

①

Chiral Anomalies

PHYSICS 230

Anomaly means symmetry of a classical system ~~fails~~ fails to survive upon quantization of the system. Anomalies can occur in field theory because of the need for regularization. It may be impossible to regulate infinities and at the same time preserve all classical symmetries. (We have already seen this happen in the case of classical scale invariance.)

The simplest example of a chiral anomaly occurs for a massless fermion in $(1+1)$ -dimensions coupled to an abelian gauge field. For a free massless fermion, the Lagrange density is

$$\begin{aligned} \mathcal{L} = \bar{\psi} i \not{\partial} \psi &= i \psi^\dagger \gamma^0 (\gamma^0 \partial_0 + \gamma^1 \partial_1) \psi \\ &= i \psi^\dagger \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \gamma_5 \right) \psi \end{aligned}$$

where " γ_5 " = $\gamma^0 \gamma^1$, $\gamma_i^2 = 1$, gamma matrices are 2×2 . We define one-component fermions by

$$\gamma_5 \psi_{R,L} = \pm \psi_{R,L}.$$

Then

$$\mathcal{L} = i \psi_R^\dagger \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \psi_R + i \psi_L^\dagger \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \psi_L,$$

and the general solution to the Dirac equation is

$$\psi_R(x, t) = \psi_R(x - t) \quad \text{-- Right mover}$$

$$\psi_L(x, t) = \psi_L(x + t) \quad \text{-- Left mover}$$

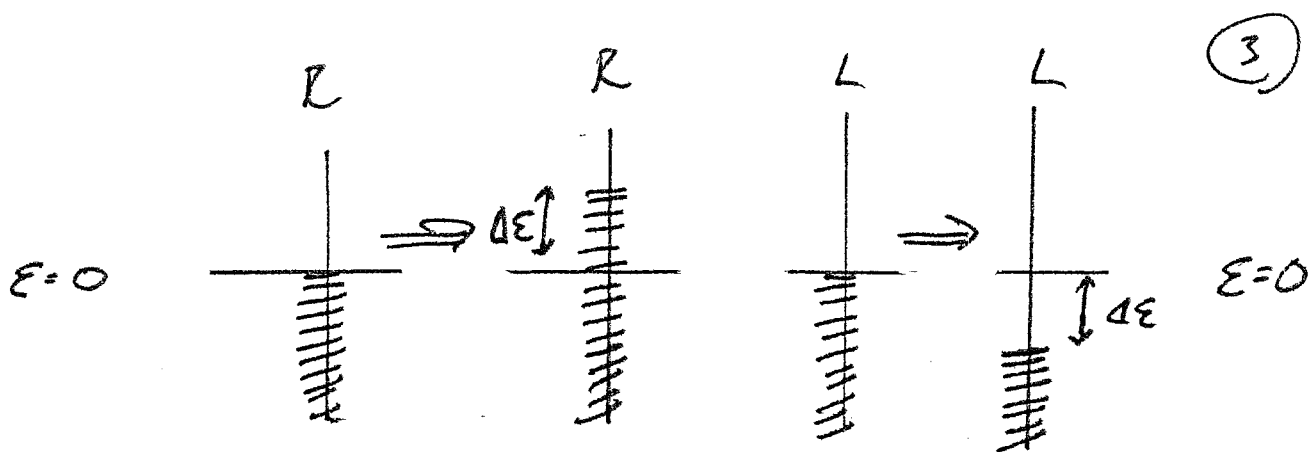
We now see what it means for a fermion to be "chiral" in $1+1$ dimensions. Unlike in $3+1$ dimensions, chirality makes no reference to helicity; there is no spin in one spatial dimension. Rather, a fermion is said to be right-handed if it propagates only to the right. This definition makes sense only for massless fermions, propagating at velocity c -- direction of propagation is same in all Lorentz frames. And it makes sense only in one spatial dimension, in which no smooth rotational symmetry (only a discrete "parity symmetry" interchanges right and left movers).

In the classical theory, R fermion number and L fermion no are independent constants of the motion; this remains true if the fermions are coupled to an abelian gauge field,

$$\mathcal{L} = \bar{\psi} i \not{D} \psi \quad D_\mu = \partial_\mu - ie A_\mu$$

