4. The Large-N Limit

A. A two-dimensional model

A lot has been said so far in this course about qualitative properties of QCD, such as confinement and chiral symmetry breakdown. But, except for perturbation theory, we have not done any quantitative calculations

What makes nonperturbative QCD so difficult is that, because of dimensional transmutation, there is no dimensionless parameter in which to expand. The scope of QCD is vast - for example, it encompasses all of nuclear physics - so we have little hope of "solving" the theory. There is no obvious way of taking a limit in which the theory simplifies, and expanding around it.

There being no obvious expansion parameter, we must look for a hidden one. It was suggested by G. 't Hooft (Nucl. Phys. B72: 461 (1974)) that QCD should simplify considerably in the limit $N \to \infty$, where N is the number of colors (i.e., the gauge group is $SU(N)$), and there is a well-defined expansion about this limit in powers of $1/N$. Of course, in the real world $N=3$, and it is not obvious that an expansion in $1/3$ will work well, but there is no a priori reason to believe that such an expansion is a bad idea.

Of course, we can expand about the $N \to \infty$ limit only if we can solve the theory in that limit. This has not yet proved possible, although it was recently shown that a four-dimensional SU(N) gauge theory exists in the $N \to \infty$ limit - this is the only non-trivial quantum field theory known more or less rigorously to exist in four dimensions.

The large-N limit is worthy of discussion because, although we are not yet able to solve QCD for $N \to \infty$, many qualitative properties of the $N \to \infty$ limit can be deduced, and correspond to observed properties of the strong interactions. It provides a useful way of thinking about QCD. And maybe some day the $N \to \infty$ limit will be solved, and we will be able to calculate spectra etc. to compare with experiment.

Gross-Neveu Model (Phys. Rev. D10: 3234 (1974))

To illustrate how a field theory can simplify when the fields have many components, we will consider a simple two-dimensional model. This model is actually pretty interesting. Not only can we solve the large-N limit, but the model has:

- Asymptotic Freedom and Dimensional Trans.
- Dynamical Breakdown of Chiral Sym.

These are features of QCD also. Here we will be able to see quite explicitly how the dynamics responsible for chiral symmetry breakdown works.

The Lagrangian is

$$\mathcal{L} = \bar{\psi}^a i\partial\!\!\!/ \psi^a + \tfrac{1}{2}\lambda_0(\bar{\psi}^a\psi^a)^2$$

where $a = 1, \dots, N$ is a "flavor" index

In addition to a $SU(N)\times U(1)$ flavor symmetry, the model has a "chiral" symmetry, which is discrete

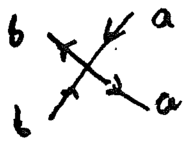
$$\psi^a \to \gamma_5 \psi^a$$
$$\bar{\psi}^a \to -\bar{\psi}^a\gamma_5$$

under which $\bar{\psi}^a\psi^a$ is odd; the symmetry prevents a mass from being generated in perturbation theory in λ_0.

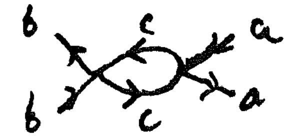
(Note: the two-dim. γ matrices satisfy $\{\gamma^\mu\gamma^\nu\} = 2g^{\mu\nu}$
E.g. choose $\gamma^0 = \sigma_z$, $\gamma^1 = i\sigma_y$, "γ^5" $= \sigma_x$.)

It is this discrete chiral symmetry which will become dynamically broken

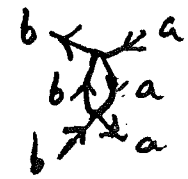
To understand the basic idea of the $1/N$ expansion, let's look at some of the low-order graphs contributing to the 4-point function.

tree: 

λ_0

One loop: 

$\lambda_0^2 N$



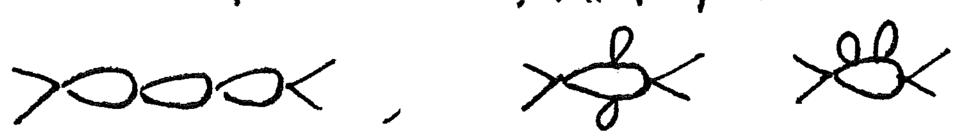
λ_0^2

The first one-loop graph has a sum over flavors, and so is proportional to N, but the second graph has no such sum and is smaller by $1/N$ from the first.

To define a large N limit, we must decide what to hold fixed as $N \to \infty$. We want an infinite class of graphs to survive, and for higher loop graphs to be comparable to the Born term. So we take

$$N \to \infty \,, \qquad g_0 \equiv \lambda_0 N \quad \text{fixed}$$

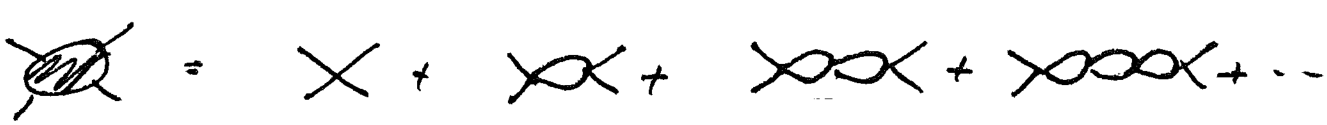
In higher order, the graphs



contribute in leading order in $1/N$

and these vanish if the bare fermion is massless

So the 4 point function, in the $N \to \infty$ limit, is given by a simple "bubble-chain" sum



Solving this theory, and, in particular, verifying that a fermion mass is dynamically generated, is simplified by the following trick. Replace

$$\mathcal{L} \to \mathcal{L} - \frac{N}{2g_0}\left(\sigma - \frac{g_0}{N}\bar{\psi}^a\psi^a\right)^2$$

There is not a kinetic term for the field σ, so there $\int d\sigma$ integral is a trivial normalization factor; the theory is not changed by the new term in the Lagrangian.

$$\mathcal{L} = \bar{\psi}^a i\partial\!\!\!/ \psi^a - \frac{N}{2g_0}\sigma^2 + \sigma\bar{\psi}^a\psi^a$$

(the σ "eqn. of motion" is $\sigma = \frac{g_0}{N}\bar{\psi}^a\psi^a$, and if we substitute this into $\mathcal{L}$, we have the same Lagrangian as before)

In this formulation, the "chiral symmetry" is

$$\sigma \to -\sigma$$
$$\bar{\psi}^a\psi^a \to -\bar{\psi}^a\psi^a,$$

which forbids a fermion mass term. The symmetry is spontaneously broken if σ acquires a vacuum expectation value

The fermions now appear bilinearly, and the fermion integration, which is Gaussian, can be done explicitly, and we obtain an effective action ---

$$\int d\psi\, d\bar{\psi}\, d\sigma\, e^{iS} = \int d\sigma\, e^{i\int d^2x\left(-\frac{N}{2g_0}\sigma^2\right)}\left[\det(\not\partial - i\sigma)\right]^N$$

The det gets raised to the Nth power because there is a determinant for each fermion flavor.

Discarding the irrelevant normalization factor $[\det(\not\partial)]^N$, the fermion determinant is

$$\det(1 - i\not\partial^{-1}\sigma) = \det(\gamma_5)^2(1 - i\not\partial^{-1}\sigma)$$
$$= \det \gamma_5(1 - i\not\partial^{-1}\sigma)\gamma_5 = \det(1 + i\not\partial^{-1}\sigma)$$

Thus

$$\det(1 - i\not\partial^{-1}\sigma) = \left[\det(1 - i\not\partial^{-1}\sigma)(1 + i\not\partial^{-1}\sigma)\right]^{\frac{1}{2}}$$
$$= \left(\det[1 + (\not\partial^{-1}\sigma)^2]\right)^{\frac{1}{2}}$$

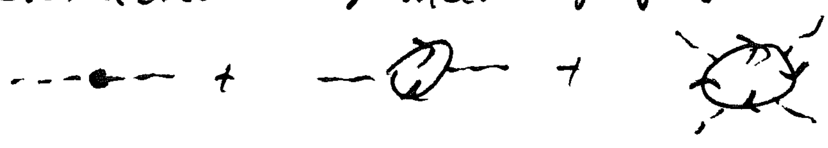
And using the identity $\det M = \exp \mathrm{Tr} \ln M$, we see that, after the fermions are integrated out, the theory has the effective action

$$S_{eff} = \int d^2x\left(\frac{-N}{2g_0}\right)\sigma^2(x) - \frac{i}{2} N\, \mathrm{Tr} \ln[1 + (\not\partial^{-1}\sigma)^2]$$

or

$$\boxed{S_{eff} = N\left[\int d^2x\left(\frac{-1}{2g_0}\right)\sigma^2(x) - \frac{i}{2}\, \mathrm{Tr} \ln[1 + (\not\partial^{-1}\sigma)^2]\right]}$$

Although we used functional integration methods to calculate S_{eff}, we would of course have obtained the same result by summing the connected Feynman graphs



Graphs with an odd no. of σ lines vanish because of the $\sigma \to -\sigma$ symmetry. And each graph is proportional to N, due to the sum over fermion flavors.

