after the interaction, the point charge is still $|q|$, but the vortex pair is now $|2q|$ and the total charge is still

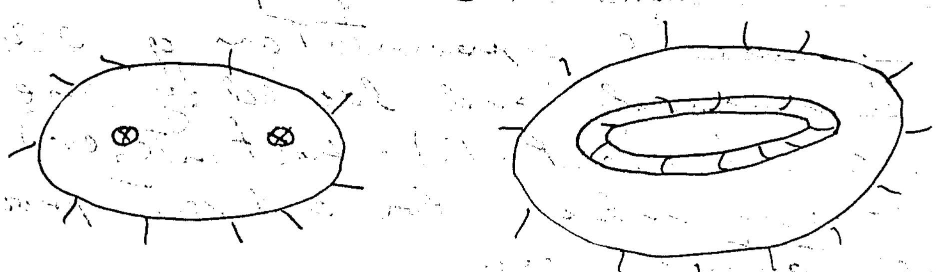
$$|q| \subset |q| \otimes |2q|$$

The irrep of the vortex pair changes while the irrep of the whole world does not, which eases any lingering suspicion that the charge transfer is a mere gauge artifact.

Lecture #11

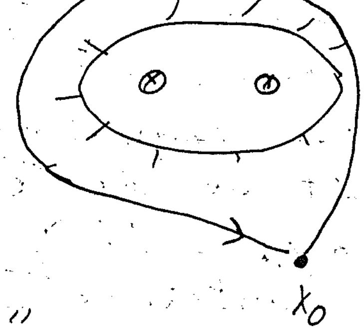
So far, our discussion of Cheshire charge has been entirely classical. We'll need the quantum theory to see (for example) that Cheshire charge is quantized.

Formally, electric charge characterizes transformation properties under global gauge transformations.

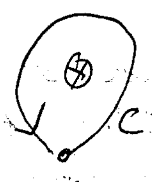


We can consider a gauge transformation that is constant outside the region that contains a pair of Alice vortices, or a loop of Alice string in three dimensions.

Global transformations characterize the interactions of an object with another object that is far away. So the "global" transformation properties of a vortex are determined by the effect of gauge transport around the vortex, beginning and ending at a basepoint x_0 that is "at infinity."



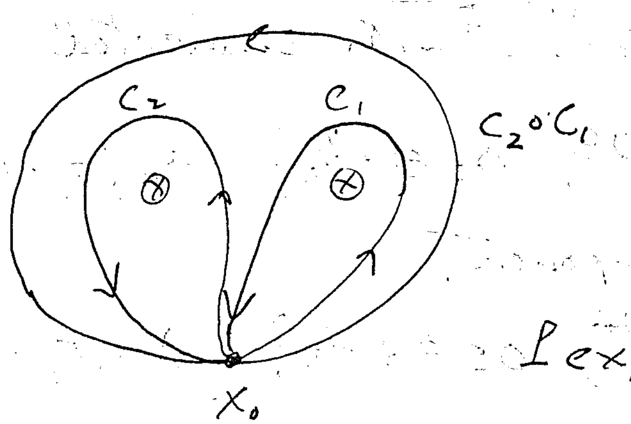
As already noted, the "flux" of a vortex



$$P\exp\left(i\oint_{C,x_0} A\right) = \qquad h(C,x_0) \in H(x_0)$$

is an element of the unbroken gauge group $H(x_0)$ that stabilizes the Higgs field at x_0. Under a gauge transformation $a \in G$, it transforms as

$$a: h \to a h a^{-1}.$$



Consider a pair of vortices, whose total flux is trivial. We have

$$P\exp\left(i\oint_{C_1,x_0} A\right) = h$$

$$\underline{P}\exp\left(i\oint_{C_2,x_0} A\right) = h^{-1}$$

$$\underline{P}\exp\left(i\int_{C_2\circ C_1,x_0} A\right) = \underline{P}\exp\left(i\oint_{C_2,x_0} A\right)\cdot\underline{P}\exp\left(i\oint_{C_1,x_0} A\right) = e$$

Here $C_2\circ C_1$ denotes the path obtained by first traversing C_1 and then traversing C_2; the order $C_2 \circ C_1$ is chosen in order to be consistent with the usual convention for composing path-ordered exponentials. If we denote the flux state of this pair by $|h\rangle$, then under a global gauge transformation $a \in H$

$$a: \; |h\rangle \to |aha^{-1}\rangle$$

Now consider the particular case of the Alice model, with Higgs symmetry breaking $SO(3) \to O(2)$. In fact, since $SO(3)$ is not simply connected, it is more convenient to describe the symmetry breaking pattern as $SU(2) \to Pin(2)$, where $Pin(2)$ is the double cover of $O(2)$, with connected component

$$H_c = \{ e^{iQ\theta}, \; 0 \le \theta \le 4\pi \}, \quad Q = \tfrac{1}{2}\sigma_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

and disconnected component

$$H_d = \{ X e^{iQ\theta}; \; 0 \le \theta \le 4\pi \} \quad X = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

If two vortices each have flux $\pm X e^{iQ\theta}$, then the total flux is

$$(-X e^{iQ\theta})(X e^{iQ\theta}) = e^{-iQ\theta} e^{iQ\theta} = I$$

Denote by $|\theta\rangle$ the state of the vortex pair, where each vortex has flux $|\theta\rangle$

