direction) the motion in the x-y plane is quantized circular motion, and the fermions propagate along the z axis like effectively (1+1)-dimensional fermions with mass

$$m^2 = (2n+1)eB - eBS_z$$

-- the levels with $n=0$ and $S_z = +\frac{1}{2}$ behave like massless 1+1 dim fermions.

For massless 3+1 dim fermions, chirality = helicity. So the RH fermion with $S_z = \frac{1}{2}$ is a R mover along z-axis, and the LH fermion is a L mover. Now, turn on E field along $\hat{z}$, and our earlier discussion applies. We saw that the density of states per unit area in a Landau level is

$$g_n = \frac{eB}{2\pi}$$

therefore, axial charge is created at a

rate

$$\dot{\rho}_A = \frac{eB}{2\pi}\frac{eE}{\pi} = \frac{e^2}{2\pi^2}\vec{E}\cdot\vec{B} \quad \text{-- Axial Anomaly}$$

An electric field applied to the vacuum causes dielectric breakdown -- i.e. pair production. And if a magnetic field is applied parallel to the electric field, the pairs created are chiral:

Fermions with charge + align their spins with $\vec{B}$, while antifermions (charge −) antialign. The pair has the quantum numbers of

$$\bar{\psi}_R \psi_L$$

and carries $N_R - N_L = 2$.

The connection between the anomaly and the need for regularization is better appreciated if we compute the anomaly by a Feynman diagram method. Consider again a 1+1 dimensional fermion in and external gauge field

$$\mathcal{L} = \bar{\psi} i \not{D} \psi$$

Classically there are two conserved currents:

$$J_\mu = \bar\psi \gamma_\mu \psi = \bar\psi_R \gamma_\mu \psi_R + \bar\psi_L \gamma_\mu \psi_L$$

$$J_{\mu 5} = \bar\psi \gamma_\mu \gamma_5 \psi = \bar\psi_R \gamma_\mu \psi_R - \bar\psi_L \gamma_\mu \psi_L$$

To define matrix elements of these currents, we need a regulator. We might use dimensional regularization, except ~~that~~ it is unclear how to continue γ_5 to $2-\epsilon$ dimensions. Instead, we'll use a Pauli-Villars regulator, chosen because it is gaurenteed to satisfy the conservation of vector current (gauge invariance):

$$\mathcal{L} = \bar\psi i \not{D} \psi + \bar\psi_{reg} [i \not{D} - M] \psi_{reg}$$

wrong statistics $\Rightarrow$ subtracts heavy loops!!

Define:

$$J_{\mu, reg} = \bar\psi \gamma_\mu \psi + \bar\psi_{reg} \gamma_\mu \psi_{reg}$$

$$J_{\mu 5, reg} = \bar\psi \gamma_\mu \gamma_5 \psi + \bar\psi_{reg} \gamma_\mu \gamma_5 \psi_{reg}$$

- then matrix elements of currents are finite.

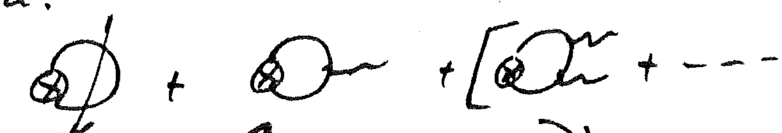
Use of eqn. of motion gives

$$\partial^\mu J_{\mu, reg} = 0$$

$$\partial^\mu J_{\mu 5, reg} = 2iM \bar\psi_{reg} \gamma_5 \psi_{reg}$$

involves only regulator field, but does not vanish compute expectation value of $\partial^\mu J_{\mu 5,reg}$ in external gauge field:

$\langle \partial^\mu J_{\mu 5,reg} \rangle$:



$\downarrow 0$ $\qquad \rightarrow O\left(\frac{1}{M}\right)$ -- power counting

Need to compute this.

$\langle \partial^\mu J_{\mu 5,reg} \rangle$ =

$2iM\,\bar\psi_{reg}\gamma_5\psi_{reg}$

$$= 2iM(i)^2 ie \int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{tr}\left[\gamma_5(\not k+\not p+M)\gamma^\mu(\not k+M)\right]}{(k^2-M^2)\left[(k+p)^2-M^2\right]} A_\mu(p)$$

Evaluate trace using $\mathrm{tr}\,\gamma_5(\underbrace{\gamma \sim \gamma}_{\text{odd number}})=0$

$\mathrm{tr}\,\gamma_5\gamma_\alpha\gamma_\beta = 2\epsilon_{\alpha\beta}$ $\quad (1=\epsilon_{01})$

$\gamma_5=\gamma_0\gamma_1$

$\mathrm{tr}[\quad] = 2M\,\epsilon^{\alpha\mu}p_\alpha$

to keep leading piece of integral for $M\to\infty$, set $p=0$

$$\langle \partial^\mu J_{\mu 5}\rangle = 4eM^2\epsilon^{\alpha\mu}p_\alpha A_\mu(p)\underbrace{\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2-M^2)^2}}$$

Integral $= \frac{i}{4\pi M^2}$ ← from Wick rotation

$$= -\frac{e}{\pi}\epsilon^{\alpha\mu}\left(-ip_\alpha A_\mu(p)\right) = -\frac{e}{\pi}\epsilon^{\alpha\mu}\partial_\alpha A_\mu$$

$$= -\left(e/2\pi\right)\epsilon^{\alpha\mu}F_{\alpha\mu}$$

we obtained

$$\partial^\mu J_{\mu 5} = -(e/2\pi)\epsilon^{\mu\nu}F_{\mu\nu}$$

which agrees with our earlier computation.