Since N now appears as a factor in front of S_{eff}, we see that the $N\to\infty$ limit is the same as the "classical" (i.e. $\hbar\to 0$) limit, in which the functional integral can be evaluated by steepest descent. So for $N\to\infty$, the expectation value of σ in the ground state is the minimum of an "effective potential". Evaluating the trace for $\sigma=$ constant in the plane wave basis, we have

$$-\frac{i}{2}\,\mathrm{tr}\,\ln\left[1+(\partial\!\!\!/^{-1}\sigma)^2\right] = -\frac{i}{2}\underset{\substack{\uparrow\\ \text{spinor}\\ \text{trace}}}{(2)}\int\frac{d^2p}{(2\pi)^2}\int d^2x\, e^{ipx}\ln\left[1+(\partial\!\!\!/^{-1}\sigma)^2\right]e^{-ipx}$$

$$= -i\int d^2x\int\frac{d^2p}{(2\pi)^2}\ln\left(1-\sigma^2/p^2\right)$$

And the potential $V(\sigma)$ is given by $S=-\int d^2x\, V(\sigma)$

$$\Rightarrow \boxed{V(\sigma) = N\left[\frac{\sigma^2}{2g_0} - \int\frac{d^2p_E}{(2\pi)^2}\ln\left(1+\sigma^2/p_E^2\right)\right]}$$

after a Wick rotation of the integral.

Let's minimize V. Notice that there is a log ultraviolet divergence in V, which is due to the graph , which requires a σ field renormalization.

Introduce a cutoff Λ:

$$-\int_0^{\Lambda^2}\frac{dp^2}{4\pi}\ln(p^2+\sigma^2)-\ln p^2$$

$$= -\frac{1}{4\pi}\left[(\Lambda^2+\sigma^2)\ln(\Lambda^2+\sigma^2)-\sigma^2\ln\sigma^2-\Lambda^2\ln\Lambda^2\right]$$

$$= \frac{1}{4\pi}\sigma^2\left[\ln\left(\sigma^2/\Lambda^2\right)-1\right] + O\left(\sigma^2/\Lambda^2\right)$$

So we have

$$V = N\left[\frac{\sigma^2}{2g_0} + \frac{1}{4\pi}\sigma^2\left(\ln \sigma^2/\Lambda^2 - 1\right)\right]$$

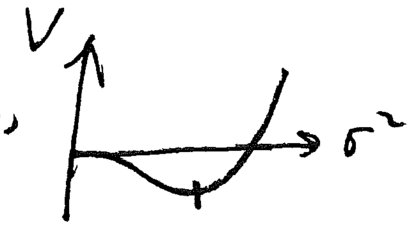
We can absorb the cutoff dependence by defining a renormalized coupling

$$\frac{1}{g_\mu} = \frac{1}{g_0} + \frac{1}{2\pi}\ln \mu^2/\Lambda^2 ,$$

and then

$$V = N\left[\frac{\sigma^2}{2g_\mu} + \frac{1}{4\pi}\sigma^2\left(\ln \sigma^2/\mu^2 - 1\right)\right]$$

Note that $g_\mu \to 0$ as $\mu \to \infty$. This theory is asymptotically free.



The potential has the shape shown, and its minimum occurs for non-zero σ^2

$$\frac{dV}{d\sigma^2} = N\left[\frac{1}{2g_\mu} + \frac{1}{4\pi}\ln \sigma^2/\mu^2\right] = 0$$

$$\Rightarrow \quad \boxed{\sigma = \mu e^{-\pi/g_\mu}} \Rightarrow V = -\frac{N}{4\pi}\sigma^2 = -\frac{N\mu^2}{4\pi}e^{-2\pi/g_\mu} < 0$$

This is the $\mu e^{-1/bg_\mu}$ behavior we expect for a spontaneously generated mass in an asymptotically free theory.

Of course, dimensional transmutation occurs in this theory. We can take advantage of our freedom to choose μ so that the minimum occurs at $\sigma_0 = \mu$. Then $g_\mu = \infty$, and

$$V = (N/4\pi)\, \sigma^2 [\ln(\sigma^2/\sigma_0^2) - 1]$$

The parameter g_μ, which was illusory, has dropped from sight. There are no dimensionless parameters in this theory.

The Chiral Gross-Neveu Model

In the Gross-Neveu model, a discrete chiral symmetry is spontaneously broken. Let us now consider a model with a continuous chiral $U(1)_A$ symmetry, which resembles the Nambu-Jona-Lasinio model (but still in two spacetime dimensions):

$$\mathcal{L} = \bar\psi^a i\not\partial \psi^a + \frac{g_0}{2N}\left[(\bar\psi^a\psi^a)^2 + (\bar\psi^a i\gamma_5\psi^a)^2\right]$$

Is a mass spontaneously generated in this case?

Actually what happens in this model is rather subtle. A mass is spontaneously generated, but $\bar\psi\psi$ does not acquire a vacuum expectation value; the $U(1)_A$ chiral symmetry is not spontaneously broken, and there is no Goldstone boson. This is in accord with general theorems about spontaneous symmetry breakdown of continuous symmetries (which cannot occur in 2 dimensions).

We discussed earlier why two is a critical dimension for breakdown of continuous symmetries. Fluctuations "of order one" of an order parameter in a region of size λ have an action which scales like

$$S \sim \int d^D x \, |\nabla \theta|^2 \sim \lambda^{D-2}$$

For $D \leq 2$, large fluctuations with $\lambda \to \infty$ have finite action and are unsuppressed. These fluctuations destroy long range order.

But, in the $N = \infty$ limit, we can calculate and minimize a potential for this model which indicates that $\langle \bar\psi \psi \rangle \neq 0$. This result cannot be correct for finite N in two dimensions, so the $1/N$ expansion must be carried out with some care.

We can introduce auxiliary scalar fields $\sigma \propto \bar\psi \psi$ and $-\pi \propto \bar\psi i \gamma_5 \psi$ as before, to write the Lagrangian in the form

$$\mathcal{L} = \bar\psi^a i \partial\!\!\!/ \psi^a - \frac{N}{2g_0}(\sigma^2 + \pi^2) + \bar\psi(\sigma + i\pi\gamma_5)\psi$$

Then, after integrating out the fermions, we obtain the effective action

$$S_{eff} = N\left[\int d^2x \left(\frac{-1}{2g_0}\right)(\sigma^2 + \pi^2) - i \operatorname{Tr} \ln\left[1 + i\partial\!\!\!/^{-1}(\sigma + i\pi\gamma_5)\right]\right],$$

which defines precisely the same effective potential, as a function now of $\sigma^2 + \pi^2$, that we calculated previously.

If we did not have a reason to be cautious, we would probably conclude that $\langle\sigma\rangle \neq 0$ (in an appropriately chosen vacuum state) and that $U(1)_A$ is spontaneously broken. But we know that we must take proper account of the long-wavelength fluctuations, which can destroy long range order.

In fact, it is wrong to expand this theory around a vacuum with nonzero σ; if we do so we encounter hopeless infrared divergences. Instead we write

$$\sigma + i\pi = \varphi e^{i\theta}$$

and argue that φ has a vacuum expectation value, but that the fluctuations in θ, the would-be Goldstone boson field, prevent θ from having a definite value in the vacuum, so that the $U(1)_A$ symmetry is not spontaneously broken. Nonetheless, since φ is nonzero, a fermion mass is spontaneously generated

To study the long-wavelength fluctuations in θ, it is sufficient to use an effective Lagrangian, which can be expanded in powers of $\nabla\theta$. Only gradients of θ, and not θ itself, can appear in the effective theory, because of the

$$\theta \to \theta + c$$

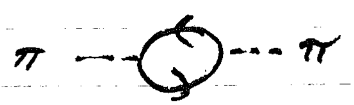
$U(1)_A$ symmetry. In fact, we need keep only the leading $(\nabla\theta)^2$ term, since others involving more powers

$\nabla\Theta$, have a negligible effect in the infrared limit

To calculate the coefficient of the leading term, we expand around the minimum of the potential $\sigma = \sigma_0$, $\pi = 0$;

so $\sigma + i\pi \doteq \sigma_0(1 + i\theta)$, or $\pi = \sigma_0\Theta$,

in perturbation theory. We calculate the two-point function for π, taking the fermion mass to be $m = \sigma_0$. The graph



generates a finite wave-function renormalization for π.

