8/20/85

Differential Geometry & Chiral Anomalies

Tricks for solving consistency condition in $D = 2n$ dimensions

First, to avoid keeping track of $(-i)$ write

$$D_\mu = \partial_\mu + A_\mu$$

$A_\mu = -i T^a A_\mu^a$
(antihermitian generators)

$$F_{\mu\nu} = [D_\mu, D_\nu]$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$(A_\mu \to \Omega A_\mu \Omega^{-1} - \partial_\mu \Omega \Omega^{-1})$
under gauge trans

We can avoid keeping track of commutators by using differential form notation

$$A = A_\mu dx^\mu$$
$$F = dA + A^2$$
$$= (\partial_\mu A_\nu + A_\mu A_\nu) \underbrace{dx^\mu dx^\nu}$$ antisym product
$$= F_{\mu\nu} \tfrac{1}{2} dx^\mu dx^\nu$$

E.g. Bianchi Identity

In commutator language Jacobi $\Rightarrow [D_\sigma [D_\mu, D_\nu]] +$ cyclic $= 0$

or $[D_\sigma, F_{\mu\nu}] + [D_\mu, F_{\nu\sigma}] + [D_\nu, F_{\sigma\mu}] = 0$

In form language,

$$d^2 = 0 \Rightarrow \quad dF = dA\,A - A\,dA$$

caution: (-1) in differentiating product of one forms

Also $[A, A^2] = 0 \Rightarrow$

$$[A, F] = A\,dA - dA\,A$$

Thus $\quad DF \equiv dF + [A, F] = 0$

or, in components $\quad [D_\sigma, F_{\mu\nu}]\, dx^\sigma dx^\mu dx^\nu = 0$

Anomaly in Form Notation:

Abelian (or $U(1)_A$) Anomaly:

minus sign differs from Zumino

$$\left[\begin{aligned} \partial^\mu J_{\mu 5} &= \frac{1}{8\pi^2}\,\frac{1}{2}\,\epsilon^{\mu\nu\lambda\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\lambda\sigma} \qquad \text{(one flavor)} \\ &= \frac{1}{4\pi^2}\,\partial_\mu\,\epsilon^{\mu\nu\lambda\sigma}\,\mathrm{tr}\left(A_\nu\partial_\lambda A_\sigma + \tfrac{2}{3}A_\nu A_\lambda A_\sigma\right) \end{aligned}\right.$$

got rid of $(-i)$

$\mathrm{tr}\,F^2$ is a four form, so write in form notation in terms of dual of J

$${}^*J_5 = \frac{1}{3!}\,\epsilon^{\mu\nu\lambda\sigma}\,J_{\mu 5}\,dx_\nu dx_\lambda dx_\sigma$$

$$\begin{aligned} d^*J_5 &= \frac{1}{4!}\,\epsilon^{\mu\nu\lambda\sigma}\,\partial^\lambda J_{\lambda 5}\,dx_\mu dx_\nu dx_\lambda dx_\sigma \\ &= \partial^\lambda J_{\lambda 5}\,dx_0 dx_1 dx_2 dx_3 \\ &= \frac{1}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}F_{\lambda\sigma}\,\underbrace{\epsilon^{\mu\nu\lambda\sigma}dx^0dx^1dx^2dx^3}_{dx^\mu dx^\nu dx^\lambda dx^\sigma} \end{aligned}$$

minus sign?

and $F = F_{\mu\nu}\,\frac{1}{2}\,dx^\mu dx^\nu$

$$\Rightarrow \mathrm{tr}\,F^2 = \mathrm{tr}\,F_{\mu\nu}F_{\lambda\sigma}\,\frac{1}{4}\,dx^\mu dx^\nu dx^\lambda dx^\sigma$$

$$\Rightarrow \quad d^*J_5 = \frac{1}{4\pi^2}\,\mathrm{tr}\,F^2$$

$$= \frac{1}{4\pi^2}\,d\,\mathrm{tr}\left(AdA + \tfrac{2}{3}A^3\right)$$

Nonabelian Anomaly:

LH current

$$[D_\mu, J^\mu]^a = \frac{1}{24\pi^2}\,\epsilon^{\mu\nu\lambda\sigma}\,\partial_\mu\,\mathrm{tr}\,T^a\left(A_\nu\partial_\lambda A_\sigma + \tfrac{1}{2}A_\nu A_\lambda A_\sigma\right)$$

no $(-i)$

or $\quad D^*J^a = \dfrac{1}{24\pi^2}\,d\,\mathrm{tr}\left(AdA + \tfrac{1}{2}A^3\right)T^a$

smaller by $\frac{1}{6} = \frac{1}{3}$ (crossing) $\times\ \frac{1}{2}$ (LH instead of axial)

But A^3 term coef. is not as easily guessed

In higher ($D=2n$) dimensions, we have, for abelian anomaly

$$d^*J_{D+1} \propto \qquad \Omega_{2n} \equiv \mathrm{Tr}\, F^n$$

we would like to
i) show Ω_{2n} is total derivative
ii) Find form of anomaly satisfying consistency condition in $D=2n$ dimensions

(i) First, note that Ω_{2n} is a closed form, $d\Omega_{2n}=0$. This follows from Bianchi identity

$$d\Omega_{2n} = n\,\mathrm{Tr}\, dF\,F^{n-1}$$
$$= n\,\mathrm{Tr}\big[(\underbrace{DF}_{\to 0 \text{ by Bianchi}}\,F^{n-1}) - \underbrace{[A,F]F^{n-1}}_{\to 0 \text{ by cyclicity of trace}}\big]$$

Thus - "Poincare Lemma"

$$\Omega_{2n} = d\omega^0_{2n-1} \quad \text{(locally)} \qquad \leftarrow \text{ Let's construct } \omega^0_{2n-1}$$

First, construct it for infinitesmal gauge field, then integrate

Consider $A \to A+\delta A$

$$F = dA + A^2 \to F + d(\delta A) + \delta A A + A\delta A$$

Then
$$\delta\Omega_{2n} = n\,\mathrm{Tr}\big([d(\delta A) + \underbrace{\delta A A + A\delta A}_{\to\ \mathrm{Tr}\,\delta A(AF^{n-1} - F^{n-1}A)}]F^{n-1}\big)$$
$$= n\,\mathrm{Tr}\,[d(\delta A)F^{n-1} - \delta A\, dF^{n-1}]$$

Here use Bianchi identity $dF = -[A,F]$
$$\Rightarrow dF^{n-1} = -[A, F^{n-1}]$$

$$\delta\Omega_{2n} = n\, d[\mathrm{Tr}(\delta A\, F^{n-1})] \qquad \text{exact differential}$$

Now, integrate along a path from 0 to A

e.g.
$$A_t = tA$$
$$F_t = t\,dA + t^2 A^2$$

Pure gauge. $A = -dgg^{-1} \Rightarrow \omega_{2n-1} = n\int_0^1 dt\, t^{n-1} (t-1)^{n-1} \ldots (-dgg^{-1})$

$dA = -dgg^{-1}dgg^{-1} = -A^2$

$$= (-1)^n\, n \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} (dg\, g^{-1})^{2n-1}$$

$$= \tfrac{1}{3}\ (n=2),\quad \tfrac{1}{10}\ (n=3)\ \text{etc}$$

(In general change in $\overset{\circ}{\omega}_{2n-1}$ under gauge trans is this closed form + an exact form)

And, thus

$$\Omega_{2n} = n\, d\int_0^1 dt\, t^{n-1}\, \mathrm{tr}\{A(dA + tA^2)^{n-1}\}$$

or
$$\boxed{\omega^0_{2n-1} = n\int_0^1 dt\, t^{n-1}\, \mathrm{tr}[A(dA + tA^2)^{n-1}]}$$

E.g. $n=2$ $(D=4)$

$$\omega_3^0 = 2\int_0^1 dt\,[t\ \mathrm{tr}\, AdA + t^2\,\mathrm{tr}\, A^3]$$
$$= \mathrm{tr}\ AdA + \tfrac{2}{3}A^3$$

$n=3$ $(D=6)$

$$\omega_5^0 = 3\int_0^1 dt\ \mathrm{tr}\,(t^2 A(dA)^2 + t^3 AdAA^2 + t^3 A^3 dA + t^4 A^5)$$

$$\boxed{\omega_5^0 = \mathrm{tr}\,(A(dA)^2 + \tfrac{3}{2}A^3 dA + \tfrac{3}{5}A^5)}$$

we note that $\Omega_{2n} = d\overset{\circ}{\omega}_{2n-1}$ holds locally; $\overset{\circ}{\omega}_{2n-1}$ need not be globally defined (if it were $\int \Omega_{2n} = 0$, by Stokes theorem) cohomology

Forms (characteristic classes) closed but not globally exact whose integrals are 'sensitive' to topologically nontrivial transition functions

(ii) Next, we wish to derive form of anomaly satisfying consistency

$$\delta_u W = G_u = \int \mathrm{tr}\, G_u$$

Consistency $\Rightarrow$ $\delta_u G_v - \delta_v G_u = i\, G_{[u,v]}$

(or eliminate i by writing $\Omega = 1 + u$
$\Rightarrow A \to A - [D, u]$

We use a trick

Consider $\Omega_{2n+2} = d(\omega^0_{2n+1})$

– which is gauge-invariant and (locally) exact

$$\delta_v \Omega_{2n+2} = d\,\delta_v(\omega^0_{2n+1}) = 0$$

why does δ_v commute with d?

