5. Lattice Gauge Theories

A. Formulation

References: Wilson, Erice '75
Kogut, RMP 55, 775 (1983)
Drouffe and Zuber, Phys. Rep. 102, 1 (1984)
Creutz, Quarks, Gluons, and Lattices (1983)

So far, our discussion of nonperturbative phenomena in QCD has been limited to qualitative considerations of the major issues — confinement and spontaneous breakdown of chiral symmetry. The lattice QCD program is an ambitious attempt to extract quantitative nonperturbative information from QCD such as mass spectrum, etc.

The main idea is to define a regularized theory with a short distance cutoff, and calculate masses etc. The quantum field theory we are really interested in is defined as the limit of a regularized theory with a short-distance (ultraviolet) cutoff a and a volume (infrared) cutoff L:

$$\text{Quantum Field Theory} = \lim_{\substack{a\to 0 \\ L\to\infty}} (\text{Cutoff Theory})_{a,L}$$

As explained in Section I.F, the $a \to 0$ limit is to be taken with correlation lengths ξ fixed. That is, the continuum limit is really the

$$\xi / a \to \infty \quad \text{limit.}$$

Quantum field theory is equivalent to the theory of 2nd (or higher)-order phase transitions, in which correlations become unbounded compared to an intrinsic length scale.

As discussed previously, continuum theories are expected to be renormalizable theories, which can be defined by tuning a finite number of relevant parameters, in accord with the concept of universality; the regulated theories can be divided into enormous classes which all give rise to the same continuum (i.e. infrared) behavior. For theories which are not asymptotically free, i.e., are infrared free, we expect that the only continuum field theory is free-field theory. Thus, interacting ϕ^4 theory and QED probably do not exist in four dimensions.

In the case of Yang-Mills theory (or QCD with massless quarks), a renormalizable theory with zero parameters, a unique continuum theory is expected to exist. To reach the continuum limit, we simply allow the gauge coupling g_0, the bare coupling defined at the lattice spacing, to approach zero, for the renormalization group eqn tells us that

$$\xi \sim a e^{+C/g_0^2},$$

so $\xi/a \to \infty$, as required, as $g_0 \to 0$.

Two different approaches to defining the regulated theory have been much discussed in the literature—

i) Euclidean Lattice:

From Feynman, we learn that any quantum theory problem can be "reduced to quadrature" - the evaluation of the functional integral. In the Euclidean lattice approach, the functional integral is approximated by ordinary integrals performed at each lattice site or link, and Green's functions are computed.

ii) Hamiltonian Lattice:

In the Hamiltonian lattice approach, time is treated as a continuous parameter, and the Hamiltonian is defined on a spatial lattice. Thus the regulated theory ($L < \infty$, $a > 0$) is an ordinary quantum mechanics problem with a finite number of degrees of freedom. A physical gauge-invariant subspace can be constructed, and the Hamiltonian acting on it diagonalized by, e.g., Rayleigh-Schrödinger perturbation Theory.

We will follow the Euclidean approach in these lectures. We will see that Euclidean Green's function can be expanded about a strong-coupling limit which has some of the qualitative features of the observed strong interactions — Quarks are confined, chiral symmetry is spontaneously broken, Deconfinement and restoration of chiral symmetry occur at finite temperature, etc. These qualitative feature are expected to survive at weak coupling (i.e., in the continuum limit).

The Hamiltonian approach will not be discussed here; see Kogut's review, cited above, or Susskind, Les Houches, 1976.

The Wilson Action

Universality suggests that any gauge-invariant action for the lattice theory is as good as any other, i.e., defines the same continuum theory. So we might as well choose the action to be as simple as possible. We define matter (e.g. quark) fields on the sites of a simple four-dimensional hypercubic lattice; the "gauge fields" occupy the links of the lattice which connect neighboring sites.

The local gauge group consists of independent transformations $\Omega(x) \in G$ at each site x. Gauge fields are defined on links so that it is possible to give the matter fields gauge-invariant nearest neighbor interactions. The gauge fields are the analogs of the continuum operators

$$U(y,x) = \underline{P}\exp[i\int_x^y dx^\mu A_\mu]$$

which transform bilocally:

$$U(y,x) \to \Omega(y)\, U(y,x)\, \Omega^{-1}(x).$$

The dynamical variables of the lattice theory are quarks, transforming as

$$q(x) \to \Omega(x)\, q(x)$$

and gauge fields $U_\mu(x) \in G$ (associated with the link ending at x and pointing in the μ direction) transforming as $\qquad U_\mu(x) \to \Omega(x+\mu)\, U_\mu(x)\, \Omega^{-1}(x)$

$\underset{x}{\bullet} \longrightarrow \underset{x+\mu}{\bullet}$ $\qquad$ ($x+\mu$ denoting the site displaced from x by one lattice spacing in the μ direction)

Quarks can have nearest-neighbor (kinetic) interactions of the form

$$\bar{q}(x+\mu)\, U_\mu(x)\, q(x),$$

consistent with the gauge symmetry.

We can take the "naive" continuum limit of a term in the lattice action by assuming that q and U change slowly on the scale of the lattice spacing a, i.e. that the change from site to site (or link to link) is of order a. This process is naive because in the quantum theory there will be fluctuations in the variables on the scale of a; it really is a classical limit. But the naive continuum limit makes some sense in an asymptotically free theory, which has nearly classical behavior at short distances.

The quark kinetic term on the lattice which has the "correct" naive continuum limit is

$$\tfrac{1}{2} i a^3\, \bar{q}(x)\, \gamma^\mu\, U_\mu^\dagger(x)\, q(x+\mu) \;+\; \text{h.c.}$$

To see this, write

$$U_\mu(x) = e^{iaA_\mu(x)} \sim 1 + iaA_\mu(x),$$

and expand in a

$$= \tfrac{1}{2} i a^3\, \bar{q}(x)\gamma^\mu \big(1 - iaA_\mu(x)\big)\big(1 + a\,\partial_\mu\big) q(x) + \text{h.c.} + \text{higher order}$$

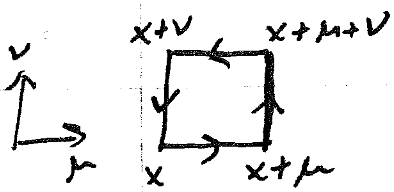
$$= \tfrac{1}{2} i a^3 \left[\bar{q}(x)\gamma^\mu q(x) + a\, \bar{q}\gamma^\mu(\partial_\mu - iA_\mu) q(x)\right] + \text{h.c.} + \cdots$$

The leading term is antihermitian, so we are left with

$$S_{\text{quark}} = \sum_x a^4 \bar{q} \gamma^\mu i(\partial_\mu - iA_\mu) q + O(a^5)$$
$$\sim \int d^4x \, \bar{q} i \not{D} q.$$

Next, consider the pure glue (Yang-Mills) lattice action. The obvious gauge-invariant operator involving only the link variables $U_\mu(x)$ is the lattice version of the Wilson loop:

$$\operatorname{tr} \prod_{(x,\mu) \in C} U_\mu(x)$$



The simplest kinetic term for the gauge field is the smallest Wilson loop, with C = a "plaquette" of the lattice.

So we define:

$$S_{\text{glue}} = \sum_{\underline{p}} S_{\underline{p}} \qquad (\underline{p} \text{ denotes a plaquette})$$

$$S_{\underline{p}} = -\beta/N \operatorname{Re} \operatorname{tr} U_\mu(x) U_\nu^\dagger(x) U_\mu(x+\nu)^\dagger U_\nu(x+\mu).$$

Here β is a coupling constant and $1/N$ is a normalization factor, inserted so that, if $G = SU(N)$, the plaquette action is $-\beta$ for all U's $= 1$. The real part is taken so that the Euclidean weight factor e^{-S} is always real.

It is obvious that the naive continuum limit of the lattice action is $\propto \operatorname{tr} F^2$, since this is the lowest-dimension gauge invariant operator, but let's show this anyway, to find the relation between β and the conventionally normalized Yang-Mills coupling.

Writing $U_\mu(x) = e^{iaA_\mu(x)}$, and expanding:

$$S_P = -\beta/N \,\mathrm{Re\,tr}\, e^{iaA_\mu(x)} e^{-iaA_\nu(x)} e^{-ia[A_\mu + a\partial_\nu A_\mu]} e^{ia[A_\nu + a\partial_\mu A_\nu]}$$

We can combine exponentials using

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y] + \text{higher order}}$$

Then,

$$S_P = -\beta/N \,\mathrm{Re\,tr} \left\{ e^{ia(A_\mu - A_\nu - \frac{ia}{2}[A_\mu, A_\nu])} e^{ia(A_\nu - A_\mu + a\partial_\mu A_\nu - a\partial_\nu A_\mu - \frac{ia}{2}[A_\mu, A_\nu])} \right\}$$

$$= -\beta/N \,\mathrm{Re\,tr}\, e^{ia^2(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]) + \cdots}$$

The product of the U's is a group element of the form $e^{i(\text{traceless})}$, so the leading term in the trace is

$$S_P = -\beta/N \frac{(ia^2)^2}{2} \mathrm{tr}\, F_{\mu\nu} F_{\mu\nu} = \frac{\beta}{2N} a^4 \,\mathrm{tr}\, F_{\mu\nu} F_{\mu\nu}.$$

This is not summed over (μ, ν), but is the term corresponding to the plaquette (x, μ, ν). The sum over plaquettes differs from independent sum over μ, ν by a factor of 2, so we have

$$S = \sum_P S_P \sim \beta/2N \int d^4x \, \tfrac{1}{2} \mathrm{tr}\, F_{\mu\nu} F_{\mu\nu},$$

with the summation convention reinstated. So we identify

$$\beta/2N = \frac{1}{g^2},$$

where g is the Yang-Mills coupling constant.

We call $2N/g^2$ β to emphasize the analogy with statistical mechanics. Small β = high temp = strong coupling implies that the link variables are disordered, i.e. fluctuate easily. For large β, the links fluctuate less.