Exercise 4.1

By calculating the graph, show that the effective Lagrangian for Θ is

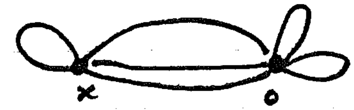
$$\mathcal{L}_{eff} = \frac{N}{4\pi}(\nabla\Theta)^2 + \cdots$$

To see whether there is really long range order, we calculate, using this effective Lagrangian,

$$\langle \bar{\psi}(1+\gamma_5)\psi(x)\,\bar{\psi}(1-\gamma_5)\psi(0)\rangle$$

$$\propto \quad \langle \rho(x)e^{-i\Theta(x)}\rho(0)e^{i\Theta(0)}\rangle \sim \sigma_0^2\langle e^{-i\Theta(x)}e^{i\Theta(0)}\rangle.$$

This operator green's function must be renormalized, but in two dimensions, the renormalization is trivial; it is only necessary to normal-order, i.e., subtract loops which are a single propagator contracted at one vertex.



To avoid infrared divergences, we will introduce a "normal-ordering mass" μ.
The loop

becomes $\propto \int \frac{d^2k}{k^2} - \int \frac{d^2k}{k^2-\mu^2}$.

Now, using the quadratic effective action, we have

$$\langle e^{-i\Theta(x)} e^{i\Theta(0)} \rangle = \langle e^{\int d^2z\, J(z)\,\Theta(z)} \rangle$$

where $J(z) = i[\delta(z) - \delta(z-x)]$

$$= N \int (d\Theta)\, e^{-\frac{1}{2}(\Theta, \Delta^{-1}\Theta) + (J,\Theta)}$$

$$= N \int d\Theta\, e^{-\frac{1}{2}(\Theta - \Delta J, \Delta^{-1}(\Theta - \Delta J)) + \frac{1}{2}(J, \Delta J)}$$

$$= e^{\frac{1}{2}(J, \Delta J)}$$

$$= \exp\,[\Delta(x) - \Delta(0)] \qquad \text{where } \Delta^{-1} = i\frac{N}{2\pi}\partial^2 = \frac{N}{2\pi}(-ip^2)$$

Normal ordering with respect to μ has the effect of replacing

$$\Delta(0;\mu) \quad \Delta(0;m) \to \Delta(0;m) - \Delta(0;\mu)$$

Here m is an infrared regulator mass which we will allow to go to zero at the end of the calculation.

Thus, $\langle N_\mu [e^{-i\theta(x)} e^{i\theta(0)}] \rangle$

$$= \lim_{m\to 0} \exp \left[\Delta(x;m) - \Delta(0;m) + \Delta(0;\mu) \right]$$

Here $\frac{N}{2\pi}\Delta(x;\mu) = \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{i}{p^2-\mu^2+i\epsilon}$

$$= \int \frac{d^2p_E}{(2\pi)^2} \frac{e^{ipx}}{p^2+\mu^2}$$

$$= \int_0^\infty ds \int \frac{d^2p}{(2\pi)^2} e^{ipx} e^{-s(p^2+\mu^2)}$$

$$= \int_0^\infty ds\, e^{-s\mu^2} \frac{1}{4\pi} s^{-1} e^{-x^2/4s}$$ ← Look up this integral in a Table

$$= \frac{1}{2\pi} K_0(\mu x)$$ ← here x means $\sqrt{x^2}$

And the asymptotic behavior of K_0 for small argument is

$$K_0(z) \sim -\left[\ln \tfrac{1}{2}z + \gamma\right] + O(z^2 \ln z)$$
$$= -\ln cz$$

Thus, we may write

$$\Delta(0;\mu) - \Delta(0;m) = \lim_{x\to 0} (\ln cmx - \ln c\mu x)\left(\frac{1}{2\pi}\right)\left(\frac{2\pi}{N}\right)$$

$$= \frac{1}{N} \ln(m/\mu) \qquad = \frac{1}{2\pi} \ln$$

And $\Delta(x;m) \to -\frac{1}{N} \ln(cmx)$ for $m \to 0$

So we finally have

$$\boxed{\langle N_\mu [e^{-i\theta(x)} e^{i\theta(0)}] \rangle = \exp\left[-\frac{1}{N} \ln(c\mu x)\right] = (c\mu|x|)^{-1/N}}$$

This is the result we expected.
If there were spontaneous symmetry breakdown we would have

$$\lim_{|x|\to\infty} \langle e^{-i\theta(x)} e^{i\theta(0)} \rangle \neq 0.$$

But instead we find that the fluctuations of θ wipe out the long range order, and cause this correlation function to drop off like a power of $1/|x|$.

There is a massless scalar particle in this theory. Otherwise we would find $\exp(-m|x|)$ asymptotic behavior. But the massless scalar is not a Goldstone boson.

As $N\to\infty$, the power-law fall-off of the two-point function is slower and slower, so that formally the symmetry appears to be spontaneously broken in the $N\to\infty$ limit. But for any finite N, spontaneous breakdown does not occur, as required by general theorems. Nonetheless, the $1/N$ expansion predicts correctly the qualitative properties of the model. In particular, it correctly predicts that a fermion mass is spontaneously generated.

The quasi-long range order we have found, the power-law decay of the correlation functions, is well-known in statistical mechanics, and is called the "Kosterlitz-Thouless" phenomenon.

Remarkably, the spectrum and S-matrix are known exactly in the chiral Gross-Neveu model! It is possible to find the exact S-matrix in the this model (and in other two-dimensional models) because the model possesses an infinite number of conservation laws which forbid particle production. Therefore, the S-matrix has only poles (no cuts); it is not hopelessly complicated as in nontrivial field theories in more dimensions.

The model can be solved for any N (Kurak and Swieca, Phys. Lett. 82B, 289 (1979); Andrei and Lowenstein 90B, 106 (1980).), so it provides us with a means of testing the accuracy of the $1/N$ expansion. For example, there are $N-1$ stable particles (which can be regarded as bound states of $1, 2, \ldots (N-1)/2$ fermions, plus antiparticles for N odd) with masses

$$m_n = \mu \sin(n\pi/N) \qquad n = 1, \ldots, N-1$$

The ratio of the two lightest masses is

$$\frac{m_2}{2m_1} = \frac{\sin(2\pi/N)}{2\sin(\pi/N)} = \cos(\pi/N) = 1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2 + \frac{1}{24}\left(\frac{\pi}{N}\right)^4 + \cdots$$

How accurate is the leading correction to the $N=\infty$ limit for e.g. $N=3$? The bound state is at threshold at $N=\infty$. Keeping just the leading correction, we predict

$$\frac{m_2}{2m_1} = 1 - \frac{1}{2}\left(\frac{\pi}{3}\right)^2 = .452, \quad \text{compared to exact value} \quad \cos(\pi/3) = .500.$$

Not bad.

B. Large-N Limit of QCD: Planar Diagrams

To define $N \to \infty$ limit of QCD, we must decide what to hold fixed as $N \to \infty$. In fact, the natural expansion parameter in QCD is

$$g^2 N / 8\pi^2$$

E.g. consider one-loop gluon propagator corrections

a  b $\propto \quad g^2 C^{acd} C^{bcd} = g^2 \delta^{ab} C_2$

N

In order to obtain a large-N limit in which an infinite number graphs survive, we should take $g^2 N \sim$ constant.

To explicitly exhibit the N-dependence of higher order graphs, it is very convenient to use a notation in which gauge fields are represented by $N \times N$ hermitian matrices with upstairs and downstairs indices

$$A_\mu{}^i{}_j$$

The gluon propagator has the group-theoretical structure

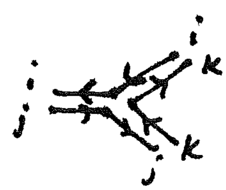
$$\langle A_\mu{}^i{}_j A_\nu{}^k{}_l \rangle \propto \left(\delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l \right)$$

removes trace

The second term on the RHS is present because $SU(N)$ generators are traceless. However, if we are interested only in leading $N \to \infty$ limit, the $1/N$ correction to the propagator can be ignored; the $SU(N)$ gauge theory becomes well-approximated by a $U(N)$ gauge theory, with N^2 gluons. So, e.g., a $U(N)$ gluon $A^i{}_j$ couples jth quark to ith quark.

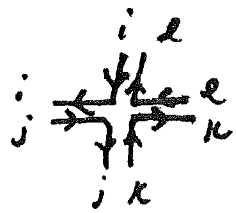
The central idea of the $N \to \infty$ limit of QCD is that gluons have more color states than quarks (N^2 instead of N) – so quark radiative corrections are suppressed relative to gluonic radiative corrections. But also, some gluon corrections are more important (more powers of N) than others.