($\delta_v dA$ is <u>defined</u> as $d\,\delta_v A$)

ω^0_{2n+1} need not be gauge invariant, but non-invariant piece is closed

Poincare $\Rightarrow$ $\delta_v\, \omega^0_{2n+1} = d\,\omega^1_{2n}$

(superscript 0, 1 signifies order in v

Now, we claim that $G_v = \int \omega^1_{2n}(v)$ satisfies the W-Z condition

Consider compactified D-dimensional space S^D
Extend to D+1 - Ball

But, is it obvious that $W[A]$ is a functional of A on S^D - independent of how A is continued into the ball? (Same as question of topo obstruction?)

$$W[A] = \int_{B^{D+1}} \omega^0_{2n+1}$$

defines a functional of A, assuming no topological obstruction prevent extending gauge field to ball (Simply contract gauge field ?)

(and if differential form of consistency condition is satisfied for such A, it will be satisfied in general)

Now $G_v = \delta_v W[A] = \int_{B^{D+1}} \delta_v \omega^0_{2n+1} = \int_{S^{2n}} \omega^1_{2n}(v)$

Not necessarily the most general solution

So consistency is automatic - G_v is the change of a functional. But note that W cannot be written as a local functional on S^D - only on the ball. If it could be, "anomaly" would be removable with a counterterm.

Now we wish to derive a formula for $\omega^1_{2n}(v)$

see α Str of $n+1$ generators

$$\leadsto \omega^1_{2n} = n(n+1)\int_0^1 dt\,(1-t)\,\mathrm{STr}\, v\, d(A F_t^{n-1})$$

E.g. 4 dim ($n=2$)

$$\omega^1_4 = 6 \int_0^1 dt (1-t)\, \mathrm{STr}\, d[A(t dA + t^2 A^2)]\, v$$

$$= 6\tfrac{1}{6}\mathrm{tr}\, v\, d(AdA) + 6(\tfrac{1}{3}-\tfrac{1}{4})\,\mathrm{tr}\, v\, dA^3$$

$$= \mathrm{tr}\, v\, d(AdA + \tfrac{1}{2}A^3)$$

6 dim $(n=3)$

$$\omega_6^1 = 12\int_0^1 dt(1-t)\,\underset{\substack{\uparrow\\ \text{symmetrization}}}{S}\mathrm{tr}\,v\,d\left[A\left(t^2(dA)^2+2t^3(dA)A^2+t^4A^4\right)\right]$$

$$= 12\;S\mathrm{tr}\,v\,d\left[\left(\tfrac{1}{3}-\tfrac{1}{4}\right)A(dA)^2+2\left(\tfrac{1}{4}-\tfrac{1}{5}\right)dAA^3+\left(\tfrac{1}{5}-\tfrac{1}{6}\right)A^5\right]$$

$$= S\mathrm{tr}\,v\,d\left[A(dA)^2+\tfrac{6}{5}\,dAA^3+\tfrac{2}{5}A^5\right]$$

Derivation of ω^1_{2n}

Consider a finite gauge transformation

$$A \to \Omega A \Omega^{-1} - d\Omega\,\Omega^{-1} = \Omega(A + dv)\Omega^{-1}$$

where $dv = -\Omega^{-1} d\Omega$ — infinitesimal gauge parameter

$$F \to \Omega F \Omega^{-1}$$

$$F_t = tF + (t^2 - t)A^2$$

This is really dv

$$\to t\,\Omega F \Omega^{-1} + (t^2-t)\,\Omega(A+v)^2\Omega^{-1}$$

$$= \Omega F_t \Omega^{-1} + (t^2-t)\,\Omega[Av + vA + v^2]\Omega^{-1}$$

Next, consider gauge transformation of

$$\omega^0_{2n-1} = n\int dt\ \mathrm{tr}\, A F_t^{\,n-1}$$

$$\to n\int dt\ \mathrm{tr}\,(A+v)\left[F_t + \{A,v\}(t^2-t) + v^2(t^2-t)\right]^{n-1}$$

Note: replace $v \to dv$

Expand to linear order in v, using symmetrized trace:

$$n\int dt\ \mathrm{Str}\left[v F_t^{\,n-1} + (t^2-t)(n-1)\,A\{A,v\}F_t^{\,n-2}\right]$$

write

$$\mathrm{Str}\ A(Av + vA)F_t^{\,n-2}$$

$$= \mathrm{Str}\ v\left[2A^2F_t^{\,n-2} - A[A, F_t^{\,n-2}]\right]$$

and note
$$F_t = t\,dA + t^2A^2 \Rightarrow dF_t = t^2(dA\,A - A\,dA)$$
$$= -t[A, F_t]$$
$$\frac{\partial F_t}{\partial t} = dA + 2tA^2$$

$$= n\int_0^1 dt\ \mathrm{Str}\ v\left[F_t^{\,n-1} + (t-1)(n-1)\left(\left(\frac{\partial F_t}{\partial t} - dA\right)F_t^{\,n-2} + A\,dF_t^{\,n-2}\right)\right]$$

$$\delta_v \omega^0_{2n-1} = n \int_0^1 dt\, \mathrm{Str}\, v\left[F_t^{n-1} + (t-1)(n-1)\left(\left(\frac{\partial F_t}{\partial t} - dA\right) F_t^{n-2} A\, d F_t^{n-2}\right)\right]$$

integrate by parts $\int_0^1 dt\, (t-1) \frac{\partial}{\partial t} F_t^{n-1} = - \int_0^1 dt\, F_t^{n-1}$

We are left with

$$n(n-1) \int_0^1 dt\, (1-t)\, \mathrm{Str}[v (dA\, F_t^{n-2} - A\, d F_t^{n-2})]$$

$$= d\omega^1_{2n-2}$$

where $\boxed{\omega^1_{2n-2} = n(n-1) \int_0^1 dt\, (1-t)\, \mathrm{Str}[v\, d(A F_t^{n-2})]}$

Evidently a symmetrized trace of $(n-1)$ generators

Should replace $v \to dv$

$$\delta v\, \omega^0_{2n-1} = n(n-1) \int_0^1 dt\, (1-t)\, \mathrm{Str}\left[(dv)\, d(A F_t^{n-2})\right]$$

Now, can pull this d outside

Summary:

Since $\delta_v \omega^0_{2n-1}(A, F) = \omega^0_{2n-1}(A + dv, F) - \omega^0_{2n-1}(A, F)$

we see that ω^1_{2n} is linear term in expansion of $\omega^0_{2n-1}(A + v)$ in powers of v.

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Anomaly Cancellation in Higher Dimensions

(J. Preskill et al., Phys. Lett. 124B, 209 (1983))

n
⇒ SU(1) . odd
does not change sign

In four dimensions, fermions in real rep have no anomalies — Pauli-Villars regularization is possible. What is the corresponding statement in $D = 2n$ dimensions

γ matrices are $2^n \times 2^n$, $[\gamma^\mu, \gamma^\nu] = 2\eta^{\mu\nu}$

Lorentz transformation:

$$\delta\psi = \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}\psi, \quad \Sigma^{\mu\nu} = \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]$$

How do we construct Lorentz invariant fermion bilinear (mass term)?

$\tilde{\psi}\psi$ is invariant, if $\tilde{\psi} = \psi^T A$

where $A^{-1}(\Sigma^{\mu\nu})^T A = -\Sigma^{\mu\nu}$

-- Because $\tilde{\psi}\psi \to \tfrac{1}{2}\epsilon_{\mu\nu}\psi^T((\Sigma^{\mu\nu})^T A + A\Sigma^{\mu\nu})\psi + \tilde{\psi}\psi$

$$\delta\tilde{\psi}\psi = \tfrac{1}{2}\epsilon_{\mu\nu}\psi^T A(A^{-1}(\Sigma^{\mu\nu})^T A + \Sigma^{\mu\nu})\psi$$

Furthermore, there is a charge conjugate spinor which transforms as ψ does:

$$\psi^c \to C^{-1}\psi^*$$

$\psi^c \to C^{-1}(\Sigma^{\mu\nu})^* C\,\psi^c$ Require $C^{-1}(\Sigma^{\mu\nu})^* C = \Sigma^{\mu\nu}$

A, C always exist, but are not necessarily unique

In $D=2n$ dimensions, can define

$$\gamma_{D+1} = (i)^{n-1}\gamma^0\gamma^1 \cdots \gamma^{D-1}$$

$$\gamma_{D+1}^2 = (-1)^{n-1}\underbrace{(-1)^{\frac{D}{2}(D-1)}}_{\text{anticommutators}}(-1)^{(D-1)} = 1$$