Measure

To define Green's functions through the functional integral we must specify not only the action, but also the integration measure. Again, we are guided by gauge invariance. At each link, we perform an integral

$$\int dU.$$

The group measure should fill the group volume uniformly, so that it is not changed by a gauge transformation; we demand

"right-invariance": $\int dU\, f(U U_0) = \int dU f(U)$
"left-invariance": $\int dU\, f(U_0 U) = \int dU\, f(U)$

where U_0 is a constant element of the group, and f is any function. If we also demand linearity,

$$\int dU \left[a f(U) + b\, g(U) \right] = a \int dU f + b \int dU g,$$

and impose a normalization condition,

$$\int dU\, 1 = 1,$$

then the measure is uniquely defined. It is called the "Haar measure" or "invariant measure."

To construct the Haar measure, begin by considering the vicinity of the identity element of the group:

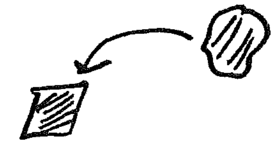
$$U = \mathbb{1} + i\alpha^a T^a.$$

In the vicinity of the identity define (up to normalization)

$$dU = d\mu(\alpha^1, \ldots \alpha^{N^2-1})$$

where $d\mu$ represents the ordinary Riemann-Lebesgue measure on the α^a's. This measure is invariant under left and right multiplication by an infinitesimal group elements which just corresponds to adding a constant to α^a

Now we can use the action of the group (e.g. on the right) to transport this measure to the vicinity of an arbitrary group element.



Write

$$U = (\mathbb{1} + i\alpha^a T^a)\, U_0$$

and again define $dU = d\mu(\alpha^1, \alpha^2, \ldots)$
that is, we can transport the neighborhood of U_0 back to the identity by right multiplication, so the measure near $\mathbb{1}$ induces a measure near U_0. This construction is obviously unique; up to normalization, there is only one right-invariant measure.

By analogous means, a left-invariant measure can be constructed. In fact, the right-invariant and left-invariant measures are the same. To see this denote the right-invariant measure by dU_r, so that

$$\int dU_r\, f(UU_0) = f(U),$$

(because $dU_r = d(UU_0)_r$)

Now define a new right-invariant measure dU_r' by

$$\int dU_r' \, f(U) = \int dU_r \, f(U, U)$$

Since the right-invariant measure is unique, we have

$$\int dU_r' \, f(U) = \int dU_r \, f(U, U) = \int dU_r \, f(U),$$

i.e., the right-invariant measure is also left-invariant.

(We've actually shown that the right-inv. and left-inv. measures give the same result for any <u>convergent</u> integral. Since the gauge group is compact, this suffices to guarantee that they are identical.)

Evaluation of group integrals is greatly simplified by the invariance of the measure; only the gauge singlet part of a function f can survive. Therefore, we won't need an explicit form for the measure to evaluate the integrals which occur in the strong-coupling expansion. First of all, we know that

$$\int dU \; U_{ij} = 0,$$

for only by being zero can this integral satisfy the invariance condition

$$\int dU \; U = \int dU \; V_1 U V_2^{+}.$$

(Therefore, a link cannot become "magnetized" and acquire an expectation value. This is a special case of Elitzur's theorem, about which we learned in Chapter 2; operators which are not gauge invariant cannot serve as order parameters in a lattice gauge theory – their vevs automatically vanish.)

The integral $\int dU\, U_{ij} U^{+}_{k\ell} = C_{ijk\ell}$ must be proportional to an invariant-tensor, so that

$$C_{ijk\ell} = V_{1ii'} V^{+}_{1\ell'\ell} V_{2j'j} V^{+}_{2kk'}\, C_{i'j'k'\ell'}$$

Normalization $\Rightarrow$ $\delta_{jk} C_{ijk\ell} = \delta_{i\ell}$, so we have

$$\int dU\, U_{ij} U^{+}_{k\ell} = \frac{1}{N} \delta_{jk} \delta_{i\ell}.$$

By analogous reasoning, we have

$$\int dU\, U_{i_1 j_1} \cdots U_{i_N j_N} = \frac{1}{N!} \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N}$$

It is clearly possible to integrate any string of U's and U^{+}'s.

Exercise 5.1

Evaluate $\int dU\; U_{ij} U^{+}_{k\ell} U_{mn} U^{+}_{pq}$

(There are two invariants with the right symmetry properties. Find them, then take traces to determine their coefficients.)

The measure in functional integrals is

$$\int (dU) = \prod_{\text{links}} \int dU_\ell$$

We haven't said anything about gauge fixing because it is unnecessary. The integration will pass over many configurations which are gauge equivalent, but the volume of the local gauge group is finite, and factors out of Green's functions.

B. The Strong-Coupling Expansion

The strong-coupling expansion is an expansion in powers of β. It is easily formulated, by expanding

$$e^{-S} = \prod_P e^{-S_P} = \prod_P \exp[\tfrac{\beta}{N} \operatorname{Re} \operatorname{tr} U_P]$$

$$\left(U_P = \prod_{\ell \in P} U_\ell \right)$$

in powers of β.

One important result is easily derived: Wilson loops show area law behavior in leading order in β. In fact, the strong coupling expansion has a finite radius of convergence (see below), so this result shows rigorously that quark confinement holds for sufficiently small β.

Let's calculate the leading contribution to $\langle A(C) \rangle$ where

$$A(C) = \frac{1}{N} \operatorname{tr} \prod_{\ell \in C} U_\ell .$$

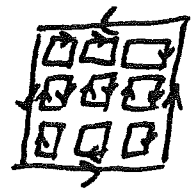
(The $1/N$ is included so that A is normalized to be 1 if all links U_ℓ are 1.)

We may write

$$e^{-S} = \prod_P \exp\left[\beta/2N \left(\mathrm{tr}\, U_P + \mathrm{tr}\, U_P^{\dagger} \right) \right]$$

← i.e. U_P traced in opposite sense

$$\langle A(C) \rangle = \frac{1}{Z} \int dU \, e^{-S} \left(\frac{1}{N} \mathrm{tr} \prod_{\ell \in C} U_\ell \right)$$



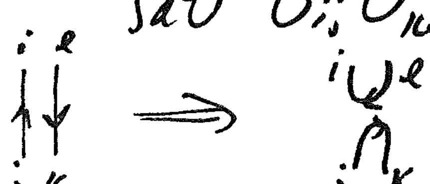
Since $\int dU \, U = 0$, we can get a nonvanishing contribution only by expanding e^{-S_P} so as to "tile" surface bounded by the loop C. The contribution requiring the minimal no. of powers of β arises from the surface with the minimal number of tiled plaquettes. Since only U_P with the correct sense can contribute, we get a factor of $\beta/2N$ from each tiled plaquette.

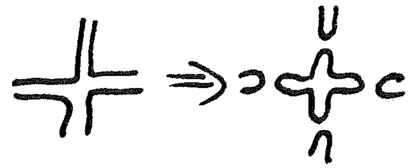
For the minimal surface, or, in fact, for any surface in which every plaquette of the lattice is tiled either one or zero times (expansion of e^{-S_P} is carried out only to order β), we obtain a contribution to $\langle A(C) \rangle$:

$$\underbrace{\left(\frac{1}{N}\right)}_{\text{Normalization}} (\beta/2N)^{\text{No. of Plaquettes}} \left(\frac{1}{N}\right)^{\text{No. of links}} (N)^{\text{No. of Sites}}$$

To see how the factors arise, it is convenient to introduce a graphical notation in which contracted indices are represented by lines; e.g.

$\int dU \, U_{ij} U^{\dagger}_{kl} = \frac{1}{N} \delta_{jk} \delta_{il}$ is represented by





Thus, when all link variables are integrated, we have a factor

$$\int dU = 1$$

for each link not included in the surface, a factor $1/N$ for each link of the surface (including the links of C) and a factor of $N = \mathrm{tr}\,\mathbb{1}$ for each site of the surface (including the sites connected by links of C) coming from contracting a string of δ's.

As we saw in Section 4.B, the number of faces (plaquettes) F, edges (links) E, and vertices (sites) V of a surface are related by

$$F - E + V = \chi$$

where χ, the Euler characteristic of the surface is

$$\chi = 1 - 2H$$

for a bounded surface with H handles and no holes. So the contribution to $\langle A(C) \rangle$ due to a given surface is

$$\frac{1}{N}(\beta/2N)^F (1/N)^E N^V = (\beta/2N^2)^F N^{\chi - 1}$$

$$= (\beta/2N^2)^{\text{No. of Plaquettes}} \left(\frac{1}{N^2}\right)^H.$$

This result is actually correct only for $N \geq 3$. For $N = 2$, the sense of the plaquette does not matter, since

$$\int dU \; U_{ij} U_{kl} = \frac{1}{2} \epsilon_{ik} \epsilon_{jl} \qquad (N = 2),$$

and we have a factor $\beta/N^2 = \beta/4$ for each plaquette (i.e., an extra factor of 2^F).

So we see that the contribution to $\langle A(C) \rangle$ due to a given surface in which each plaquette is tiled at most once depends only on the "area" of the surface (the no. of plaquettes) and its topology. We also see that "planar" surfaces dominate for large N, as in the continuum theory.

If the loop C lies in a plane, then there is a unique surface of lowest area ending on C, and we have

$$A(C) = (\beta/2N^2)^{\text{Area}/a^2} [1 + \cdots], \quad N \geq 3$$
$$= e^{-K(\text{Area})}$$

where $\boxed{K = -\frac{1}{a^2} \ln(\beta/2N^2) + \cdots, \quad N \geq 3}$

(For $N=2$, we have $K \sim -\frac{1}{a^2} \ln(\beta/4)$.)

The calculation of K in leading order is essentially trivial. At small β (high temperature) fluctuations of the link variables U_ℓ are unsuppressed, but fluctuations of the surface bounded by C are suppressed by powers of β — an example of order-disorder duality.

We have found it to be very simple to calculate the contribution to $A(C)$ due to an arbitrary surface (with each plaquette tiled at most once) But two complications make it difficult to carry out the strong-coupling expansion to very high order in β:

i) It becomes difficult to enumerate all surfaces with a given area.

ii) A given plaquette may be tiled many times. The corresponding $\int dU$ becomes difficult to evaluate.

Although calculation becomes difficult, it is possible to obtain estimates of high order terms which ensure that the expansion in β has a finite radius of convergence.