In the "double-line" notation, gluon vertices can be represented by



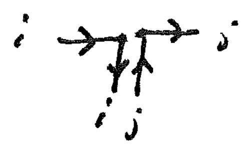
$$A_\mu{}^i{}_j \, A_\nu{}^j{}_k \, (\partial_\mu A_\nu)^k{}_i$$

Arrows indicate flow of color indices – i.e., distinguish up/down indices



$$A_\mu{}^i{}_j \, A_\nu{}^j{}_k \, A_\mu{}^k{}_l \, A_\nu{}^l{}_i$$

and there is also the quark gluon vertex:



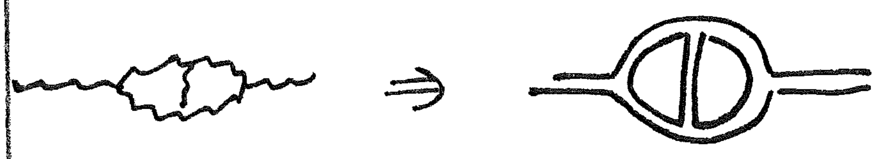
$$\bar{q}_i \, \gamma^\mu A_\mu{}^i{}_j \, q^j$$

In this notation, one-loop correction to gluon propagator is represented by

$= O(g^2 N)$

the factor of N arises from the closed "index loop", i.e., the sum over the free index k. It arises because there are N gluons coupling to any given gluon.

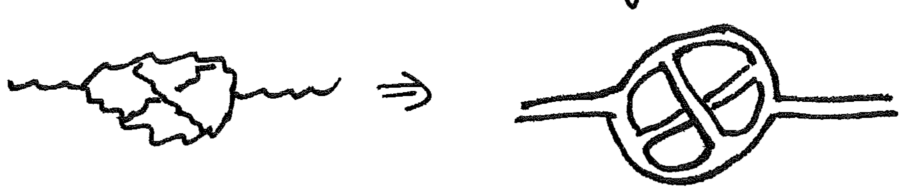
In next order



$\sim (g^2 N)^2$

two independent index loops $\Rightarrow$ two powers of N; this graph survives when $N \to \infty$ with $g^2 N$ fixed.

But in order g^6 there is



$\sim (g^2 N)^3 \frac{1}{N^2}$

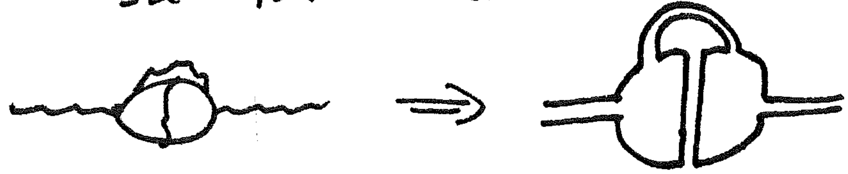
This graph has just one closed index loop, and so is formally suppressed by $1/N^2$ in $N \to \infty$ limit with $g^2 N$ fixed. The suppression arises because the graph is nonplanar, gluon lines cross, so it is not possible for all gluons which couple to given gluon to appear inside graph.

There are also corrections to gluon propagator in which quark loops appear

 $\sim g^2 = (g^2 N)\frac{1}{N}$

No index loop appears, so this graph is suppressed by $\frac{1}{N}$.

In higher order

 $\sim (g^2)^3 \sim (g^2 N)^3 \frac{1}{N^3}$

is suppressed by an additional power of $1/N^2$, because gluon lines cross when graph is drawn with quark loop on boundary.

These examples illustrate the basic rules of the $1/N$ expansion.

i) Quark loop cost a factor of $1/N$

ii) Nonplanar gluon exchanges cost a factor of $1/N^2$ (where gluon connection to quark loop is considered nonplanar when it cannot be drawn as planar <u>inside</u> the loop)

Let us give a systematic derivation of these rules for the case of vacuum bubbles, connected graphs with no external lines.

First, redefine fields so that N scales out in front of $\mathcal{L}$.

$$\mathcal{L} = \frac{N}{g^2}\left(-\tfrac{1}{4}F^2 + \bar{q}\, i\not{D} q - m\bar{q}q\right)$$

(where no powers of g or N appear in D or F, except of course, for N-dependence implicit in no. of components.)

Thus, gluon/quark propagator $\sim 1/N$
vertex $\sim N$
index loop $\sim N$

counting has a simple geometric interpretation if we associate 2 dimensional surface with each vacuum bubble.

index loop $\Longrightarrow$ face (polygon)
gluon propagator $\Longrightarrow$ edge (where two polygons meet)
quark propagator $\Longrightarrow$ boundary of a hole (also an edge of a polygon)

If F = no. of faces (index loops)
E = no. of edges (propagators)
V = no. of vertices

Then no. of powers of N associated with graph is

$$\boxed{N^{F-E+V} \equiv N^{\chi}}$$

χ is called the Euler characteristic. It depends only on the topology of the surface, and not on how the surface is covered with polygons. (χ is a "topological invariant") χ can also be written

$$\boxed{\chi = 2 - 2H - B}$$

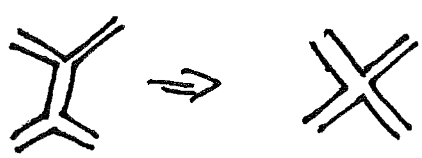
H = no. of "handles" on surface
B = no. of "holes" in surface (i.e., boundaries)

To show that $\chi = F - E + V = 2 - 2H - B$, we first show that χ is unchanged if we alter the way the surface is covered by polygons, then show that χ decreases by 2 if a handle is added to surface, and by 1 if a hole is punched in it.

i) To show that χ does not depend on how a given surface is covered by polygons, we note that one covering of the surface can be deformed into another by carrying out two elementary operations (and their inverses) a finite number of times.

- Edge $\rightarrow$ Vertex

e.g.



$E \rightarrow E-1$ (edge shrunk)
$V \rightarrow V-1$ (two vertices merge)
$F \rightarrow F$

Thus $\chi = F - E + V$ unchanged

- Face $\rightarrow$ Vertex

e.g.

$F \rightarrow F-1$ (face shrunk)
$E \rightarrow E-m$ (m sided polygon removed)
$V \rightarrow V-(m-1)$

$\chi = F - E + V$ unchanged

other operations (e.g. Face $\rightarrow$ Edge) are composites of above operations and inverses. Thus---

χ is a topological invariant

ii) To show $\chi = 2 - 2H - B$, we now know we can cover surface with polygons anyway we please, and then count $\chi = F - E + V$

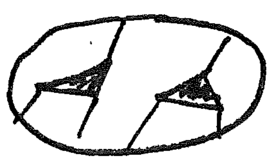
- Sphere

Divide into two polygons
$F = 2 \quad E = V$
$\Rightarrow \chi = 2$

- Hole

cut a hole in surface by removing one face
$$\left.\begin{array}{l} F \to F-1 \\ E \to E \\ V \to V \end{array}\right\} \chi \text{ reduced by one for each hole}$$

- Handle



Add a hole by cutting out two faces, and <u>identifying</u> edges of holes
$F \to F-2$
$$\left.\begin{array}{l} E \to E-m \\ V \to V-m \end{array}\right\} \text{if face is a } m\text{-gon}$$
$\Rightarrow$ χ reduced by two for each handle

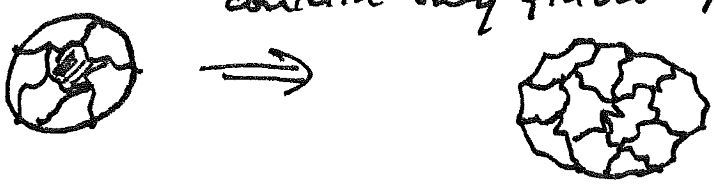
Hence, by induction,
$$\chi = 2 - 2H - B.$$

Leading graphs for $N \to \infty$ define surface which is topologically a sphere, and scale like N^2. Holes (quark loops) cost a factor of $1/N$, and handles cost a factor of $1/N^2$

That is, the leading vacuum bubbles involve gluons only:

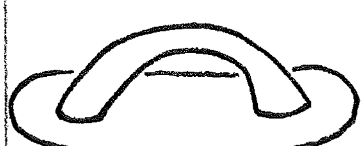
<u>$N \to \infty$</u>

- Leading connected vacuum bubbles $= O(N^2)$
  contain only gluon propagators



These graphs can be drawn on a plane (by removing one face of sphere and projecting) without any <u>crossing</u> of gluon lines

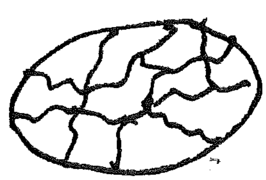
i.e. these graphs are <u>planar</u>



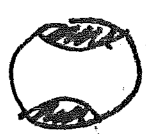
Nonplanar gluon exchanges, i.e. gluons which <u>cross</u>, are down by $1/N^2$

- Leading connected vacuum bubbles involving quark loops

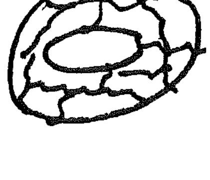
correspond to sphere with one hole, and are $O(N)$.