Under a gauge transformation in H_c:

$$e^{i\omega Q} : \; X e^{i\theta Q} \to e^{i\omega Q}(X e^{i\theta Q}) e^{-i\omega Q}$$
$$= X e^{i\theta Q} e^{-2i\omega Q}$$

or $\quad U(e^{i\omega Q}) |\theta\rangle = |\theta - 2\omega\rangle$

Under a gauge transformation in H_d:

$$X e^{i\omega Q} : \; X e^{i\theta Q} \to (X e^{i\omega Q})(X e^{i\theta Q})(e^{-i\omega Q} X)$$
$$= X e^{i\omega Q} e^{-i\theta Q} e^{+i\omega Q}$$

or $\quad U(X e^{i\omega Q}) |\theta\rangle = |-\theta + 2\omega\rangle \quad$ (up to a phase)

We can construct superpositions of the $|\theta\rangle$ states that transform irreducibly under $Pin(2)$. The irreducible representations are two-dimensional and contain states with charge $Q = \pm|q|$

$$U(e^{i\omega Q}) |q\rangle = e^{i\omega q} |q\rangle$$
$$X |q\rangle = |-q\rangle \quad \text{for } q = \text{integer}$$
$$X |q\rangle = i |-q\rangle \quad \text{for } q = \text{integer} + \tfrac{1}{2}$$

(i.e. $X^2 = -\underline{1}$ in the irreps of $Pin(2)$ contained in the half-odd-integer irreps of $SU(2)$.)

The charge eigenstates are obtained from flux eigenstates as:

$$|q\rangle = \int_0^{4\pi} \frac{d\theta}{\sqrt{4\pi}}\, e^{i\theta q/2}\, |\theta\rangle .$$

We verify that: Note: q is an integer for this to be a smooth wave function on $\theta \in [0, 4\pi)$

$$U(e^{i\omega Q})\,|q\rangle = \int_0^{4\pi} \frac{d\theta}{\sqrt{4\pi}}\, e^{i\theta q/2}\, |\theta - 2\omega\rangle$$

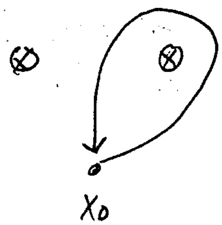
$$= e^{i\omega q}\,|q\rangle$$

$$X\,|q\rangle = \int_0^{4\pi} \frac{d\theta}{\sqrt{4\pi}}\, e^{i\theta q/2}\, |-\theta\rangle = |-q\rangle$$

The charge q is restricted to integer values. The q = integer + $\frac{1}{2}$ irreps of Spin(2) do not arise, because the vortex is invariant under the nontrivial element $-I$ in the center of SU(2). This is so because the Higgs field, in the (5) irrep of SO(3), is invariant under the center.

We note that, for a vortex pair (or string loop) the flux eigenstate basis and the charge eigenstate basis are complementary — the pair cannot be a simultaneous eigenstate of flux and charge. The flux states are states of a "charge rotor" with a definite orientation, and the charge states are the states of the rotor with definite "angular momentum."

What happens when a particle with charge q passes through a pair of vortices that initially carried no charge (or charge $\tilde{q}$.)?



If the charged particle passes through a vortex pair in a flux eigenstate, the phase of the wave function is modified according to

$$|\theta\rangle\, |q\rangle \Rightarrow |\theta\rangle\, U(Xe^{iQ\theta})\, |q\rangle$$

$$= e^{iq\theta}\, |\theta\rangle\, |-q\rangle \quad \text{(up to an overall phase)}$$

So for a superposition of flux eigenstates

$$|\tilde{q}\rangle |q\rangle = \left(\int_0^{4\pi} \frac{d\theta}{\sqrt{4\pi}}\, e^{i\theta\tilde{q}/2}\, |\theta\rangle\right) |q\rangle$$

$$\rightarrow \left(\int_0^{2\pi} \frac{d\theta}{\sqrt{4\pi}}\, e^{i\theta\tilde{q}/2} e^{i\theta q}\, |\theta\rangle\right) |-q\rangle$$

$$= |\tilde{q} + 2q\rangle\, |-q\rangle$$

— the charge of the particle flips, and the charge of the vortex pair shifts by a compensating amount, just as we found in our classical discussion.

"Holonomy interaction" of Nonabelian Vortices

Now that the Alice model has helped us to grasp the concept of Cheshire charge, let's consider the more general setting of a symmetry breaking pattern

$$G \to H$$

where H is nonabelian. Furthermore, let's suppose that H is a discrete group, so that the model contains no massless gauge fields.

For example, suppose now that the Higgs potential of the Alice model is minimized by

$$\Phi_0 = v\begin{pmatrix} 1+\delta & 0 & 0 \\ 0 & 1-\delta & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

— the degeneracy of the eigenvalues is broken for $\delta \neq 0$. This breaks $SO(3)$ to the discrete group of rotations Ω such that

$$\Omega \Phi_0 \Omega^T = \Phi_0$$ — rotations that are diagonal in the same basis that diagonalizes Φ_0

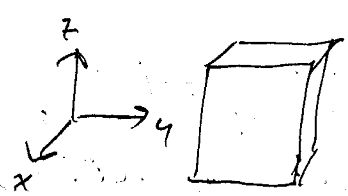
There are

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Omega = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Omega = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ as well as $\Omega = \mathbb{1}$

i.e. 180° rotations about the x-y-z axes

This is actually the abelian group $D_2 \cong Z_2 \times Z_2$, (the rotations that leave invariant a rectangular prism)



But when lifted to $SU(2)$, the double cover of D_2 is the nonabelian quaternionic group

$$Q = \{\pm I, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\}$$
$$\Downarrow$$
$$D_2$$

– elements of Q commute up to sign – i.e. the commutator of two elements is

$$aba^{-1}b^{-1} \in \{\pm I\} \text{ ; the center of the group}$$

(the center of G is the subgroup of G containing those elements that commute with everything in G.)