Since the plaquette action is bounded, and the group volume is finite, the contribution to $\langle A(C)\rangle$ due to any given surface which arises in order β^K (including surfaces with plaquettes tiled many times) satisfies a bound of the form

$$|\text{contribution}| < (C\beta)^K \quad , \text{ for some constant } C$$

To establish a finite radius of convergence, we need only show that the no. of surfaces contributing in order K grows no faster than exponentially

$$(\text{no. of surfaces})_{\text{order } K} < (C')^K,$$

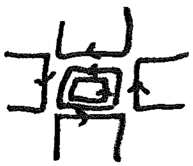
for then we have

$$|\langle A(C)\rangle|_{\text{order } K} < (CC'\beta)^K,$$

and the series converges for $\beta < (CC')^{-1}$

Such a bound on the number of random surfaces is derived by Osterwalder + Seiler, Ann. Phys. 110, 440 (1978).

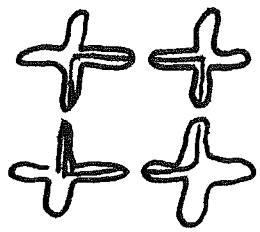
To illustrate contributions from nonminimally tiled plaquettes, we consider the $O(\beta)$ correction to the string tension, specializing to the case of $G=SU(3)$.



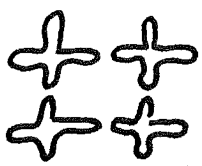
This contribution comes from tiling any plaquette of the surface twice in the same sense. Using

to denote $\int dU\, U_{ij} U_{kl} U_{mn} = \frac{1}{3!}\epsilon_{ikm}\epsilon_{jln}$,

this plaquette contributes



instead of



The $(\frac{1}{6})^4$ from the link integrations is cancelled by $(6)^4$ from contracting the ϵ's, so the correction has the form:

$$A(C) = (\beta/18)^{\text{No. of Plaq.}}\left[1 + \tfrac{1}{2}(\text{No. of Plaq.})(\beta/6) + \cdots\right]$$

from expanding e^{-S_P} to quadratic order

since plaq with extra tiling can occur anywhere

from $\beta/2N$ in plaq. action

$$\sim (\beta/18)^{\text{Area}/a^2}\, e^{\frac{1}{2}(\beta/6)\text{Area}/a^2} = e^{-\kappa\,\text{Area}}$$

where $\boxed{a^2\kappa = -\ln(\beta/18) - \beta/12 + \cdots}$

Character Expansion

In higher-order calculations, it is convenient to sum up all contributions where S_P acts on the same plaquette more than once. This can be done by a Fourier transform method, called a character expansion.

We have $e^{-S} = \prod_P e^{-S_P}$ where S_P is a gauge-invariant function of U_P, $S_P(VU_PV^\dagger) = S_P(U_P)$; S_P is a function of only the eigenvalues of U_P, said to be a "class function" because it depends only on the conjugacy class of U_P.

Any gauge-invariant (class) function can be expanded in characters:

$$\boxed{e^{-S_P(U_P)} = N(\beta)\left[1 + \sum_{\nu\neq 1} C_\nu(\beta)\, \chi^{(\nu)}(U_P)\right]}$$

$\chi^{(\nu)}$ is the character (i.e. trace) of the irreducible representation $R^{(\nu)}$ of the group. The characters are a complete orthonormal basis for gauge-invariant functions. ($N(\beta)$ is factored out, because it drops out of correlation functions.) It follows from Shur's lemma that the matrix elements $D_{ij}^{(\nu)}$ of the rep $R^{(\nu)}$ satisfy

$$\int dU\, D_{ij}^{(\nu)*} D_{kl}^{(\nu')} = \frac{1}{(\dim)_\nu}\, \delta^{\nu\nu'} \delta_{ik}\delta_{jl},$$

and, since $\chi^{(\nu)} = \sum_i D_{ii}^{(\nu)}$, we have

$$\boxed{\int dU\, \chi^{(\nu)*}(U)\, \chi^{(\nu')}(U) = \delta^{\nu\nu'}}$$

Moreover, since the D's furnish a representation, we have

$$\int dU\, D_{ij}^{(\nu)*}(U)\, D_{kl}^{(\nu')}(UU_0)$$

$$= \left[\int dU\, D_{ij}^{*(\nu)}(U)\, D_{km}^{(\nu')}(U)\right] D_{ml}^{(\nu')}(U_0) = \frac{\delta^{\nu\nu'}}{d_\nu} \delta_{ik} D_{jl}^{(\nu)},$$

or

$$\boxed{\int dU\, \chi^{(\nu)*}(U)\, \chi^{(\nu')}(UU_0) = \frac{\delta^{\nu\nu'}}{d_\nu} \chi^{(\nu)}(U_0)}$$

Since a direct product of irreducible representations can be decomposed into irreducible representation by a unitary change of basis, which does not change the trace, we have

$$\chi^{(\nu_1)}(U) \cdots \chi^{(\nu_k)}(U) = \sum_\lambda \chi^{(\lambda)}(U)$$

where λ runs over all irr. reps. $R^{(\lambda)}$ such that

$$R^{(\lambda)} \subset R^{(\nu_1)} \otimes \cdots \otimes R^{(\nu_k)}$$

It follows that

$$\boxed{\int dU\, \chi^{(\nu_1)}(U) \cdots \chi^{(\nu_k)}(U) = n_k = \text{no. of singlets in } R^{(\nu_1)} \otimes \cdots \otimes R^{(\nu_k)}}$$

In some cases (e.g. $SU(2)$) the coefficients $c(\beta)$ in the character expansion of e^{-S_B} can be written out in closed form, but for the purpose of the strong coupling expansion, it suffices to expand them in powers of β:

$$N(\beta) = \int dU\, e^{-S_B(U)}$$

$$N(\beta) c_\nu(\beta) = \int dU\, \chi^{(\nu)*}(U)\, e^{-S_B(U)}$$

Consider the case $G = SU(3)$, for which

$$e^{-S_P} = \exp(\beta/6)(\chi^{(3)} + \chi^{(\bar{3})})$$
$$= 1 + \beta/6(\chi^{(3)} + \chi^{(\bar{3})}) + \beta^2/72(\chi^{(3)} + \chi^{(\bar{3})})^2 + \cdots$$

and thus

$N(\beta) = 1 + \beta^2/36 + \cdots$ ($(3+\bar{3})^2$ contains the singlet twice)

$N(\beta) C_3(\beta) = N(\beta) C_{\bar{3}}(\beta) = \beta/6 + \beta^2/72 + \cdots$

(e.g. $(3+\bar{3})^2$ contains 3 once)

$N C_6 = N C_{\bar{6}} = \beta^2/72 + \cdots$

$N C_8 = \beta^2/36 + \cdots$

Higher representations, the 10, 15, etc. enter in higher order in β.

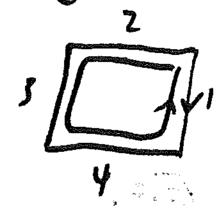
We can now carry out the strong coupling expansion as before, except that each plaquette is covered by the character of some irreducible rep. of the group, associated with the coefficient $C_r(\beta)$. The $O(\beta)$ contribution to the string tension which we computed earlier arises now from the $O(\beta^2)$ correction to C_3. The leading contribution is

$$A(C) = \left(\tfrac{1}{3} C_3\right)^{\text{No. of Plaq}} = \left[\beta/18\left(1 + \beta/12\right)\right]^{\text{Area}/a^2},$$

in agreement with our previous result. The factors of 1/3 arise from link integrals:

$$\int dU \, \chi(U)^* \chi(U U_0) = \tfrac{1}{3} \chi(U_0).$$

But why are there only as many factors of $1/3$ as plaquettes? It is because most of the link integrations are redundant, i.e., give identically 1 because the integrand is independent of U_e. For example, consider the one-plaquette loop:

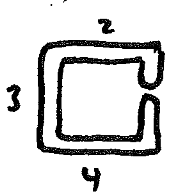


$$\int dU_1 dU_2 dU_3 dU_4 \; \chi^{(3)}(U_1 U_2 U_3 U_4) \; \chi^{(\bar{3})}(U_1 U_2 U_3 U_4)$$

using the identity,

$$\int dU \, \chi^{(\nu)*}(U U_1)\, \chi^{(\nu')}(U U_0) = \frac{\delta^{\nu\nu'}}{d_\nu} \chi^{(\nu)}(U_1^+ U_0),$$

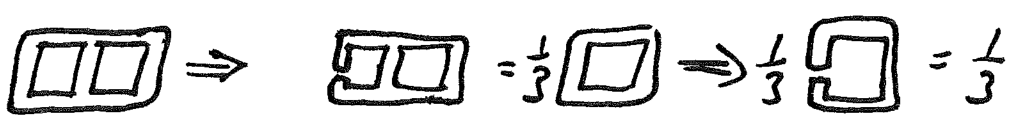
this becomes, after performing the U_1 integral,



$$= \frac{1}{3} \int dU_2 dU_3 dU_4 \; \chi^{(3)}(U_4^+ U_3^+ U_2^+ U_2 U_3 U_4) = 1$$

the U_1 integration reconnects the link integrals as shown and the closed index loop becomes $\frac{1}{3} \mathrm{tr}\, \mathbb{1} = 1$.

Similarly, to calculate two plaquette loop, we need do only two integrals:



In general, we need only perform the minimal number of link integrations which reduce the graph to a (possibly disconnected) tree; at that point, the remaining integrand is trivial. This redundance of the links is a consequence of gauge invariance; we have the freedom to fix the links of a tree by performing gauge transformations.

E.g.,

The remaining links are "pure gauge." We can start at one of the ends and transform each link in turn, as long as there are no closed loops.

Our factor of $N^{-2\,(\text{No. of handles})}$ now occurs because two more link integrations are required to reduce a surface with an extra handle to a tree, compared to a surface with the same no. of plaquettes.