Projected onto a plane, this is a graph with one closed quark loop on boundary of graph, and planar gluon exchanges inside.



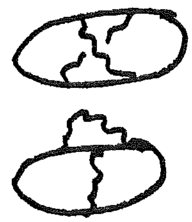
Down by $1/N$ is sphere with two holes, which, projected on plane becomes



i.e. one quark loop inside another, connected by planar gluon exchanges.

Down by still another power of $1/N$ (i.e., order N^{-1}) are ---

two quark loops inside



Nonplanar gluon exchanges (or gluons outside quark loop)

subtract away $U(1)$ "photon" exchanges (the "photon" is decoupled from gluons, couples only to quarks)

The above "counting rules" are all we will need to deduce the qualitative features of meson phenomenology in the large N limit.

C. Meson/Glueball Phenomenology for $N \to \infty$

We will now show that QCD for $N \to \infty$ has the following general features:

- In leading order, it is a theory of an infinite number of noninteracting stable mesons and glueballs.
- The decay amplitudes of mesons (glueballs) are $O(\frac{1}{\sqrt{N}})$ [$O(\frac{1}{N})$]. Amplitude for meson-meson (glueball-glueball) scattering is order $1/N$ ($1/N^2$).

- The "OZI rule" holds –
  - Quark annihilation into gluons
  - meson–glueball mixing
  - splitting of "nonets"

  are suppressed.

To demonstrate these results we will actually make an

Assumption: Color confinement holds for $N \to \infty$
– this is a reasonable assumption, and were it to fail, the $1/N$ expansion would be of no use in hadron phenomenology.

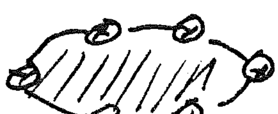
Our method is to study Green's functions of irreducible gauge-invariant operators (irreducible means cannot be written as product of two gauge singlets) which can create mesons or glueballs when acting on vacuum.

First consider mesons: Quark bilinears which create mesons have the generic structure

$$J = \left(\frac{N}{g^2}\right)\bar{q}\,\Gamma q$$

where Γ is a matrix acting on spin and flavor indices. The N/g^2 is included in J so that it is a conventionally normalized bilinear (recall $\mathcal{L}_{kin} \doteq \frac{N}{g^2}\bar{q}\, i\not{D} q$)

It is obvious that our counting of powers of N in vacuum bubbles applies also to Green's functions with insertions of J placed arbitrarily on closed quark loops.



Thus, to leading order in the $1/N$ expansion

$$\langle T(\underbrace{J - - J}_{m})\rangle_{\text{connected}} \sim N$$

Consider the two point function. $\langle T(JJ)\rangle \sim N$
If we assume confinement, we can show that the only singularities in this Green's functions are poles, in order N; i.e. the only intermediate states which contribute are one-particle states.

The key assumption is really that the bilinear J does not couple to degenerate states with degeneracy of order N (as it would if quarks were unconfined) Since the singularities of $\langle T(JJ)\rangle$ have strength of order N, they must then couple to states with coupling of order $\sqrt{N}$. That is, the singularities (or imaginary part) of the Green's functions are due to physical intermediate states

$$N \sim \text{Im}\langle TJJ\rangle \sim \sum_n |\langle 0|J|n\rangle|^2$$

So $\langle 0|J|n\rangle \sim \sqrt{N}$, if degeneracy is $O(1)$.

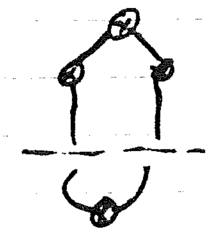
Thus, we redefine normalization of bilinears, so coupling is order one,

$$\tilde{J} = \frac{1}{\sqrt{N}} J \qquad \langle 0|\tilde{J}|n\rangle = O(1)$$

And now powers of N in scattering amplitudes of mesons coupling to $\tilde{J}$ are determined, through the reduction formula, by

$$\langle T(\underbrace{\tilde{J} - - \tilde{J}}_{m})\rangle_{\text{connected}} \sim N^{1-m/2}$$

Moreover, we conclude that, in leading order, $\hat{J}$ couples to one-particle states only. For suppose that $\hat{J}$ couples in $O(1)$ to e.g. a two-particle state. That state appears as an intermediate in



$$\langle T(\hat{J}\hat{J}\hat{J}\hat{J})\rangle_{\text{con}},$$

and is expected to couple to $T(\hat{J}\hat{J}\hat{J})$ with strength of order one, since it is kinematically allowed for the two particles to each scatter off a current and then be absorbed by the other current. All vertices are $O(1)$, since the vertices with spacelike momentum transfer are related by crossing to those with timelike momenta. Thus $\langle T(\hat{J}\tilde{J}\tilde{J}\tilde{J})\rangle$ should have a singularity with strength of order 1, but large N counting says it is order $1/N$

We conclude that only one-particle states couple to $\hat{J}$ in $O(1)$. The only other way satisfy the large N counting rules is to have N-fold degeneracy; i.e., unconfined quarks (Similar resoning shows that the only singularities of n-point connected amplitude are poles, in leading order.)

For the two-point function, we have

$$\langle T\tilde{J}(k)\tilde{J}(-k)\rangle = \sum_n \frac{a_n^2}{k^2-m_n^2} \qquad \langle 0|\hat{J}|n\rangle = a_n = O(1)$$

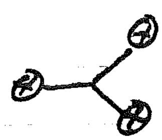
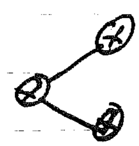
$\sum \otimes\!\!-\!\!-\!\!-\!\!\otimes$ — A sum of meson poles. But the sum must be infinite. If the sum terminated, we would have $\langle T\, \hat{J}(k)\hat{J}(-k)\rangle \xrightarrow[|k|\to\infty]{} \frac{1}{k^2}$

We know, however, from asymptotic freedom and perturbation theory, that the correct asymptotic behavior is $\sim k^2 \ln k^2$. This behavior can be generated only by an infinite sum.

In leading order, these mesons are noninteracting. The connected m-point amplitude scales like

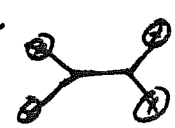
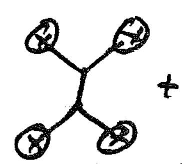
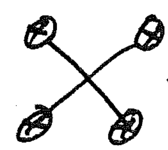
$$\langle T(\underbrace{\tilde{J}\tilde{J}\tilde{J}\cdots\tilde{J}}_{m})\rangle \sim N^{1-m/2}.$$

Consider, e.g., the three point function:

$\frac{1}{\sqrt{N}} \sim \langle T\,\hat{J}\hat{J}\tilde{J}\rangle_{con} = \Sigma$  $+\ \Sigma$  + higher order

Thus, the vertex  (meson decay amplitude) and also coupling of $\tilde{J}$ to two mesons is $O(\frac{1}{\sqrt{N}})$

Four-point function:

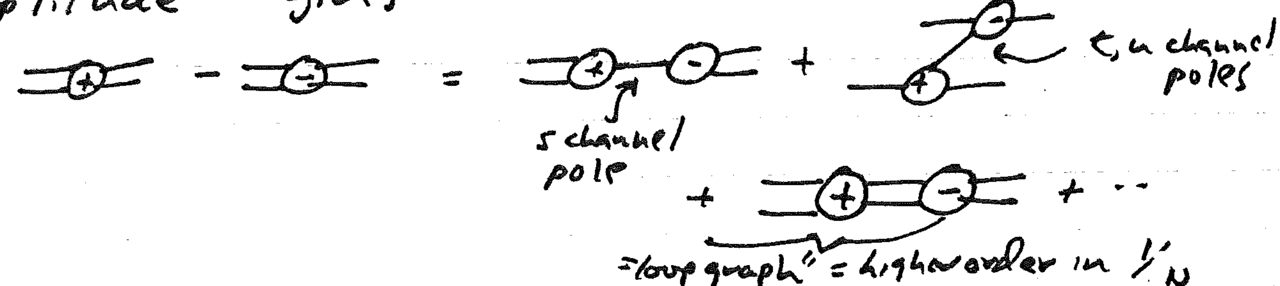
$\frac{1}{N} \sim \langle T\,\hat{J}\hat{J}\hat{J}\tilde{J}\rangle_{con} = \Sigma\Big($  +  +  +  $\Big)$

+ higher order

Two-body meson scattering amplitude is order $1/N$

All the leading m-point connected meson amplitudes can be expressed as tree graphs in an effective field theory. This follows from large-N counting and unitarity.