$[\gamma_{D+1}, \Sigma_{\mu\nu}] = 0$, so Dirac spinors split into two IR's of Lorentz group, which are

inequivalent reps of Lorentz group

γ_{D+1} eigenstates, eigenvalues ± 1 (2^{n-1}-dim Weyl spinors) ψ_+, ψ_-

Write invariant bilinears in terms of ψ_+, ψ_-

Find commutation properties of A, C with γ_{D+1}

$$\gamma_{D+1} = 2^n (i)^{n-1} \Sigma^{01}\Sigma^{23} \cdots \Sigma^{D-2,D-1}$$

Thus $A^{-1}(\Sigma^{\mu\nu})^T A = -\Sigma^{\mu\nu}$

$$\Rightarrow A^{-1}(\gamma_{D+1})^T A = (-1)^n \gamma_{D+1}$$

and $C^{-1}(\Sigma^{\mu\nu})^* C = \Sigma^{\mu\nu} \Rightarrow$

$$C^{-1}(\gamma_{D+1}^*) C = (-1)^{n-1}\gamma_{D+1}$$

Now, define projections $P_{(\pm)} = \frac{1}{2}(1 \pm \gamma_{D+1})$, $\psi_\pm = P_{(\pm)}\psi$

$$\tilde{\psi} = \psi^T A \Rightarrow \tilde{\psi}_\pm = \psi^T P(\pm)^T A = \tilde{\psi} A^{-1} P(\pm)^T A$$
$$= \tilde{\psi} P(\pm(-1)^n)$$

Mass Term is $\tilde{\psi}_+\psi_+$, $\tilde{\psi}_-\psi_-$ – n even

$\tilde{\psi}_+\psi_-$, $\tilde{\psi}_-\psi_+$ – n odd

As for charge conjugation ...

$$(\psi_\pm)^c = C^{-1}(\underline{P}(\pm))^* \psi^* = C^{-1}\underline{P}(\pm)^* C\, \psi^c$$
$$= \underline{P}(\pm(-1)^{n-1})\, \psi^c$$

$\psi_\pm^c \sim \psi_\pm$ n odd

$\psi_\pm^c \sim \psi_\mp$ n even — charge conjugation flips chirality

Additional mass terms

$\psi_-^c \psi_+ \quad \psi_+^c \psi_-$ n even

$\psi_-^c \psi_+ \quad \psi_+^c \psi_-$ n odd

For n odd, we can make a mass term only by coupling fermions of opposite chirality

Pauli Villars regulator exists only if $+,-$ fermions are in identical reps

To summarize: $D = 2n$ dimensions

n even: ψ_- in rep R is equivalent to $\psi_+ \sim \psi_-^c$ in rep $\overline{R}$

n odd: ψ_- in rep R is equivalent to $\psi_- \sim \psi_-^c$ in rep $\overline{R}$

Anomaly $\propto K\, \epsilon^{n+1}$ is same for $R, \overline{R}$ and cancellation occurs only between ψ_- and ψ_+

Note: A symmetric in $D = 8K$ dimensions $\Rightarrow$ no Majorana mass allowed by Fermi stat. for a single Weyl fermion)

Consider LH Weyl fermions ψ_+
coupled to gravity in $D=2n$ dimensions

i) n even

$\Rightarrow$ there is a Lorentz invariant mass term $\tilde{\psi}_+ \psi_+$
(Provided no of fermions is even or A is antisymmetric)
Thus Pauli-Villars regularization is possible,
and there are no one-loop anomalies spoiling
general covariance

(Related: CPT changes sign of γ_{D+1} — particle and
antiparticle have opposite chirality — so couplings
to gravity are necessarily nonchiral)

ii) n odd

$\Rightarrow$ No Lorentz invariant mass term $\Rightarrow$ No Lorentz
invariant regulator $\Rightarrow$ Anomalies
("Gravitational Anomalies") can (and do) occur.
Energy momentum tensor $T_{\mu\nu}$ not conserved,
reflecting failure of general covariance.

(Related: CPT preserves sign of γ_{D+1} — particle and
antiparticle have same chirality,
and couplings to gravity can be chiral)

See L. Alvarez-Gaume and E. Witten,
Nucl. Phys. B234 (1984) 269.

(For $D=8K$ or $8K+1$, the dimensions in which
P-V regularization does not work for a single Weyl
fermion, there is a sign ambiguity in the effective
action, and a resulting nonperturbative
anomaly analogous to the SU(2) anomaly
in four dimensions)

<u>Anomaly Cancellation</u>

Recall 4 dim

$$\mathrm{tr}\{\lambda_R^{a}\lambda_R^{b}\lambda_R^{c}\} = A(R)\, d^{abc}$$ ← same tensor for all reps R

Thus, anomaly cancellation is <u>one</u> algebraic condition on the fermion representation

This happens because there is only one completely symmetric tensor with 3 adjoint indices (3rd order Casimir)

In general there may be more such invariants — a <u>leading</u> (irreducible) Casimir + <u>non leading</u> (reducible) Casimirs

E.g. 6 dimensions

$$S\,\mathrm{tr}(\lambda_R^{a}\lambda_R^{b}\lambda_R^{c}\lambda_R^{d}) = A_1(R)\, d^{abcd} + A_2(R)\, S(\delta^{ab}\delta^{cd})$$

i.e. $\mathrm{tr}\, F^4$ and $(\mathrm{tr}\, F^2)^2$ are independent invariants

There are thus as many algebraic conditions on the fermion rep as there are independent invariants in the sym. product of $(n+1)$ adjoint reps

In general, the number of irreducible Casimirs is equal to rank of group (dimension of Cartan subalgebra = no of independent eigenvalues of a diagonal generator)

E.g. $SU(N)$ rank $n-1$
Casimirs of order $2, 3, 4, \ldots, N$

Number of Conditions:
If l_i = rank of ith Casimir,
no of ways of choosing a_i such that
$$\sum_i a_i l_i = n+1$$

i.e. no of "partitions" of $n+1$ into integers $2, \ldots, N$

E.g.	$n=2$	$N \geq 3$	1 partition (3)
	$n=3$	$N \geq 4$	2 partition (4, 2+2)
	$n=4$	$N \geq 5$	2 partitions (5, 3+2).
	$n=5$	$N \geq 6$	4 partitions (6, 4+2, 3+3, 2+2+2)
	$n=6$	$N \geq 7$	4 partitions (7, 5+2, 4+3, 2+2+3)

etc

E.g., in 10 dimensions, express anomaly of rep R of $SU(N)$ in terms of anomalies of defining rep.

$$S\,tr\,\lambda_R^{a_1} \cdots \lambda_R^{a_6} = S\big[A_6\, tr\,\lambda^{a_1}\cdots\lambda^{a_6} + A_6^{2,4}\, tr\,\lambda^{a_1}\lambda^{a_2}\, tr\,\lambda^{a_3}\cdots\lambda^{a_6}$$
$$+ A_6^{3,3}\, tr\,\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\, tr\,\lambda^{a_4}\lambda^{a_5}\lambda^{a_6}$$
$$+ A_6^{2,2,2}\, tr\,\lambda^{a_1}\lambda^{a_2}\, tr\,\lambda^{a_3}\lambda^{a_4}\, tr\,\lambda^{a_5}\lambda^{a_6}\big]$$

These anomalies have been computed in special cases --

S. Okubo + J. Patera, PRD 31, 2669 (1985).
T. Kephart, PL 151B, 269 (1985).

Are these nonleading anomalies genuine? i.e., can they be removed with a local counterterm?

E.g., in eight dimensions, there is a nonleading abelian anomaly

$$(\mathrm{tr}\, F^2)^2$$

Associated with it is a consistent nonabelian anomaly in 6 dimensions:

$$(\mathrm{tr}\, F^2)^2 = d(\omega_3^0\, \mathrm{tr}\, F^2) \qquad \text{since } d\,\mathrm{tr}(F^2) = 0$$

and $\delta_v\, \omega_3^0\, \mathrm{tr}\, F^2 = d(\omega_2^1\, \mathrm{tr}\, F^2)$

$$G_u = \int \omega_2^1\, \mathrm{tr}\, F^2$$

An obvious suggestion is, introduce counterterm

$$\int \omega_3^0\, \omega_3^0$$

then $\delta_v \int \omega_3^0 \omega_3^0 = \int (d\omega_2^1\, \omega_3^0 + \omega_3^0\, d\omega_2^1)$

integrate by parts $= \int (\omega_2^1\, \mathrm{tr}\, F^2 - \mathrm{tr}\, F^2\, \omega_2^1) = 0$

crucial minus sign

Doesn't work

cf $\mathrm{tr}\, dA\, \mathrm{tr}(dA)^4$; $\mathrm{tr}(dA)^3\, \mathrm{tr}\, dA^2$ (Twice as many perms)

determined by Bose sym? No - each is Bose sym

Similarly

$$\mathrm{tr}\, F^2\, \mathrm{tr}\, F^4 \to (a\, \omega_2^1\, \mathrm{tr}\, F^4 + b\, \omega_6^1\, \mathrm{tr}\, F^2)$$

in ten dimensions

and $\delta_v \int \omega_3^0 \omega_7^0 = \int (\omega_2^1\, \mathrm{tr}\, F^4 - \mathrm{tr}\, F^2\, \omega_6^1)$

— cannot change $a + b$

(Difference $a - b$ is convention dependent)

To cancel nonleading anomaly, need something new (Cf. Schwarz + Green, PL 149B, 117 (1984))

The Schwarz-Green trick is to introduce a two-form B with the gauge transformation property

$$\delta_u B = -\omega^1_{2,\mathrm{Tr}_R} \text{ for Yang-Mills} \quad (dB + \omega^0_{3,\mathrm{Tr}} \text{ is invariant})$$