It is convenient to express the strong-coupling series as an expansion in

$$\boxed{t \equiv \tfrac{1}{3} C_3(\beta)}$$

a presumably known function of β. From the expansions above, we have

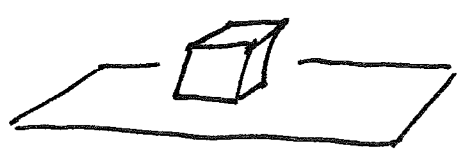
$$C_6 = \tfrac{9}{2} t^2 + O(t^3)$$

$$C_8 = 9t^2 + O(t^3)$$

In leading order, the string tension is

$$a^2 K = -\ln t$$

The next correction is $O(t^4)$, due to a bump on the surface, as shown:

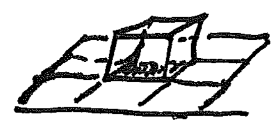


$$A(C) = (t\ )^{\text{Area}/a^2}\left[1 + 4(\text{Area}/a^2)\, t^4 \quad + \cdots\right]$$

The 4 is for possible orientations of the bump; the surface can fluctuate in two transverse directions, and either up or down. And the bump can occur on any plaquette.

Thus, $$a^2 K = -\ln t - 4t^4 + \cdots$$

In order t^5, there is a contribution from a bump with a floor. After doing link integrals which open up all plaquettes except the floor plaquette, we are left with a contribution to K



$$(Ka^2)_{O(t^5)} \sim -4t^4 (3t) \int dU_1 dU_2 dU_3 dU_4 \; \chi^{(3)}(U_P)\, \chi^{(3)}(U_P)\, \chi^{(3)}(U_P)$$

(where $U_P = U_1 U_2 U_3 U_4$) $= -12t^5$

So this contribution differs from that from the bump without floor by only the factor $c_3 = 3t$

$$a^2 K = -\ln t - 4t^4 - 12t^5 + \cdots$$

In order t^6, there are three types of contributions

i) The nonminimal bumps, which has $2 \times 4 = 8$ possible orientations.

ii) the disconnected graphs do not get completely divided out by Z, since some of the cubes contributing to Z contain plaquettes of the minimal surface.

iii) Bumps with floor, where floor is a character in rep 6 or 8.

To calculate these floored bumps, we need to evaluate

$$\int dU \; \chi^{(6)}(U)\,\chi^{(\bar 3)}(U)\,\chi^{(\bar 3)}(U)$$
$$\int dU \; \chi^{(8)}(U)\,\chi^{(3)}(U)\,\chi^{(\bar 3)}(U)$$

We can do these integrals by tensor methods. E.g., the 6 is a symmetric tensor; the rep acting on the 6 can be denoted

$$D^{(6)}_{ik,jl}(U) = \frac{1}{2}\left(U_{ij}U_{kl} + U_{kj}U_{il}\right)$$

which has trace

$$\chi^{(6)}(U) = \frac{1}{2}\left(\mathrm{tr}^2 U + \mathrm{tr}\, U^2\right)$$

Similarly, the 8 representation can be written

$$D^{(8)}_{ik,jl}(U) = U_{ij}U^*_{kl} - \frac{1}{3}\delta_{ik}\delta_{jl} \qquad \text{(Traceless tensor)}$$

which has trace

$$\chi^{(8)}(U) = \mathrm{tr}\, U\, \mathrm{tr}\, U^* - 1.$$

Now we can use our results for

$$\int dU \; U U^\dagger U U^\dagger \qquad \text{(Exercise 5.1)}$$

to evaluate the integrals.

Exercise 5.2

Calculate the string tension K in the strong coupling expansion to order t^6.

(The contribution of order t^6 is $+10t^6$. The calculation for SU(3) has been done to order t^{12} by Munster + Weisz, Phys. Lett. 96B, 119 (1980).)

Is it possible to extract continuum physics from the strong-coupling expansion? Perhaps, but it is a delicate matter; the continuum limit is the weak-coupling limit

At weak coupling, observables, like the string tension should scale as dictated by the renormalization group equation. If the two loop β function is

$$\beta = -g\left[\beta_0 (g^2/16\pi^2) + \beta_1 (g^2/16\pi^2)^2 + \cdots\right],$$

then (see page 1.103)

$$\ln \mu^2/\Lambda^2 = \int \frac{dg^2}{g\beta(g)} = \frac{1}{\beta_0 (g_\mu^2/16\pi^2)} - \frac{\beta_1}{\beta_0^2} \ln \frac{16\pi^2}{g_\mu^2} + \cdots$$

or $\quad \Lambda/\mu = (16\pi^2/g_\mu^2)^{\beta_1/2\beta_0^2} e^{-8\pi^2/\beta_0 g_\mu^2}$

For $SU(3)$, we have $\quad \beta_0 = \frac{11}{3}N = 11$

$$\beta_1 = \frac{34}{3}N^2 = 102$$

and the RG-invariant mass scale Λ is expressible in terms of the lattice coupling $g^2 = 6/\beta$ as

$$\boxed{\Lambda a = (16\pi^2/g^2)^{51/121} e^{-8\pi^2/11g^2}}$$

All renormalization group invariants should scale this way at weak coupling. So, for the string tension K, we should have

$$\sqrt{K}\, a = C\, (16\pi^2/g^2)^{51/121} e^{-8\pi^2/11g^2},$$

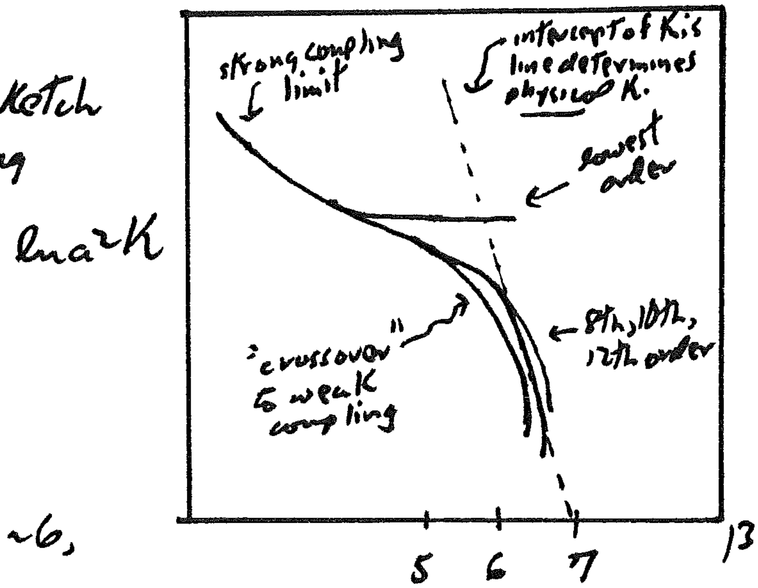
where $\quad C = \sqrt{K}/\Lambda$ is a physical quantity. To compare calculation and experiment, we should relate Λ appropriate for lattice coupling to e.g. $\Lambda_{\overline{MS}}$.

By a lattice weak-coupling perturbation theory calculation, it is found that

$$\boxed{\Lambda_{\overline{MS}} = 29\,\Lambda_{EL}}$$

(EL for "Euclidean Lattice")

A very rough sketch of the strong coupling expansion for K to order t^8, t^{10}, t^{12} is shown. The expansion seems to converge well up to $\beta \sim 5$. For $\beta \sim 6$, higher order terms are becoming important, and the



curve is bending over, as though trying to simulate (continuum) scaling behavior. In each order, there is a narrow band in β in which the slope of the curve agrees with the slope predicted by the one-loop β function. To get a crude prediction of the continuum value of K, we can find a straight line with this slope tangent to curve of strong coupling expansion to given order, and thus extract $C = \sqrt{K}/\Lambda$. Using 10th and 12th order in t, we find

$$\text{10th order} \quad \Lambda_{EL}/\sqrt{K} = 3.7 \times 10^{-3}$$
$$\text{12th order} \quad \Lambda_{EL}/\sqrt{K} = 3.5 \times 10^{-3}$$

Since $\sqrt{K} \sim 420$ MeV is measured (extracted from Regge slope), this calculation can be regarded as a nonperturbative determination of $\Lambda_{\overline{MS}}$ (ignoring renormalization of the string tension by quark loops), which we can compare to the values obtained earlier using perturbative QCD. The result, from the 12th order fit is...

$$\Lambda_{\overline{MS}} \sim 43 \text{ MeV};$$

a bit low, but more or less consistent with the perturbative estimate, given the errors. Recent Monte Carlo calculations give $\Lambda_{EL}/\sqrt{K} \sim (9 \pm 1) \times 10^{-3}$, or

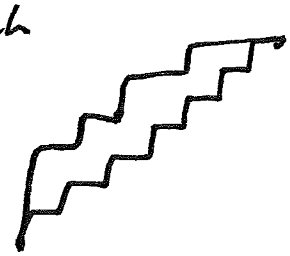
$$\Lambda_{\overline{MS}} \sim 110 \text{ MeV},$$

a more plausible value.

Roughening

Can strong coupling calculations be systematically improved by going to higher order, or is there an intrinsic limitation which plagues these calculations?

A serious limitation becomes evident when we consider "off-axis" Wilson loops which do not lie in a plane. Because the minimal area surface bounded by the loop must be composed of lattice plaquettes, the potential energy of quark sources at $(0,0)$ and (x,y) is

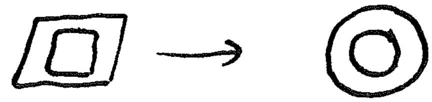


$$V = K(|x|+|y|)$$

instead of $K\sqrt{x^2+y^2}$; rotational invariance is badly broken by the leading strong coupling expansion on the lattice.

Moreover, the minimal surface contributing to an off-axis loop is highly degenerate; transverse quantum fluctuations of the surface, with amplitude comparable to the radius of the loop, are completely unsurpressed at strong coupling.

Rotational invariance should be restored as we approach the continuum limit; i.e. at sufficiently weak coupling.



Equipotentials

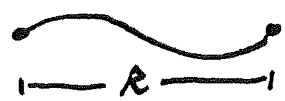
The thickness of the string eventually exceeds the lattice spacing, and the static equipotentials become isotropic. (This can be seen explicitly in a degenerate perturbation theory calculation in the Hamiltonian framework – see Kogut, op. cit.) This occurs at $\beta \sim 6$, at the "cross over" of $K(\beta)$. But restoration of rotational invariance means that the surface bounded by an on-axis Wilson loop must fluctuate wildly, so very high order terms in the strong coupling expansion of $\langle A(C) \rangle$ become important. Doubts are thus raised about the reliability of the strong coupling expansion in the region of interest, on the weak coupling side of the cross over.