E.g., unitarity applied to $2\to 2$ scattering amplitude gives



Here ⊕, ⊖ are connected parts of S, S^+ and "internal lines" represent intermediate states which are summed over. The leading (order $1/N$) contributions to the discontinuity of the amplitude are one-particle (δ-function) singularities; these are the imaginary parts of pole graphs - i.e. tree graphs which are determined by $O(1/\sqrt{N})$ 3 meson amplitudes.

There is also the term X, which has no discontinuity; it is an $O(1/N)$ vertex in the effective field theory

Similarly, the leading (order $1/N^{m/2-1}$) contribution to the m-point meson amplitude can be expressed in terms of tree graphs with local m'-point vertices ($m'<m$) plus a local m-point vertex

Glueballs

It is easy to extend the above analysis to glueballs. As interpolating fields for glueballs, we use bilinears of the form

$$K = \left(\frac{N}{g^2}\right) F^2 \quad \text{spin structure suppressed}$$

(K is normalized to produce a gluon pair with the prop $= O(1)$.)

From large N counting rules for glue vacuum bubbles, we know that

$$\langle T(\underbrace{K \cdots K}_{m})\rangle_{\text{connected}} \sim N^2$$

Define $\tilde{K} = \frac{1}{N} K$, which has probability of order 1 to create a glueball state

$$\langle T(\underbrace{\tilde{K} \cdots \tilde{K}}_{m})\rangle_{\text{con}} \sim N^{2-m}$$

The two-point function is $O(1)$, and by the same reasoning as before

$$\langle T(\tilde{K}\tilde{K})\rangle_{\text{con}} = \sum_n \frac{b_n^2}{k^2 - m_n^2} \qquad \langle 0|\tilde{K}|n\rangle = b_n = O(1)$$

– an infinite sum over glueball poles

Since the 3-point amplitude is order $1/N$, the heavy glueballs have decay amplitudes of order $1/N$ and widths of order $1/N^2$ (narrower than mesons). The m-point connected scattering amplitude is order $1/N^{m-2}$. (They interact more weakly than mesons.)

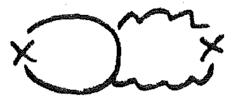
Glueball-Meson Mixing

Greens functions with n $\tilde{J}$'s and m $\tilde{K}$'s scale (for $n=0$) like

$$\langle T(\underbrace{\tilde{J} - \tilde{J}}_{n}\ \underbrace{\tilde{K} - \tilde{K}}_{m})\rangle_{\text{con}} \sim N N^{-n/2} N^{-m} \sim \frac{1}{N^{m+n/2-1}}$$

In particular

$\langle T(\tilde{J}\tilde{K})\rangle \sim \frac{1}{\sqrt{N}}$ So mesons and glueballs cannot mix in leading order (even if they have the same quantum numbers).

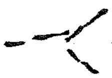
 Mixing is forbidden because it requires quark/antiquark in meson to annihilate into gluons, which costs a factor of $\frac{1}{\sqrt{N}}$.

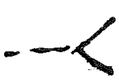
In general $1/N^{m+\frac{n}{2}-1}$ behavior of connected amplitude means that, in meson-meson scattering, producing a final state glueball instead of meson costs a factor of $1/\sqrt{N}$

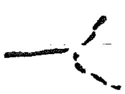
E.g., we have

---- glueball
—— meson

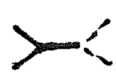
 $\sim N^{-\frac{1}{2}}$

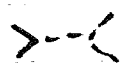
 $\sim N^{-1}$

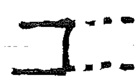
 $\sim N^{-1}$

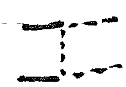
 $\sim N^{-3/2}$

 $\sim N^{-2}$

 $\sim N^{-2}$

 $\sim N^{-2}$

 $\sim N^{-2}$

 $\sim N^{-3}$

Decay amplitude of a glueball into mesons is also of order $1/N$. And amplitude for meson production in glueball-glueball scattering is $O(\frac{1}{N^2})$, comparable to amplitude for glueball final state.

The difficulty of producing glueballs in meson scattering in the $1/N$ expansion may help to explain that glueballs are not readily seen experimentally.

Summary: Attractive features of the $N=\infty$ limit

1) Simplicity:

The large N limit (if confining) is a theory of free, stable mesons and glueballs. It is sufficiently simple that one can imagine solving it exactly. Finding the exact S-matrix means finding all hadron masses (an infinite number of them).

Mesons (glueballs) have decay amplitudes of order $N^{-\frac{1}{2}}$ (N^{-1}) and interactions which to leading nontrivial order are given by the tree graphs of an effective field theory.

2) Narrow Resonances:

Pole dominance in low energy scattering of hadrons is observed, and predicted (for mesons) in the $1/N$ expansion. That resonances are very conspicuous in phenomenology is an indication that resonances are in some sense narrow. Also, expansion of amplitudes in an infinite number of tree graphs with narrow resonances may indicate Regge asymptotic behavior and linearity of Regge trajectories, and hence, perhaps, a field theoretic basis for the string picture of hadrons.

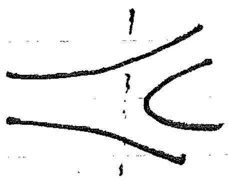
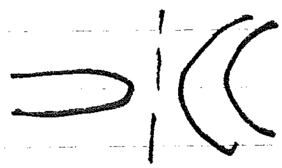
3) Suppression of exotics and the $q\bar{q}$ sea:

Since quark loops are down by $1/N$ in the expansion the $q\bar{q}$ "sea" content of a meson is suppressed by $1/N$. This provides a field theoretic understanding of the observed success of the valence quark model description of hadrons

Similarly, mesons with "exotic" quantum numbers, like $\bar{q}q\bar{q}q$ do not occur in the leading $1/N$ expansion, because the interactions of $\bar{q}q$ mesons are too weak (order $1/N$) to cause meson boundstates. And, indeed, exotics are very inconspicuous in phenomenology

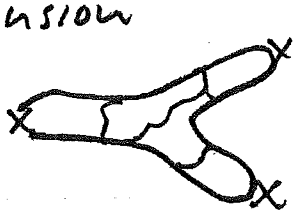
4) Zweig's Rule:

Zweig's rule (or the OZI rule) is a phenomenological principle which applies in various contexts. According to this rule, meson

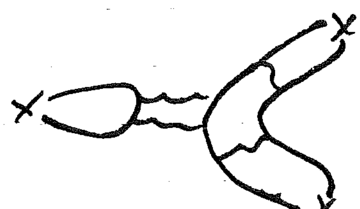
allowed   forbidden

amplitudes for which graphs can be divided into two parts by a boundary which cuts no quark lines are suppressed.

Zweig's rule is satisfied by the $1/N$ expansion



allowed



forbidden

The Zweig-forbidden amplitudes involve an extra quark loop, and so are suppressed by $O(1/N)$

Zweig's rule "explains", for example, the tendency of ϕ (which is mostly an $\bar{s}s$ state) to decay to $K\bar{K}$ rather than e.g. $\pi\rho$, and the general tendency of mesons to come in nonets; for example, the small ρ-ω splitting.

The 1/N expansion is not only an expansion around a simple limit (free field theory); it gives a reasonable qualitative description of many of the properties of real mesons.

The above discussion of mesons in the 1/N expansion follows closely that of —

E. Witten, Nucl. Phys. B160: 57 (1979)
S. Coleman, Erice '79

D. The η' mass in the 1/N expansion

(E. Witten, Nucl. Phys. B156, 269 (1979)

Does it make sense to regard the η' meson as the pseudo-Goldstone boson associated the the anomalous $U(1)_A$ symmetry? That is, can we think of $U(1)_A$ as being both spontaneously broken and explicitly broken (by the anomaly) so that the anomaly gives mass to a would-be Goldstone boson, which can be identified as the η'?

This question makes sense only if there is a systematic approximation scheme in which the anomaly can be "turned off" or treated as a small perturbation. The 1/N expansion is such a scheme.

The anomalous divergence of the axial current is

$$\partial^\mu J_{\mu 5} = \frac{1}{16\pi^2}\frac{g^2}{N} n_f F^a_{\mu\nu}\tilde{F}^{\mu\nu a}$$

where n_f is the number of flavors, and $g/\sqrt{N}$ is the conventionally normalized coupling, i.e. $g = O(1)$ in the $1/N$ expansion. So, formally, the anomaly turns off as $N \to \infty$.