As another example, consider

$$G = SU(3) \longrightarrow H = S_3$$

S_3 is the group of permutations of three objects, with defining 3×3 representation

$$\left\{ I,\ (123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ (132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\ (13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

This defining representation of S_3 is not contained in $SU(3)$, because the transpositions are represented by unitary matrices with $\det = -I$. However, with the replacement

$$(ij) \to -(ij), \text{ e.g. } (12) \to \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

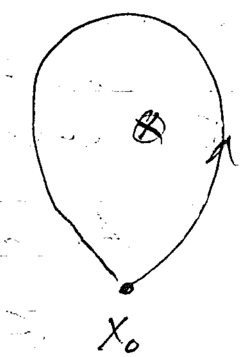
we obtain matrices with $\det = 1$, and the group multiplication properties are unaffected. The pattern

$$SU(3) \to S_3$$

can be realized with a suitable expectation value for Higgs fields Φ_{ab} transforming as the symmetric tensor (six-dimensional) irrep of $SU(3)$.

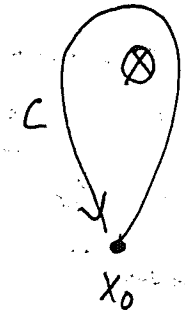
Since the group is discrete, each flux sector with

$$P\exp\left(i\oint_{C,x_0} A\right) = h(C, x_0)$$



is topologically distinct.
— infinite energy barriers separate the different flux sectors. This does not ensure that there is a stable vortex in each sector; rather the lowest energy configuration could be several widely separated vortices (as would be the case in the sector of a type II superconductor with winding $|n| \geq 2$).

The way flux sectors "fuse" is determined by the group multiplication law in H, but we need to establish careful conventions to fix the order of multiplication if the group is nonabelian



To assign a "flux" (i.e. group element) to a vortex, we specify basepoint x_0 and path C surrounding the vortex. Even so the flux is fixed only up to conjugacy — the flux h transforms under $a \in H(x_0)$ as

$$h \to a h a^{-1}$$

If we specify a particular element of a (nontrivial) conjugacy class, we are implicitly "fixing the gauge"

Physically, "fixing the gauge" corresponds to establishing a convention about "who's who" among charged particles that comprise a basis for an irrep of dimension greater than one — it is like establishing a "bureau of standards" at x_0 where standard "red, yellow, and blue" quarks are kept, in QCD.

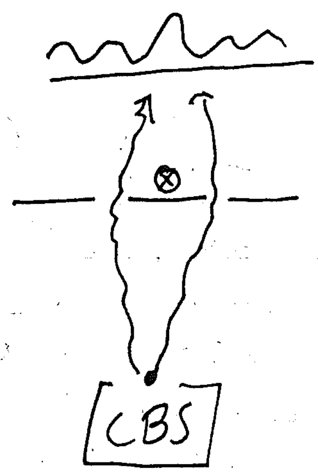
Let our charged particles transform as irrep R of H, and let

$$\{ |R; i\rangle, \; i = 1, \ldots, \dim R \}$$

denote our standard basis. Then $a \in H$ is represented by $D^{(R)}(a)$, with matrix elements

$$D_{ij}^{(R)}(a) = \langle i | D^{(R)}(a) | j \rangle$$

i.e. $\quad a : |j\rangle \longrightarrow \sum_i |i\rangle D_{ij}^{(R)}(a)$



with standardized particles, we can measure the flux in a double-slit interference experiment, if

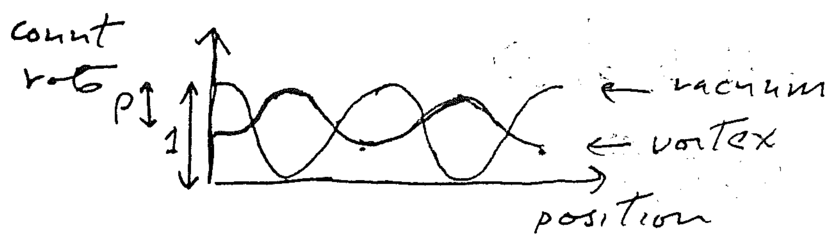
$$|u\rangle = \sum u_i |i\rangle$$

is the charged particle state, then there is an Aharonov-Bohm contribution to the relative phase of the two beams

$$\langle u | D^{(R)}(a) | u \rangle = \rho e^{i\phi}$$

The phase of this quantity causes a detectable shift of the interference fringes, relative to the "vacuum" case where there is no vortex behind the screen. The modulus of the quantity is also measurable, as it determines the "visibility" (i.e. amplitude) of the interference pattern

$$\| |u\rangle + e^{-i\alpha} D(a)|u\rangle \|^2 = 2 + e^{-i\alpha}\langle u| \ D(a)|u\rangle + e^{-i\alpha}\langle u|D(a)|u\rangle$$
$$= 2\left[1 + \rho\cos(\alpha - \phi)\right]$$



The visibility is less than unity in general, because carrying the charge around the vortex might, with nonzero probability, cause the charge to "flip" to an orthogonal state

In that case, the charge carries "which-way" information about the path followed by the particle and there can be no interference.