In fact, the unbounded transverse fluctuations of the surface may generate a singularity in $K(\beta)$ at a critical value of β (the "roughening transition"). This is believed to be an essential singularity ("infinite-order" phase transition, see Drouffe-Zuber, op cit.) which blocks the continuation of the strong-coupling series to the weak-coupling region.

It may be possible to avoid the roughening problem by deriving the strong coupling expansion for the <u>off-axis</u> Wilson loop; the surface is already rough in lowest order, so no phase transition need occur. The off-axis expansion is more difficult than on-axis (there are many more surfaces to keep track of), but not completely intractable, perhaps.

We note in passing that the large transverse vibrations of the string generate a universal $1/R$ correction to the static potential. This prediction can be used as a check of scaling in e.g. Monte Carlo calculations.

This universal correction is the one-dimensional analog of the Casimir effect. We compare the energy of the zero-point transverse vibrations of a string with fixed ends a distance R apart with the energy of the zero-point transverse vibrations of a length R of string not subject to fixed-end boundary conditions.



$$E_0 = \sum_n \tfrac{1}{2}\omega_n = \sum_{n=1}^{\infty} \frac{1}{2}\frac{\pi n}{R}$$

is infinite, but if we subtract away energy of a free string,

$$\Delta E(R) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\pi n}{R} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{R\,dk}{2\pi}|k|$$

$$= \frac{1}{2}\frac{\pi}{R}\left(\sum_{n=1}^{\infty} n - \int_0^{\infty} n\,dn\right)$$

To evaluate the factor in parantheses, introduce a convergence factor:

$$\lim_{\lambda\to 0}\left[\sum_{n=1}^{\infty} n e^{-\lambda n} - \int_0^{\infty} dn\, n e^{-\lambda n}\right]$$

$$= \lim_{\lambda \to 0}\left[-\frac{d}{d\lambda}\left(\frac{1}{e^{\lambda}-1}\right) - \frac{1}{\lambda^2}\right] = -\frac{1}{12}$$

seen by expanding in powers of λ

So we have
$\Delta E(R) = \frac{-\pi}{24R}$ for each transverse degree of freedom of the string. In D dimensional spacetime, the string can vibrate in $D-2$ directions (i.e. there are $D-2$ directions transverse to Wilson's surface), and therefore

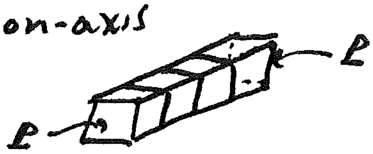
$$\boxed{V(R) = KR - \frac{D-2}{24\pi}\frac{1}{R} + O\left(\frac{1}{R^2}\right)}$$

the Casimir effect generates the potential $-1/12\pi R$ in 4-dim, independent of K.

Mass Gap

In the strong coupling expansion, we compute hadron (e.g. glueball) masses by finding asymptotic large distance behavior of appropriate correlation functions. Since expansion has finite radius of convergence, we know rigorously that there is a mass gap for sufficiently small β.

For example, consider the on-axis plaquette-plaquette correlation function:



$$\langle \mathrm{Re\,tr}\, U_P \ \mathrm{Re\,tr}\, U_{P'} \rangle_{\text{connected}}$$

the leading contribution comes from a tube running between the plaquettes.

there is a factor of $(\beta/2N^2) = \frac{1}{N} c_N \equiv t$ associated with each plaquette of the tube, so we have

$$\langle \mathrm{Re\,tr}\, U_P \;\mathrm{Re\,tr}\, U_{P'} \rangle_{\mathrm{con}} = t^F + \cdots .$$

And, on-axis, the minimal no. of plaquettes of a tube connecting plaquettes separated by distance R is

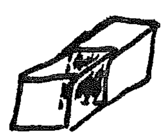
$$F = 4\frac{R}{a},$$

so

$$\langle \mathrm{Re\,tr}\, U_P \;\mathrm{Re\,tr}\, U_{P'} \rangle_{\mathrm{con}} \sim e^{-mR},$$

where

$$\boxed{m = -\frac{1}{a} 4 \ln t \; , \quad N \geq 3.}$$



In $SU(3)$, there is an $O(t)$ correction from filling in an interior plaquette of the tube with a triplet character

As for the corresponding correction to the string tension, this costs a factor of $c_3 = 3t$ without a compensating $1/3$ from the integrations, so we have

$$\boxed{am = -4\ln t - 3t + \cdots.}$$

To carry out high-order strong coupling calculations of the masses, we must sum over the location of P and P' (with separation fixed) to project out zero momentum states, and sum over their orientations to project out states of definite "spin" (i.e., irreducible representations of the cubic symmetry group). Masses have been calculated to 8th order in t. Using method described for the string tension, we can attempt to extract masses for $J^{PC} = 0^{++}, 1^{+-}, 2^{++}$ glueballs

Using $\sqrt{K} \sim 420$ MeV to set the mass scale, one finds (Drouffe-Zuber, op. cit.):

$$m_{0^{++}} \sim 900 \text{ MeV}$$

$$\frac{m_{2^{++}}}{m_{0^{++}}} \sim 1 \qquad \frac{m_{1^{+-}}}{m_{0^{++}}} \sim 2 ,$$

results which are in reasonable agreement with Monte Carlo calculations.

Again, the reliability of the strong coupling expansion might be enhanced by working "off-axis," at some cost in complexity.

C. Duality

Let us now consider gauge theories with the gauge group Z_N. The link variables are

$$U_\ell = e^{(2\pi i/N)m} \qquad m = 0, 1, \dots, N-1$$

and the invariant measure becomes

$$\frac{1}{N}\sum_{m=0}^{N-1}.$$

The Z_N gauge transformations act at the sites. For the action, we may choose any local gauge invariant functional; e.g.,

$$S = \sum_P S_P \qquad S_P = -\beta \operatorname{Re} U_P \qquad \text{– the Wilson action.}$$

Since the link variables take discrete values, the Z_N theory doesn't have a naive continuum limit, in which the $U_{\square}$'s are smooth. But it could have a quantum continuum limit; a second (or higher) order phase transition in which the correlation length diverges.

The Z_N gauge theories are interesting to study for several reasons. First, although they are abelian theories, they can help us to understand the dynamics of confinement, in which, as we saw in Chapter 2, Z_N, the center of $SU(N)$, plays an important role. Second, we will see that, for Z_N theories, there are convergent expansions about both strong and weak coupling limits, so the analysis is on a firmer footing than in the $SU(N)$ gauge theories. And the $U(1)$ gauge theory can also be studied as the $N \to \infty$ limit of the Z_N theory.

The convergent weak coupling expansion is derived by rewriting the theory in terms of new variables, performing what is known as a "duality transformation." We'll carry out the duality transformation explicitly in the simplest case, $N=2$.

Z_2 Gauge Theory

Link variables: $U_\ell = \pm 1$

Measure : $\frac{1}{2} \sum_{\pm 1}$ at each link

Action : $\sum_{P} -\beta U_P$, $\quad U_P = \prod_{\ell \in P} U_\ell$

The character expansion for the Z_2 theory is formulated by expanding

$$e^{-S} = \prod_{P} e^{\beta U_P},$$

using $$e^{\beta U_P} = \cosh\beta + U_P \sinh\beta, \quad \text{since } U_P^2 = 1$$
$$= \cosh\beta\,(1 + U_P \tanh\beta)$$
(1 and U_P are a complete set of "characters" for Z_2.)
the partition function is
$$Z = (\cosh\beta)^{N_P} \sum_{\{U_\ell\}} \prod_P (1 + U_P \tanh\beta),$$
(N_P = no. of plaquettes of lattice.)

which can be expanded in powers of $\tanh\beta$, the strong-coupling (small β) expansion.

Associated with each contribution in the strong coupling expansion is a closed surface — each plaquette of the surface is "occupied" by U_P, and all other plaquettes are "unoccupied"; there is a power of $\tanh\beta$ for each occupied plaquette. The idea of the duality transformation is to regard the binary choice occupied/unoccupied as the new dynamical variable carried by a plaquette.

We can write
$$Z = (\cosh\beta)^{N_P} \sum_{\{n_P=0,1\}} \sum_{\{U_\ell=\pm1\}} \prod_P (\tanh\beta\, U_P)^{n_P}.$$

where $n_P = 1$ for an occupied plaquette and zero for an unoccupied plaquette. We prefer a variable which takes values ±1, so we write
$$n_P = \tfrac{1}{2}(1 - V_P)$$
where $V_P = -1$ for an occupied plaquette, $+1$ for an unoccupied plaquette.

Thus,

$$Z = (\cosh\beta)^{N_P} \sum_{\{V_P=\pm1\}} \sum_{\{U_\ell=\pm1\}} \prod_P \left(U_P \tanh\beta\right)^{\frac{1}{2}(1-V_P)}$$

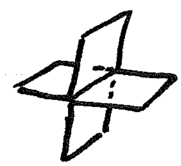
$$= \left[(\cosh\beta)(\sinh\beta)\right]^{N_P/2} \sum_{\{V_P\}} \sum_{\{U_\ell\}} \prod_P U_P (\tanh\beta)^{-\frac{1}{2}V_P}$$

Now, the sum over the U_ℓ's is easy to perform. Since

$$\tfrac{1}{2}\sum U_\ell = 0$$
$$\tfrac{1}{2}\sum 1 = 1$$

this sum imposes the <u>constraint</u> that each link must be contained in an even number of occupied plaquettes, or

$$\prod_{P \ni \ell} V_P = 1 \quad \text{for each } \ell .$$



$2(d-1)$ plaq. contain each link.

Thus,

$$Z = (\cosh\beta \sinh\beta)^{N_P/2} \sum_{\{V_P\}}{}' (\tanh\beta)^{-\frac{1}{2}V_P} .$$

where $\sum'$ indicates a sum subject to the above constraint.