If we regard the η' as a pseudo-Goldstone boson, how should we expect the η' mass to scale with N. There are two possible pictures of how the η' mass is generated:

i) <u>Instantons</u>

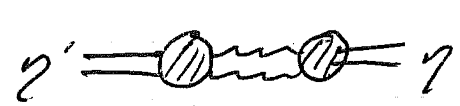
In our discussion of the $U(1)_A$ problem we emphasized that instanton effects broke the $U(1)_A$ symmetry. These scale with the instanton density, which is proportional, for weak coupling, to

$$\left[\left(\frac{1}{g^2}\right)^3 e^{-8\pi^2/g^2}\right]^N,$$

which is <u>exponentially</u> small for large N. But semiclassical reasoning breaks down for $g^2 = O(1)$, and we are left with no definite expectation for how the η' mass behaves as a function of N

ii) <u>Gluons</u>

There is also a



simple quark model explanation for how the η' becomes split from the other pseudoscalar mesons. Since it is an isospin singlet, the η' can annihilate into gluons; this mechanism can push up its mass.

Since the anomaly is a total derivative, it cannot generate the splitting to any finite order of perturbation, but, of course, there is no η', or any spontaneous breaking of the "$U(1)_A$ symmetry" to any finite order of perturbation theory. If QCD is confining in the large-N limit, as we have been assuming, then large gauge field fluctuations must be implicit in the sum of planar graphs, and it is reasonable to expect that the anomaly will have physical effects in the 1/N expansion.

Let us, then, assume that the vacuum energy of pure Yang-Mills theory is Θ-dependent in the large-N limit. (Gauge field fluctuations implicit in planar diagrams.) We will find support for the picture that the η' mass is generated by annihilation into gluons; it is $O(1/N)$, since annihilation into gluons is a Zweig-suppressed process.

Θ-dependence in Large-N Yang-Mills theory:

Consider $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_\Theta$

$$\mathcal{L}_\Theta = \Theta B \qquad B = \frac{g^2}{16\pi^2 N} F\tilde{F}$$

The vacuum energy density $E(\Theta)$ is

$$e^{-iE(\Theta)VT} \sim Z = \int dA\, e^{i[S_{YM} + \int d^4x\, \Theta B(x)]}$$

Therefore,

$$\frac{d^2}{d\Theta^2} E(\Theta) = \frac{i}{VT}\frac{d^2}{d\Theta^2}\ln Z = \frac{-i}{VT}\int d^4x\, d^4x' \langle T\, B(x)\, B(x')\rangle_{\Theta,\text{ connected}}$$

$$= -i\int d^4x \langle T\, B(x)\, B(0)\rangle_{\Theta,\text{ connected}}$$

To define this quantity, introduce an external momentum as an infrared cutoff:

$$\frac{d^2E}{d\Theta^2} = \lim_{K\to 0} -i\int d^4x\, e^{iKx} \langle T B(x) B(0) \rangle_{\Theta,\,\mathrm{con.}}$$

Because $B(x)$ is a total divergence, this limit vanishes in any finite order of perturbation theory:

$$\sim \lim_{K\to 0} K^2(\ln K^2 + (\ln K^2)^2 + (\ln K^2)^3 + \cdots)$$

But higher order terms vanish more and more slowly. Our assumption is that, if we sum up all planar diagrams for nonzero K, the accumulation of logarithms generates a quantity with a nonzero $K\to 0$ limit. That is, we assume that $d^2E/d\Theta^2$ is <u>not</u> identically zero for $N\to\infty$

Inclusion of Quarks:

Now, when we introduce massless quarks, physics must become independent of Θ; in particular $d^2E/d\Theta^2$ must vanish. But how is this possible, if the effects of quark loops are formally suppressed by $1/N$ in the $1/N$ expansion?

Assuming confinement, we know that the leading contribution to $d^2E/d\Theta^2$ due to quarks can be expressed as a sum over meson poles. By the analysis on p (4.31), we also know that

$$\langle 0 | B | \text{meson} \rangle = O(N^{-\frac{1}{2}})$$

So ---

$$\frac{d^2E}{d\Theta^2} = \left.\frac{d^2E}{d\Theta^2}\right|_{\substack{\text{no}\\ \text{quarks}}} + \lim_{K^2\to 0} \sum_{\substack{\text{mesons}\\ n}} \frac{N^{-1}c_n^2}{K^2 - M_n^2} + O(1/N^2),$$

where $\frac{1}{\sqrt{N}} C_n = \langle 0 | B | n \rangle$; C_n is order 1.

The two terms on the right hand side must cancel, which is possible only if there is a meson which couples to B and has a mass of order $1/N$. (Because of the $1/N$ in B, $d^2\mathcal{E}/d\theta^2|_{\text{no quarks}}$ is $O(1)$). Assuming there is only one such meson, the η', we have

$$\left.\frac{d^2\mathcal{E}}{d\theta^2}\right|_{\substack{\text{no}\\ \text{quarks}}} = \frac{C_{\eta'}^2}{N m_{\eta'}^2}$$

We can put this result in a more suggestive looking form by using the divergence equation

$$\partial^\mu J_{\mu A} = 2 n_f B$$

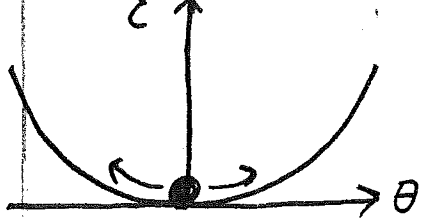
Thus
$$\frac{1}{\sqrt{N}} C_{\eta'} = \frac{1}{2n_f} \langle 0 | \partial^\mu J_{\mu A} | \eta' \rangle$$

and $\langle 0 | \partial^\mu J_{\mu A} | \eta' \rangle = -ip^\mu i p_\mu \sqrt{n_f}\, f_{\eta'} = m_{\eta'}^2 \sqrt{n_f}\, f_{\eta'}$

(the $\sqrt{n_f}$ is inserted so that $f_{\eta'}$ is normalized like f_π)

and we may write

$$\boxed{m_{\eta'}^2 = \frac{4 n_f}{f_{\eta'}^2} \left.\frac{d^2\mathcal{E}}{d\theta^2}\right|_{\text{no quarks}}}$$



This is a familiar looking kind of formula; it is of the type we would obtain in chiral perturbation theory for the mass of a pseudo Goldstone boson. It is clear what is going on. The η' is the PGB associated with the $U(1)_A$ current which rotates θ. But the "symmetry" $\Theta \to \Theta + \phi$ is explicitly broken by effects of pure Yang-Mills theory, and these effects give the η' its mass. The η' is light in the $1/N$ expansion because its f coupling is large, order N.

Physics is independent of Θ because, whatever the value of Θ, the η' field slides to the minimum of the potential $\mathcal{E}(\Theta)$. In this respect, the η' behaves exactly like an <u>axion</u>; it is really the axion of large-N QCD with massless quarks. Like the usual axion, it is weakly coupled because its f coupling is large

Couplings of the η' to glueballs can be determined by analogous current algebra reasoning. In leading order, glueball masses and couplings are functions of Θ. When quarks are introduced, Θ becomes replaced by an appropriately normalized η' field, the deviation from the minimum of the potential $\mathcal{E}(\Theta)$:

$$\Theta \to 2\sqrt{n_f}\,(\eta'/f_{\eta'}) .$$

So the Θ-dependence in leading order determines the couplings of the η'

To check the reasonableness of our formula for $m_{\eta'}$, lets estimate

$$\left.\frac{d^2\mathcal{E}}{d\Theta^2}\right|_{\text{no quarks}} \equiv \mathcal{M}^4 .$$

For $f_{\eta'}$ we use f_π, since these are equal in leading order in $1/N$ (they are generated by identical sets of graphs):

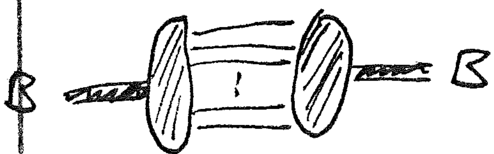
$$\mathcal{M}^4 \sim \frac{f_\pi^2 m_{\eta'}^2}{4n_f} \sim \frac{(.1\,\text{GeV})^2(1\,\text{GeV})^2}{12} \sim (200\,\text{MeV})^4 .$$

Seems reasonable. (η' is surprisingly <u>heavy</u> because f_π is <u>small</u>.)

E. Baryons in the 1/N Expansion

(E. Witten, Nucl. Phys. B160, 57(1979).)

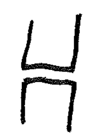
We conclude our analysis of the "phenomenology" of the large-N limit by considering baryons. The large-N limit of a baryon is slightly delicate, since a baryon necessarily contains N quarks.