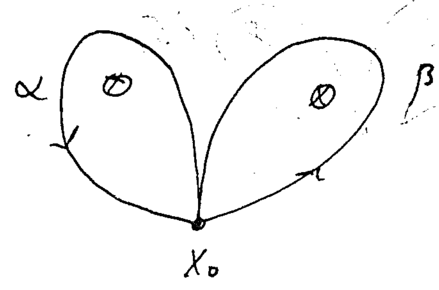
Using the states $|i\rangle$ as projectiles, we can measure diagonal matrix elements $D_{ii}^{(R)}(a)$, and choosing $\frac{1}{\sqrt{2}}(|i\rangle + |j\rangle)$ or $\frac{1}{\sqrt{2}}(|i\rangle + i|j\rangle)$ we can measure real and imaginary parts of $D_{ij}^{(R)}(a)$

E.g. $\frac{1}{2}(\langle i| + \langle j|)\, D^{(R)}(a)\, (|i\rangle + |j\rangle)$

$$= \frac{1}{2}\left(D_{ii} + D_{jj} + D_{ij} + D_{ji} \right)$$
$$= \frac{1}{2}\left(D_{ii} + D_{jj}\right) + \frac{1}{2}\left(D_{ij} + D_{ij}^{*}\right).$$

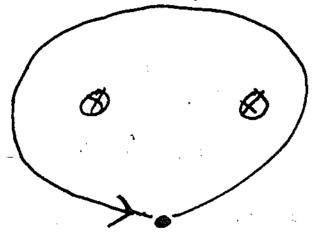
Thus the matrix $D_{ij}^{(R)}(a)$ is completely determined by our measurements, and if the irrep R is faithful (represents no element other than e by the identity), this determines a in the gauge corresponding to our conventions

The reason that "fixing a gauge" is quite useful is that when there is more than one vortex present, the choice of the conjugacy class representative assigned to a vortex is not purely a convention. Consider two vortices, and paths α and β with a common basepoint x_0 that wind around their positions. In a particular gauge, the fluxes associated with



these paths are $a, b \in H$. In the same gauge, the flux associated with the composite path $\alpha \circ \beta$ that winds around both vortices is ab. Now if we perform a gauge transformation $h \in H(x_0)$, all of these fluxes are conjugated by a common element

$\alpha \circ \beta$:



x_0

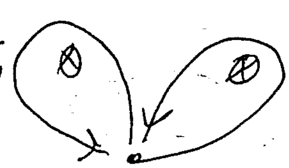
$$
\begin{aligned}
h:\quad & a \to hah^{-1} \\
& b \to hbh^{-1} \\
& ab \to h(ab)h^{-1}
\end{aligned}
$$

– in particular, the conjugacy class of the product flux is gauge-independent

Now multiplication of conjugacy classes is not well defined. That is, if $a \sim a'$ and $b \sim b'$ are conjugate, it need not be the case that $ab \sim a'b'$. Thus there may be inequivalent ways to patch together vortices chosen from classes $[a]$ (the class containing a) and $[b]$ (the class containing b).

For example, in Q, $\pm i\sigma_1$ are conjugate – when we patch two vortices from the class together, the composite flux can be either

$\pm i\sigma_1$



$i\sigma_1$

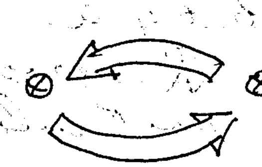
$$(i\sigma_1)(-i\sigma_1) = I$$

or $$(i\sigma_1)(i\sigma_1) = -I$$

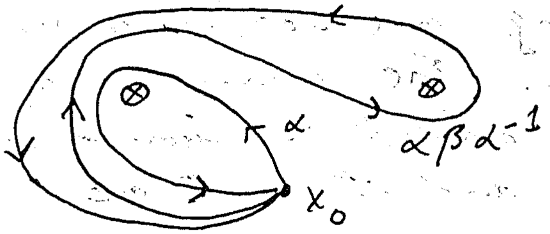
In the former case, we can bring the vortices together, and they can annihilate — in the latter case they can not annihilate.

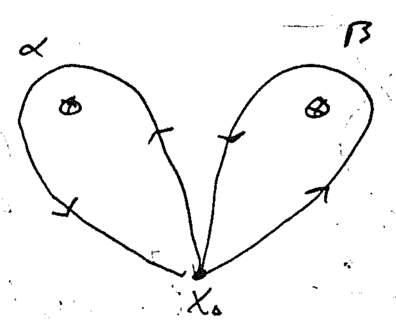
In the nontrivial $(-I)$ sector, the $(i\sigma_1)$ vortices might form a bound state, so there is a stable $(-I)$ vortex. Or they might repel so that the energy is minimized when they are ∞'ly far apart. This is a dynamical question, and can't be answered with topology alone.

Now we will discuss a new phenomenon. Namely, the vortices themselves can have mutual Aharonov-Bohm interactions. In fact, they can realize an exotic nonabelian version of quantum statistics.

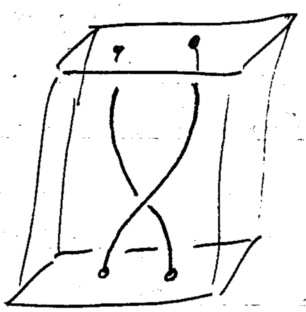


Let's consider how the quantum numbers of two vortices are modified if we perform a CCW exchange





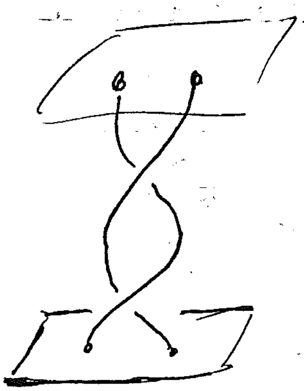
As we perform the exchange, the two paths $\alpha\beta\alpha^{-1}$ and α shown on the left, are "dragged" to the standard paths α and β, shown on the right, the paths we use to define the vortex flux. This means that the flux aba^{-1} associated with path $\alpha\beta\alpha^{-1}$ before the exchange, must be the flux associated with the path α after the exchange. Similarly a associated with α before the exchange must be the flux associated with path β after the exchange.



