Physics 230bc
Elementary Particle Theory

Lecture #1

John Preskill

Topic: nonperturbative methods in quantum field theory

- Topological defects: vortices, monopoles, instantons
- Quark confinement:
  Wilson loops, 't Hooft loops, magnetic disorder
- Chiral symmetry and anomalies:
  Triangle anomaly, U(1) problem, global anomalies, realizations of chiral symmetry.

If time allows:
- Strongly coupled supersymmetric gauge theories and duality.

Prerequisite: Quantum field theory, including
Feynman rules
Quantization of gauge theories
Path integrals

Requirements: Problem sets

Recommended Book: S. Weinberg
Quantum Theory of Fields, Vol II (~\$60)

Grader: Costin Popescu

Grading: Pass - Fail

Meetings: 469 Lauritsen W 4:30 - 6:00
F 4:00 - 5:30

Object of Today's lecture: deepen our grasp of the concept of a local symmetry

The (nonabelian) prototype: QCD

QCD: qft's greatest triumph!

qft: causality, unitarity, gauge inv, renormalizability] qualitatively new consequences: asymptotic freedom, quark confinement

Paving the way

- Quark model: Hadron resonances $\hookrightarrow$ eightfold way $\hookrightarrow$ quarks

  $\bar{q}q$, qqq] "no exotics" $\to$ Are quarks fundamental? Current algebra

- Color:

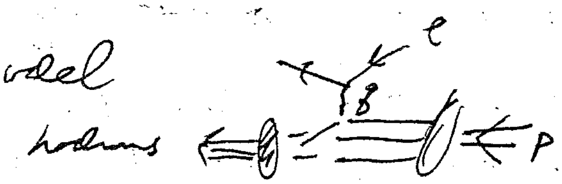
  $SU(3)_c$ singlets are light

  $\bar{q}q \supset 1$, $qqq \supset 1$ $\qquad$ $qq \not\supset 1$, q

  statistics $\Delta^{+} = uuu$ Fermi stat

  Nowadays $\sigma(e^+e^- \to \text{hadrons}) \propto N_c$

- Parton model



$d\sigma \sim \frac{1}{|q|^2} f(q^2/2p\cdot q)$

($p^2 = m^2$ invalid)

No resonances $\to$ AF

In 4D NA gauge theories

A unique theory! Can it explain

Qualitative feature
- confinement (eg. Regge)
- SSB of χS (eg pion)
- spectrum, structure functions, scaling violations

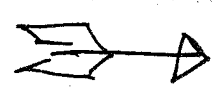
What is local symmetry?

Global symmetry: two different objects behave the same way

Local symmetry: a redundant description with unphysical degrees of freedom

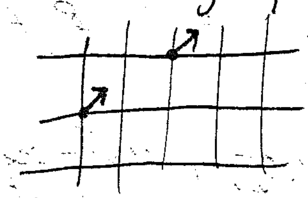
Better: a language for describing geometrical properties

Warm up: Riemannian geometry
A race of people in two dimensions
They can parallel transport a tangent vector
e.g. a swinging pendulum

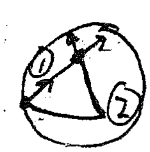


Arrow // to ground

Locally flat



Path dependent



e.g. 2 paths from equator to north pole

Don't need to visit cone to know there is curvature there

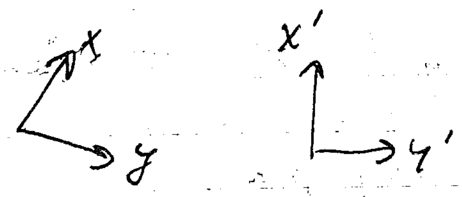
Example: cone is flat "almost everywhere" (except tip)

$\boxed{\text{Rot} \angle = \text{enclosed solid angle}}$

Rot $\angle$ = "deficit $\angle$" of cone
$\Rightarrow$ "conical singularity"

The core of geometry: "orientational democracy"
= "local symmetry"
(eg. on a nonrotating earth)

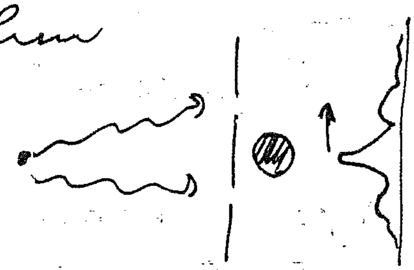
A convention



"local"
because residents at different locations can orient axes differently

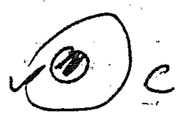
Another geometrical effect:
Aharonov-Bohm

"phase democracy"



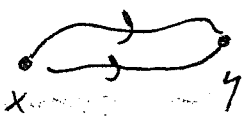
Fringe shift

$$e^{ie\Phi/\hbar c} = \exp\left[i\oint_C \vec{A}\cdot d\vec{x}\right]$$



⇓
convention w/ no observable consequences

$\vec{A}$ = connection:
// transport of phase



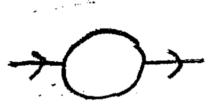
Magnetic field:
curvature

Aharonov - Bohm = conic

Don't need to visit the solenoid to know there is B field there

E.g. Webb experiment ~'85

electrons



cold ring in strong field, w/
Two leads
Resistance oscillates

Note: scattering in wave guide is elastic ⇒ no loss of coherence

Consider world on scales small compared to scale of confinement

The heart of QCD:
"color democracy"

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

3 colors eg RYB (arbitrary)

Now: an abstract 3-dim color vector space

Reorientates in color space

Bring eg $q\bar{q}$ together?
Can compare the color

$q(x) \Rightarrow \Omega(x)\, q(x)$

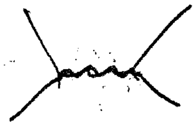
Inner product
$|(q_{ref}(x), q_{unknown})|$

E.g. couple to em • • ⇝ $\gamma\gamma$

Prob of annihilation: $\mathrm{tr}(Q^\dagger Q)$

But: we can do nondestructive measurements, and we don't need em, since quarks couple to gluons

More later on evaluating this diagram



⟹ Coulomb interaction [see later]

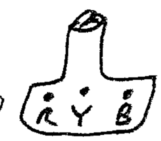
$E(r) \propto \frac{1}{r}$

⇑ An invariant for 2 objects at the same place

i.e. invariant under simultaneous rotations

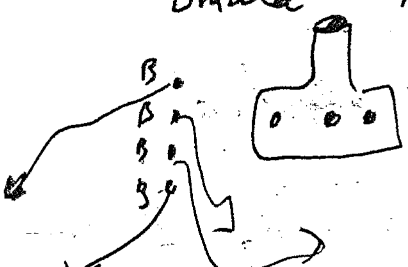
Go quark watch: "Oh, what a beautiful ? quark"

Establish a quark bureau of standards

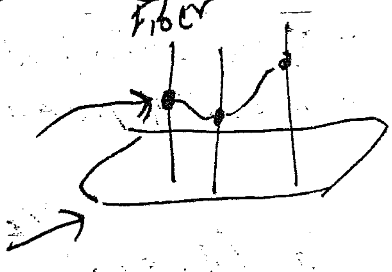


Encounter a "wild" quark — carry carefully → Take it to QBS for analysis

Except:
Drama — Rupert Murdoch — Having bought Dodgers, as a promotional campaign — All q's are Dodger blue



carry quarks, being careful not to rotate — works for a (isospin) global symmetry, but not for local



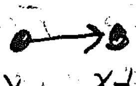
More on mathematics of color parallel transport

Fiber bundle

Lie group acts on fibers

section: specify q(X) at each point

Connection: A way to parallel transport the color



E.g. I can carry R quark with me, and be sure it stays red.
Same for Y B

Transport basis $\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \Rightarrow \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix}$ new basis

$q(x)$ is a quark field,
in original basis ($\rightarrow \Omega(x)\, q(x)$ if we rotate basis locally)

Condition for parallel transport

$$q(x+\varepsilon) = \Omega(\varepsilon)\, q(x)$$

$\Omega(\varepsilon)$: element of $SU(3)_c$ close to identity

Expand to linear order

$$\Omega(\varepsilon) = \mathbb{1} + i\varepsilon^\mu A_\mu$$

$A_\mu \in$ Lie algebra
8-dimensional vector space

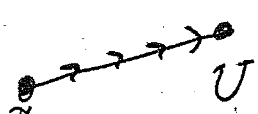
Ω unitary $\Rightarrow$ A_μ hermitian
$\det \Omega = 1$ $\Rightarrow$ A_μ traceless

Aside: Lie algebra
Continuous (Lie) group $\rightarrow$ Lie algebra

$$\begin{aligned} \Omega_1 &= \mathbb{1} + \omega_1 + \cdots \\ \Omega_2 &= \mathbb{1} + \omega_2 + \cdots \end{aligned} \Rightarrow \Omega_1\Omega_2 = \mathbb{1} + \omega_1 + \omega_2 + \cdots$$

Linear (vector) space, with dim = # of parameters

Build up finite transformation from many infinitesimals



$$\lim (\mathbb{1} + \varepsilon\omega)^{1/\varepsilon} = e^{\omega}$$

also in group
(converges for finite dim matrix)

Consider

$$e^{\omega_1} e^{\omega_2} e^{-\omega_1} e^{-\omega_2} = \mathbb{1} + [\omega_1, \omega_2] + \text{h.o.}$$

so $[\omega_1, \omega_2]$ also in the linear space can represent group multiplication in vicinity of identity

$$e^{\omega_1} e^{\omega_2} = \exp\left[\omega_1 + \omega_2 + \tfrac{1}{2}[\omega_1, \omega_2] + \cdots\right]$$

Baker-Campbell-Hausdorff] - Finite dim case

Full power series from commutators] - aside from global topology (discuss later)

Back to parallel transport

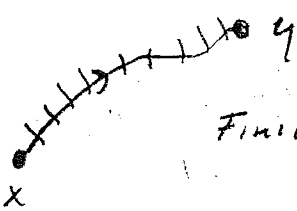
$$q(x+\varepsilon) = (\mathbb{1} + i\varepsilon^\mu A_\mu)\, q(x)$$

$$\varepsilon^\mu \partial_\mu q = \varepsilon^\mu i A_\mu q$$

$$\partial_\mu q = i A_\mu q \Big|_{\text{along path}}$$

Characterize in terms of structure constants

$$\underbrace{\begin{matrix}\omega_1 = i\varepsilon_1 T^a \\ \omega_2 = i\varepsilon_2 T^b\end{matrix}}_{\text{Basis for space}} \qquad [T^a, T^b] = i C^{abc} T^c$$



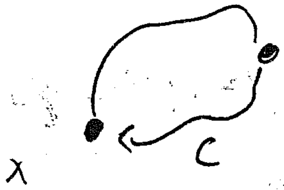
Finite path $\quad q(y) = \left[\mathcal{P} \exp i \int_x^y dx^\mu A_\mu\right] q(x)$

$$\underline{\mathcal{P}}\, e^{i\int_x^y A}:$$

Ex finite SU(3) trans associated with parallel transport from x to y

But: this just relates arbitrary bases in color space at two different points ⟶ cf rotating an arrow

Geometry = path dependence



$$U = \underline{\mathcal{P}}\, e^{i\oint_x^x A}$$

still $U \to \Omega(x)\, U\, \Omega(x)^{-1}$

of course -- depends on axes

But now: has some invariant content:

E.g. angle or $W(C) = \mathrm{tr}\left(e^{i\oint A}\right)$ Don't have to specify start point

Wilson loop: An invariant notion associated with path dependence of parallel transport (e.g. = 0 if color rotates to orthogonal color) [– roughly speaking]