The quarks in a baryon can interact by exchanging gluons. But, since the baryon must remain in a color singlet, the gluon exchange (order 1/N) is not compensated by an N from an index sum. On the other hand, there are many ways of choosing the connected cluster of quarks which interact.

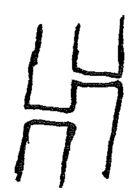
2 quark interactions:

$(N^2 \text{ pairs}) \times (\frac{1}{N} \text{ for each pair interaction}) \Rightarrow$ interaction energy $O(N)$



3 quark interactions:

$(N^3 \text{ triplets}) \times (\frac{1}{N^2} \text{ for two gluons}) = O(N)$



Etc.

All connected n-quark amplitudes contribute an interaction energy of order N. So the baryon mass can have a smooth $N \to \infty$ limit if it is $O(N)$:

$$M_{Baryon} \propto N,$$

which is also how M_{Baryon} would scale in the nonrelativistic quark model, in which the quarks have constituent masses.

Since the baryon is completely antisymmetric in color the quarks are effectively bosons — their (space) × (spin) × (flavor) wave functions are symmetric. So, in the large-N limit, the baryon consists of a large number of constituents interacting weakly. This is the natural domain of the Hartree approximation; that is, we should be able to regard each quark as moving in a potential self-consistently generated by the other N-1 quarks.

For simplicity (not because it is really essential to the logic) suppose that the quarks are sufficiently heavy that the baryon can be regarded as a nonrelativistic many-particle bound state, with binding energy small compared to NM_{quark}. The Hamiltonian for this system has the form --

$$H = NM + \sum_a \frac{\nabla_a^2}{2M} + \frac{1}{2N}\sum_{a\neq b} V^{(2)}(\vec{r}_a,\vec{r}_b) + \frac{1}{6N^2}\sum_{a\neq b\neq c} V^{(3)}(\vec{r}_a,\vec{r}_b,\vec{r}_c) + \cdots$$

where M is the quark mass, and the explicit factors of $1/N$ have been included in the 2, 3, -- -body interactions.

In the Hartree approximation, the baryon wave function can be factorized into one-body wavefunctions. Consider the baryon completely symmetrized in spin and flavor (the analog of the $\Delta^{3/2++}$). Its spatial wave function is completely symmetric, so we have

$$\psi_{\text{Baryon}}(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N) = \prod_a \phi_0(\vec{r}_a),$$

in Hartree approximation.

Now the one-body ground state ϕ_0 is determined variationally. We calculate

$$\langle B | H | B \rangle = N[\mathcal{M} + O(1)].$$

Here the m-body interaction terms give a contribution of order N, because each is a sum of N^m terms of order N^{1-m}. Since N factors out, the one-body wave-function $\phi_0(\bar{r})$ which minimizes $\langle B|H|B\rangle$ is completely <u>independent of N</u>. So we have reached our main conclusion, from which all else follows: In the large-N limit,

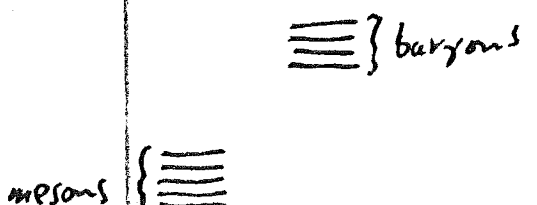
$$\text{Baryon Mass} = O(N)$$
$$\text{Baryon Size, Shape} = O(1)$$

In retrospect, this result justifies the use of the Hartree approximation — For large N the quark density of the baryon becomes arbitrarily large, so each quark interacts simultaneously with many others.

Now we can deduce how various properties of the baryon scale with N.

• <u>Excited Baryons: Splittings</u>

In the lowest excitations, one quark is in an excited state of the self-consistent potential (which is $O(1)$) while all others are in the ground state



$$\Psi^*_{Bary} = \frac{1}{\sqrt{N}} \sum_a \phi_1(\bar{r}_a) \prod_{b \neq a} \phi_0(\bar{r}_b).$$

Thus, the excitation energy is $O(1)$. The baryon spectrum starts high, but is as closely spaced as the meson spectrum.

Excited Baryons: Decay Amplitudes

An excited baryon can decay to a ground state baryon and a meson. To estimate the amplitude, consider

$$\langle B^* | \hat{J} | B \rangle$$

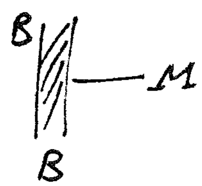
where $\tilde{J} = \frac{1}{\sqrt{N}} \sum_a \bar{q}^a q^a$ is normalized to create a meson with probability of order one. We have

$$\langle B^* | \tilde{J} | B \rangle \sim \underbrace{\left(\frac{1}{\sqrt{N}}\right)}_{\text{normalization of } \hat{J}} \underbrace{\left(\frac{1}{\sqrt{N}}\right)}_{\text{normalization of } B^*} \underbrace{N}_{\text{sum over colors}} = O(1)$$

The decay amplitude, and width, is $O(1)$, the same order as the splitting.

Baryon-Meson Coupling

Above, we saw that the on-shell coupling of an excited baryon to a baryon and meson is $O(1)$. But if we consider the off-shell coupling, the meson can couple to any one of the N quarks in the baryon - it need not deexcite the one excited quark.

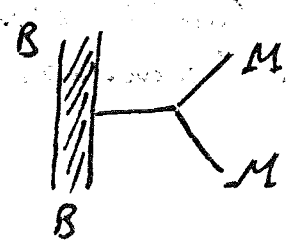


So the off-shell amplitude scales like

$$\langle B | \tilde{J} | B \rangle \sim \underbrace{\frac{1}{\sqrt{N}}}_{\text{normalization of } \hat{J}} \underbrace{N}_{\text{sum over colors}} = O(\sqrt{N})$$

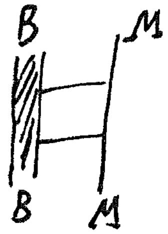
Baryon-Meson Scattering

From the baryon-meson coupling off-shell, we can infer the baryon-meson scattering amplitude.



If we consider scattering with fixed momentum transfer as $N \to \infty$ (so virtual meson propagator is $O(1)$), the scattering amplitude scales like

$$A(BM \to BM) \sim \underbrace{\sqrt{N}}_{\bar{B}BM \text{ vertex}} \; \underbrace{\frac{1}{\sqrt{N}}}_{MMM \text{ vertex}} = O(1) \quad \text{(fixed momentum transfer)}$$

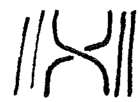


Since the amplitude is $O(1)$, scattering at fixed momentum transfer, unlike meson-meson scattering, is not pole dominated. The contribution from exchange of many mesons is unsurpressed.

Baryon-Baryon Scattering

The interaction between two baryons is $O(N)$

Gluon exchange scales like N by the same argument we gave to show that the baryon mass is $O(N)$



It is also possible for two baryons to exchange quarks without exchanging gluons. This is also $O(N)$ because there are only N such exchanges possible if the baryons are to remain color singlets.

Since the baryon mass is also $O(N)$, we see that N scales out, if we consider scattering at fixed velocity, scattering angle instead of fixed momentum transfer.

$$A(BB \to BB) = O(1) \quad \text{(fixed velocity, angle)}$$

Some processes involving baryons ~~do~~ not occur in the large-N limit; in fact, they are exponentially forbidden, forbidden in every finite order of the $1/N$ expansion. See Witten, op. cit., for details.

Summary

In the $N \to \infty$ limit mesons are narrow resonances; their interactions are weak and dominated by the exchange of single resonances.

But the decay amplitudes of baryon resonances are large; they interact strongly, and their interactions are not pole-dominated.

Thus, in the $1/N$ expansion, baryons are not at all like mesons. This is not a very successful prediction. In the real world baryon resonances are qualitatively similar to meson resonances. On the other hand, it is unsurprising that a baryon made of $N \to \infty$ quarks is qualitatively different from a meson made of one quark and one antiquark. In this respect, $N = \infty$ is apparently much different than $N = 3$.

Since N scales out of the baryon Hamiltonian, the $N \to \infty$ limit for baryons resembles the semiclassical $\hbar \to 0$ limit (compare our discussion of the Gross-Neveu model). In fact, as $N \to \infty$, the baryon mass diverges while its size stays fixed, so it becomes very large compared to its Compton wavelength, and becomes a nearly classical object.

This observation suggests a connection between the large N limit and the Skyrme model, in which a semiclassical description of the baryon is also given. Perhaps results from the Skyrme model which follow from semiclassical reasoning alone can be regarded as predictions of the large-N limit.