Thus the effect of a counterclockwise braiding of the vortices can be expressed as the action of a braid operator

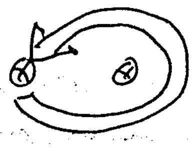
$$R: |a, b\rangle \longrightarrow |aba^{-1}, a\rangle$$

– the nontrivial content of which is the conjugation of b by a. The asymmetry of the action of R arises from our conventions – in particular the way we have chosen to place the base point x_0

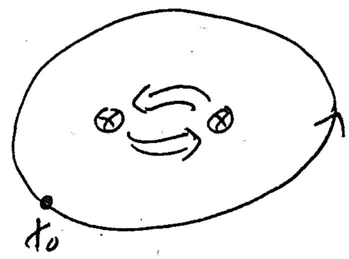


If we perform the CCW exchange twice, we are carrying one vortex around another. The corresponding operator R^2 is called the monodromy operator and it acts as

$$R^2: |a, b\rangle \rightarrow |aba^{-1}, a\rangle$$
$$\rightarrow |aba(ab)^{-1}, (ab)b(ab)^{-1}\rangle$$



– each vortex is conjugated by the "total flux" ab

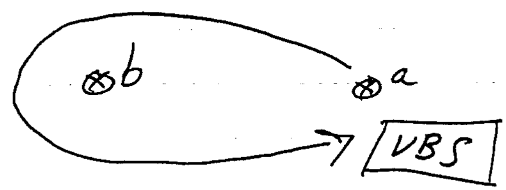


Naturally, performing an exchange cannot alter the total flux of a pair, which could in principle be detected via the interaction of the composite object with a charge that is very far away. Indeed, after braiding the total flux becomes

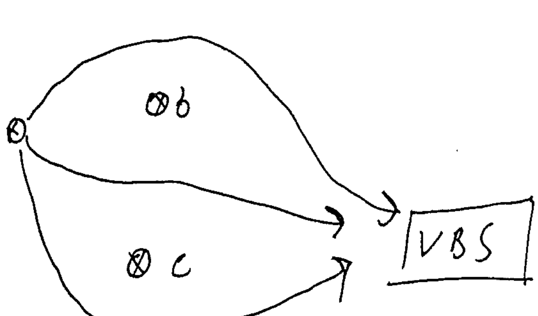
$$ab \to (aba^{-1})\cdot a = ab$$

– it is unchanged.

If a and b don't commute with ab, then carrying one vortex around another changes the quantum numbers of each.



The physical meaning is that we can assemble a vortex bureau of standards (VBS) where fluxes are calibrated and select an (a) vortex. We carry it around a (b) vortex and return it to the VBS for recalibration. Now it is an $a' = (ab)a(ab)^{-1}$ vortex. In particular, it will not annihilate when fused with a (a^{-1}) vortex.



When other vortices are present, then the flux of an object when we arrive at the VBS depends on the path chosen – i.e. how we thread the object through the other vortices

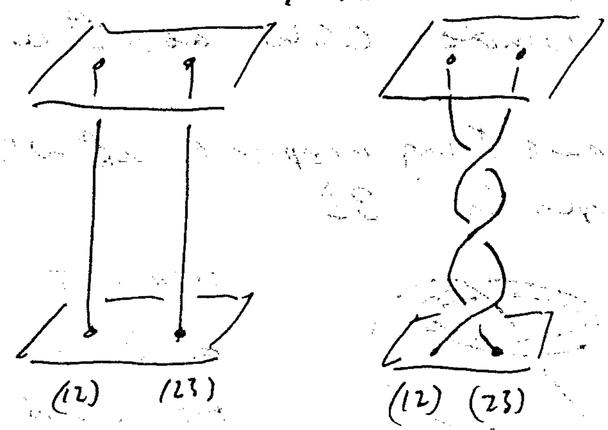
Example: For the case $H = S_3$

$$R: \quad |(12),(23)\rangle \to |(13),(12)\rangle$$
$$\to |(23),(13)\rangle$$
$$\to |(12),(23)\rangle$$

The orbit of the braid operator closes after three exchanges. There are two distinct such orbits – we also have

$$R: \quad |(23),(12)\rangle \to |(13),(23)\rangle$$
$$\to |(12),(13)\rangle$$
$$\to |(23),(12)\rangle$$

Because these orbits have odd length, the two-cycle vortices in S_3 must be regarded as indistinguishable objects. E.g., in the two vortex sector



The histories with a triple exchange can interfere with histories with no exchange – in both cases the final quantum numbers of the vortices are $|(12),(23)\rangle$

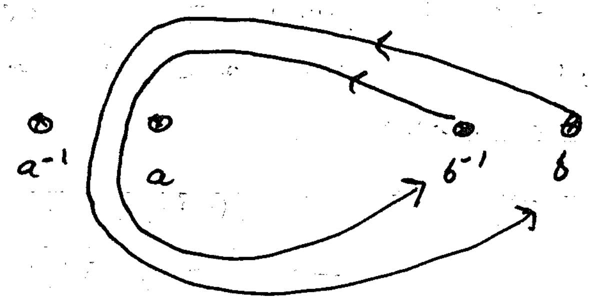
But this means that we can't keep track of "who's who" – in the triple exchange, the two vortices have changed places. Thus we have encountered the central oddity of nonabelian statistics — the (12) and (23) vortices are indistinguishable, yet they are not the same!