Covariant derivative $\quad q(x) \quad q(x+\varepsilon)$

$x \quad x+\varepsilon$

How quark field varies relative to parallel transported basis

$$D_\mu q(x) = (\partial_\mu - i A_\mu(x))\, q(x)$$

] This rotates away the varying dependence of basis with position

$$\Longrightarrow D_\mu \to \Omega(x) D_\mu \Omega(x)^{-1}$$

= "transforms covariantly"

From this, we can infer how the connection itself transforms

$$D_\mu = \partial_\mu - i A_\mu \Longrightarrow \partial_\mu - i A_\mu^\Omega = \Omega(\partial_\mu - i A_\mu)\Omega^{-1}$$

$$i A_\mu^\Omega = \Omega(i A_\mu)\Omega^{-1} - \Omega \partial_\mu \Omega^{-1}$$

$$= \Omega(i A_\mu)\Omega^{-1} + (\partial_\mu \Omega)\Omega^{-1}$$

$q(x)$

$x \quad x+\varepsilon$

Find q – in the basis obtained by transporting from $q(x)$

$$[\mathbb{1} + \varepsilon^\mu D_\mu]\, q$$

Transport ν, μ ; ε, ε' around small closed path

$$e^{\omega_2} e^{-\omega_1} e^{\omega_2} e^{\omega_1} = \mathbb{1} + [\omega_2, \omega_1] + \cdots$$

$\omega_1 = i\varepsilon^\mu D_\mu$

$\omega_2 = i\varepsilon^\nu D_\nu$

$$\Longrightarrow \mathbb{1} + \varepsilon^\mu \varepsilon'^\nu [D_\mu, D_\nu]$$

$$[D_\mu, D_\nu] = [\partial_\mu - iA_\mu, \partial_\nu - iA_\nu]$$

$$= -i[\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$\equiv -iF_{\mu\nu}$$

The YM field strength, or <u>curvature</u>

Comment: Jacobi identity = Bianchi identity

$[D_\sigma,[D_\mu, D_\nu]] +$ cyclic $= 0$

$[D_\sigma, F_{\mu\nu}] +$ cyclic $= 0$

cf GR and source free Maxwell.

How $q(x)$ changes around infinitesimal path

$$F_{\mu\nu}(x) \to \Omega(x) F_{\mu\nu}(x) \Omega(x)^{-1}$$

again, a covariant property

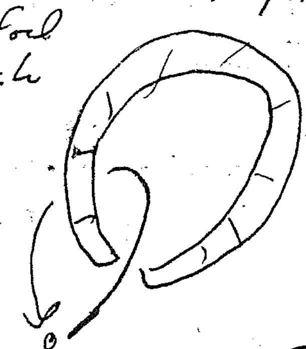
For a <u>finite</u> path -- not obtained by locally integrating a density (at least not simply)

Best described as

$$U(x,c) = P\exp\left(i\int_x^x A\right)$$

Can't, in general, impose a global color convention, because of color magnetic field

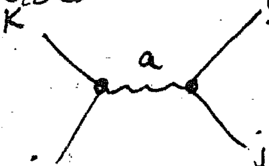
How we fool murdoch



Carry B quark around solenoid — it is now red

This is a puzzle — what happens to the color? (thinking about it reinforces our appreciation of the subtlety of the local symmetry concept

Like Coulomb, except for color factor



$$E(r) = \frac{g^2}{4\pi r}(T_1{}^a)_{ki}(T_2{}^a)_{kl} \qquad \begin{pmatrix} R \\ Y \\ B \end{pmatrix}$$

Diagonal generators

$$T^3 = \mathrm{diag}\left(\tfrac{1}{2}\ -\tfrac{1}{2}\ 0\right)$$

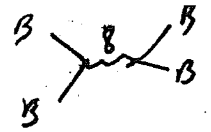
$$T^8 = \frac{1}{\sqrt{3}}\left(\tfrac{1}{2}\ \tfrac{1}{2}\ -1\right)$$

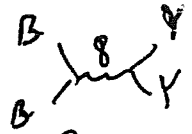
$$\mathrm{Tr}\, T^a T^b = \tfrac{1}{2}\delta^{ab}$$

only ones that contribute to interaction

RR YY

RY YR





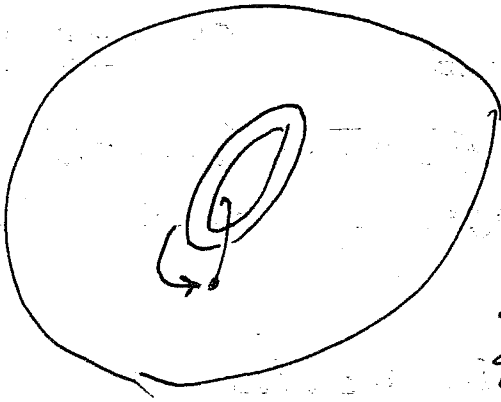
$\frac{1}{3}(-\frac{1}{2}) = -\frac{1}{6}$ $\frac{1}{2}(-\frac{2}{3}+\frac{1}{3}) = -\frac{1}{6}$ symmt antisymm

$\frac{1}{3}(1) = \frac{1}{3}$ symmetric

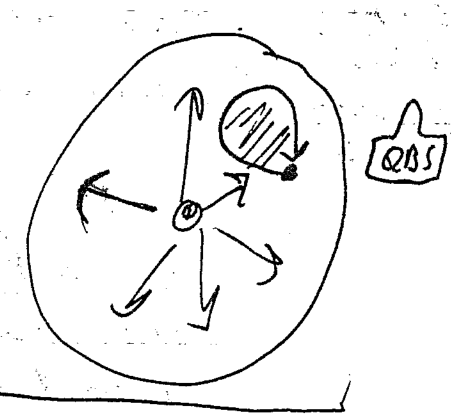
(We'll describe this in a more invariant way shortly)

RR: $\frac{1}{4}(1+\frac{1}{3}) = 1/3$ Repulsive

RY: $\frac{1}{4}(-1+\frac{1}{3}) = -\frac{1}{6}$ Attractive



Draw large sphere around whole system: flux tube and colored quark



So .. we can establish a global color standard on surface of the sphere ⟸ If there is no magnetic charge, then fields fall off with distance

But color magnetic monopoles change the story

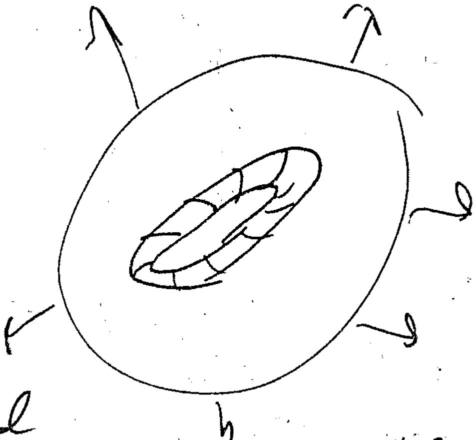
convention that can be established for $H \subseteq SU(3)$, commuting with the magnetic charge

More about this later

Inescapable conclusions

- The total "redness" of the system, detectable far away can't be changed by the process
- Yet the red quark really has become a blue quark — If isolated, it would have a long-range blue field!

- So --
there has to be some color electric charge somewhere, and where else but the solenoid



[Ignore subtlety concerning emission of NA radiation

= "Cheshire charge"

- there is no localized source
- there is a LR color electric field!

More about theory of Cheshire charge within a few weeks!

We ought not to be shocked by color charge with no localized source

cf: the gravitational pull of the sun is also more than the sum of it's parts — the gravitational self energy contributes: Why? because gravitons themselves carry mass-energy, and can be sources for other gravitons

Similar: gluons themselves carry color, and can be sources for other gravitons

- Note: In GR there is a conserved energy -- but not an integral of a local density — a covariantly conserved tensor

- Similar: In YM, there is a conserved color charge, but not integral of a local density (again, more later)

Gauss Law $[D_i, F^{0i}] = +J^0$

$D_\mu F^{\mu\nu} = J^\nu \Rightarrow D_\nu D_\mu F^{\mu\nu} = 0 = D_\nu J^\nu$

GR $T^{\mu\nu}_{eff} = T^{\mu\nu}_{matter} + t^{\mu\nu}$ ⇐ not a tensor — No geom meaning (ind of coords)

$J^{\mu}_{eff} = J^{\mu}_{matter} + J^{\mu}_{YM}$ ⇐

Lecture #2 Some history: (Gauge principle and Aharonov-Bohm effect)

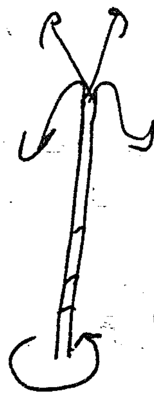
Weyl 1918 Geometric theory of em (before QM!) parallel transport of length] Einstein comment

meterstick or clock recalibration

London/Fock 1927 parallel trans of phase

Weyl 1929: Modern formulation

Dirac 1931



A monopole is a flux tube -- with one end in isolation

"One would be surprised if Nature made no use of it"

$e^{ie\Phi/\hbar c} \Rightarrow \Phi = \frac{2\pi\hbar c}{e} n$

Ehrenberg-Siday 1949
Aharonov-Bohm 1959] clarify that nonlocal gauge inv. quantities can have observable effects

Yang-Mills 1954
Did not know of Weyl's geom ideas, only Pauli's review

⇑ local sym constrains dynamics

Yang: late 60's: YM geometry

Wu-Yang: 1975 Fiber bundles] Geometric ideas have remained decisive for next 25 yrs

Hermann Weyl had a good idea in 1918!

Why did gauge invariance turn out to be so important

- Nature reads the book of geometry
- Covariant descriptions require redundancy — for spin $1, \frac{3}{2}, 2$ massless particles

A rather mundane answer..

Wouldn't Weyl/Einstein have loved YM theory? (They died in '55)

Mismatch between

fields	particles
Finite dim nonunitary irreps of Lorentz	∞ dim unitary irreps of Poincaré

e.g. A^μ for 2 helicity states of photon

Redundant components needed $\Rightarrow$ gauge symmetry (local transf. that don't change the physics)

Don't really need to know geometrical jazz to quantize a gauge theory $\Rightarrow$

Back to: one-gluon-exchange

choose basis for Lie algebra with

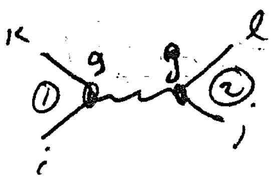
In defining 3 dim irrep

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

$$[T^a, T^b] = +i\, C^{abc}$$

$$\Rightarrow \quad C^{abc} = -2i\, \mathrm{tr}\, [T^a, T^b]\, T^c$$

totally antisymmetric



spacetime structure same as Coulomb potential in QED

$$E(r) = \frac{g^2}{4\pi r} (T_1^a)_{ki} (T_2^a)_{lj} \quad (a \text{ summed})$$

Invariant under simultaneous SU(3) color rotations of the objects 1 and 2

$\Rightarrow$ SU(3)$_{color}$ symmetry

Diagonal in basis of irrep of SU(3)

$$R_1 \otimes R_2 = \bigoplus_\mu R_\mu$$

cf $\vec{L}\cdot\vec{S}$ coupling $\quad \vec{J} = \vec{L} + \vec{S}$

$$T_1^a T_2^a = \tfrac{1}{2}\left[(T_1^a + T_2^a)(T_1^a + T_2^a) - T_1^a T_1^a - T_2^a T_2^a\right]$$

irrep $\quad T_R^a T_R^a = C(R)\,\mathbb{1}$

why? $\quad [T^aT^a, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a$

$$= i\,c^{abc}(T^aT^c + T^cT^a) = 0$$

T^b preserves eigenvalues of T^aT^a

$$T_1^a T_2^a = \tfrac{1}{2}\left[C(R) - C(R_1) - C(R_2)\right]$$

How to evaluate?