Thus

$$\delta_u\, B\, \mathrm{tr} F^4 = -\omega^1_{2,\mathrm{Tr}}\, \mathrm{tr} F^4$$

– and a linear combination of

$$B\, \mathrm{tr} F^4 , \quad \omega^0_3\, \omega^0_7$$

can cancel the nonleading gauge anomaly associated with $\mathrm{tr} F^2\, \mathrm{tr} F^4$

Thus, only $\mathrm{tr} F^6$ anomaly needs to cancel, for gauge-invariance to be assured

Gravitational Anomalies

Also have a structure related to abelian anomaly in $D+2$ dimensions $\propto \mathrm{tr} R^{n+1}$

(R = Tensor valued curvature 2-form)

10 dimensions – 3 independent invariants

$$\mathrm{tr} R^6 , \quad \mathrm{tr} R^2\, \mathrm{tr} R^4 , \quad (\mathrm{tr} R^2)^3$$

(why no $\mathrm{tr} R^3$? No odd order Casimirs for $O(n)$)

Fields with gravitational anomalies

3 types: spin $\frac{1}{2}$, spin $3/2$, self-dual Antisym tensor — But only two are linearly independent, so anomaly cancellation is possible

E.g. $\quad (\frac{1}{2})^- + (\frac{3}{2})^+ + (A)^+$ has cancelling anomalies
— low energy limit of type IIB string theory

In 14 or more dimensions, cancellation of gravitational anomalies is impossible (except the trivial case: nochiral theory)

The Green-Schwarz trick can be applied to gravitational anomalies as well. In fact, in low energy limit of type I superstring there is two form B such that

$$\delta_\alpha B = -\omega^1_{2,Y} \quad (\text{Yang-Mills})$$
$$\delta_\epsilon B = \omega^1_{2,L} \quad (\text{Lorentz})$$

these are consistent 2 dim anomalies, defined by

$$d\omega^1_{2,Y} = \delta_\alpha \omega^0_{3,Y} \qquad d\omega^0_{3,Y} = \operatorname{tr} F^2$$
$$d\omega^1_{2,L} = \delta_\epsilon \omega^0_{3,L} \qquad d\omega^0_{3,L} = \operatorname{tr} R^2$$

Thus $\quad H = dB + \omega^0_{3,Y} - \omega^0_{3,L}$ — transforms covariantly

Nonleading abelian gravitational anomaly
$\operatorname{tr} R^2 \operatorname{tr} R^4$ has consistent form

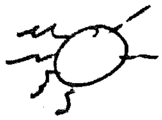
$$a\, \omega^1_{2,L} \operatorname{tr} R^4 + b\, \omega^1_{6,L} \operatorname{tr} F^2$$

Linear combination of $B \operatorname{tr} R^4 \quad \omega^0_{3,L}\, \omega^0_{7,L}$ cancels anomaly

$\operatorname{tr}(R^2)^3$ has consistent form
$$\omega^1_{2,L} \operatorname{tr}(R^2)^2$$
cancelled by $\quad B(\operatorname{tr} R^2)^2$

So only cancellation of $\operatorname{tr} R^6$ anomaly is nontrivial

Mixed Anomalies



Hexagon with external graviton, gluon lines

12 dimensional abelian anomalies:

$$\mathrm{tr}R^2\,\mathrm{tr}F^4 \qquad \mathrm{tr}R^2(\mathrm{tr}F^2)^2$$
$$\mathrm{tr}F^2\,\mathrm{tr}R^4 \qquad \mathrm{tr}F^2(\mathrm{tr}R^2)^2$$

Consistent anomalies

Gauge: $\omega^1_{6,Y}\,\mathrm{tr}R^2$, $\omega^1_{2,Y}\,\mathrm{tr}F^2\,\mathrm{tr}R^2$ $\quad \omega^1_{2,Y}\,\mathrm{tr}R^4$

$\omega^1_{2,Y}\,\mathrm{tr}(R^2)^2$

Lorentz: $\omega^1_{6,L}\,\mathrm{tr}F^2$, $\omega^1_{2,L}\,\mathrm{tr}F^2\,\mathrm{tr}R^2$ $\quad \omega^1_{2,L}\,\mathrm{tr}F^4$

$\omega^1_{2,L}(\mathrm{tr}F^2)^2$

Possible counterterms, <u>not</u> involving B

$$\left.\begin{matrix}\mathrm{tr}R^2\\ \mathrm{tr}F^2\end{matrix}\right\}\ \omega^0_{3,L}\,\omega^0_{3,Y} \to \left(\omega^1_{2,Y}\,\mathrm{tr}R^2 - \omega^1_{2,L}\,\mathrm{tr}F^2\right)\ \begin{matrix}\mathrm{tr}R^2\\ \mathrm{tr}F^2\end{matrix}$$

$$\omega_{3,L}\,\omega_{7,Y} \to \omega^1_{2,L}\,\mathrm{tr}F^4 - \omega^1_{6,Y}\,\mathrm{tr}R^2$$
$$\omega_{3,Y}\,\omega_{7,L} \to \omega^1_{2,Y}\,\mathrm{tr}R^4 - \omega^1_{6,L}\,\mathrm{tr}F^2$$

So far, 4 possible counterterms for 8 anomalies

If we also introduce the two form B, $\delta B = \omega^1_{2,L} - \omega^1_{2,Y}$

there are counterterms

$B(\mathrm{tr}F^2)^2$ $\quad B\,\mathrm{tr}F^4 \to (\omega^1_{2,L} - \omega^1_{2,Y})\,\mathrm{tr}F^4$

$B(\mathrm{tr}R^2)^2$ $\quad B\,\mathrm{tr}R^4 \to$ " $\quad \mathrm{tr}R^4$

we already fixed these, to cancel gauge, grav anomalies

$B\,\mathrm{tr}F^2\,\mathrm{tr}R^2 \to$ " $\quad \mathrm{tr}F^2\,\mathrm{tr}R^2$ ← one new counterterm

what about?

Ignore these

$\mathrm{tr}F^2\,\omega^0_{3,L}\,\omega^0_{3,L} \to \omega^1_{2,L}\,\mathrm{tr}R^2\,\mathrm{tr}F^2$

$\mathrm{tr}R^2\,\omega^0_{3,Y}\,\omega^0_{3,Y} \to \omega^1_{2,Y}\,\mathrm{tr}R^2\,\mathrm{tr}F^2$

Now we have 5 counterterms

<u>NO</u>, this doesn't work — Variation vanishes as on page (14)

we should really do the counting differently. there are four distinct anomalies, each with 2 consistent forms

$$\mathrm{tr} R^2 \,\mathrm{tr} F^4 \rightarrow \quad \omega^1_{2,L}\,\mathrm{tr} F^4,\ \mathrm{tr} R^2 \omega^1_{6,Y}$$
$$\mathrm{tr} R^2 (\mathrm{tr} F^2)^2 \rightarrow \quad \omega^1_{2,L}(\mathrm{tr}F^2)^2,\ \mathrm{tr} R^2 \omega^1_{2,Y}\,\mathrm{tr} F^2$$
$$(\mathrm{tr} R^2)^2\,\mathrm{tr} F^2 \rightarrow \quad \omega^1_{2,L}\,\mathrm{tr} R^2\,\mathrm{tr} F^2,\ (\mathrm{tr} R^2)^2 \omega^1_{2,Y}$$
$$\mathrm{tr} R^4\ \mathrm{tr} F^2 \rightarrow \quad \omega^1_{6,L}\,\mathrm{tr} F^2,\ \mathrm{tr} R^4 \omega^1_{2,Y}$$

None of the four counterterms affects the overall normalization of these anomalies, just the relative normalization of the two terms

$B\,\mathrm{tr} F^2\,\mathrm{tr} R^2$ is the only "real" counterterm available — So we have one counterterm for 4 anomalies

By introducing the twoform we (i.e. Green-Schwarz) reduce the number of algebraic conditions to 5:

Leading Grav. Anomaly
Leading Gauge Anomaly
Mixed Anomalies (3 conditions)

Remarkably, all conditions are satisfied in the Green-Schwarz (Type I superstring) theory, with Yang-Mills gauge group either SO(32) or $E_8 \times E_8$. (Actually must be "heterotic string" to gauge $E_8 \times E_8$

The Wess-Zumino Effective Lagrangian

Back on page 5, we derived an expression for the fermion effective action as an integral over a $2n+1$ dimensional disk with compactified Euclidean space S^{2n} as its boundary

$$W[A] = c\int_{B^{2n+1}} \omega^{0}_{2n+1}(A)$$

This expression has the right variation under an infinitesimal gauge transformation. We would like to explore further the consequences of this formula.