Reinterpreting this calculation, we have a statement about theory of free fermions:

$$\int d^2x\, e^{ip\cdot x}\,\partial^\mu \langle 0|T^* J_{\mu 5}(x) J_\nu(0)|0\rangle = \frac{1}{\pi}\epsilon_{\alpha\nu}p^\alpha$$

Or, defining

$$\Gamma_{\mu\nu} = \int d^2x\, e^{ip\cdot x}\langle 0|T^* J_{\mu 5}(x) J_\nu(0)|0\rangle$$

$$\left[\begin{array}{l} p^\mu\Gamma_{\mu\nu} = \frac{i}{\pi}\epsilon_{\alpha\nu}p^\alpha \\ p^\nu\Gamma_{\mu\nu} = 0 \end{array}\right.$$

because Pauli-Villars regularization preserves vector symmetry

This was the result with Pauli-Villars regularization. The regulator broke the axial symmetry, and a finite remnant of the symmetry breaking remained in the $M\to\infty$ limit. But is it conceivable that there is some other regularization scheme that preserves both the vector and axial symmetries.

How does the calculation depend on the choice of regulator? We can always expand a graph in powers of external momentum, and each successive term in the expansion has improved UV behave compared to the preceding term. Thus, after expanding to some finite order, the remainder is independent of the regulator. Therefore, the dependence on the choice of regulator enters in only some polynomial in external momentum. Thus, given a calculation of Γ using one regulator, we can obtain the result of calculating Γ using another regulator by merely adding some polynomial in external momentum. To explore whether there is any choice of regulator for which both

$$p^\mu \Gamma_{\mu\nu} = 0 \text{ and } p^\nu \Gamma_{\mu\nu} = 0$$

are satisfied, we therefore ask whether there is any choice of <u>local</u> counterterm that can be added to Γ that restores both conservation equations.

This search for a local counterterm can be viewed in another way. The need for a regulator to define Γ indicates that Γ is ill-defined, because of the singular nature of a product of currents at short distance $(x=0)$. We need a definition of Γ, some renormalization

convention. But our freedom to redefine Γ order-by-order in perturbation theory is restricted by unitarity, which requires imaginary part of Γ in each order to be related to Γ in lower orders.
The only part of Γ that can be convention-dependent in each order is a piece with no imaginary part -- a <u>local</u> <u>counterterm</u>, or polynomial in momentum.

We had

$$p^\mu \Gamma_{\mu\nu} = \frac{-i}{\pi} \epsilon_{\nu\alpha} p^\alpha \,, \quad p^\nu \Gamma_{\mu\nu} = 0$$

Now consider adding a counterterm

$$\Gamma'_{\mu\nu} = \Gamma_{\mu\nu} + \Gamma^{c.t.}_{\mu\nu}$$

But $\Gamma^{c.t.}_{\mu\nu}$ is required by parity and Lorentz invariance to be a dimensionless "pseudotensor"; the only possibility is

$$\Gamma^{c.t.}_{\mu\nu} = c\, \epsilon_{\mu\nu}$$

Now $$p^\mu \Gamma'_{\mu\nu} = -\frac{i}{\pi} \epsilon_{\nu\alpha} p^\alpha - c\, \epsilon_{\nu\alpha} p^\alpha = 0$$

if $$c = i/\pi$$

But, then,

$$p^\nu \Gamma'_{\mu\nu} = \frac{i}{\pi} \epsilon_{\mu\alpha} p^\alpha$$

we succeed at restoring axial symmetry only at the cost of spoiling vector symmetry.

Now we understand what is really meant by an anomoly -- that there is no choice of local counterterms such that all classical symmetries are preserved.

It is evident that anomolies are always finite, for the infinite part of a Feynman graph is always a polynomial in external momentum, and can be removed by a local counterterm. It is also clear that the piece of $\Gamma_{\mu\nu}$ that contributes to the anomoly is not a polynomial in external momentum. What, precisely, is the structure of it?