$$T_R^a T_R^a = C(R)\,\mathbb{1}$$

$$(\#\text{ of generators})\, T(R) = C(R)\,(\dim \text{ of } R)$$

$\mathrm{tr}(T^aT^b) = T(R)\,\delta^{ab}$ $\quad$ e.g. $T(R_{fund}) = \frac{1}{2}$

$$C(R) = \frac{\#\text{ of generators}}{\dim R}\, T(R)$$

$$\mathrm{tr}(T^3 T^3)_R = T(R) \qquad T^3 = \mathrm{diag}\left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right)$$

$$T(R) = \sum_{\substack{SU(2)\\ \text{ireps}\\ S}} T(S)$$

E.g. $R = 8$ of $SU(3)$

$8 = 3 \otimes \bar{3} - 1$

$(2+1)\otimes(2+1) - 1 = 3 + 2 + 2 + 1$

$(1 \oplus 1)$

$T(R=8) = 2 + \frac{1}{2} + \frac{1}{2} = 3$

$C(R=8) = \frac{8}{8}\, 3 = 3$

$C(R=3) = \frac{8}{3}\, T(R=3) = \frac{4}{3}$

$6 = (3 \times 3)_{\text{sym}}$

$\rightarrow (2+1)\times(2+1) = 3 + 2 + 1$ of $SU(2)$

$T(R=6) = 2 + \frac{1}{2} = \frac{5}{2}$

$C(R=6) = \frac{8}{6}\,\frac{5}{2} = \frac{10}{3}$

quark-antiquark

$$\frac{g^2}{4\pi}\,\frac{1}{r}\,\frac{1}{2}(C - C_1 - C_2) = \frac{g^2}{4\pi r}\left[-\frac{4}{3}\right] \quad \text{singlet}$$

$$\frac{1}{2}\left[3 - \frac{8}{3}\right] = \frac{1}{6} \quad \text{octet}$$

could also obtain from Kaluza ness

$3 \times 3 = 6 + \bar{3}$

$\frac{1}{3}, -\frac{2}{3}$

A small step in the right direction

Gluon exchange is attractive in color singlet channel

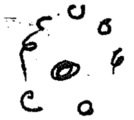
$$E(r) = \frac{g^2}{4\pi r}\, C$$

can guess by dimensional analysis in pure YM theory

g^2 is only (classical) parameter, $g^2 \sim$ energy · distance

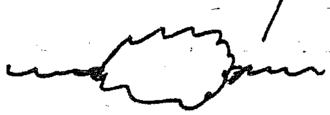
and $\rightarrow$ $E \sim \frac{1}{r}$ because of classical scale invariance

Scale invariance: <u>not</u> a property of the quantum theory

Physically:  Virtual gluons $\Rightarrow$ polarizable medium

$$E(r) = \frac{C\, g^2(\mu r)}{4\pi r}$$

A scale μ appears

Mathematically:  Log UV divergences $\Rightarrow$ must subtract at (Euclidean) $p^2 = -\mu^2 \neq 0$ (not at $\mu^2 = 0$ $\Rightarrow$ IR divergences, poles, cuts

In classically scale invariant, we must choose a scale μ to renormalize, but physics is <u>independent</u> of μ

$\Rightarrow$ An anomaly: a classical symm spoiled by a quantum effect

Does the spontaneously generated scale allow us to understand why quarks are permanently confined in hadrons

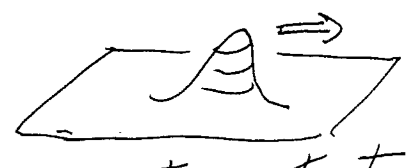
A very instructive analogy: superconductivity
- spectacular & bewildering properties
- eventually explained by a microscopic theory

Topological ideas are very helpful -- we'll spend some time developing a theory of topological defects

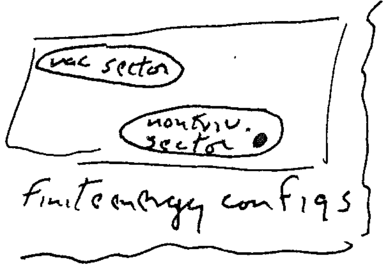
The Abelian Higgs Model

In some models, the particle spectrum contains excitations that are not seen in any order of the Feynman diagram expansion. But these particles can be studied, as nondissipative classical solutions to the (nonlinear) field equation.

Here is a static solution — a lump of localized energy density — and a Lorentz boosted solution that describes a particle moving at constant velocity.



Since the object is "soft and squishy" rather than pointlike, it is comprised of many quanta (and has a large size compared to its Compton wavelength.



A general strategy for finding these solutions: If the space of finite energy field configurations has disconnected components, find a nontrivial static solution by minimizing energy in a sector that does not contain the vacuum (zero-energy config).

Example: In two spatial dimensions

$$\mathcal{L} = |D_\mu \phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2}\left(|\phi|^2 - v^2/2\right)^2 \qquad \begin{cases} D_\mu = \partial_\mu - ie A_\mu \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{cases}$$

For $v^2 > 0$ and weak coupling, Higgs phase

Perturbative spectrum: $\phi = \frac{1}{\sqrt{2}}(v + \phi')$

$$|D_\mu \phi|^2 = \frac{1}{2} e^2 v^2 A_\mu A^\mu + \cdots \Rightarrow \mu = ev \quad \text{Massive vector}$$

$$|\phi|^2 = \frac{1}{2} v^2 + v\phi' + \cdots$$

$$-\frac{\lambda}{2}\left(|\phi|^2 - v^2/2\right)^2 = \frac{\lambda}{2} v^2 \phi'^2 + \cdots \Rightarrow m_H = \sqrt{\lambda}\, v \quad \text{Massive (Higgs) scalar}$$

First — nature of semiclassical limit

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - \frac{1}{2}\lambda\left(|\phi|^2 - \frac{1}{2}v^2\right)^2$$

Rescale $A \to \frac{1}{e}\tilde{A}$

$\phi \to \frac{1}{e}\tilde{\phi}$

$$\mathcal{L} = \frac{1}{e^2}\left[-\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + |D_\mu\tilde{\phi}|^2 - \frac{1}{2}\frac{\lambda}{e^2}\left(|\tilde{\phi}|^2 - \frac{1}{2}e^2v^2\right)^2\right]$$

Mass scale $\mu = ev$
Dimensionless coupling λ/e^2

$e^{iS/\hbar}$ semiclassical limit
$e^2 \to 0$
ev, λ/e^2 fixed

Length scale ev fixed
Energy scale $\propto \frac{1}{e^2}$ $\Rightarrow$ So — mass diverges as size stays fixed

Finite energy
$V \to 0$ $\qquad \phi(r,\theta) \xrightarrow[r\to\infty]{} \frac{v}{\sqrt{2}}e^{i\sigma(\theta)}$

$\sigma(\theta)$ $\quad \theta \in [0, 2\pi)$
Mapping $S^1 \to S^1$ $\qquad$ Has winding number

$$n = \frac{1}{2\pi}\left[\sigma(2\pi) - \sigma(0)\right]$$

Discrete $\Rightarrow$ cannot change continuously

So -- config of finite energy
$\to$ sectors labeled by n

"Topological conservation law"

Minimize energy for $n = 1$

What about $|D_r\phi|^2$ $\quad \int \left| \left(\frac{1}{r}\frac{\partial}{\partial \theta} - ieA_\theta \right) \phi \right|^2 d^2 r$

$A =$ constant $\Rightarrow$ logarithmic divergence

$$\phi = \frac{v}{\sqrt{2}} e^{i\sigma(\theta)}$$

$$A_\theta \to \frac{1}{er}\frac{d\sigma}{d\theta} + \cdots$$ corrections fall off faster

Parallel transport is trivial, but not topologically trivial

"a pure gauge" at $r = \infty$, but not everywhere $\quad \Phi = \int r\, d\theta\, A_\theta = \frac{1}{e}\left[\sigma(2\pi) - \sigma(0)\right] = \frac{2\pi}{e} n$

Minimum energy soln — has a core $\qquad$ Lecture #3

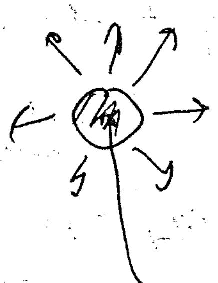
ϕ must vanish somewhere

Otherwise $r = \infty \Rightarrow r = 0$ deforms σ to constant $(n = 0)$

Choose $\phi = 0$ at origin

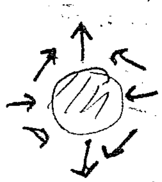
(Minimizes cost in energy)

"Vortex"



Lump of potential energy

Antivortex

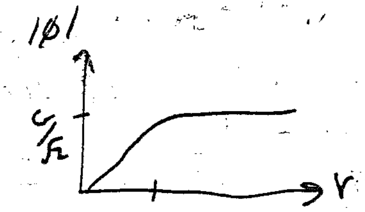


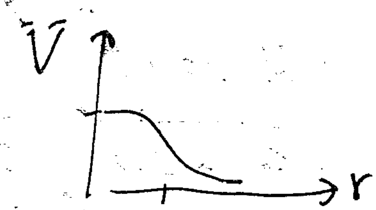
Two characteristic lengths

They can annihilate (Vacuum sector)

Produces Higgs and gauge fields

r_s





r_M: where gauge field reaches asymptotic value

Suppose $r_M > r_s$ $\quad$ Determine variationally

$$\phi(r, \theta) = \frac{v}{\sqrt{2}} f(r) e^{i\theta} \; ; \; f(0) = 0 \quad f(\infty) = 1$$

$$A_\theta(r, \theta) = \frac{1}{er} a(r) \; ; \; a(0) = 0 \quad a(\infty) = 1$$

∘ Magnetic flux $\Phi = \frac{2\pi}{e}$ $\qquad \frac{1}{2}\left(\frac{2\pi}{e r_M^2}\right)^2 \pi r^2 \sim \frac{\pi}{e^2 r_M^2}$

– Flux wants to spread out

∘ Core energy $\qquad \frac{\lambda}{8} v^4 \pi r_S^2 \sim \pi v^2 r_S^2 m_H^2$

– Core wants to collapse

What ties the scales together? Gradient energy

$\sim \pi v^2 \ln\left(r_M/r_S\right)$

No pert because "bound state" of $1/\alpha$ particles

$\Downarrow$

Mass

$\sim \pi v^2 \ln(\lambda/e^2)$

$\sim \frac{\mu^2}{e^2}\ln(\lambda/e^2)$

$$E \sim \pi v^2 \left[\frac{1}{\mu^2 r_M^2} + \ln r_M/r_S + m_H^2 r_S^2\right]$$

Radial gradient energy $= \pi v^2 \times O(1)$ It is scale invariant

$\Rightarrow \quad r_M \cong \mu^{-1}$ $\qquad$ (if $m_H > \mu$)

$r_S \cong m_H^{-1}$ $\qquad$ (Known as "Type II")

The case $\mu = m_H$ is interesting! (exercise)

– What about $m_H \to 0$?