But consider now the quantum mechanics of a Dirac fermion (R and L one-component fermions) in an external electric field. If an electric field E pointing right acts for time t , all right-movers gain and left-movers lose energy

$$\Delta E = eEt$$



If the initial state before the Electric field turns on is the Dirac vacuum, with all negative energy levels filled, then after time t , the Rth "Fermi level" has increased by ΔE and the Lth Fermi level has decreased by ΔE . Thus, R fermions are created, and so are L antifermions (holes). Since the one-dimensional density of states is $dp/2\pi$, the number density per unit length of R and L fermions are

$$\rho_R = \frac{eE}{2\pi} t \quad \rho_L = -\frac{eE}{2\pi} t$$

Thus, the "vector" fermion number is preserved

$$\dot{\rho}_V = \dot{\rho}_R + \dot{\rho}_L = 0.$$

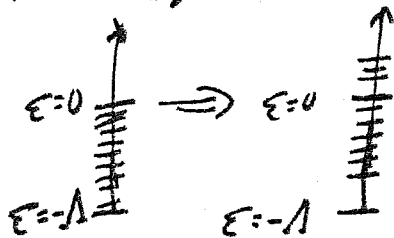
But the "axial" fermion number, also conserved at the classical level, changes, according to

$$\dot{\rho}_A = \dot{\rho}_R - \dot{\rho}_L = \frac{eE}{\pi} \quad \text{-- Axial Anomaly}$$

When an electric field is applied to the vacuum, it suffers dielectric breakdown, and fermion-

antifermion pairs are produced. The fermions produced are right movers and the antifermions are left movers, so the difference between R no. and L no. increases.

I said that lurking behind the anomaly are the infinities of field theory. Where are the infinities in this discussion? The only infinity is the depth of the Dirac sea.



If we regularize by cutting off the bottom of the sea at $E = -1$, then fermions are not produced. That is, as

many fermions pop out of the sea as holes open up in the sea (the total number of states is finite), so no net R no. is generated. The anomaly occurs only because the number of modes filling the sea is infinite, so a finite number of fermions can check out without any vacancies opening up. (The Dirac sea is a "Hilbert Hotel".)

Does an analogous breakdown of axial charge conservation occur in 3+1 dimensions? The above discussion seems peculiarly one-dimensional, relying as it does on the notion of R mover and L mover.

But the same discussion applies in $3+1$ dimensions to massless fermions in a magnetic field. We've seen (page 1.92) that massless fermions in a constant magnetic field occupy Landau levels labeled by integer n , with energy

$$H^2 = p_z^2 + (2n+1)eB - eB 2S_z$$

(for $\vec{B} = B \hat{z}$ in the $\hat{z}$ direction)

The motion in the x - y plane is quantized circular motion, and the fermions propagate along the z axis like effectively $(1+1)$ -dimensional fermions with mass

$$m^2 = (2n+1)eB - 2eB S_z$$

-- the levels with $n=0$ and $S_z = +\frac{1}{2}$ behave like massless $1+1$ dim fermions.

For massless $3+1$ dim fermions, chirality = helicity. So the RH fermion with $S_z = +\frac{1}{2}$ is a R mover along z -axis, and the LH fermion is a L mover. Now, turn on E field along $\hat{z}$, and our earlier discussion applies. We saw that the density of states per unit area in a Landau level is

$$g_n = \frac{eB}{2\pi}$$

Therefore, axial charge is created at a

(6)

rate

$$\dot{\rho}_A = \frac{eB}{2\pi} \frac{eE}{\pi} = \frac{e^2}{2\pi^2} \vec{E} \cdot \vec{B} \quad \text{-- Axial Anomaly}$$

An electric field applied to the vacuum causes dielectric breakdown -- i.e. pair production. And if a magnetic field is applied parallel to the electric field, the pairs created are chiral:

$$\begin{array}{ccc} B \Rightarrow & \Leftarrow (-) & (+) \Rightarrow \\ E \Rightarrow & & \end{array}$$

Fermions with charge + align their spins with $\vec{B}$, while antifermions (charge -) anti-align. The pair has the quantum numbers of

$$\bar{\psi}_R \psi_L$$

and carries $N_R - N_L = 2$.

The connection between the anomaly and the need for regularization is better appreciated if we compute the anomaly by a Feynman diagram method. Consider again a 1+1 dimensional fermion in an external gauge field

$$\mathcal{L} = \bar{\psi} i \not{D} \psi$$

Classically there are two conserved currents:

$$J_\mu = \bar{\psi} \gamma_\mu \psi = \bar{\psi}_R \gamma_\mu \psi_R + \bar{\psi}_L \gamma_\mu \psi_L$$

$$J_{\mu 5} = \bar{\psi} \gamma_\mu \gamma_5 \psi = \bar{\psi}_R \gamma_\mu \psi_R - \bar{\psi}_L \gamma_\mu \psi_L$$

To define matrix elements of these currents, we need a regulator. We might use dimensional regularization, except that it is unclear how to continue γ_5 to 2- ϵ dimensions. Instead, we'll use a Pauli-Villars regulator, chosen because it is guaranteed to satisfy the conservation of vector current (gauge invariance):

$$\mathcal{L} = \bar{\psi} i \not{D} \psi + \bar{\psi}_{\text{reg}} i \not{D} \psi_{\text{reg}} - M \bar{\psi}_{\text{reg}} \psi_{\text{reg}}$$

$\Rightarrow$ wrong statistics
 $\Rightarrow$ subtracts "heavy loops!"