If we consider the change in $W[A]$ under a finite gauge transformation Ω, we write

$$\Gamma(\Omega, A) = W[A^{\Omega}] - W[A]$$

$$= c\int_{B^{2n+1}} [\omega^{0}_{2n+1}(A^{\Omega}) - \omega^{0}_{2n+1}(A)]$$

e.g. $G=SU(N)$ $N \geq n+1$

(It is implicit that $\pi_{2n}(G)=0$, so that we can contract the gauge transformation on S^{2n} to the disk) If we know regard $\Omega(x)$ as a dynamical variable, this expression can be regarded as a local functional in gauge fields and Goldstone boson fields parametrized nonlinearly by $\Omega = \exp(2i\pi^a\lambda^a/f)$

formally locally exact, but not

E.g., if we gauge an anomaly-free subgroup of $SU(N)_L \times SU(N)_R$ and regard Ω as a $SU(N)_L$ transformation, $\Gamma(\Omega, A)$ has local $SU(N)_L \times SU(N)_R$ symmetry. It is called the Wess-Zumino effective Lagrangian.

globally exact (Expand in powers of π ⇒ local)

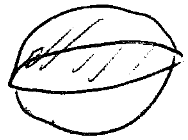
E.g. suppose we set $A = 0$. Then we have an effective action with the local symmetry

$$\Omega(x) \to \Omega'(x)\,\Omega(x)$$

which can be written

$$\Gamma(\Omega) = c' \int_{B^{2n+1}} \mathrm{tr}(\Omega^{-1} d\Omega)^{2n+1},$$

recalling the form of the Chern-Simons form for a "pure gauge" field configuration (page 4). Here we note a possible ambiguity, since this form $(\Omega^{-1} d\Omega)^{2n+1}$ is closed but not exact. We can choose the disk B^{2n+1} in more than one way; consistency requires



$$c' \int_{S^{2n+1}} \mathrm{tr}(\Omega^{-1} d\Omega)^{2n+1} = 2\pi\, m$$

(cf magnetic monopole). In fact, the normalization of the anomaly found in perturbation theory satisfies this condition - m turns out to be the no. of colors in the case of QCD. (See E. Witten, Nucl. Phys. B223 (1983) 422.)

With $\Omega = e^{2i\pi/f}$, we have $\Omega^{-1} d\Omega \propto d\pi$ + higher order, and, expanded in powers of π, Γ becomes

$$\Gamma = c' \int_{B^{2n+1}} d\,\mathrm{tr}\left(\pi (d\pi)^{2n}\right)$$

$$= c' \int_{S^{2n}} \mathrm{tr}\, \pi (d\pi)^{2n}$$

which, in the case of four dimensions becomes

$$\propto \int d^4x\, \epsilon^{\mu\nu\lambda\sigma} \mathrm{tr}\, \pi\, \partial_\mu \pi\, \partial_\nu \pi\, \partial_\lambda \pi\, \partial_\sigma \pi$$

In the case of spontaneously broken $SU(3)_L \times SU(3)_R$, this effective Lagrangian describes couplings

$$K^+ K^- \to \pi^+ \pi^- \pi^0$$

These couplings, like $\pi \to \gamma\gamma$, are generated by the anomaly

We could reintroduce an electromagnetic gauge field to infer other anomalous couplings, not only $\pi \to \gamma\gamma$, but also $\gamma \to 3\pi$, etc. (In fact, we could derive the Wess-Zumino Lagrangian by starting with the $\pi\gamma\gamma$ coupling, then integrating the consistency condition to find a consistent anomaly.

Topological Interpretation of the Nonabelian Anomaly

We've found connection between Nonabelian chiral anomaly in D dimensions and abelian anomaly in D+2 dimensions by purely algebraic means. Is there a deeper mathematical connection lurking here? In particular, can we relate nonabelian anomaly to index theorem? (Alvarez-Gaume and Ginsparg, Nucl. Phys B (1984))

Define effective action

$$e^{-\Gamma[A]} = \det[i\hat{D\!\!\!/}_{2n}(A)]$$

$\hat{D\!\!\!/}_{2n} = \gamma^\mu(\partial_\mu + \frac{1}{2}(1+\gamma_{D+1})A_\mu)$ Gauge field couples to LH fermions only.

(Well-defined eigenvalue problem, although $D\!\!\!/$ not hermitian)

Compactify to S^{2n} coordinates $x^1 \dots x^{2n}$

Suppose $\pi_{2n+1}(G) \supset Z \Rightarrow$

$\exists$ noncontractible loop $g(\theta, x)$ $\theta \in [0, 2\pi]$

Consider loop of gauge fields

$$A^\theta = g^{-1}(\theta,x)(A + d_x)g(\theta,x)$$

$(d_x = dx^\mu \frac{\partial}{\partial x^\mu})$ Now we show $\det[i\hat{D\!\!\!/}_{2n}(A^\theta)]$ has nontrivial winding number around the loop $\Longrightarrow$ a topological reason for $\Gamma[A]$ to fail to be gauge invariant

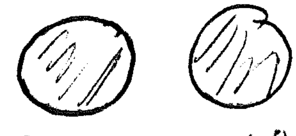
Extend $\mathcal{A}$ to $2n+2$ dimensional gauge field on disk bounded by $S^{2n}_x \times S^1_\theta$

A_+

$$\mathcal{A} = t\, g^{-1}(\theta,x)\,(d_x + d_\theta)\, g(\theta,x) \qquad t \in [0,1]$$

$2n+2$ dimensional Dirac operator $i\not{D}_{2n+2}(\mathcal{A})$ has index given by Atiyah-Singer theorem.

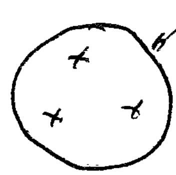
(On boundary $S^{2n}_x \times S^1_\theta$, g can be thought of as transition function of nontrivial bundle – then $\mathcal{A} = A$ on "lower patch"



$$\mathrm{ind}[i\not{D}_{2n+2}(\mathcal{A})] \propto \int_{D^2\times S^{2n}} \mathrm{tr}\, \mathcal{F}^{n+1}$$

$$= \int_{S^1\times S^{2n}} \omega^0_{2n+1}(A^\theta + g^{-1} d_\theta g, F^\theta)$$

Now, this index can be shown to be the winding number of $\det[i\not{D}_{2n}(A^\theta)]$ (Alvarez-Gaumé and Ginsparg, op. cit.



winding no. = sum of winding nos of (isolated) zeros of $\det\not{D}_{2n}$ inside disk

(use adiabatic argument similar to that relating zero modes of $\not{D}_{D+1}$ to modes which cross zero of $\not{D}_D$)

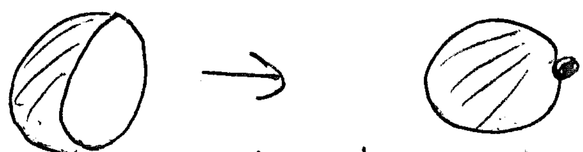
If we write $g^{-1} d_\Theta g = v \, d\Theta$ (infinitesimal gauge trans)

$$\operatorname{ind}(i \not{D}_{2n+2}) = \int_{S^1} d\Theta \int_{S^{2n}} \omega^1_{2n}(v, A^\Theta, F^\Theta)$$

(expand to linear order in $d\Theta$)

$\Rightarrow \delta_v \Gamma \sim \int \omega^1_{2n}$, as we found earlier

Another way to say what we have found:



Our disk in $\mathcal{Q}^{(2n)}$ (space of gauge field configurations) is a sphere in $\mathcal{Q}^{(2n)}/\mathcal{G}^{(2n)}$ — configuration space of gauge equivalence classes. We find that $e^{-\Gamma}$ cannot be smoothly defined on this two-sphere (A "monopole bundle")

But we had to assume $\pi_{2n+1}(G) \supset \mathbb{Z}$

Not always true when there are anomalies in perturbation theory —

E.g. $\pi_5(U(1)) = 0 = \pi_4(SU(3))$

However --- (conjecture)

$\pi_{2n+1}(G) \supset \mathbb{Z}$ iff G has an order $(n+1)$ Casimir invariant

We have thus found a topological basis for irreducible anomalies

In general, should choose a compactification suited to form of anomaly.

E.g. $SU(3)$ in $D=6$ anomaly $\sim$ $\mathrm{tr}\, \lambda^a \lambda^b\, \mathrm{tr}\, \lambda^c \lambda^d$

Compactify $\longrightarrow$ $S_x^4 \times S_y^2$

Consider
$$A^\theta = \underbrace{g^{-1}(\theta, y)}_{\text{winding no.}} [\, \underbrace{A_1(x)}_{\text{instanton}} + d_y]\, g(\theta, y)$$

$$\mathrm{ind}(i \not{D}_8) \sim \int_{S^4 \times S^2 \times S^1} \omega_2{}^0\, \mathrm{tr}\, F^2$$

consistent with

$$\delta_u \Gamma \propto \int \mathrm{tr}\, F^2 \omega_2{}^1$$

– consistent form of anomaly

<u>All anomalies</u> can be interpreted as a topological obstruction to globally defining $e^{-\Gamma}$ on configuration space $\mathcal{A}/\mathcal{G}$

Chiral Solitons — Lectures by H. Sonoda.

§1. Chiral Lagrangian

Ref. ph 234 lecture notes

Let's consider QCD with massless quarks. As we learned in the second term, the strong interaction induces chiral symmetry breaking. The low energy sector of the theory consists of Goldstone bosons.

For n massless flavors, the phenomenological Lagrangian is written as

$$\mathcal{L} = \frac{f^2}{8} \mathrm{Tr}\, \partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger \quad + \quad \text{higher derivative terms}$$

Σ is $SU(n)$ matrix, which transforms like

$$\Sigma \to L \Sigma R^\dagger$$

under $SU(n)_L \times SU(n)_R$. In terms of Goldstone boson fields, π^a

$$\Sigma = \exp\left(\frac{\sqrt{2}i}{f} \pi^a \lambda^a \right).$$

Experimentally ($\pi \to \mu \bar{\nu}$), f is given by ~ 130 MeV.