Comment here on monopoles in a superconductor

String: $M_H \to$ energy/length

Meissner $\Rightarrow$ confinement

Type I vs Type II

Comment: Bogomol'nyi bound for $\lambda = e^2$: $\geq \pi v^2 |n|$

A Vortex in "QCD"

Consider E.g. $G = SU(2)_c$ with triplet Higgs field

$$\Phi = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \qquad SU(2)_c \to U(1)$$

Two triplet Higgs fields that misalign

Aside: Bogomol'nyi Bound

An exercise to show (for $\lambda = e^2$):

$$E = \int d^2x \left[\tfrac{1}{2} B^2 + |D_1 \phi|^2 + |D_2 \phi|^2 + \tfrac{\lambda}{2}\left(|\phi|^2 - \tfrac{v^2}{2}\right)^2 \right]$$

(Note: $A_0 = 0$, $\vec{E} = 0$, $D_0 \phi = 0$)

$$= \int d^2x \left[|D_1 \phi \pm i D_2 \phi|^2 + \tfrac{1}{2}\left[B \pm e\left(|\phi|^2 - \tfrac{v^2}{2}\right)\right]^2 \pm \tfrac{1}{2} e v^2 B + \text{curl } \vec{J} \right]$$

curl $\vec{J}$ → A surface term that vanishes in finite-energy config.

$$\pm \tfrac{1}{2} e v^2 \int d^2x\, B = \pm \tfrac{1}{2} e v^2 \frac{2\pi}{e} n$$

$$= \pm \pi v^2 n$$

By choosing + or - sign as appropriate

$E \geq \pi v^2 |n|$. ("Bogomol'nyi bound")

Condition for equality

$$0 = D_1 \phi \pm i D_2 \phi$$
$$0 = B \pm e\left(|\phi|^2 - \tfrac{v^2}{2}\right)$$

+ for $n > 0$
- for $n < 0$

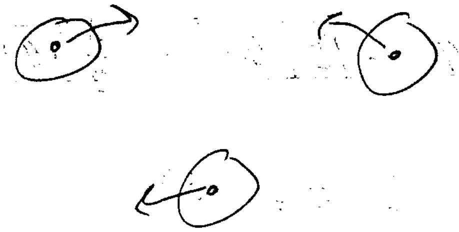
Eqns become 1st order, and can be solved!

[Comment: in general ($\lambda \neq e^2$)

Energy = same + $\tfrac{1}{2}(\lambda - e^2) \int d^2x \left(|\phi|^2 - v^2/2\right)^2$

and hence $E > \pi v^2 |n|$ for $\lambda > e^2$.]

Solution has $2|n|$ real parameters



the positions of the $|n|$ zeros of ϕ (vortex positions) which are unconstrained. Classically, the vortices are noninteracting (the boundary between "Type I" and "Type II").

If there is enough supersymmetry, they remain noninteracting in the full quantum theory — they are "BPS" (Bogomolnyi-Prasad-Sommerfield) states. More on BPS later...

Comment: in the extreme type I case $\lambda/e^2 \to 0$ there is just one core size r, chosen to minimize

$$E \sim \pi v^2 \left[\frac{1}{\mu^2 r^2} + m_H^2 r^2 \right]$$

$$\Rightarrow r^2 \sim \frac{1}{\mu m_H}$$

and $\quad M_{\text{vortex}} \sim \pi v^2 \frac{m_H}{\mu} = \pi v^2 \sqrt{\lambda/e^2}$

E.g. if $e^2 \to$ large with λ fixed -- magnetic flux becomes uncostly — and core can shrink to minimize potential energy. What about radial gradient??

$$\phi(r) = \frac{v}{\sqrt{2}} \left[\frac{\log(r/\varepsilon)}{\log(R/\varepsilon)} \right] \Rightarrow \frac{d\phi}{dr} = \frac{v}{\sqrt{2}} \frac{1}{r \log R/\varepsilon} \Rightarrow 2\pi \int_\varepsilon^R r\,dr \left(\frac{d\phi}{dr} \right)^2 = \frac{\pi v^2}{\log(R/\varepsilon)} \xrightarrow[\varepsilon \to 0]{} 0.$$

$$\Phi_1 = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} \quad \Phi_2 = \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix}$$

$$SU(2) \longrightarrow Z_2$$

$$(\text{or } SO(3) \longrightarrow I)$$

$$\Phi(\theta) = \Omega(\theta)\,\Phi_0$$

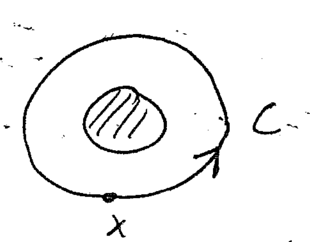
$$D_\mu \Phi = (\partial_\mu \Omega\, \Omega^{-1} - ie A_\mu)\,\Omega \Phi_0 = 0 \Rightarrow$$

$$\partial_\mu \Omega = ie A_\mu \Omega \Rightarrow$$

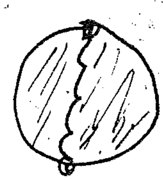
$$\Omega(2\pi) = P\exp\left(i \oint A_\mu dx^\mu\right)\Omega(0)$$

Again: parallel transport of condensate about vortex <u>must</u> be trivial

$$U(x,C) = P\exp\left(i\int A\right) = \pm I$$

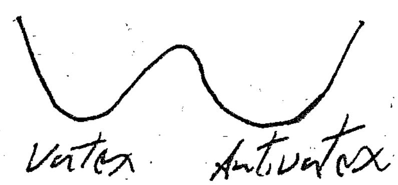


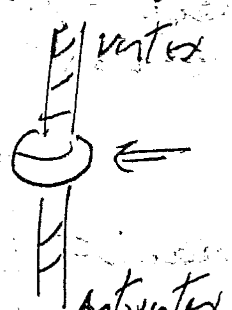
There are two topological sectors — comment: the "coffee-cup trick"



Now, conserved quantities take Z_2 values

So — vortex and antivortex are topologically equivalent. This means that they are in same sector, but not that there is no barrier between them





There can be a bead on a string. In 2+1 dim, it is an instanton: coherent tunneling under <u>barrier</u>

More about beads later

In two spatial dimensions: quantum tunneling, and 2 distinct "low-lying" states..

...ment on adjoint representation $\phi = \sum_a \alpha_a \sigma^a$ {Lecture #4}
...to SU(N): $\rightarrow U\phi U^+ = \sum_a \alpha_a U \sigma^a U^+$ (24)
$\rightarrow \sum_a \alpha_a U T^a U^+ = \sum_a \alpha_a R(U)^a{}_b T^b$ $= \sum_a \alpha_a R(U)^a{}_b \sigma^b$

Generalize $SU(N)/Z_N \Longrightarrow I$

E.g. adjoint rep. Higgs fields $SU(N) \rightarrow Z_N$

Z_N vortices

$P\exp(i\int A) = e^{(2\pi i/N)K} I$

$e^{2\pi i/3} I$

E.g. In $SU(3)$

$I, e^{\pm 2\pi i/3} I$

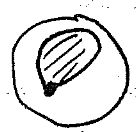
Commute with the generators, but act nontrivially on triplet

...g $(1\ 1\ 1)\cdot \#$

$\frac{1}{e^2} \rightarrow 0$

Note: Just one Higgs field
$SU(2) \rightarrow U(1)$
$\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$ Takes values on a 2-sphere

"You can't lasso a basketball"



Not topologically stable vortex

A general classification

...tion of ...symmetry $\underbrace{G \rightarrow H}$ Are there topologically stable vortices?

$\Rightarrow$ there may be global syms as well, but they are not relevant to finite-energy configs

Except—see theory of "skyrmions" below

Φ: Takes values in G/H — i.e. these are values that are related by gauge parallel transport

...topologically ...trivial ...s, outside

a core?

on circle at ∞

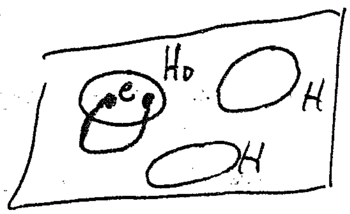
$$\Phi(\theta) = \Omega(\theta)\,\Phi_0$$

Φ single valued

$\Omega(0) = e$

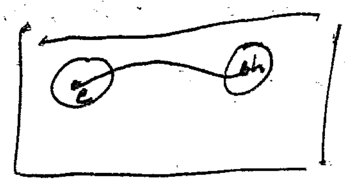
$\Omega(2\pi) \in H$

[Comment: If $\Phi(\theta)$ is continuous, the $\Omega(\theta)$ can be chosen to be continuous.]



G_0

Identity component of G — contains several components of H



$(h \notin H_0)$ H_0 = identity component

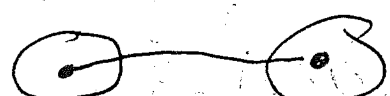
Question -- Can path be smoothly deformed to one contained entirely in H?

If G is simply connected:

- $h \in H_0$

obviously yes

- $h \notin H_0$



obviously not

The paths have a group structure

$\pi_1(G/H)$

this tells us how to "fuse" the sectors together

If G is simply connected --

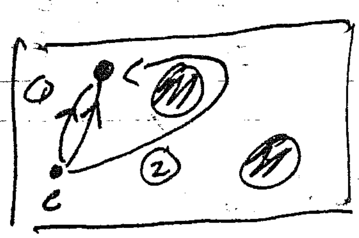
$$\pi_1(G/H) = \pi_0(H)/\pi_0(G)$$

special case: $\pi_1(G) = \pi_0(G) = 0$, H discrete, $\Rightarrow \pi_1(G/H) = H$

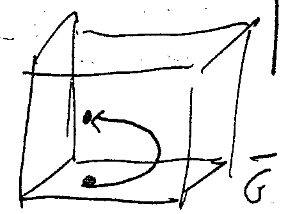
And ... we can always choose G to be simply connected (though that might not be convenient)

It has a universal covering group

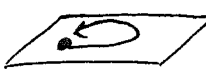
How is covering (group) constructed?



Consider the space of homotopic equivalence classes of paths from e to $g \in G$



$\bar{G}$



G

This is simply connected by construction

This is a closed path in the group, but not in the covering group

E.g.

Homomorphism $\bar{G} \to G$

Kernel is $\pi_1(G)$

$$\begin{array}{c} \pi_1(G) \to \bar{G} \\ \downarrow \\ G \end{array}$$

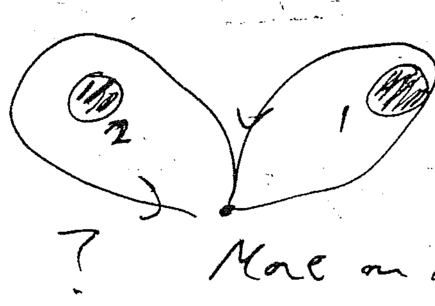
Note: equivalent to say that $\pi_0(H)$ classifies topology of Higgs field, or flux carried by gauge field

E.g. $SU(N) \to SU(N)/Z_N$ kernel Z_N

$\mathbb{R} \to U(1)$ kernel Z

$$\begin{array}{c} Z \to \mathbb{R} \\ \downarrow \\ S^1 \end{array} \qquad \begin{array}{c} Z_N \to SU(N) \\ \downarrow \\ SU(N)/Z_N \end{array}$$

Remark: $\pi_1(G/H)$ could be a noncommutative group



There is more than one possible way to patch vortices together

? More on this later

"Local Discrete Symmetry"

Consider again the pattern of gauge symmetry breaking

$$SU(2) \to Z_2 \quad \text{or} \quad SU(N) \to Z_N$$

But now suppose that, in addition to the (adjoint) Higgs fields that drive the symmetry breakdown, the theory also contains a field ψ that transforms as the fundamental representation of $SU(N)$ – e.g. the doublet representation of $SU(2)$. Now we really have no choice but to say that the gauge group is $SU(2)$ (rather than $SO(3) = SU(2)/Z_2$). Formally, a discrete subgroup $Z_2 \subset SU(2)$ remains unbroken, although all gauge fields have become massive. Does this unbroken discrete subgroup of the local symmetry group have physical consequences?