Three Dimensions

Lecture #12

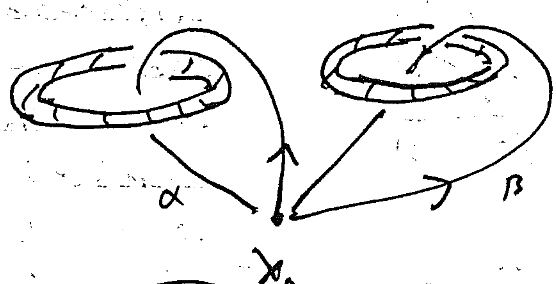
The "holonomy interactions" among vortices in 2D correspond to interactions among strings in 3D

For example, consider two pairs of vortices, each with trivial total flux

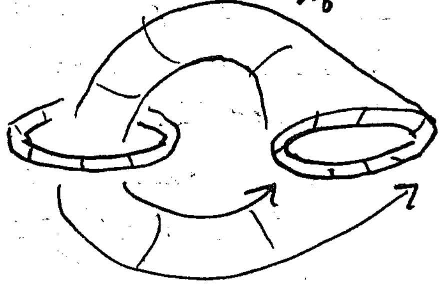


Suppose that we pull the (b^{-1}, b) pair through the (a^{-1}, a) pair. The "outside pair" is unaffected, since the total flux pulled through is $b^{-1}b = e$. But the "inside pair" is conjugated by the flux a, since both vortices it winds CCW about a

Just the same thing happens if we have 2 string loops in 3D



We pick a basepoint and paths α, β the wind around the loops, the corresponding fluxes are a and b

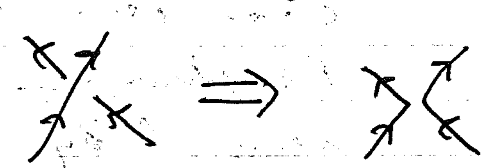


Then wind the b string loop through the a string loop as shown then it is $\alpha \beta \alpha^{-1}$ that is "dragged" to β by this process, so

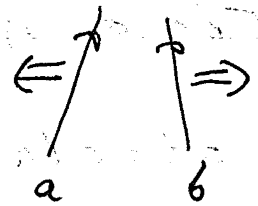
$$|a, b\rangle \rightarrow |a, aba^{-1}\rangle$$

Is there also a 3D analog of winding one vortex around another? There is, and it involves the "entangling" of two strings:

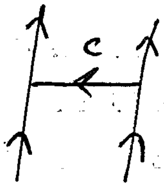
Suppose that an a string and a b string cross

If $a = b$, 

they could change partners, or intercommute (if $a = b^{-1}$, they could intercommute the "other way".) This is not possible otherwise.



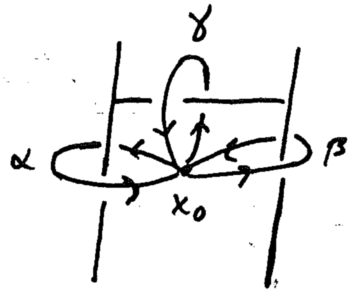
Can the strings "pass through" one another, then, and continue on their way? <u>NO</u> -- not if a and b do not commute:



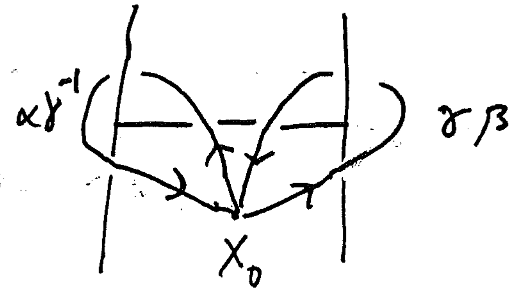
They get connected by the "commutator flux"

$$c = aba^{-1}b^{-1}$$

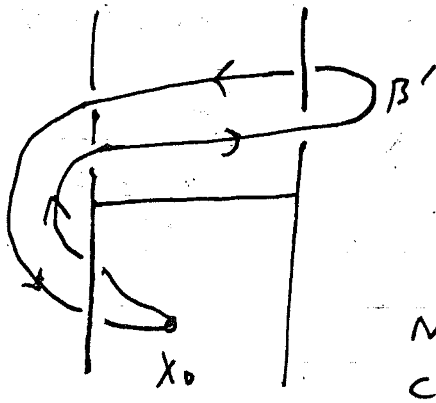
To see this:



choose basepoint and standard paths.



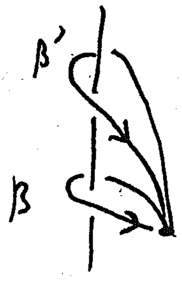
these loops are obtained by composing paths



and we see this loop is

$$\beta' = \alpha^{-1}\gamma\beta\alpha$$

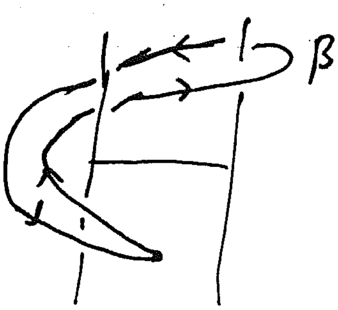
Now -- suppose the segment c has been created by crossing the strings -- then it can be removed by "uncrossing them."