$\mathcal{L}$ is invariant under $SU(n)_L \times SU(n)_R$. Higher derivative terms are suppressed by powers of $\frac{E}{f} \ll 1$.

(E is the typical energy of the interaction.)

§2. Derivation of the Anomalous Baryon Number

Let us consider QCD with n massless flavors :

$$\mathcal{L} = -\frac{1}{4} F^2 + i \bar{\Psi} \not{D} \Psi .$$

The symmetry of this Lagrangian is

$$G_f = SU(n)_L \times SU(n)_R \times U(1)_V .$$

We can introduce the external gauge fields iA_μ^L, iA_μ^R, iB_μ which couple to G_f currents :

$$\mathcal{L} = -\frac{1}{4} F^2 + i \bar{\Psi} \not{D} \Psi$$
$$- \bar{\Psi}_L \gamma^\mu A_{\mu L} \Psi_L - \bar{\Psi}_R \gamma^\mu A_{\mu R} \Psi_R - \bar{\Psi} \gamma^\mu B_\mu \Psi .$$

The functional integral

$$e^{i\Gamma[iA_L, iA_R, iB]} = \int [d\Psi d\bar{\Psi}]\, e^{i\int \mathcal{L}\, d^4x}$$

is not invariant under gauge transformations of the G_f external gauge fields. For infinitesimal transformations

$$\delta A_{\mu,L} = -\partial_\mu \varepsilon_L + i[\varepsilon_L, A_\mu^L]$$
$$\delta A_{\mu,R} = -\partial_\mu \varepsilon_R + i[\varepsilon_R, A_\mu^R]$$
$$\delta B_\mu = -\partial_\mu \varepsilon ,$$

the change of Γ is given by

$$\delta\Gamma = \int d^4x \Big[\quad \langle \bar\Psi_L \gamma^\mu (\partial_\mu \varepsilon_L + i [A^L_\mu, \varepsilon_L]) \psi_L \rangle$$
$$+ \langle \bar\Psi_R \gamma^\mu (\partial_\mu \varepsilon_R + i [A^R_\mu, \varepsilon_R]) \psi_R \rangle$$
$$+ \langle \bar\Psi \gamma^\mu \partial_\mu \varepsilon \Psi \rangle \Big]$$

$$= - \int d^4x \Big[\varepsilon^a_L (\partial_\mu \delta^{ac} + f^{abc} A^{L,b}_\mu) \langle \bar\Psi_L \gamma^\mu T^c \psi_L \rangle$$
$$+ \varepsilon^a_R (\partial_\mu \delta^{ac} + f^{abc} A^{R,b}_\mu) \langle \bar\Psi_R \gamma^\mu T^c \psi_R \rangle$$
$$+ \varepsilon \; \partial_\mu \langle \bar\Psi \gamma^\mu \Psi \rangle \Big]$$

The covariant divergence of the current is given by the anomaly equation.

$$\therefore \quad \delta\Gamma = - \overset{\text{color}}{3} \frac{1}{24\pi^2} \int d^4x \, \varepsilon^{\mu\nu\alpha\beta}$$

$$\cdot \mathrm{Tr} \Big[(\varepsilon_L + \varepsilon) (\partial_\mu A^L_\nu \partial_\alpha A^L_\beta + \tfrac{1}{2} i \, \partial_\mu (A^L_\nu A^L_\alpha A^L_\beta)$$
$$+ 2 \partial_\mu A^L_\nu \partial_\alpha B_\beta + \tfrac{1}{2} i \, \partial_\mu (A^L_\nu A^L_\alpha B^L_\beta))$$
$$- (\varepsilon_R + \varepsilon) (\partial_\mu A^R_\nu \partial_\alpha A^R_\beta + \tfrac{1}{2} i \, \partial_\mu (A^R_\nu A^R_\alpha A^R_\beta)$$
$$+ 2 \partial_\mu A^R_\nu \partial_\alpha B_\beta + \tfrac{1}{2} i \, \partial_\mu (A^R_\nu A^R_\alpha B_\beta)) \Big] \qquad (1)$$

For simplicity, let us consider the case $n = 2$ and $A^R_\mu = 0$. Then the $SU(2)^3_L$ anomaly vanishes and the only anomaly is $SU(2)^2_L \times U(1)$:

$$\frac{1}{3}\delta\Gamma = -\frac{1}{24\pi^2}\int d^4x\ \varepsilon^{\mu\nu\alpha\beta} \cdot \mathrm{Tr}\Big[\ \varepsilon\ (\ \partial_\mu A^L_\nu\, \partial_\alpha A^L_\beta + \frac{1}{2} i\ \partial_\mu (A^L_\nu A^L_\alpha A^L_\beta)\ ) + \varepsilon_L\ (\ 2\, \partial_\mu A^L_\nu\, \partial_\alpha B_\beta + \frac{1}{2} i\ \partial_\mu (\ A^L_\nu A^L_\alpha B_\beta\ )\ )\ \Big]. \tag{2}$$

Having understood the non-invariance of Γ, let us consider the functional integral of the chiral Lagrangian

$$e^{i\Gamma[iA_L, iA_R, iB]} = \int [d\Sigma]\ e^{i\int d^4x\, \mathcal{L}_{chiral}}.$$

By définition of the chiral Lagrangian, this Γ must be the same as Γ_{QCD}. In particular it transforms non-trivially under G_f transformations in the manner given in (1), (2). Usually when we introduce gauge fields, we replace ordinary derivatives by covariant derivatives. This is not sufficient in our case, because the Lagrangian constructed by the replacements would be G_f gauge invariant. We have to add extra terms to satisfy (1), (2).

The extra term which transforms like (1) is called "Wess-Zumino term". We will come back to the general discussion in the third lecture. Here we just state the answer for (2):

$$\Gamma_{W-Z} = -3 \, B_\mu J^\mu$$

where
$$J^\mu = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \mathrm{Tr}\Big[\Sigma^+(\partial_\nu + i A^L_\nu)\Sigma \cdot \Sigma^+(\partial_\alpha + i A^L_\alpha)\Sigma \cdot \Sigma^+(\partial_\beta + i A^L_\beta)\Sigma$$
$$- \frac{3}{2} i \, \Sigma^+ F^L_{\nu\alpha} (\partial_\beta + i A^L_\beta)\Sigma$$
$$+ \frac{1}{2} \left(- A^L_\nu F^L_{\alpha\beta} + i A^L_\nu A^L_\alpha A^L_\beta \right) \Big] .$$

In particular when $A^L_\mu = 0$, we find

$$J^\mu = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \mathrm{Tr}\left[\Sigma^+ \partial_\nu \Sigma \cdot \Sigma^+ \partial_\alpha \Sigma \cdot \Sigma^+ \partial_\beta \Sigma \right] . \qquad (3)$$

We recall that B_μ is coupled to the quark number current.

$\therefore$ $3 J^\mu$ is the quark number current.

$\Rightarrow$ J^μ defined by (3) can be interpreted as the baryon number current.

Thus we see that even meson field can carry baryon number.

At each time t, Σ gives a map from S^3 to $SU(n)$. In order to have finite energy, Σ approaches a fixed value at $r \to \infty$. Therefore, the Σ configuration can be classified by $\pi_3(SU(n)) = \mathbb{Z}$. (3) is the density of the corresponding winding number:

$$\text{winding number} = \int d^3x \; J^0 .$$

Baryons can be regarded as solitons of the chiral field Σ.

§3. Method of Goldstone-Wilczek to calculate the baryon number

In the previous section, we derived the anomalous baryon current by considering the flavor anomalies. In this section we show another way of deriving the same expression due to Goldstone and Wilczek.
(PRL 47 (1981) 986)

Let us consider the following phenomenological Lagrangian which also has nucleon fields $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ in it:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\,\psi - m\,(\bar{\psi}_L \Sigma \psi_R + \bar{\psi}_R \Sigma^{+} \psi_L)$$
$$+ \frac{f^2}{8} \mathrm{Tr}\, \partial_\mu \Sigma\, \partial^\mu \Sigma^{+} + \text{higher derivative terms} .$$

We regard Σ as a background field and compute the expectation value of the baryon number current

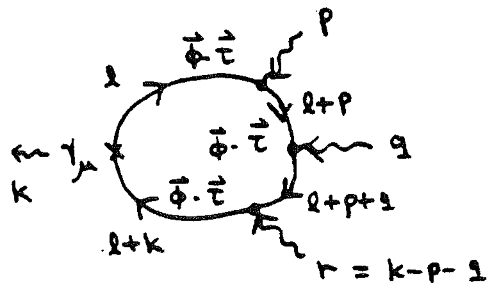
$J_\mu = \bar{\psi}\gamma_\mu \psi$.

The SU(2) matrix Σ can be written as

$$\Sigma = \phi_0 + i\vec{\phi}\cdot\vec{\tau} \qquad . \ (\phi_0^2 + \vec{\phi}^2 = 1)$$

Suppose $\phi_0 = 1$ and $\vec{\phi}$ is a small perturbation.