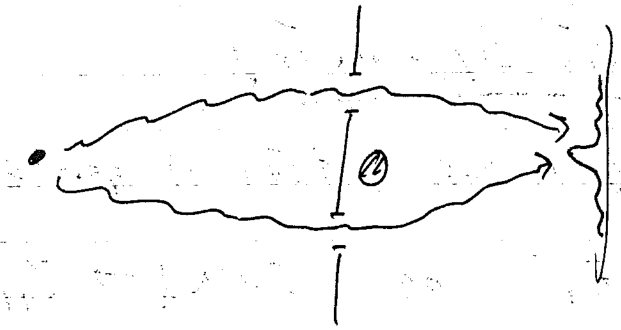
Yes – consider the effect of covariantly transporting the fundamental doublet in the gauge potential far from the vortex. Since

$$P\exp\left(ie\oint A\right) = -I,$$

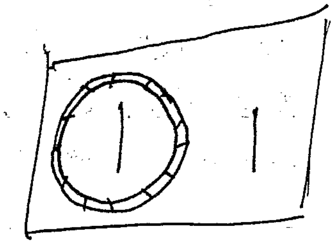
the wavefunction of the doublet particle (but not the triplet) changes sign. There is an Aharonov-Bohm interaction between vortex and charge – and remarkably, this interaction has ∞ range, even though the theory has a "mass gap" (its spectrum includes no zero mass particles).

[Comment: an abelian example –

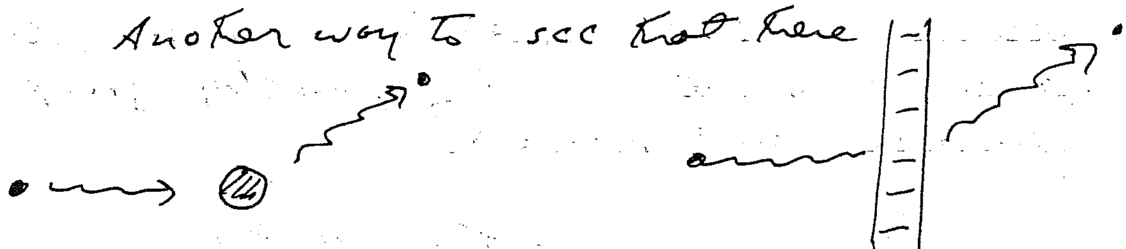
$U(1) \to Z_N \Rightarrow \Phi = \frac{2\pi}{Ne}$ if a charge N Higgs field condenses cf. — Cooper pairs!]

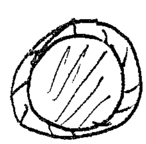


We can do a double-slit experiment with a "quark". The two slits can be far apart, but the presence or absence of a Z_2 flux behind the screen can be determined by the position of the central interference fringe. There is a similar phenomenon in 3 spatial dimensions — we can tell whether a loop of string winds around one of the slits



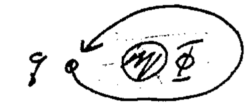
Another way to see that there are ∞-range interactions is to consider the scattering of the Z_2 charge by a Z_2 flux (in 2 or 3 dimensions) — the scattering cross section has an ∞ peak in the forward direction, reflecting that the charge and flux interact at long range





or — total cross section of loop ~ area of loop. —

The calculation of the Aharonov-Bohm scattering cross section is somewhat delicate, but the main idea is easy to understand semiclassically...



The phase acquired by charge q transported about flux Φ is
$$e^{iq\Phi} \equiv e^{2\pi i\alpha}$$

We perform a partial wave decomposition of incoming wave packet — the $L = m > 0$ components receive a phase shift $e^{i\pi\alpha}$ and the $L = m < 0$ components receive a phase shift $e^{-i\pi\alpha}$

Let ϕ denote the scattering angle. An unscattered wave has support at $\phi = 0$, i.e., its angular dependence is
$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} = \delta(\phi)$$

Nonnegative angular momentum: $\displaystyle\sum_{m=0}^{\infty} e^{im\phi} = \frac{1}{1-e^{i\phi}}$

Negative angular momentum: $\displaystyle\sum_{m=-\infty}^{-1} e^{im\phi} = e^{-i\phi}\sum_{m=0}^{\infty} e^{-im\phi} = \frac{e^{-i\phi}}{1-e^{-i\phi}} = \frac{-1}{1-e^{i\phi}}$

Applying phase shift $e^{i\pi\alpha}$ for $m \geq 0$
$e^{-i\pi\alpha}$ for $m<0$

we obtain the angular dependence of the scattered wave
$$\sim \frac{1}{1-e^{i\phi}}\left(e^{i\pi\alpha} - e^{-i\pi\alpha}\right) = -e^{i\phi/2}\,\frac{\sin\pi\alpha}{\sin\phi/2}$$

$$\Rightarrow \frac{d\sigma}{d\phi} \sim \frac{\sin^2\pi\alpha}{\sin^2(\phi/2)}$$

On dimensional grounds $\dfrac{d\sigma}{d\phi} \propto \dfrac{1}{k}$ (in 2 spatial dim.)

The answer is $\frac{d\sigma}{d\phi} = \frac{1}{2\pi K} \frac{\sin^2 \pi\alpha}{\sin^2(\phi/2)}$

Naturally, it vanishes for α = integer (no A-B phase)

The singularity as $\phi \to 0$ arises because the interaction is long range — i.e. substantial phase shifts persist for arbitrarily large partial waves

Another way to describe the physical effects of the unbroken discrete gauge symmetry is in terms of superselection sectors

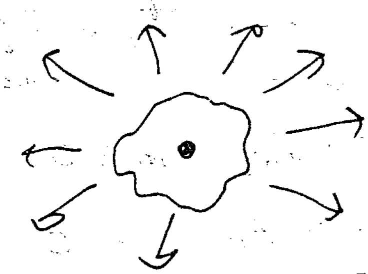
Decomposition of Hilbert space

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n$$

where $\langle m | \mathcal{O} | n \rangle = 0$, $m \neq n$,

for any local operator $\mathcal{O}$

Example: charge superselection rule in QED

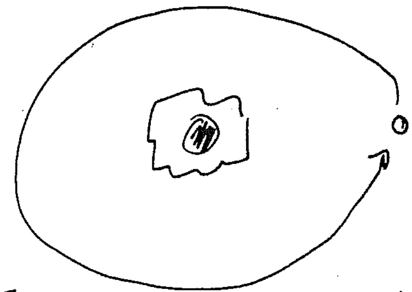


An operator acting in a compact region cannot create or destroy charge — a charge has an ∞-range electric field

[More formally: the physical observables of QED are gauge-invariant and smeared in a compact region $\Rightarrow$ they have charge = 0]

More generally... superselection sectors classify the kinds of long range "hair" that a localized object can carry.

Similarly, if there is an unbroken Z_N local symmetry, no local operator can create or destroy Z_N charge — it can have ∞ range interactions with other objects — there are N distinct sectors, labeled by their distinct Z_N charges. But in this case (in contrast to the charge sectors of electrodynamics) the long range hair is invisible classically — it is quantum hair.



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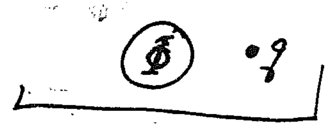
The theory of anyons

Consider: the angular momentum of a (two-dimensional) vortex, bound to a charge. We will argue that

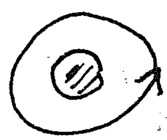
$$L = -\frac{q\Phi}{2\pi} + \text{integer}$$



(where $\hbar = c = 0$) — the angular momentum spectrum is displaced from integer values

Imagine a "flux tube" or solenoid, with enclosed flux Φ — the charged particle stays strictly outside the solenoid

- First: imagine "turning on" the flux — there is a torque on the charge



$$\int \text{curl}\, E = -\frac{d\Phi}{dt} \quad \text{or} \quad 2\pi R E = -\frac{d\Phi}{dt}$$

$$\text{Torque} = RqE = -\frac{q}{2\pi}\frac{d\Phi}{dt}$$

$$\Delta L = \int dt\,(dL/dt) = -\frac{q}{2\pi}\Phi$$

This quantity is an "angular variable" describing displacement $\Theta = q\Phi$ is periodic mod 2π

- Another viewpoint:

Suppose we rotate the composite object by 2π -- in doing so, we are transporting the charge in the gauge potential of the flux

we obtain

$$U(2\pi) = e^{iq\Phi} = e^{i\Theta}$$

what is going on with the (-1)? Should be $e^{-i2\pi L} = e^{i\Theta}$

$$\Rightarrow L = -\frac{q\Phi}{2\pi} + \text{integer} \quad (?)$$

[There is an issue here concerning whether L is "active" or "passive". We get the usual angular momentum algebra is $U(\theta) = e^{-i\theta L}$..]

$$\exp\left[-\varphi\frac{\partial}{\partial\tilde\varphi}\right]\psi(\tilde\varphi)=\psi(\tilde\varphi-\varphi)$$

$U(\varphi)\,|\tilde\varphi\rangle=|\tilde\varphi+\varphi\rangle$

$U(\varphi)\int d\tilde\varphi\,\psi(\tilde\varphi)|\tilde\varphi\rangle$
$=\int d\tilde\varphi\,\psi(\tilde\varphi)|\tilde\varphi+\varphi\rangle$
$=\int d\tilde\varphi\,\psi(\tilde\varphi-\varphi)|\tilde\varphi\rangle$

$L=-i\frac{\partial}{\partial\varphi}$

- Yet another viewpoint:

$L=-i\frac{\partial}{\partial\varphi}\Rightarrow\quad e^{i(m-\theta/2\pi)\varphi}$ has $L=m-\frac{\theta}{2\pi}$

Consider the "singular gauge" where $A_\varphi=0$, and $\psi(\varphi)$ becomes multi-valued.

E.g. $\quad A_\varphi=\Phi/2\pi$

$D_\mu\psi=(\partial_\mu-iqA_\mu)\psi$ invariant $\quad\psi\to e^{iq\Lambda}\psi$
$A\to A+\partial\Lambda$

So we gauge away A with
$\partial_\theta\Lambda=-\Phi/2\pi$

or $\Lambda(2\pi)=\Lambda(0)-\Phi$

$\psi(2\pi)=e^{-iq\Phi}\psi(0)$ in the singular gauge

$\Rightarrow\psi\sim e^{-iq\Phi\varphi/2\pi}$

or $L=-\frac{q\Phi}{2\pi}+$ integer is the orbital angular momentum

---

Superselection sectors

But shouldn't we have $e^{-i2\pi J}=1$?
— a rotation by 2π should act trivially
We already know this is not so, from our experience with spinors, in 3 spatial dimensions
We can have $e^{-i2\pi J}=(-1)^F$

This is allowed because there is a $(-1)^F$ superselection rule --- no local observable operator can change value of $(-1)^F$

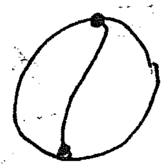
$e^{-2\pi iJ}(a|+\rangle+b|-\rangle)=a|+\rangle-b|-\rangle$ — It's okay, because no local operator detects relative phase

[The observables are bilinear in fermi fields.]