In that case after the uncrossing, β′ and β are homotopically equivalent -- hence both paths must be assigned the same flux, i.e.

$$b = a^{-1}cba \Rightarrow ab = cba, \text{ or } \boxed{c = aba^{-1}b^{-1}}$$

-- which of course vanishes if a and b commute



If we can remove the segment by crossing the other way (tipping the strings out of the plane in the opposite sense) then

$$\beta'' = \alpha\beta\gamma\alpha^{-1} \sim \beta \text{ or}$$

$$b = abca^{-1} \Rightarrow ba = abc \Rightarrow \boxed{c = b^{-1}a^{-1}ba}$$

In the former case — the flux of the upper string on right (path $\gamma\beta$) is $cb = aba^{-1}$

and flux of upper left is $ac^{-1} = aba(ab)^{-1}$

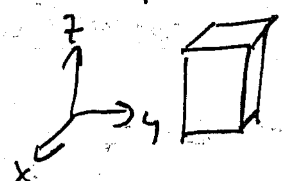
— Both are conjugated, relative to lower strings, by "total flux" ab — cf ccw winding of vortices.

In the latter case

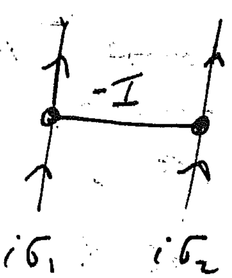
$$cb = (ab)^{-1}b(ab)$$
$$ac^{-1} = b^{-1}ab = (ab)^{-1}a(ab)$$

— cf cw winding of vortices

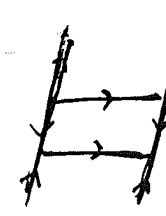
Strings that entangle in this way are observed in liquid crystals — E.g. the pattern $SU(2) \to Q$ is realized by a phase of aligned molecules with $Z_2 \times Z_2 = D_2$ rotational symmetry (invariance under 180° rotations about 3 axes)



Since $(i\sigma_1)(i\sigma_2)(-i\sigma_1)(-i\sigma_2) = -I$

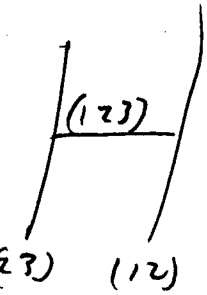


— two strings labeled by $i\sigma_1$, $i\sigma_2$ cannot cross — then become connected by $-I$ flux (which might, for energetic reasons, prefer to separate into two segments).



In the case of our S_3 model — crossing of 2-cycle strings produces a segment of 3-cycle string e.g.

$$(23)(12)(23)(12) = (123)$$

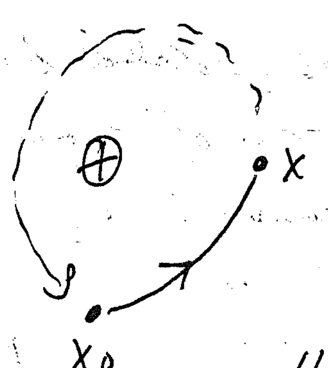


Flux - charge Composites

$\otimes\ \bullet$ In the case where H is a (discrete) nonabelian group, what are the properties of flux-charge composites?

First of all -- we need to be careful about what is meant by a charge on the background of a flux. Charge is formally defined in terms of transformation properties under global gauge transformations.

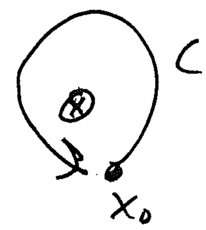
But as we found in the case of the Alice vortex — some global gauge transformations are unrealizable on vortex background.



The unbroken group $H(x)$ at point x is obtained from $H(x_0)$ at point x_0 via covariant gauge transport

$$H(x) = U(x, x_0)\, H(x_0)\, U^{-1}(x, x_0)$$

$$U = P \exp\left(i \int_{x_0}^{x} A\right)$$

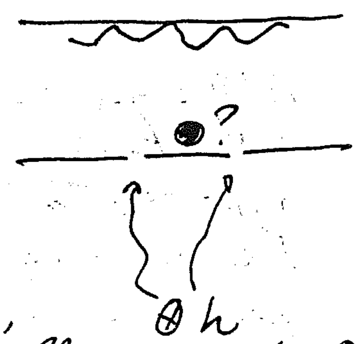


when the path circumnavigates the vortex, the group returns to the original group, but group elements do not. If the flux is $U(C, x_0) = a$

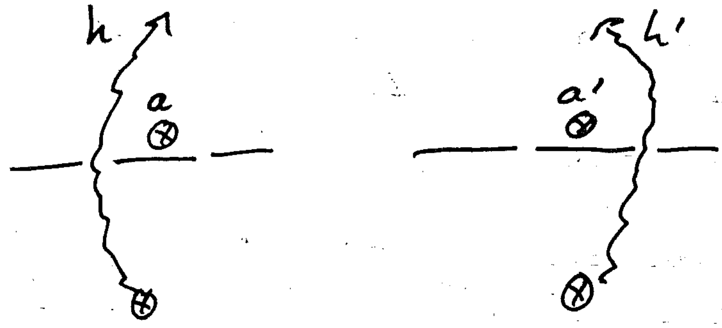
then transformation $h \to aha^{-1}$; i.e. h has a twist on the vortex background — there is no such thing as a global a transformation — if h does not commute with the flux a the charge carried by a vortex with flux a is not described by a representation of a, but rather a representation of $N(a)$
$N(a)$, the normalizer of a (in H) is the subgroup of elements that commute with a

$$N(a) = \{a \in H,\ ah = ha\}$$

Physically, why can only normalizer charges be defined? Just as we can measure a flux through its Aharonov-Bohm interactions with charges, we measure charge through AB interaction with flux. E.g., we do double slit interference experiment with flux as a projectile, and mystery object behind screen. If the projectile is a flux h, and the charge has state $|u\rangle$, the visibility and fringe shift tell us $\langle u|D(h)|u\rangle$.