Define:

$$J_{\mu, \text{reg}} = \bar{\psi} \gamma_\mu \psi + \bar{\psi}_{\text{reg}} \gamma_\mu \psi_{\text{reg}}$$

$$J_{\mu 5, \text{reg}} = \bar{\psi} \gamma_\mu \gamma_5 \psi + \bar{\psi}_{\text{reg}} \gamma_\mu \gamma_5 \psi_{\text{reg}}$$

- then matrix elements of currents are finite.

Use of eqn. of motion gives

$$\partial^\mu J_{\mu, \text{reg}} = 0$$

$$\partial^\mu J_{\mu 5, \text{reg}} = 2iM \bar{\psi}_{\text{reg}} \gamma_5 \psi_{\text{reg}}$$

involves only
 regulator field, but does not vanish

compute expectation value of $\partial^\mu J_{MS, reg}$ in external gauge field:

$$\langle \partial^\mu J_{MS, reg} \rangle = \text{diagram 1} + \text{diagram 2} + [\text{diagram 3} + \dots] \rightarrow O(1/M)$$

need to compute this.

-- power counting

$$\langle \partial^\mu J_{MS, reg} \rangle = \int \frac{d^2 k}{(2\pi)^2} \text{diagram} \frac{2iM \gamma_{reg} \gamma_5 \gamma_{reg}}{(k^2 - M^2) [(k+p)^2 - M^2]} A_\mu(p)$$

$$= 2iM(i)^2 e \int \frac{d^2 k}{(2\pi)^2} \frac{k [\gamma_5 (k+p+M) \gamma^\mu (k+M)] A_\mu(p)}{(k^2 - M^2) [(k+p)^2 - M^2]}$$

Evaluate trace using $k \gamma_5 (\gamma \sim \gamma) = 0$
odd number

$$k \gamma_5 \gamma_\alpha \gamma_\beta = 2 \epsilon_{\alpha\beta} \quad \gamma_5 = \gamma_0 \gamma_1 \quad \rightarrow \quad \epsilon_{11} = 1 = \epsilon_{00}$$

$$k [\quad] = 2M \epsilon^{\alpha\mu} p_\alpha$$

to keep leading piece of integral for $M \rightarrow \infty$, set $p=0$

$$\langle \partial^\mu J_{MS} \rangle = 4eM^2 \epsilon^{\alpha\mu} p_\alpha A_\mu(p) \int \frac{d^2 k}{(2\pi)^2} \frac{1}{(k^2 - M^2)^2}$$

$$\text{Integral} = \frac{i}{4\pi M^2} \leftarrow \text{from Wick rotation}$$

$$= -\frac{e}{\pi} \epsilon^{\alpha\mu} (-ip_\alpha A_\mu(p)) = -\frac{e}{\pi} \epsilon^{\alpha\mu} \partial_\alpha A_\mu$$

$$= -\left(\frac{e}{2\pi}\right) \epsilon^{\alpha\mu} F_{\alpha\mu}$$

(9)

we obtained

$$\partial^\mu J_{\mu 5} = -(\epsilon/2\pi) \epsilon^{\mu\nu} F_{\mu\nu}$$

-which agrees
computation.

with our earlier

Reinterpreting this calculation, we
have a statement about theory
of free fermions:

$$\int d^2x e^{ip \cdot x} \partial^\mu \langle 0 | T^* J_{\mu 5}(x) J_\nu(0) | 0 \rangle$$

$$= \frac{1}{\pi} \epsilon_{\alpha\nu} p^\alpha$$

Or, defining

$$\Gamma_{\mu\nu} = \int d^2x e^{ip \cdot x} \langle 0 | T^* J_{\mu 5}(x) J_\nu(0) | 0 \rangle$$

$$\begin{cases} p^\mu \Gamma_{\mu\nu} = \frac{i}{\pi} \epsilon_{\alpha\nu} p^\alpha \\ p^\nu \Gamma_{\mu\nu} = 0 \end{cases}$$

because Pauli-Villars
regularization
preserves
vector symmetry

This was the result with Pauli-Villars
regularization. The regulator broke the
axial symmetry, and a finite remnant
of the symmetry breaking remained
in the $M \rightarrow \infty$ limit. But is it
conceivable that there is some other regularization
scheme that preserves both the vector
and axial symmetries.