The leading contribution to $\langle J_\mu \rangle$ comes from the diagram

$\langle J_\mu \rangle \simeq$ 

fermion loop

$$= \overset{\downarrow}{-} \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu \frac{i}{\not{l}+\not{k}-m} \, m\vec{\phi}(r)\cdot\vec{\tau}\gamma^5 \right.$$

$$\left. \cdot \frac{i}{\not{l}+\not{p}+\not{q}-m} \, m\vec{\phi}(q)\cdot\vec{\tau}\gamma^5 \cdot \frac{i}{\not{l}+\not{p}-m} \, m\vec{\phi}(p)\cdot\vec{\tau}\gamma^5 \frac{i}{\not{l}-m} \right]$$

$$= -\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l+k)^2-m^2} \frac{1}{(l+p+q)^2-m^2} \frac{1}{(l+p)^2-m^2} \frac{1}{l^2-m^2}$$

$$\cdot m^3 \, \mathrm{tr}\, \vec{\phi}(r)\cdot\vec{\tau} \ \vec{\phi}(q)\cdot\vec{\tau} \ \vec{\phi}(p)\cdot\vec{\tau}$$

$$\cdot \mathrm{Tr}\left[\gamma_\mu (\not{l}+\not{k}+m)\gamma^5(\not{l}+\not{p}+\not{q}+m)\gamma^5(\not{l}+\not{p}+m) \right.$$

$$\left. \cdot \gamma^5 (\not{l}+m) \right]$$

$$= \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \, \mathrm{tr}\, i\partial^\nu \vec{\phi}(r)\cdot\vec{\tau} \ i\partial^\alpha \vec{\phi}(p)\vec{\tau} \ i\partial^\beta \vec{\phi}(q)\cdot\vec{\tau}$$

$$\Rightarrow \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \, \mathrm{tr}\, \Sigma^\dagger \partial^\nu \Sigma \cdot \Sigma^\dagger \partial^\alpha \Sigma \cdot \Sigma^\dagger \partial^\beta \Sigma \qquad //$$

Ex 1. Complete the calculation.

§4. Skyrme's Model

Ref. Adkins, Nappi, Witten NP B228 (1983) 552.

In the previous sections, we have learned that the meson field Σ can carry baryon number current

$$J_\mu = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \operatorname{Tr} \Sigma^+ \partial^\nu \Sigma \, \Sigma^+ \partial^\alpha \Sigma \, \Sigma^+ \partial^\beta \Sigma .$$

This is topological. Its conservation $\partial_\mu J^\mu = 0$ is automatic. We expected this sort of topological conservation, since we know $\pi_3(SU(n)) = \mathbb{Z}$.

Skyrme's model corresponds to a particular choice of the chiral Lagrangian

$$\mathcal{L} = \frac{f^2}{8} \operatorname{Tr} \partial_\mu \Sigma^+ \partial^\mu \Sigma + \frac{1}{16 e^2} \operatorname{Tr} [\Sigma^+ \partial_\mu \Sigma, \Sigma^+ \partial_\nu \Sigma]^2 .$$

I remind you of the necessity of the second term. In order to stabilize the soliton, higher order derivatives are necessary. (Derick's theorem)

The soliton solution can be found easily as follows. First we make an Ansatz

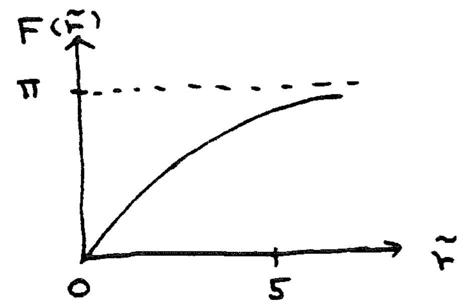
$$\Sigma_0(\vec{x}) = \exp[\, i F(r) \hat{\tau} \,] .$$

Σ_0 is invariant under simultaneous rotation of space $SU(2)_{rot}$ and $SU(2)_V$.

Then as an equation for $F(r)$, we obtain the second order differential equation

$$\left[\frac{\tilde{r}^2}{4} + 2\sin^2 F\right] F'' + \frac{\tilde{r}F'}{2} + \sin 2F \cdot F'^2$$
$$- \frac{\sin 2F}{4} - \frac{\sin^2 F \sin 2F}{\tilde{r}^2} = 0 \; . \qquad (\tilde{r} \equiv e f r)$$

We impose
$$F(0) = 0$$
$$F(\infty) = n\pi$$
as boundary conditions. For $n = 1$, the solution looks like



The baryon number is calculated to be

$$B = \int d^3x \; J^0$$
$$= \int_0^\infty 4\pi r^2 dr \; \frac{1}{2\pi^2} \frac{\sin^2 F}{r^2} F'$$
$$= \frac{2}{\pi} \int_{F(0)}^{F(\infty)} dF \, \sin^2 F = \frac{1}{\pi} F(\infty) = \underline{n} .$$

i.e. n is the baryon number.

Next we quantize the soliton. We introduce collective coordinates.

$$\Sigma(\vec{x},t) = A(t)\,\Sigma_0(\vec{x})\,A^\dagger(t).$$

$A(t)$ is a time dependent $SU(2)$ matrix. It is independent of space, i.e. we ignore meson excitations.

The Lagrangian $L = \int d^3x\, \mathcal{L}$ has two symmetry transformations. One is isospin $SU(2)_V$ and the other is space rotation $SU(2)_{rot}$. Under $SU(2)_V$,

$$\Sigma \to V\Sigma V^\dagger \quad \text{i.e.} \quad A \to VA.$$

Under $SU(2)_{rot}$,

$$\Sigma \to AR\Sigma_0 R^\dagger A^\dagger \quad \text{i.e.} \quad A \to AR.$$

With respect to A, $SU(2)_V$ is left multiplication and $SU(2)_{rot}$ is right multiplication.

We quantize the soliton by canonical quantization. The Lagrangian is calculated to be

$$L = -M + \lambda\, \mathrm{Tr}(\partial_0 A\, \partial_0 A^\dagger),$$

$$\text{where } \lambda = \frac{4\pi}{3}\frac{1}{e^3 f}\Lambda$$

$$\Lambda = \int_0^\infty d\tilde{r}\, \tilde{r}^2 \sin^2 F\left[1 + 4\left(F'^2 + \frac{\sin^2 F}{\tilde{r}^2}\right)\right]$$

$$\simeq 50.9 \text{ (numerically)}.$$

An SU(2) matrix A is a function of three local coordinates a_i $(i=1,2,3)$.

$$L = \lambda \, Tr \, \partial_0 A \, \partial_0 A^{+}$$

$$= \lambda \, Tr \frac{\partial A}{\partial a_i} \frac{\partial A^{+}}{\partial a_j} \, \dot{a}_i \dot{a}_j$$

We define

$$C_{ai} \equiv Tr \, T^a A^{+} i \frac{\partial A}{\partial a_i} \, . \qquad (Tr \, T^a T^b = \delta^{ab})$$

Then

$$L = \lambda \sum_a C_{ai} C_{aj} \, \dot{a}_i \dot{a}_j \, .$$

The canonical momenta are defined by

$$\pi_i = \frac{\partial L}{\partial \dot{a}_i} = 2 \lambda \sum_{a,j} C_{ai} C_{aj} \, \dot{a}_j \, .$$

The Hamiltonian is

$$H = \sum_i \dot{a}_i \pi_i - L \, .$$

$$= \frac{1}{4\lambda} \sum C^{-1}{}_{ib} \, C^{-1}{}_{jb} \, \pi_i \pi_j \, .$$

We introduce the generators of right multiplications :

$$[R^a , A] = - A T^a$$

Let $R^a = f^{ai} \pi^i$, then

$$[R^a, A] = f^{ai} [\pi_i , A]$$

$$= - i f^{ai} \frac{\partial A}{\partial a_i} = - i f^{ai} A \cdot A^{+} \frac{\partial A}{\partial a_i} = - f^{ai} A \, C_{bi} T^b$$

$$\therefore \quad f^{ai} C_{bi} = \delta^a{}_b \quad \Rightarrow \quad f^{ai} = C^{-1}{}_{ia}$$

$$\therefore \quad \underbrace{R^a = C^{-1}{}_{ia}\,\pi_i}_{} = 2\lambda\, C_{ai}\, \dot{a}_i$$

Therefore, we obtain

$$H = \frac{1}{4\lambda} (R^a)^2 .$$

This is manifestly $SU(2)_L \times SU(2)_R$ invariant.
(R^a is left invariant.)

The physical states are classified by their $SU(2)_V \times SU(2)_{rot}$ quantum numbers. In general we can write

$$\psi(A) = \sum_{\substack{r \\ ab}} c^{(r)}_{ab}\, D^{(r)}_{ab}(A) . \qquad \left(D^*_{ba}(A) \equiv \langle b | A | a \rangle \right)$$

a, b correspond to $SU(2)_V$, $SU(2)_{rot}$, resp.

Examples.

r	(a,b)	
$\frac{1}{2}$	$(\frac{1}{2}, \pm\frac{1}{2})$	p
	$(-\frac{1}{2}, \pm\frac{1}{2})$	n
$\frac{3}{2}$	$(+\frac{3}{2}, \pm\frac{3}{2}, \pm\frac{1}{2})$ Δ^{2+}	$(-\frac{1}{2},\ ``\ )$ Δ^0
	$(+\frac{1}{2},\ ``\ )$ Δ^+	$(-\frac{3}{2},\ ``\ )$ Δ^-

We can also consider integer spin states. However their Hilbert space is orthogonal to half-odd-integer spin states, i.e. meson excitations do not excite integer spin states to half-odd-integer spin states. So we can forget about them.