Here, too, in 2D, we must have $e^{-2\pi i J}$ = phase in each sector

$$e^{-2\pi i J} = e^{i\Theta}$$] superselection sectors labeled by angle $\Theta \in [0, 2\pi)$

In 3D, only $\Theta = 0, \pi$ are allowed
J is integer or half-integer] Why?

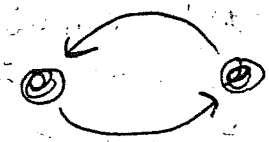


Because $\pi_1(SO(3)) = Z_2$
— the coffee cup trick] Since a path in SO(3), beginning at I, ending at 4π rotation is contractible — Necessary that $e^{-4\pi i J} = 1$

Not so in 2D
(Covering group of SO(2) is $\mathbb{R}$)

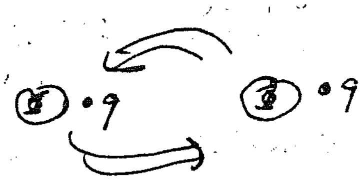
Statistics → there is a spin-statistics connection?
Then — fractional spin implies -- what?

Fractional statistics!



Consider the effect of a counterclockwise exchange of two particles — what happens to the many body wavefunction

In the case of the charge-flux composites — each charge is transported by π in the gauge potential of each flux



$$\Longrightarrow \exp i\left(\tfrac{1}{2} q \Phi + \tfrac{1}{2} q \Phi\right) = e^{iq\Phi} = e^{i\Theta}$$

The spin-statistics connection is respected, in the form

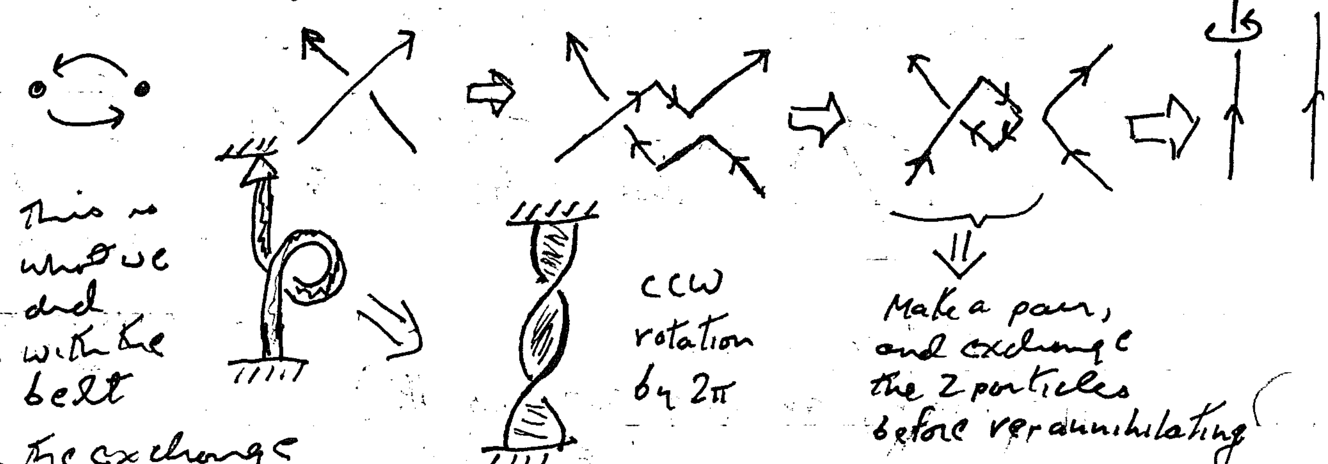
$$e^{-2\pi i J} = e^{i\Theta} = \text{exchange phase}$$

Spin and Statistics:

Why is the connection so robust -- the belt trick

What does the belt mean ---
There is a sense in which exchanging two particles is equivalent to rotating one by 2π

This doesn't require relativity -- but rather the existence of antiparticles will suffice



Comment -- The exchange phase violates P and T, as does the fractional J

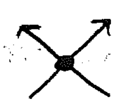
In 2D, a reflection is not equivalent to inversion and rotation..

Violation due to B field

Quantum Statistics in 2D

It is profoundly different from in 3D or 1D

- In 1D --



To exchange two particles, we move them past one another

So "fermi statistics" is equivalent to a phase shift of π due to their short-range interaction.

Consider n particles in 2D that are forbidden to occupy coincident positions -- And consider computing a quantum transition amplitude by the path integral method -- The histories for fixed initial and final configuration divide into distinct sectors

We have the freedom to weight the sectors contributing to the path integral in distinct ways — there are physically inequivalent ways to quantize the classical theory (this will be a recurring idea in our continuing discussion of topological methods)

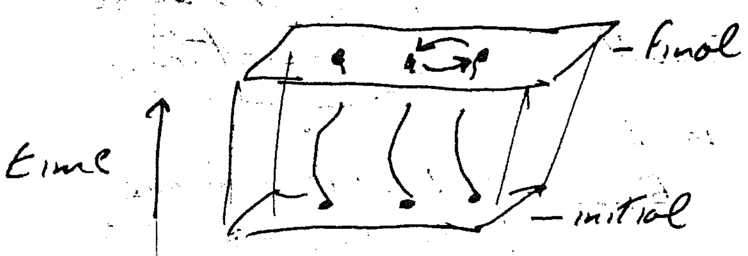
How much freedom? We must insist that the amplitudes we obtain obey the principle of conservation of probability.

If C_n = config space for n indistinguishable particles (in the plane, it is

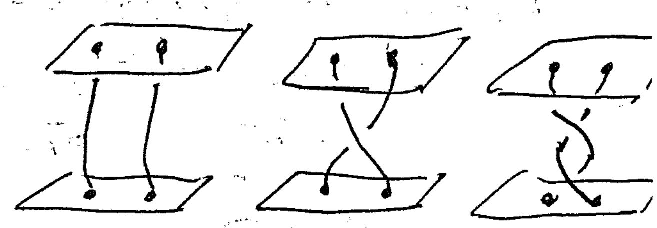
$$C_n = [(R^2)^n - D^{(n)}]/S_n$$

where $D^{(n)}$ is subset of R^{2n} where 2 or more positions coincide, and S_n is the group of perms acting on the n positions;

Consider closed loops in this space i.e. $\pi_1(C_n)$, acting on the final configuration ("exchange operators")

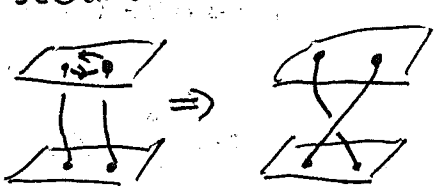


Lets start with 2 particles.



Here are three distinct sectors.

Exchanges mix the sectors:



Amplitude is a sum over sectors

$$A = \sum_{\substack{\text{sector} \\ i}} \alpha_i A_i$$

The amplitude itself need not be invariant under an exchange (a subtlety to which we will return) — but probabilities must be

And — π_1, π_2 are exchanges

$\Rightarrow$ $\pi_2 \circ \pi_1$ is equivalent to π_1 first, then π_2

A unitary representation

$$\pi_1 : A_i \to U(\pi_1)^{-1}_{ij} A_j$$
$$\pi_2 : A_i \to U(\pi_2)^{-1}_{ij} A_j$$

Some subtleties
- antiunitary?
- reps up to a phase.

$$\Rightarrow \pi_2 \circ \pi_1 : A_i \to U(\pi_1)^{-1}_{ij} A_j \to U(\pi_1)^{-1}_{ij}\, U(\pi_2)^{-1}_{jk} A_k$$
$$= \left[U(\pi_2)\, U(\pi_1)\right]^{-1}_{ik} A_k$$

$$U(\pi_2 \circ \pi_1) = U(\pi_2) \cdot U(\pi_1)$$ ⤳ a unitary representation of the group of exchange operations

What is this group?

$B_n \equiv$ Braid group on n strands:

An ∞ group with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$

The elements are words, subject to defining relations, strings of the generators $\{\sigma_i\}$ in this presentation

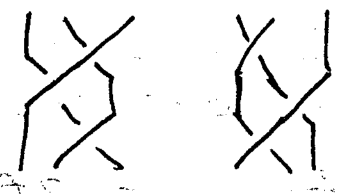
These are complete defining relations for B_n ---

$$\sigma_j \sigma_k = \sigma_k \sigma_j \qquad |j-k| \geq 2$$

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \qquad j = 1, \ldots, n-2$$

= the "Yang-Baxter relation"

Yang - Baxter relation:



$\sigma_2\sigma_1\sigma_2$ $\quad$ $\sigma_1\sigma_2\sigma_1$

In both cases, the guy moving right passes in front of everything and the guy moving left passes behind everything

What are the 1D irreps?
$\sigma_j : e^{i\theta_j}$ (unitary)

YB relation $\Rightarrow e^{i\theta_2}e^{i\theta_1}e^{i\theta_2} = e^{i\theta_1}e^{i\theta_2}e^{i\theta_1}$

$\Rightarrow e^{i\theta_2} = e^{i\theta_1}$

$$\boxed{\sigma_j = e^{i\theta}}$$

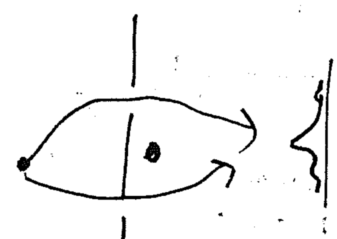
- a common phase for all exchanges

Anyon statistics, with statistical phase θ

The nonabelian irreps are very intricate — we'll learn more about them later

Anyons have a long range statistical interaction (not associated with light dynamical degrees of freedom)

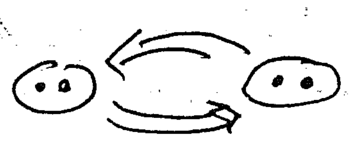
winding one around another $\Rightarrow e^{2i\theta}$



$$\boxed{\frac{d\sigma}{d\phi} = \frac{\sin^2\theta}{2\pi K}\left[\frac{1}{\sin(\phi/2)} + \frac{1}{\cos(\phi/2)}\right]^2 = \frac{2\sin^2\theta}{\pi K\sin^2\phi}}$$

Correspondingly, there is nontrivial A-B scattering for $\theta \neq 0, \pi$

Composite of Anyons



If we exchange 2 bound states of 2 anyons — what is the statistical phase of the composite?

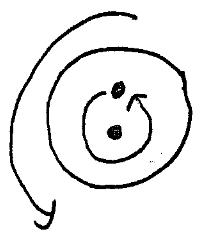
Each anyon in one composite has an exchange interaction with each anyon in the other

$$\Theta_{comp} = 2\Theta \cdot 2\Theta = 4\Theta$$

or for a composite of n anyons $\Theta_{comp} = n^2 \Theta$

But what about the spin statistics connection? Don't the spins just add?

$$J = -\frac{\Theta}{2\pi} \Rightarrow J_{comp} = -\frac{2\Theta}{2\pi} \text{ ? NO!}$$



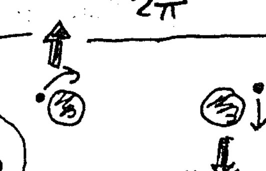
If we rotate the composite by 2π, not only do the constituents rotate, but one winds about the other:

$$\Theta_{comp} = \Theta + \Theta + 2\Theta$$

Another way to understand this: an anyon pair has fractional angular momentum

$$\psi(\varphi) = e^{i(m - \frac{2\Theta}{2\pi})\varphi} \Rightarrow L = m - \frac{2\Theta}{2\pi}$$

Antianyon has same Θ



Turn on both fluxes

$$J_{comp} = S_1 + S_2 + L_{12}$$
$$= -\frac{1}{2\pi}(\Theta + \Theta + 2\Theta) = -\frac{1}{2\pi} 4\Theta$$

Comment: Anyon-antianyon pair has $J = 0$

(Abelian) Chern-Simons theory

Describe anyons in field theory

A mass for a gauge field $-A_\mu A^\mu-$ is forbidden by gauge invariance — in gauge theories, gauge boson masses are ordinarily introduced by the Higgs phenomenon

But -- in 2 spatial dimensions, the situation is different — a gauge invariant + Lorentz invariant gauge boson mass is possible — as a "Chern Simons term" —

$$S_{CS} = -\frac{\mu}{2}\int d^3x\, \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \Big]$$ Comment: odd under P, T

Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$

$F_{\nu\lambda} \to F_{\nu\lambda}$

$\Delta\, \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda$

$= \varepsilon_{\mu\nu\lambda}\partial^\mu \Lambda\, \partial^\nu A^\lambda = \partial^\mu\left[\varepsilon_{\mu\nu\lambda}\Lambda\, \partial^\nu A^\lambda\right]$

- the Lagrangian changes by a total derivative, the action by a surface term —
(Can the surface term really be neglected — surely yes for $\Lambda \to 0$ at ∞ — more later!)