We would like to find a simple way of writing this sum which automatically takes into account the constraint. The natural way to do this involves working on the <u>dual lattice</u>

The sites of the dual lattice are displaced by one-half lattice spacing in all directions, relative to the sites of the original lattice. Thus, in 4 dimensions, we have the one-to-one correspondences

Lattice		Dual Lattice	
Site	$\longleftrightarrow$	Hypercube	(site becomes body center.)
Link	$\longleftrightarrow$	Cube	(Link connecting two sites is dual to cube contained by two hypercubes.)
Plaquette	$\longleftrightarrow$	Plaquette	(Plaquette containing four links is dual to plaquette common to four cubes.)
Cube	$\longleftrightarrow$	Link	
Hypercube	$\longleftrightarrow$	Site	

The 6 plaquettes which contain a given link correspond on the dual lattice to the 6 plaquettes which make up the cube dual to the link. Thus, if we regard the V_P's as being carried by the plaquettes of the dual lattice, then the constraint becomes

$$\prod_{P^* \in C^*} V_{P^*} = 1.$$

(We use ℓ^*, P^*, C^* to denote link/plaquette/cube of dual lattice dual to C, P, ℓ of original lattice.)

This constraint is automatically satisfied if we regard V_{P^*} as the plaquette variable of a Z_2 gauge theory,

$$V_{P^*} = \prod_{\ell^* \in P^*} V_{\ell^*},$$

expressible in terms of link variables V_{ℓ^*} defined on dual lattice. The constraint is satisfied because each link ℓ^* of the cube C^* is shared by two plaquettes. (This is a discrete version of the identity $\mathrm{div}(\mathrm{curl}) = 0$.)

We may now rewrite the partition function as

$$Z = (\cosh\beta \sinh\beta)^{N_\ell/2} \sum_{\{V_\ell^* = \pm 1\}} \prod_{P^*} (\tanh\beta)^{-\frac{1}{2}V_{P^*}}$$

Except for the analytic factor out in front, which does not affect the thermodynamic properties, this has the same form as the original partition function

$$Z \propto \sum_{\{V_\ell\}} \prod_{P} e^{\beta' V_P}$$

where $\boxed{\beta' = -\frac{1}{2}\ln\tanh\beta}$.

We have discovered that the Z_2 gauge theory is <u>self-dual</u>, but the duality transformation interchanges strong and weak coupling.

For β small, there is a convergent strong-coupling expansion in $\tanh\beta$. Because of duality, there is also a convergent expansion for weak coupling (large β) in $\tanh\beta' \sim e^{-2\beta}$. (Inspite of the essential singularity at $1/\beta \to 0$, there is a sensible expansion in $e^{-2/(1/\beta)}$.)

Self-duality implies that phase transitions occur in pairs, at values of β related by duality, or at the self-dual point

$$\beta = \beta' = -\tfrac{1}{2}\ln\tanh\beta$$

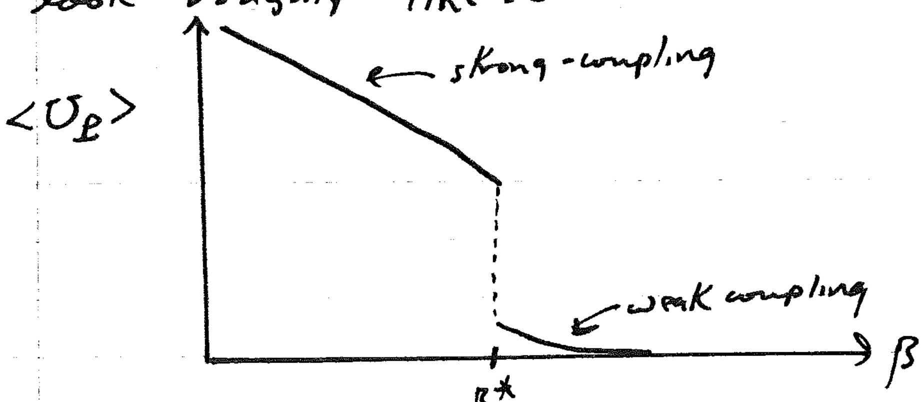
$$\Rightarrow (\beta = \ln x) \qquad x^{-2} = \tanh\ln x = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} \Rightarrow x^2 = 1 + \sqrt{2}$$

$$\boxed{\beta^* = \tfrac{1}{2}\ln(1+\sqrt{2}) \simeq .441 = \text{self-dual}}$$

A quantity which is easily calculated in the strong coupling expansion, and which can be used to locate phase transitions of the models, is the "average plaquette" $\langle U_p \rangle$. It is related to the free energy density, since

$$\langle U_p \rangle = \frac{1}{N_p} \frac{\partial}{\partial \beta} \ln Z$$

(and $N_p = \frac{1}{2} D(D-1) V = 6V$ where V = lattice volume).
The strong and weak coupling expansions for U_p look roughly like --



They do not agree at the self-dual point, and Monte Carlo calculations show that there is in fact a strongly first-order phase transition at $\beta = \beta^*$, and that the strong and weak coupling expansions are accurate on the left and right sides respectively of β^*.

Wilson Loop and 't Hooft Loop

Let us now consider how correlation functions (i.e., Wilson loops) are affected by the duality transformation in the Z_2 gauge theory

For the Wilson loop $A(C) = \prod_{\ell \in C} U_\ell$, we have

$$\langle A(C)\rangle = \frac{1}{Z}\sum_{\{U_\ell\}}\left(\prod_P e^{\beta U_P}\right)\left(\prod_{\ell\in C} U_\ell\right)$$

$$= \frac{1}{Z}(\cosh\beta\sinh\beta)^{N_P/2}\sum_{\{\tilde{V}_P=\pm1\}}\ \sum_{\{U_\ell=\pm1\}}\left(\prod_P (U_P\tanh\beta)^{-\frac{1}{2}\tilde{V}_P}\right)\left(\prod_{\ell\in C} U_\ell\right)$$

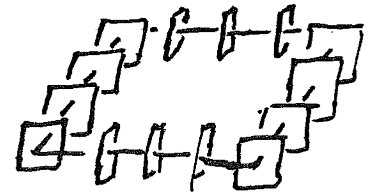
by the same manipulations as on pp 5.34-5.35. But now when we perform the sum over $\{U_\ell\}$ the constraint imposed on the $\tilde{V}_P$ (occupied and unoccupied plaquettes) is different than before: The links on C must be contained in an odd no. of occupied plaquettes. We have

$$\langle A(C)\rangle = \frac{1}{Z}(\cosh\beta\sinh\beta)^{N_P/2}\sum_{\{\tilde{V}_P\}}{}'\ \prod_P(\tanh\beta)^{-\frac{1}{2}\tilde{V}_P}$$

where the sum Σ' is subject to the constraint

$$\prod_{P\ni\ell}\tilde{V}_P = \begin{cases} 1 & \ell\notin C \\ -1 & \ell\in C\end{cases}$$

On the dual lattice there is a chain of disconnected cubes dual to the links of C, and the constraint becomes



$$\prod_{P^*\in c^*}\tilde{V}_{P^*} = \begin{cases} 1 & c^*\notin C^* \\ -1 & c^*\in C^*\end{cases}$$

(Bad notation: lower case c^* indicates cube dual to link; captial C^* denotes chain of cubes dual to loop C.)

If we regard $\tilde{V}_P{}^*$ as a Z_2 magnetic flux through P^* of a Z_2 gauge theory, we can interpret the constraint as meaning that the total flux through all cubes of the dual lattice is trivial (as required by div curl = 0) except for the cubes dual to the links of C, which contain Z_2 magnetic charges. Dual to the Wilson loop, which describes the world line of a static "electric" source, is an operator describing the world line of a of a static "magnetic" source. This is the lattice version of the 't Hooft loop (see Chapter 2).

In order to rewrite the sum over the $\tilde{U}_P$ in a way which automatically takes into account the constraint, we proceed as follows. First we arbitrarily choose a surface S (a set of plaquettes) which is bounded by the loop C. This surface has the property ---

i) Each $l \notin C$ is contained in an even no. (e.g., zero or two) of the $P \notin S$

ii) Each $l \in C$ is contained in an odd no. (e.g., one) of the $P \in S$

And thus, on the dual lattice ---

i) Each $c^* \notin C^*$ contains an even no. of $P^* \notin S^*$
ii) Each $c^* \in C^*$ contains an odd no. of $P^* \in S^*$

To impose the constraint, define

$$V_{P^*} = \prod_{l^* \in P^*} V_{l^*}$$

and

$$\tilde{V}_{P^*} = \begin{cases} V_{P^*} & P^* \notin S^* \\ -V_{P^*} & P^* \in S^* \end{cases}$$

or $\tilde{V}_{P^*} = (-1)^{K_{P^*}} V_{P^*}$ where $K_{P^*} = \begin{cases} 0 & P^* \notin S^* \\ 1 & P^* \in S^* \end{cases}$

the minus signs cancel for all cubes except those dual to the links of C, which contain an odd number of the plaquettes in S^*. It obviously does not matter how S is chosen, as long as it is bounded by C. And so we have

$$\langle A(C)\rangle = \frac{1}{Z}(\cosh\beta \sinh\beta)^{N_\ell/2} \sum_{\{V_\ell=\pm1\}} \prod_{P^*} e^{\beta'(-1)^{k_{P^*}} V_{P^*}}$$

where $\beta' = -\frac{1}{2}\ln\tanh\beta$. Or

$$\boxed{\langle A(C)\rangle_\beta = \langle B(C^*)\rangle_{\beta'}}$$

The Wilson loop is dual to the 't Hooft loop.