How does the calculation depend on the choice of regulator? We can always expand a graph in powers of external momentum, and each successive term in the expansion has improved UV behavior compared to the preceding term. Thus, after expanding to some finite order, the remainder is independent of the regulator. Therefore, the dependence on the choice of regulator enters in only some polynomial in external momentum. Thus, given a calculation of Γ using one regulator, we can obtain the result of calculating Γ using another regulator by merely adding some polynomial in external momentum. To explore whether there is any choice of regulator for which both

$$p^\mu \Gamma_{\mu\nu} = 0 \text{ and } p^\nu \Gamma_{\mu\nu} = 0$$

are satisfied, we therefore ask whether there is any choice of local counterterm that can be added to Γ that restores both conservation equations.

This search for a local counterterm can be viewed in another way. The need for a regulator to define Γ indicates that Γ is ill-defined, because of the singular nature of a product of currents at short distance ($x=0$). We need a definition of Γ , some renormalization

(11)

convention. But our freedom to redefine Γ order-by-order in perturbation theory is restricted by unitarity, which requires imaginary part of Γ in each order to be related to Γ in lower orders. The only part of Γ that can be convention-dependent in each order is a piece with no imaginary part -- a local counterterm, or polynomial in momentum.

We had

$$p^\mu \Gamma_{\mu\nu} = -\frac{i}{\pi} \epsilon_{\nu\alpha} p^\alpha, \quad p^\nu \Gamma_{\mu\nu} = 0$$

Now consider adding a counterterm

$$\Gamma'_{\mu\nu} = \Gamma_{\mu\nu} + \Gamma_{\mu\nu}^{\text{c.t.}}$$

But $\Gamma_{\mu\nu}^{\text{c.t.}}$ is required by parity and Lorentz invariance to be a dimensionless "pseudo tensor"; the only possibility is

$$\Gamma_{\mu\nu}^{\text{c.t.}} = c \epsilon_{\mu\nu}$$

$$\text{Now } p^\mu \Gamma'_{\mu\nu} = -\frac{i}{\pi} \epsilon_{\nu\alpha} p^\alpha - c \epsilon_{\nu\alpha} p^\alpha = 0$$

$$\text{if } c = -i/\pi$$

But, then,

$$p^\nu \Gamma'_{\mu\nu} = \frac{i}{\pi} \epsilon_{\mu\alpha} p^\alpha$$

We succeed at restoring axial symmetry only at the cost of spoiling vector symmetry.

Now we understand what is really meant by an anomaly -- that there is no choice of local counterterms such that all classical symmetries are preserved.

It is evident that anomalies are always finite, for the infinite part of a Feynman graph is always a polynomial in external momentum, and can be removed by a local counterterm. It is also clear that the piece of $\Gamma_{\mu\nu}$ that contributes to the anomaly is not a polynomial in external momentum. What, precisely, is the structure of Γ ?

The most general possible form consistent with Lorentz invariance and parity is

$$\Gamma_{\mu\nu}(p) = f(p^2) \epsilon_{\mu\nu} + g(p^2) p_\mu \epsilon_{\nu\alpha} p^\alpha + h(p^2) p_\nu \epsilon_{\mu\alpha} p^\alpha$$

The two conditions

$$p^\mu \Gamma_{\mu\nu} = -\frac{i}{\pi} \epsilon_{\nu\alpha} p^\alpha, \quad p^\nu \Gamma_{\mu\nu} = 0$$

$$\text{require } -f(p^2) + p^2 g(p^2) = -\frac{i}{\pi}$$

$$f(p^2) + p^2 h(p^2) = 0$$

So we have ---

$$\Gamma_{\mu\nu}(p) = f(p^2) \left(\epsilon_{\mu\nu} + \frac{p_\mu}{p^2} \epsilon_{\nu\alpha} p^\alpha - \frac{p_\nu}{p^2} \epsilon_{\mu\alpha} p^\alpha \right) - \frac{i}{\pi} \frac{p_\mu}{p^2} \epsilon_{\nu\alpha} p^\alpha$$

-- the expression multiplying $f(p^2)$ has both symmetries; the additional term violates the axial symmetry. It indeed is nonpolynomial -- it has a pole at $p^2 = 0$.

It is rather remarkable that we can infer from the anomaly equation that Γ has a $1/p^2$ pole. Since only massless particles can generate a singularity at zero momentum, we see that the anomaly really arises only in theories with massless particles.

What happens if fermions have explicit mass

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi$$

then axial symmetry is spoiled even at the classical level

$$\partial^\mu J_{\mu 5} = 2im \bar{\psi} \gamma_5 \psi$$

And, in the limit $p \ll m$

$\partial^\mu J_{\mu 5}$  cancels exactly the corresponding regulator loop, so there is no term linear in momentum, for $p \rightarrow 0$