Is this really a mass term? We should check that
- the pole in the photon propagator shifts from $k^2 = 0$
- EM fields decay exponentially (screening)

Consider
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\mu}{4}\varepsilon_{\mu\nu\lambda}A^\mu F^{\nu\lambda} + \mathcal{L}_{g.f.}$$

A homework exercise: Pole is at $k^2 = \mu^2$.

In fact -- there is also a pole at $k^2 = 0$, whose interpretation is subtle. (see later: ∞ range Aharonov-Bohm scattering, despite mass gap).

Anyway, consider a current distribution coupled to gauge potential with a CS term that is infinitely heavy (nondynamical)

$$\mathcal{L} = eA_\mu J^\mu - \frac{\mu}{2}\varepsilon_{\mu\nu\lambda}A^\mu \partial^\nu A^\lambda$$

what is the A_μ field equation

$$\frac{\delta S}{\partial A_\mu} = 0 = + eJ^\mu - \mu\,\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$$

$$\Rightarrow \quad eJ^\mu = \frac{\mu}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}$$

And so, in particular

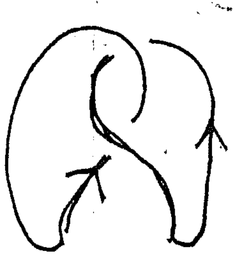
$$eJ^0 = \mu B$$

$$\int d^2x\, eJ^0 = \mu\int d^2x\, B$$

or $\boxed{Q = \mu\Phi}$

Relates the particle number (and charge density) to the magnetic field

This means that gauge field, which is completely constrained by the charge distribution (is not an independent dynamical field) associates flux with charge (and charge with flux).



This contributes a phase, if charges have linking world lines

e^{iS} — and for a soln to the field eqn

$$\mathcal{L} = eA_\mu J^\mu + \frac{-\mu}{2}\varepsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda$$
$$= eA_\mu J^\mu - A^\mu\left(\frac{e}{2}J_\mu\right) = e/_2\, A_\mu J^\mu$$

So the phase is $\exp\left(i \int d^3x \, \frac{1}{2} e A_\mu J^\mu \right)$

In the exchange of two particles, we accumulate the phase

$$\Theta = \frac{1}{2} q \Phi$$

(where $q = \mu \Phi$)

$$= \frac{\mu}{2} \Phi^2 = q^2/2\mu$$

(where we had $\Theta = q\Phi$ for our charge flux tube composites)

Why the factor of $\frac{1}{2}$ — it is because the charge and flux distributions actually coincide — e.g.

– If we consider the accumulated torque when we turn on the flux in a solenoid

$$\text{Torque} = -\frac{q}{2\pi} \frac{d\Phi}{dt} \qquad q = \mu \Phi$$

$$= -\frac{\mu}{2\pi} \Phi \frac{d\Phi}{dt} = -\frac{\mu}{4\pi} \frac{d}{dt} \Phi^2 \Rightarrow$$

$$\Delta L = -\frac{1}{2\pi} \frac{\mu}{2} \Phi^2$$

Lecture #6

Mathematically, the result can be obtained by doing a (carefully regulated) Gaussian integral — subtle because there is a "self-linking" contribution, too.

$$\left\langle e^{iq_1 \oint_{C_1} A} \, e^{iq_2 \oint_{C_2} A} \, e^{-i\int d^3x \frac{\mu}{2} \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda} \right\rangle$$

$$= \exp\left[+\frac{i}{2} \left(q_1 \Phi_2 + q_2 \Phi_1 \right) L(C_1, C_2) \right]$$ ← Linking number

$\Phi_{1,2} = q_{1,2}/\mu$

$$= \exp\left[i \left(\frac{q_1 q_2}{\mu} \right) L(C_1, C_2) \right]$$

A homework exercise!

What is tricky about the evaluation of this Gaussian integral is that it is singular, and therefore ambiguous. Completing the square we have

$$\langle e^{i\int A_\mu J^\mu}\rangle = N\int(dA)\,\exp\left[-\tfrac{1}{2}\int A_\mu(\Delta^{-1})^{\mu\nu}A_\nu + i\int A_\mu J^\mu\right]$$

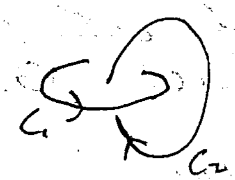
$$= \exp\left[-\tfrac{1}{2}\int J^\mu \Delta_{\mu\nu} J^\nu\right]$$

where $\Delta_{\mu\nu}$ is the gauge field propagator

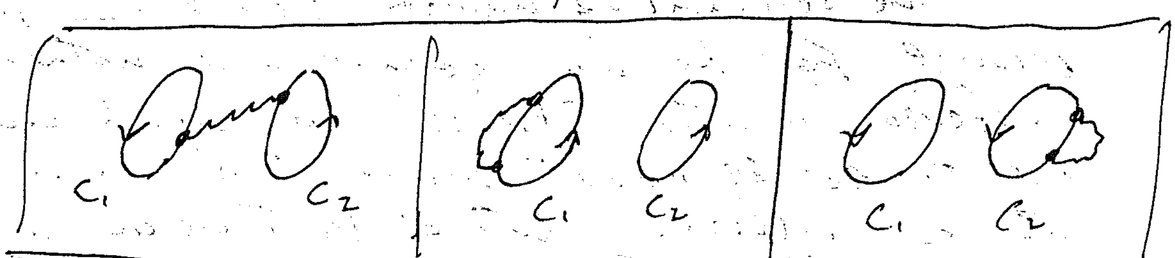
Let $\int A_\mu J^\mu = \oint_C dx^\mu A_\mu$ (where C is a sum of closed loops).

$$\Rightarrow \langle e^{i\oint_C A_\mu dx^\mu}\rangle = \exp\left[-\tfrac{1}{2}\oint_C dx^\mu \oint_C dy^\nu\, \Delta_{\mu\nu}(x,y)\right]$$

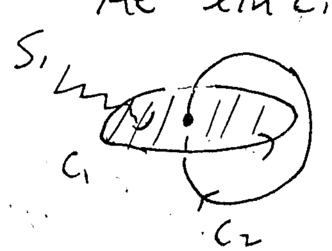
Suppose $C = C_1 + C_2$, where the loops C_1 and C_2 might link



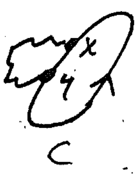
There are three contributions to the argument of the exponential -- in terms of Feynman diagrams:



You will find that the first diagram is proportional to the linking number $L(C_1, C_2)$

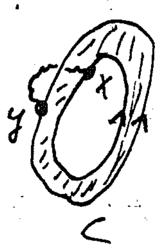


If one loop is the boundary of a non-self-intersecting surface — this is the (signed) number of times that C_2 crosses the surface S_1 — which inherits an orientation from C_1, defined by the RH rule.

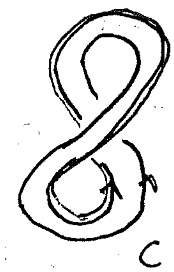


But this diagram becomes singular where $|x-y| \to 0$

One way to define the integral is to replace C by a ribbon of finite thickness, and integral x over one edge of the ribbon and y over the other.



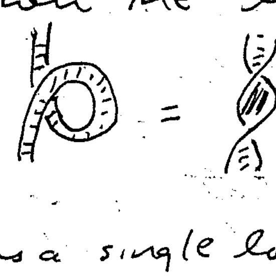
Then this integral is the linking number of the edges, which counts how many times the ribbon twists by 2π as C is traversed. In this sense, the Gaussian integral knows that anyons obey $e^{-2\pi i J} = e^{i\theta}$



[Diagram has a symmetry factor of $\frac{1}{2}$, which explains why the phase associated with winding one particle around another is twice θ.]

The self-linking number is still ambiguous, because there is no general and natural prefered way to "frame" a closed curve — i.e. to decide how it should twist when blown up into a ribbon. But at least we can make a definite statement about how the self-linking changes when we change the framing, or when the loop crosses itself

E.g.



means that

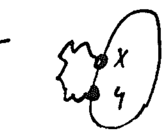
$$\langle \circlearrowleft \rangle = e^{i\theta} \langle \uparrow \rangle, \quad \theta = \frac{q^2}{2\mu}$$

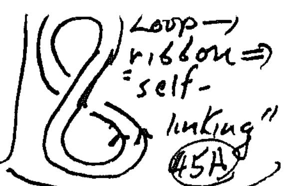
If C is a single loop $\langle \circlearrowright \rangle = e^{-i\theta} \langle \uparrow \rangle$

$\langle W(C) \rangle = (e^{i\theta})^r$ where r is the "writhe" — it counts how many (signed) times the loop crosses itself, when flattened.

$L(C_1, C_2)$ = # of signed crossings of S_1 by C_2

need to regulate as $|x-y| \to 0$





How can we *use* the CS theory ---

For one thing, we can obtain a useful phenomenological description of fermions (or anyons) by representing them as bosons with $\theta = \pi$ (or other value of θ). For example, a 2D electron gas in a strong B field is known to exhibit phases with distinctive properties that we can understand in qualitative terms using a CS description

E.g. for fermions in an electromagnetic field:

$$\mathcal{L} = |D_\mu \phi|^2 - U(|\phi|) - \frac{1}{4} F_{\mu\nu}^2 - \frac{\mu}{2} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda$$

where $D_\mu = \partial_\mu - ieA_\mu - ia_\mu$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Need to discuss: A dependence of J in minimal coupling

A_μ: (dynamical) em field (but we can regard it as fixed external field)

a_μ: (constrained) statistical gauge field

To describe fermions

$$\theta = \tfrac{1}{2} q \Phi = q^2/2\mu = \pi(2m+1), \quad m = \text{integer}$$

Choose $q = 1$

$$\mu = \frac{1}{2\pi} \cdot \frac{1}{(2m+1)}$$

<u>Vacuum solution</u>:

Suppose an external em field is applied. We can minimize the energy with

$$D_\mu \phi = 0$$

or $\quad eA_\mu + a_\mu = 0$, and ϕ = constant.