From the convergent strong-coupling expansion, we know that the Wilson loop has area-law behavior for small β. Now, by duality, we also know that the 't Hooft loop has area law behavior for large β. We won't show it in detail here, but it can be shown that our analysis of the phases of a gauge theory in Chapter 2 applies to this lattice theory. (See Ukawa-Windey-Guth, Phys. Rev. D 21, 1013 (1980).) In particular, it is not possible for the 't Hooft and Wilson loops to both show area law behavior, so we know that there must be (at least) two phases:

Small β	Large β
Wilson Area Law	Wilson Perimeter Law
't Hooft Perimeter Law	't Hooft Area Law

Now we understand the nature of the phase transition at the self-dual point $\beta = \beta^*$; it separates a strong-coupling "confinement" phase from a weak coupling "Higgs" phase

Actually, we do not need our old analysis to show that the 't Hooft loop has perimeter law behavior and strong coupling (and thus, by duality, that the Wilson loop has perimeter law behavior at weak coupling); we can show it explicitly.

$$< B(C^*) >_\beta = \frac{1}{Z} \sum_{\{V_{\ell^*}\}} \prod_{P^*} e^{\beta (-1)^{K_{P^*}} V_{P^*}}$$

$$= \frac{\sum_{\{V_{\ell^*}\}} \prod_{P^*} \left[1 + (-1)^{K_P^*} V_P^* \tanh\beta\right]}{\sum_{\{V_\ell\}} \prod_{P} \left[1 + V_P \tanh\beta\right]}$$

Both numerator and denominator can be expanded in powers of $\tanh\beta$, each contribution corresponding to a closed surface of occupied plaquettes.
Expanding both to 1st nontrivial order (order $(\tanh\beta)^6$) we have

$$= \frac{1 + (N_c - L)(\tanh\beta)^6 - L(\tanh\beta)^6 + \cdots}{1 + N_c(\tanh\beta)^6 + \cdots},$$

where N_c is the number of cubes of the lattice and L is the number of links in the contour C (i.e. its perimeter in lattice units)
The two terms in the numerator arise from the $N_c - L$ cubes with $\prod_{P^* \in C^*} (-1) \tilde{V}_{P^*} = +1$ and the L cubes with $\prod_{P^* \in C^*} \tilde{V}_{P^*} = -1$ respectively.

So the leading approximation is

$$\langle B(C^*)\rangle_\beta = 1 - 2L(\tanh\beta)^6$$

which is an approximation to $\exp[-2L(\tanh\beta)^6]$; in higher orders, the contributions from disconnected surfaces made up of independent cubes exponentiate, except for excluded volume effects which can be systematically taken into account. This is perimeter law behavior. The Wilson loop behaves similarly at weak coupling, but with $\tanh\beta$ replaced by $\tanh\beta' \sim e^{-2\beta}$.

Incidentally, it is clear that the 't Hooft loop can be constructed in an $SU(N)$ gauge theory. We simply replace the plaquette variable U_{P^*} by

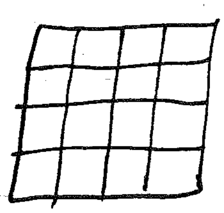
$$U_{P^*} \to z\, U_{P^*} \qquad z \in \text{center of } SU(N)$$

on a "surface" S^* dual to an arbitrary surface S bounded by a loop C. This corresponds to a Z_N magnetic source flowing along the contour C. But the $SU(N)$ theory is not self-dual, so the duality relation has not been very usefully applied to it.

We found that the Z_2 gauge theory has a deconfining phase transition at $\beta = \beta^*$. We would be very distressed if the $SU(N)$ theory had a similar transition, for that would mean that the strongly-coupled lattice theory bears no relation to the continuum theory; that, in fact, the continuum theory is not even confining

It is therefore important to consider the physics responsible for the Z_2 transition, and, in particular, whether this physics has an analog in the SU(N) theory.

Confinement in the strongly-coupled Z_2 theory arises as a consequence of "magnetic disorder" as described in Section 2B. The magnetic flux through a surface bounded by C is the product of magnetic flux through all plaquettes of the surface:



$$\prod_{\ell \in C} U_\ell = \prod_{P \in S} U_P \qquad (\text{"Stokes theorem"}).$$

Confinement occurs for small β because U_P fluctuates randomly between ± 1.

But for sufficiently large β, these fluctuations in U_P become too costly and freeze out. These fluctuations at the scale of the lattice spacing are the only magnetic excitations in the theory which are capable of disordering the system. Once they freeze out, the theory is magnetically ordered, and becomes deconfined.

Here is where the nonabelian theory is different. If the group is nonabelian, the contribution to the Wilson loop is <u>not</u> just the product of plaquette variables over a surface bounded by the loop. At large β, magnetic fluctuations at the scale of the lattice spacing freeze out, but fluctuations over a larger length scale (the hadronic scale) can still survive. These magnetic fluctuations continue to disorder the system, and cause confinement.

D. Phase Diagrams of Lattice Systems

In this section, we summarize a little of what is known about the phase structure of various lattice gauge theories. Good references are....

Ukawa-Windey-Guth, op. cit.

Fradkin-Shenker, Phys. Rev. D19, 3682 (1979).

Z_N gauge theory

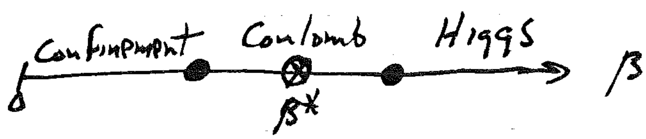
The Z_N gauge theory with the Wilson action is self-dual for only $N = 2, 3, 4$. But there is another form for the action (the Villain form) which is self dual for all N. (See Ukawa et al.) The effect of the duality transformation on the coupling is

$$\beta' = \left(\frac{N}{2\pi}\right)^2 \beta^{-1},$$

so the self-dual point is $\beta^* = \frac{N}{2\pi}$

From the strong-coupling expansion and duality we know that the theory is in a confinement phase for small β and a Higgs phase for large β. Furthermore estimates indicate that the theory is in a "Coulomb" phase with massless excitations in the vicinity of the self-dual point for $N \geq 5$ (Ukawa et al.) So there are (at least) two phase transitions (at values of β related by duality) for $N \geq 5$.

Z_N phases, $N \geq 5$:



All three of the possible phases consistent with the 't Hooft analysis occur in this theory.

As N increases to infinity, the "$U(1)$ limit" of the Z_N theory is approached. One finds (e.g. by Monte Carlo calculations) that the value of β at which the confinement/Coulomb transition occurs approaches a fixed position, independent of N, and the self-dual point $\beta^* = O(N)$ chases the other transition out to $\beta \to \infty$. So in the $U(1)$ limit, the Higgs phase disappears. There is a strong coupling confinement phase and a weak-coupling Coulomb phase.

$U(1)$ Higgs Model

Next, we consider the lattice version of the abelian Higgs model. We have link variables (gauge fields)

$$U_\mu(x) = e^{i\theta_\mu(x)}$$

as well as charge-n scalar fields defined on sites

$$\phi(x) = e^{in\theta(x)}.$$

Under a gauge transformation, $\Omega(x) \in U(1)$,

$$U_\mu(x) \to \Omega(x+\mu)\, U_\mu(x)\, \Omega(x),$$
$$\phi(x) \to \Omega^n(x)\, \phi(x).$$

A gauge invariant action is

$$S = -\beta \sum_P \operatorname{Re} U_P - \gamma \sum_{x,\mu} \operatorname{Re} \phi^+(x+\mu)\, U_\mu^n(x)\, \phi(x),$$

and the (Haar) measure is

$$\prod_{x,\mu}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_\mu(x)\right]\prod_x\left[\frac{n}{2\pi}\int_{-\pi/n}^{\pi/n} d\theta(x)\right].$$

What phase transitions does this system undergo as a function of β and γ? First consider the boundaries of the phase diagram. $\beta, \gamma \to 0, \infty$.

i) For $\beta = 0$, the gauge action is trivial; it leaves the $U_\mu(x)$'s unconstrained. Therefore, the nearest neighbor "spin" variables $\phi(x)$ and $\phi(x+\mu)$ are actually uncoupled, for $\phi^+(x+\mu)\, U_\mu(x)\, \phi(x)$ can be 1, regardless of the value of $\phi(x+\mu)$ and $\phi(x)$. So the whole system is trivial, and there can be no phase transition along the axis $\beta = 0$.

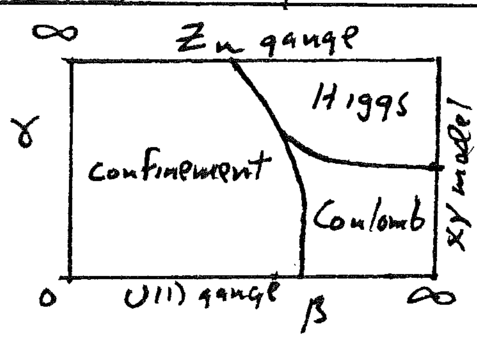
ii) For $\gamma = 0$, we have a pure $U(1)$ gauge theory, which we claimed has a phase transition separating a small-β confinement phase from a large-β Coulomb phase.

iii) For $\beta = \infty$, the plaquette variables U_P are frozen at 1, and, in an appropriate gauge, so are the $U_\mu(x)$'s. In this gauge, the model reduces to the XY-model, a model of planar spins with nearest neighbor interactions. The four-dimensional XY-model has a phase transition separating a large-γ magnetized phase from a small-γ unmagnetized phase.

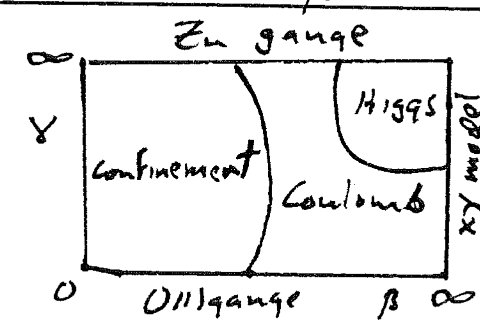
iv) For $\gamma = \infty$, the link variables $\phi^\dagger(x+\mu)\, U_\mu^n(x)\, \phi(x)$ are frozen at 1, and, in the unitary gauge, $\phi = 1$, U_μ^n is frozen at one. In this gauge, the values of the $U_\mu(x)$'s are restricted to the nth roots of unity, and the model reduces to a Z_n gauge theory. The Z_n gauge theory has either one phase transition $(n = 2, 3, 4)$ or two $(n \geq 5)$.