We obtain an important relation from this picture :

$$\text{spin} = \text{isospin} .$$

This is what we observe in nature.

Let us look at the masses of the baryons. For states of (iso)spin I, $(R^a)^2 = 2\, I\,(I+1)$.

(The factor of 2 comes from the normalization $\mathrm{Tr}\, T^a T^b = \delta^{ab}$.)

Therefore, the mass is given by

$$M_N = M + \frac{1}{4\lambda}\frac{3}{2} \quad \text{and} \quad M_\Delta = M + \frac{1}{4\lambda}\frac{15}{2} .$$

We can use the observed masses to fit f and e, which are the only parameters of the model.

The result is

$$e = 7.71 \quad , \quad f = 91.2 \text{ MeV} .$$

f is 30 % off the observed value.

Having understood the quantization of the soliton, we can calculate any matrix elements. I refer the reader to the paper by Adkins, et. al (op. cit.) for predictions of charge radii, magnetic moments, and current matrix elements. I only remark that the skyrmion model is at least as good as the static quark model.

§5. Wess-Zumino term

Ref. Wess & Zumino, PL 37B (71) 95; Witten, NPB223 (83) 42

As I promised in §2, we proceed to general discussions of the extra terms in the chiral Lagrangian.

We look for $\Gamma_{W\text{-}Z}(\Sigma$, external fields$)$ which transforms like (1) in §2 upon gauge transformations.

The derivation is rather involved. So I only sketch it.

Let us consider the Lagrangian

$$\mathcal{L}(\psi_L, \psi_R; iA_L, iA_R) = i\bar{\psi}\not\partial\psi - \bar{\psi}_L\gamma^\mu A^L_\mu \psi_L - \bar{\psi}_R\gamma^\mu A^R_\mu \psi_R .$$

($\psi_{L \atop R}$ is in the fundamental rep. of $SU(n)_{L \atop R}$. We absorb B_μ by $A_{\mu L}$ and $A_{\mu R}$.) Define the functional integral

$$e^{i\Gamma[iA_L, iA_R]} = \int [d\psi d\bar{\psi}]\, e^{i\int d^4x\, \mathcal{L}(\psi_L, \psi_R; iA_L, iA_R)} .$$

3 (color) $\times\, \Gamma$ transforms in the same way as (1) in §2.

We want to extract the anomalous part of Γ in the following. We introduce the change of variables:

$$\psi_L' = \Sigma^+\psi_L \quad , \quad \psi_R' = \psi_R .$$

Then we find

$$e^{i\Gamma} = \int [d\psi' d\bar{\psi}']\, e^{iW(\Sigma^+; iA_L, iA_R)}\, e^{i\int d^4x\, \mathcal{L}(\Sigma\psi_L', \psi_R'; iA_L, iA_R)}$$

$$= \int [d\psi d\bar{\psi}]\, e^{iW(\Sigma^+; iA_L, iA_R)}\, e^{i\int d^4x\, \mathcal{L}(\psi_L, \psi_R;\, \Sigma^+\partial_\mu\Sigma + \Sigma^+ iA_{\mu L}\Sigma,\, iA_{\mu R})} .$$

Here $e^{iW(\Sigma^\dagger; iA_L, iA_R)}$ is a Jacobian for the change of variables. It is independent of the fermion variables.

(Recall Fujikawa's derivation of the anomaly. $W \neq 0$ means that the change of variable, which is $SU(n)_L$, is anomalous.)

Now we have

$$e^{i\Gamma} = e^{iW(\Sigma^\dagger; iA_L, iA_R)} \underbrace{\int [d\psi d\bar{\psi}]\, e^{i\int d^4x\, \mathcal{L}(\psi_L, \psi_R;\, \Sigma^\dagger\partial_\mu\Sigma + i\Sigma^\dagger A_\mu^L \Sigma,\, iA_\mu^R)}}_{\equiv\, e^{i\Gamma'}} .$$

Let us see how the functional integral Γ' on the RHS changes under $G_f = SU(n)_L \times SU(n)_R \times U(1)_V$ transformation:

Since
$$\delta(\Sigma^\dagger(iA_\mu^L)) = -i\partial_\mu \varepsilon_R - [\varepsilon_R, \Sigma^\dagger(A_\mu^L)] - i\partial_\mu \varepsilon$$
$$\delta(iA_\mu^R) = -i\partial_\mu \varepsilon_R - [\varepsilon_R, A_\mu^R] - i\partial_\mu \varepsilon ,$$

(We defined $\Sigma^\dagger(iA_\mu^L) \equiv \Sigma^\dagger \partial_\mu \Sigma + i\Sigma^\dagger A_\mu^L \Sigma$)

G_f acts as a vector. It is known that the vector anomaly can be canceled by counter terms.

They have been computed by Bardeen (PR 184 (69) 1848) as follows (for simplicity we replace $\Sigma^\dagger(iA_L)$ by iA_L):

$$\Delta\Gamma[iA_L, iA_R]$$
$$= \frac{1}{96\pi^2}\int d^4x\, \varepsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\Big[(A_\mu^L A_\nu^R - A_\mu^R A_\nu^L)(F_{\alpha\beta}^R + F_{\alpha\beta}^L)$$
$$+ i(-A_\mu^R A_\nu^L A_\alpha^R A_\nu^L - 2A_\mu^L A_\nu^R A_\alpha^R A_\beta^R + 2A_\mu^R A_\nu^L A_\alpha^L A_\beta^L)\Big] .$$

Therefore, $\Gamma' + \Delta\Gamma[\Sigma^{+}(iA^{L}), iA^{R}]$ is invariant under G_f transformations. Now we are done.

Because

$$\begin{aligned} \Gamma &= W + \Gamma' \\ &= W - \Delta\Gamma[\Sigma^{+}(iA^{L}), iA^{R}] + (\Gamma' + \Delta\Gamma), \end{aligned}$$

$\Gamma_{W-Z} \equiv W - \Delta\Gamma$ transforms in the same way as Γ.

This is what we wanted !!

The tedious part is the calculation of W, which I will not show here. It is straightforward. You repeat infinitesimal transformations to construct the global rotation by Σ^{+}. (Recall we know Jacobians for the infinitesimal transf.) Unlike in the $n=2$ case, Γ_{W-Z} does not vanish even if the external fields are zero. It is given by

$$\Gamma_{W-Z} = \frac{1}{48\pi^2}\int_0^1 ds \int d^4x\, \varepsilon^{\mu\nu\alpha\beta}\, \mathrm{Tr}\left[M\, \Sigma_s^{+}\partial_\mu \Sigma_s \cdot \Sigma_s^{+}\partial_\nu \Sigma_s \cdot \Sigma_s^{+}\partial_\alpha \Sigma_s \cdot \Sigma_s^{+}\partial_\beta \Sigma_s\right],$$

where $\Sigma \equiv e^{iM}$, $\Sigma_s \equiv e^{iMs}$.

This can be written in a five-dimensional form as

$$\Gamma_{W-Z} = \frac{-i}{240\pi^2}\int d\Sigma^{ijklm}\, \mathrm{Tr}\left[\Sigma_s^{+}\partial_i \Sigma_s \cdots \Sigma_s^{+}\partial_m \Sigma_s\right].$$

It contains a term describing $K^{+}K^{-} \to \pi^{+}\pi^{-}\pi^{0}$, etc.

When we introduce $U(1)_{EM}$, we get additional terms:

$$-\frac{1}{48\pi^2}\int d^4x\, \varepsilon^{\mu\nu\alpha\beta} A_\mu \operatorname{Tr} \underset{\left[\frac{2}{3} -\frac{1}{3} -\frac{1}{3}\right]}{Q} \left(\partial_\nu\Sigma\cdot\Sigma^+ \partial_\alpha\Sigma\cdot\Sigma^+ \partial_\beta\Sigma\cdot\Sigma^+ + \Sigma^+\partial_\nu\Sigma\cdot\Sigma^+\partial_\alpha\Sigma\cdot\Sigma^+\partial_\beta\Sigma \right)$$

$$+\frac{i}{24\pi^2}\int d^4x\, \varepsilon^{\mu\nu\alpha\beta}\, \partial_\mu A_\nu \cdot A_\alpha \cdot \operatorname{Tr}\left[Q^2\Sigma^+\partial_\beta\Sigma + Q^2\partial_\beta\Sigma\cdot\Sigma^+ + Q\Sigma Q\Sigma^+\partial_\beta\Sigma\cdot\Sigma^+ \right].$$

The first term is an ordinary $-J_\mu A_\mu$ coupling.

The second term describes, e.g. $\pi^0 \to 2\gamma$ decay.

It has a piece

$$-\frac{e^2}{48\pi^2\sqrt{2}\, f}\,\pi^0\,\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\,.$$

We should not be surprized by this, since we expect Γ_{W-Z} takes account of the effects of the anomalies. $\pi^0 \to 2\gamma$ is one of them.

The Wess-Zumino term plays an important role for the soliton as well. It determines that the soliton should be a fermion. Moreover it gives a constraint to the quantization of the soliton. I refer the interested reader to

Witten, NP. B 223 (83) 433

Guadagnini, NP. B2 (84)

Bijnens, et al, PLB to be published (CALT-68-1096).