What about the field equation for a_μ? Note that

$$J^\mu = \frac{\partial \mathcal{L}_{mat}}{\partial A_\mu} = e\frac{\partial \mathcal{L}_{matt}}{\partial a_\mu} = e j^\mu$$

(because A_μ, a_μ enter $\mathcal{L}_{matt}$ only in combination $(eA_\mu + a_\mu)$). J^μ is the em current, and j^μ is the particle number current coupled to the statistical gauge field. Recall that

$$j^0 = \mu b \qquad (b = \text{statistical gauge field})$$

and hence $\quad eA_\mu + a_\mu = 0 \Rightarrow b = -eB$

implies

$$j^0 = -\mu e B$$

The ratio of j^0 to B can be related to the "filling factor" of a Landau level, the number of electrons per quantum of magnetic flux

$$\Phi_0 = \frac{2\pi}{e}$$

The "filling factor" is

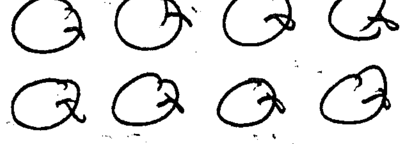
$$\nu = -j^0\left(\frac{\Phi_0}{B}\right) = \mu e \Phi_0 = 2\pi\mu = \frac{1}{2m+1}$$

(the minus sign because electrons are negatively charged)

The electrons can consistently "manufacture" a statistic field that cancels the applied field

But only if filling is one over an odd integer!

For $\nu = 1$ $(m=0)$, this is just a way to say that electrons are happy if they can fill a Landau level.



(an incompressible state — there is a gap if we try to squeeze in just one more electron)

But for $\nu = \frac{1}{3}$, what has happened is remarkable — through a complicated collective phenomenon, the electrons have found a state in which they can be happy, a very different incompressible state (now incompressible because squeezing the electrons squeezes the applied B field, too)

At least if the potential $U(|\phi|)$ is such as to favor $|\phi| \neq 0$ in the vacuum, this collective state can be described as a type of Higgs condensate — if we think of the gauge field as having a conventional kinetic term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} f_{\mu\nu}^2$$

then $D_\mu = \partial_\mu - ieA_\mu - i\tilde{e}\, a_\mu$

where $\tilde{e} \to \infty$. So the l.c. of A_μ and a_μ that gets an (∞) mass from the Higgs mechanism is a_μ, and the photon A_μ stays massless.

Physically — what does the condensing field ϕ represent?

Since $\Phi = -\frac{}{e}\Phi_{stat} = \frac{}{\mu e} \quad \frac{}{e}(2m+1) = -\Phi_0(2m+1)$
$2m+1$ flux quanta tied to each scalar particle
Condensate = electron + $(2m+1)$ flux quanta

The Hall conductivity:

What happens when we apply an electric potential across the 2DEG? Just consider again the eqns

$$J^\mu = e j^\mu = e\frac{\mu}{2}\varepsilon^{\mu\nu\lambda} f_{\nu\lambda} = -\frac{e^2\mu}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda}$$

i.e. $$J^1 = e^2\mu E_2$$
$$J^2 = -e^2\mu E_1$$

No longitudinal conductivity
Only a transverse current flows with $J = \sigma_{xy} E$

$$\sigma_{xy} = e^2\mu = \frac{e^2}{2\pi}\frac{1}{2m+1} = \frac{e^2}{2\pi}\nu$$ ← filling factor

If we restore $\hbar$: $\frac{e^2}{2\pi} \to \frac{e^2}{2\pi\hbar} = e^2/h$

$$\boxed{\sigma_{xy} = e^2/h \, \nu}$$

Vortices:

What are the low-lying excitations? Apart from the em photon, which stays massless, we have the (only) massive massive statistical gauge field, and the massive Higgs field — none of these excitations carry em charge. What are the quasiparticles responsible for carrying the Hall current.

What else is in the spectrum of the Higgs model? The vortex. It has fractional charge and statistics

Recall $j^0 = \mu b \Rightarrow q = \mu \Phi_{stat}, \quad \Phi_{stat} = 2\pi$

$J^0 = e j^0$

$$\Theta = \tfrac{1}{2} q \Phi_{stat} = \tfrac{\mu}{2} \Phi_{stat}^2 = \frac{1}{2}\frac{1}{2\pi}\frac{1}{2m+1}(2\pi)^2 = \frac{\pi}{2m+1} = \pi\nu$$

and electric charge

$$Q = eq = e\mu\Phi_{stat} = e\frac{1}{2\pi}\frac{1}{2m+1}2\pi = \frac{e}{2m+1} = e\nu$$

A "quasihole" with fractional spin and statistics

(But vortex does not carry any "real" magnetic flux – only the statistical flux.)

The Hierarchy of FQHE states

Values of Θ other than $\frac{\pi}{2m+1}$ are seen, as are FQH states for $\nu \neq \frac{1}{2m+1}$. These arise as collective states of Laughlin quasiparticles!

To describe the Laughlin quasiparticles, we choose

$$\Theta = \pi\nu + 2\pi n = \frac{1}{2\mu}, \text{ or}$$

$$2\pi\mu = (\nu + 2n)^{-1}$$

The new collective state has

$$\nu_{new} = 2\pi\mu, \quad \sigma_{xy} = \tfrac{e^2}{h}\nu, \quad \Theta_{new} = \pi\nu_{new}$$

We could also choose the other sign for $2n$, or for $\pi\nu$ (a condensate of holes rather than particles) or we can say: $\dfrac{\Theta_{new}}{\pi} = \left(\pm \dfrac{\Theta_{old}}{\pi} \pm 2n\right)^{-1}$

E.g. $\nu_{new} = \left(\frac{1}{3} + 2\right)^{-1} = 3/7$

$-\left(\frac{1}{3} - 2\right)^{-1} = 3/5$

$\left(\frac{-1}{5} + 2\right) = 5/9$

etc

"Adiabatic Principle"

This comes about because

$\nu^{-1} = \dfrac{\text{flux q.}}{\text{electrons}} = \dfrac{1}{q} + 2m$

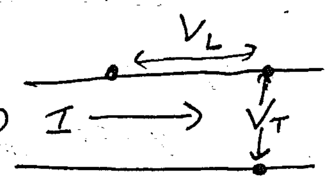
start w/ q. filled Landau levels

2m flux quanta on each electron spread to give uniform field

Get states with odd denominators

Naive QHE

2D Electron Gas



$\rho_L = \sigma_L = 0$

$\vec{E} = \underset{\sim}{\sigma} \vec{J}$

$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = 0$

$\vec{J} = ne\vec{v}$ (n = # density)

$\vec{E} = -\frac{1}{nec} \vec{J} \times \vec{B}$

$\sigma_H = I/V_T$

$V_L = 0$

$\sigma = \begin{pmatrix} 0 & \sigma_H \\ \sigma_H & 0 \end{pmatrix}$

or $\sigma_H = \dfrac{nec}{B}$

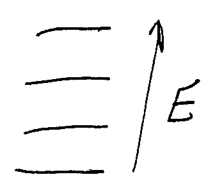
$\Phi_0 = \dfrac{2\pi\hbar c}{e}$

$\dfrac{n}{B} = \dfrac{e}{2\pi\hbar c}\nu$, $\nu = \dfrac{\text{no of electrons}}{\text{no of flux quanta}} = \dfrac{n\Phi_0}{B}$

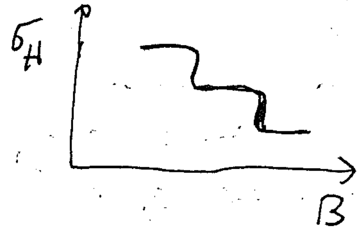
$\boxed{\sigma_H = \dfrac{e^2}{2\pi\hbar}\nu}$ → Free classical charges

Quantum theory: Landau levels

Larmor frequency $\omega = \dfrac{eB}{mc}$, $E = \hbar\omega(n + \frac{1}{2})$

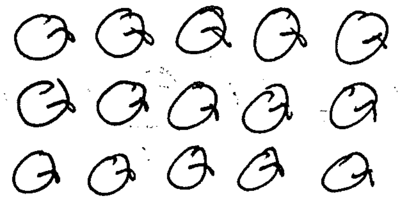


Degeneracy: $\dfrac{\text{Number}}{\text{Area}} = \dfrac{eB}{2\pi\hbar c} = \dfrac{B}{\Phi_0}$ Filled Landau level has $\nu = 1$



For free electrons, σ_H jumps by e^2/h when E_F crosses a Landau level

High B field
Low Temperature
Pure sample



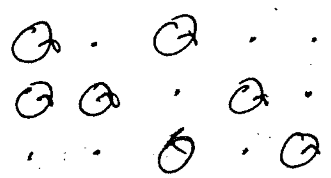
But -- what happens when we introduce perturbations due to dirt, interactions among electrons, etc.. Why is the IQHE robust?

The key point is:

① The filled Landau level is an <u>incompressible</u> state - there is an energy gap; it costs finite energy to excite it. Hence perturbations do not alter its qualitative properties if the perturbations are weak (large energy denominators)

② The property $\sigma_H = \frac{e^2}{h} \cdot$ integer has a <u>topological</u> origin. I'll postpone the explanation, since it involves the Berry phase and the Dirac quantization condition. But the key point is that a "topological invariant" of the system should not change as we vary the Hamiltonian, as long as the correlation length remains finite (no "gap collapse")

As we add electrons, they remain <u>localized</u>, and don't contribute to σ_H, until a phase transition occurs



Now, the FQHE is a completely different phenomenon. A partially filled Landau level is <u>compressible</u>; it is highly degenerate, and therefore very sensitive to weak perturbations. As we increase the number of charge carriers, σ_H should change smoothly... but it does not -- there are still sharp jumps, with values $\sigma_H = \frac{e^2}{h}\nu$ preferred for special filling factors ν

Apparently (since the FQHE is seen only in very pure high mobility samples) electron interactions (Coulomb repulsion) drive the electrons to a new kind of collective state that is again incompressible (it has a gap). Thus, while the IQHE can be understood without reference to interactions among electrons (it is only important to understand how each electron is affected by the disorder in the sample), the interactions among electrons are critical for the formation of the FQHE phase.

We won't discuss the microscopic description of the FQHE phases (Laughlin states). But the point is that, being incompressible, these states are stable w.r.t. perturbations. Their σ_H has a topological origin (to be explained) and so is unmodified as the electron density increases. (Additional electrons remain localized and do not contribute to the Hall current) — until a phase transition occurs to a new FQHE state, with another preferred value of ν

One way to describe what happens is that each electron "forms a composite" with $2m$ quanta of flux, these composite electrons form a $\nu = 1$ state that fills a Landau level, if $\nu = 1/(2m+1)$ to begin with.

Another way to say this — imagine a filled Landau level ($\nu = 1$) and put $2m$ flux quanta on top of each electron. Now let this flux "spread" to become a uniform B field. If there is no gap collapse — we obtain an

incompressible state with $\nu = \frac{1}{2m+1}$
this is the state that our
Chern-Simons theory describes.

The main task of (Laughlin's) microscopic theory is to explain how, for special values of the filling factor ν, the electrons manage to find an incompressible collective state with a gap. Then the low momentum behavior is well described by the Chern-Simons field theory. It is similar to the Ginzburg-Landau theory of superconductivity (abelian Higgs model) which describes a superconductor well, but can be justified only by a microscopic (Bardeen-Cooper-Schrieffer) theory that explains the origin of the gap.

Topological Degeneracy — Lecture #7

As we will discuss in more detail later, the phases of a gauge theory cannot be distinguished by means of a local order parameter — a nonlocal criterion is needed. One such criterion is the degeneracy of the groundstate on a space of nontrivial topology.

In fact, we can see that any system with anyonic excitations (whether a gauge theory or not) has such a topological degeneracy, if the anyons can be pair created.