The most general possible form consistent with Lorentz invariance and parity is

$$\Gamma_{\mu\nu}(p) = f(p^2)\,\epsilon_{\mu\nu} + g(p^2)\,p_\mu \epsilon_{\nu\alpha} p^\alpha + h(p^2)\,p_\nu \epsilon_{\mu\alpha} p^\alpha$$

The two conditions

$$p^\mu \Gamma_{\mu\nu} = -\tfrac{i}{\pi}\,\epsilon_{\nu\alpha} p^\alpha, \qquad p^\nu \Gamma_{\mu\nu} = 0$$

require

$$-f(p^2) + p^2 g(p^2) = -\tfrac{i}{\pi}$$

$$f(p^2) + p^2 h(p^2) = 0$$

So we have ----

$$\Gamma_{\mu\nu}(p) = f(p^2)\left(\epsilon_{\mu\nu} + \frac{p_\mu}{p^2}\epsilon_{\nu\alpha}p^\alpha - \frac{p_\nu}{p^2}\epsilon_{\mu\alpha}p^\alpha\right)$$
$$- \frac{i}{\pi}\,\frac{p_\mu}{p^2}\epsilon_{\nu\alpha}p^\alpha$$

-- the expression multiplying $f(p^2)$ has both symmetries; the additional term violates the axial symmetry. It indeed is nonpolynomial -- it has a pole at $p^2 = 0$.

It is rather remarkable that we can infer from the anomaly equation that Γ has a $1/p^2$ pole. Since only massless particles can generate a singularity at zero momentum, we see that the anomaly really arises only in theories with massless particles.

What happens if fermions have explicit mass

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \quad ?$$

Then axial symmetry is spoiled even at the classical level

$$\partial^\mu J_{\mu 5} = 2im\bar{\psi}\gamma_5\psi$$

And, in the limit $p \gg m$

$\partial^\mu J_{\mu 5}$  cancels exactly the corresponding regulator loop, so there is no term linear in momentum for $p \to 0$

Schwinger Model

It is interesting to apply the (1+1)-dimensional axial anomoly to (1+1)-dim. massless electrodynamics (Schwinger model). We can solve this model exactly.

Now we include gauge field kinetic term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

In terms of component fields $\psi_{R,L}$

$$\bar{\psi} i \not{D} \psi = i\psi^{+}\left(\frac{\partial}{\partial t} + \gamma_0\gamma_1\frac{\partial}{\partial x} - ieA_0 - ieA_1\gamma_0\gamma_1\right)\psi$$

$$= i\psi_R^{+}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\psi_R + i\psi_L^{+}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\psi_L$$

$$+ (\psi_R^{+}\psi_R + \psi_L^{+}\psi_L)eA_0 + (\psi_R^{+}\psi_R - \psi_L^{+}\psi_L)eA_1$$

Since R charge density propagates only to the right, and L charge density only to the left, we may write

$$\psi_R^{+}\psi_R = \rho_R(x-t)$$
$$\psi_L^{+}\psi_L = \rho_L(x+t)$$

In terms of $\rho_{R,L}$, the vector and axial current can be represented as

$$J^0 = \rho_R(x-t) + \rho_L(x+t)$$
$$J^1 = \rho_R(x-t) - \rho_L(x+t)$$
$$J_5^0 = \rho_R(x-t) - \rho_L(x+t)$$
$$J_5^1 = \rho_R(x-t) + \rho_L(x+t)$$

So if we write
$$\rho_R(z) = \frac{d}{dz} F(z)$$
$$\rho_L(z) = \frac{d}{dz} G(z),$$

then we have
$$J_5^0 = -\frac{\partial}{\partial t}\left[F(x-t) + G(x+t) \right]$$
$$J_5^1 = \frac{\partial}{\partial x}\left[F(x-t) + G(x+t) \right]$$

or, if $\phi(x,t) = F(x-t) + G(x+t)$,
then
$$J_5^\mu = -\partial^\mu \phi$$

and
$$J^0 = J_5{}^1 \qquad \text{or} \quad J_\mu = \epsilon_{\mu\nu} J_5^{\nu}$$
$$J^1 = J_5{}^0 \qquad = -\epsilon_{\mu\nu}\partial^\nu \phi$$

The anomaly equation is
$$\partial_\mu J_5^\mu = -\partial^2 \phi = -\frac{e}{\pi} F_{01}$$

We have replaced the pair of chiral fermions $\psi_{L,R}$ by the scalar ϕ (which propagates both left and right) -- this procedure is called Bosonization.

Now, we can fix the gauge $A_1 = 0$, and the gauge kinetic term is
$$-\tfrac{1}{2}(\partial_1 A_0)(\partial^1 A^0)$$
- A_0 is merely a constrained variable

The constraint equation is

$$-\partial_1{}^2 A_0 = eJ_0 = e\partial_1 \phi$$

or $$F_{01} = -\partial_1 A_0 = e\phi$$

And the anomoly equation becomes

$$\partial^2 \phi = \frac{e^2}{\pi}\phi$$

So ϕ is a free scalar field with mass $e/\sqrt{\pi}$. The massless pole in the fermion - antifermion channel has been "eaten" by the gauge field, and the resulting spectrum contains only a massive scalar.