The phase transitions on the boundaries are expected to extend into the phase diagram, because the perturbations respect the same symmetries as the theories on the axes $\beta = 0$, $\gamma = 0$, $\beta = \infty$, $\gamma = \infty$. Thus, the simplest guess for the qualitative features of the phase diagram is ---

U(1) Gauge Theory Coupled to Charge-n Higgs:

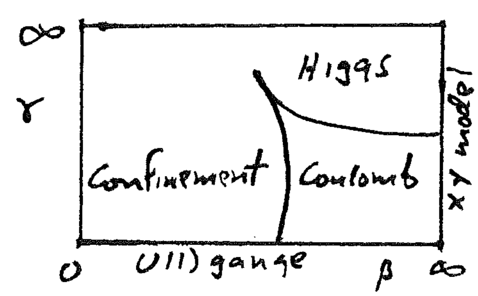


$n = 2, 3, 4$



$n = 5, 6, \ldots$

For $n = 1$, the $\gamma = \infty$ theory is trivial, and there is no phase transition along that axis. Thus, no phase boundary separates the "Higgs" and "Confinement" regions.



$n = 1$

This is a special case of the observation we made back in section 2G. If there is "matter" in the fundamental representation, then the string can break and the Wilson loop has perimeter law behavior. There is no mathematical criterion for distinguishing a "confinement phase" from a "Higgs phase." It is not surprising that it is possible to continue analytically from one to the other.

SU(N) Higgs Model

The SU(N) Higgs system can be analyzed similarly. Let's choose the Higgs field, defined on sites, to be an SU(N) matrix, so that it develops a "magnetization" the SU(N) symmetry is completely broken. We may choose the action to be

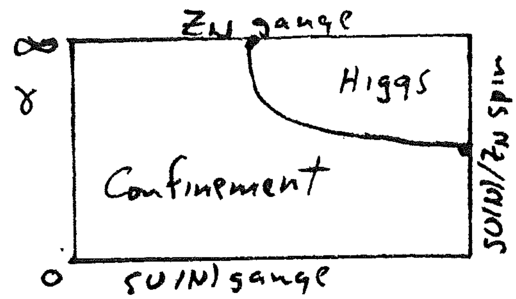
$$S = -\beta \sum_{P} \frac{1}{N} \qquad - \gamma \sum_{x,\mu} \chi^{(\nu)}[\phi^{+}(x+\mu)\, U_\mu(x)\, \phi(x)]$$

where $\chi^{(\nu)}$ is the character of the rep. $R^{(\nu)}$, which we will choose to be an $SU(N)/Z_N$ rep; that is, an SU(N) rep. with trivial N-ality

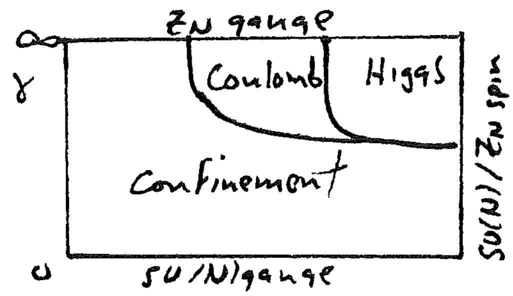
Now, considering limits as before...

i) $\beta = 0 \Rightarrow$ trivial
ii) $\gamma = 0 \Rightarrow$ pure SU(N) gauge theory $\Rightarrow$ no transitions
iii) $\beta = \infty \Rightarrow$ $SU(N)/Z_N$ spin system $\Rightarrow$ (presumably one) magnetic trans
iv) $\gamma = \infty \Rightarrow$ Z_N gauge theory

we can guess that the phase diagram looks like:



$N = 2, 3, 4$



$N = 5, 6, \ldots$

while, if the Higgs were in the fundamental representation, the transition line separating the Higgs and Confinement "phases" would terminate before reaching the $\gamma = \infty$ axis.

The Fundamental-Adjoint SU(2) Gauge Model

A test of whether a strong-coupling or Monte Carlo calculation has provided information about continuum physics is suggested by universality — the continuum physics should not depend on the detailed form of the lattice action. It is of interest, then, to do calculations using alternatives to the Wilson action.

There are two obvious ways of generalizing the action: we can include loops containing more than one plaquette (e.g., two plaquette loops), or we can take characters of U_P in higher representations. The latter possibility is considered here.

We may choose the plaquette action to have the form ...

$$S_P = -\sum_\nu \frac{\beta_\nu}{d_\nu} \mathrm{Re}\, \chi^{(\nu)}(U_P)$$

where $\chi^{(\nu)}$ is the character of the representation $R^{(\nu)}$, d_ν is its dimensionality, and β_ν is a coupling constant, normalized so that the minimum value of the νth term is $-\beta_\nu$. The Wilson action corresponds to keeping only the character of the fundamental representation.

Taking the classical continuum limit as on p. 5.7, we find

$$\frac{1}{g^2} = \sum_\nu \frac{t_\nu}{2d_\nu}\beta_\nu \quad \text{, where } t_\nu = \frac{\mathrm{tr}[T^{(\nu)}]^2}{\mathrm{tr}[T^{\mathrm{Fund}}]^2}$$

The naive continuum limit $g^2 \to 0$ is approached as any one of the $\beta_\nu \to \infty$, or as all β_ν scale to ∞ together with their ratios fixed. According to the notion of universality, continuum behavior should not depend on which β_ν gets large, or on the values of the fixed ratios.

A simple testing ground for universality is the fundamental-adjoint SU(2) model, in which the action is a linear combination of the characters of the doublet and triplet representations of SU(2):

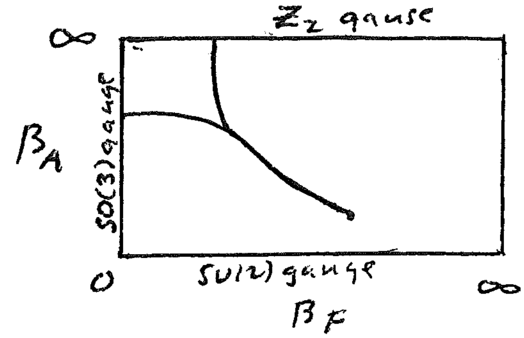
$$S_P = -\frac{\beta_F}{2}\chi^{(F)}(U_P) - \frac{\beta_A}{3}\chi^{(A)}(U_P),$$

and

$$\frac{1}{g^2} \cong \frac{\beta_F}{4} + \frac{2\beta_A}{3}.$$

The phase structure of this model has been studied by renormalization group and Monte Carlo methods.

Roughly, the phase diagram looks like this:



Let's try to understand this diagram. First, it is easy to understand the first order phase transition on the $\beta_A=\infty$ boundary. The action can be written

$$S_P = -\frac{\beta_F}{2}\,\mathrm{tr}\,U_P - \frac{\beta_A}{3}\left[(\mathrm{tr}\,U_P)^2-1\right];$$

the adjoint term is of course invariant under $U_P\to -U_P$. For $\beta_A=\infty$, U_P is frozen at the values ± 1, and the model reduces to the Z_2 gauge model, which has a first-order transition.

But what about the first-order phase transition on the $\beta_F=0$ axis, i.e., in the pure $SO(3)$ gauge theory? It turns out that this transition, like the Z_2 transition, can be understood as due to topological excitations localized at the lattice spacing; in this case, magnetic monopoles

Roughly, we can identify a Z_2 magnetic flux emanating from a cube of the lattice by defining

$$\eta_P = \mathrm{sign}\,\mathrm{tr}\,U_P$$

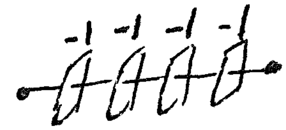
and

$$\eta_C = \prod_{P\in C}\eta_P .$$

We say the cube contains a monopole if $\eta_C=-1$.

Such monopoles can occur only because of the nonabelian structure of the gauge group.

If the gauge group is $SU(2)$, then plaquettes with $\eta_P = -1$ are energetically costly, and monopoles are connected together by strings which carry finite energy per unit length. But if the gauge group is $SO(3)$, U_P and $-U_P$ are degenerate, and a monopole can appear singly, without a string.



In the $SO(3)$ theory, monopoles can be expected to "condense" for small β_A, but to freeze out for sufficiently large β_A. The freezing out of monopoles probably drives the first-order transition.

The two first-order transitions on the boundary extend into the phase diagram and eventually meet. From this point there extends another first order line pointing toward the $\beta_A = 0$ (Wilson) axis. This line terminates before reaching the Wilson axis, but points toward the region of this axis ($\beta_F \sim 2$) where the rapid crossover from strong-coupling to weak-coupling behavior occurs. Perhaps this rapid crossover can be interpreted as a remnant of this transition.

If the weak-coupling limit is approached, starting with β_F, β_A small, then allowing $g^2 \to 0$ with β_A/β_F fixed, we see from the phase diagram that for some values of β_A/β_F, a first-order phase transition separates the weak-coupling and strong-coupling regions. Strictly speaking, this phase transition does not cast any doubt on the validity of the notion of universality, but it does raise serious concern about the relevance for continuum physics of strong-coupling expansions and Monte Carlo calculations done at intermediate coupling.

There may be many first-order phase transitions lurking about in a complicated multiparameter space, exerting an unpredictable influence in the intermediate coupling region. Such transitions are unsurprising if the plaquette action has nontrivial local minima. In fact, expanding around $U_P = -1$ in the fundamental-adjoint model, we find

$$\frac{1}{g^2} = -\frac{\beta_F}{4} + \frac{2\beta_A}{3},$$

so this is a local minimum ($g^2 > 0$) for $\beta_A > \frac{3}{8}\beta_F$. And all the phase transitions in this model occur in this region.

Even for the Wilson action, some nontrivial elements of the center are local minima for $N > 4$. Expanding around $U_P = z \in Z_N$, we have

$$-\mathrm{Re}\,\mathrm{tr}\, U_P = -\mathrm{Re}\,\mathrm{tr}\, z e^{i\epsilon T} = -\mathrm{Re}\,\mathrm{tr}\, z\left(1 + i\epsilon T - \tfrac{1}{2}\epsilon^2 T^2 + \cdots\right)$$
$$= -\left(N - \tfrac{1}{4}\epsilon^2\right)\cos(2\pi m/N) \quad \text{where } z = e^{i2\pi m/N}\,\mathbb{1}.$$

So $U_P = z$ is a local minimum for $\cos(2\pi m/N) > 0$.

And, indeed, Monte Carlo calculations have demonstrated the existence of first-order phase transitions in $SU(N)$ theories with the Wilson action for $N \geq 4$.

An improved understanding of the nature and location of such transitions will better enable us to extract continuum physics from strong-coupling and Monte Carlo calculations.