But what if the charge also has flux a, where a, h don't commute?



then there is no interference, as the "which way" information gets imprinted on the fluxes

e.g. they are: a, h if projectile passes on left

$(ah)a(ah)^{-1}$, $(ah)h(ah)^{-1}$, if projectile passes on right

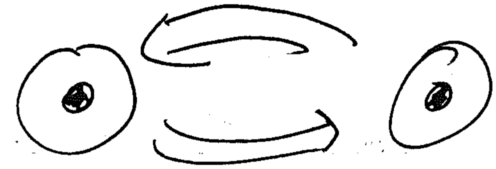
But since we can measure $\langle u | D(h) | u \rangle$ for $h \in N(a)$, the charge associated with normalizer is measurable.

The superselection sectors of theory are labeled by quantum nos. detectable at long range.

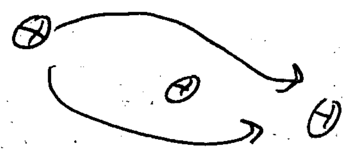
- A conjugacy class $[a]$ of H ($[a]$ denotes class containing all elements hah^{-1} conjugate to a)
- An irrep R of $N(a)$

(the abstract group $N(a)$ is the same for each $a \in [a]$ — only its embedding in H differs)

Although as we have discussed, when we fuse together two fluxes, not all class representatives are equivalent, still only the conjugacy class is an invariant property of an isolated flux



The superselection sectors characterize the properties of an isolated object, far away from other objects.
— But if we do bring the objects together to compare them — the flux itself can depend on the path taken — while the conjugacy class is an invariant property.



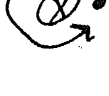
$\otimes\circ$ Let the flux of an isolated vortex be a, in a particular gauge
And let it transform as irrep R of $N(a)$
How is a itself represented in R?
a commutes with any $b \in N(a)$ (definition of $N(a)$) — so $D^R(a)$ commutes with the irrep; by Schur's lemma it is

$$D^R(a) = e^{i\theta} I$$

This means that two such objects, with the same flux and charge a, R are like (abelian) anyons

E.g. if one is carried CCW about the other, the wave function is modified by phase $e^{2i\theta}$ (each charge is transported about the other flux) — And a CCW rotation gives phase

$$e^{-2\pi i J} = e^{i\theta}$$



(the charge rotates about the flux)
So $D^R(a)$ determines the angular momentum, and the usual spin-statistics connection is satisfied. The nonabelian behavior of the anyons can be manifested only in a system of 3 or more particles (for 2 bodies, we can diagonalize the braid operator $\mathcal{R}$.)

But what happens to charge as we fuse two fluxes — this is a subtle algebraic problem.

$$\underset{R^{(a)}}{\underset{a}{\otimes}} \longrightarrow \longleftarrow \underset{R^{(b)}}{\underset{b}{\otimes}}$$

We want to combine irreps of two different groups (in general) $N(a)$ and $N(b)$, and decompose into irreps of a third, $N(ab)$

$$|a, R^{(a)}\rangle \otimes |b, R^{(b)}\rangle$$
$$= \bigoplus_R |ab, R^{(ab)}\rangle$$

Note that performing this decomposition actually diagonalizes the braid operator $\mathcal{R}$ (and the monodromy operator $\mathcal{R}^2$).

why -- because transporting one object around another is equivalent to rotating the composite by 2π, and each constituent by -2π

Hence $\alpha^2 = \exp\left[i\left(\theta_{R(ab)} - \theta_{R(a)} - \theta_{R(b)}\right)\right]$

where
$$D^{R(ab)}(ab) = e^{i\theta_{R(ab)}}$$
$$D^{R(a)}(a) = e^{i\theta_{R(a)}}$$
$$D^{R(b)}(b) = e^{i\theta_{R(b)}}$$

An important special case -- fuse flux and antiflux — so $ab = e$, and the composite provides a representation of H

$$\underset{a^{-1}}{\otimes} \qquad \underset{a}{\otimes}$$

under a global gauge transformation $h \in H$, the (a^{-1}, a) pair is not invariant...
But we can decompose the action of H on $[a]$ by conjugation

$$a \to hah^{-1}$$ into irreps of H.

Thus, we have pairs of fluxes that carry Cheshire charge: uncharged fluxes can be combined to obtain charged states.

For any conjugacy class of any group H, there is an invariant state

$$|0;[a]\rangle = \frac{1}{\sqrt{n}} \sum_{a' \in [a]} |a'\rangle$$ (n = # elements in the class)

($a' \to ha'h^{-1}$ just permutes elements of class).

This state has vacuum quantum numbers, no net flux or charge. This state can annihilate, or be pair created.

States orthogonal to $|0;[a]\rangle$ have charge

Example: 2-cycles of S_3

$n=3$ members of the class: (12) (23) (13)

Conjugation by (12): $(23) \leftrightarrow (13)$ etc

Conjugation by (123): $(12) \to (132)(12)(123) = (13)$

$(23) \to (132)(23)(123) = (12)$

$(13) \to (132)(13)(123) = (23)$

Characters

	e	(12)	(123)
this irrep	3	1	0
cf	1	1	1
	1	-1	1
	2	0	-1

$3 \to R^{(0)} + R'$ ⤳ the 2-dim irrep

$$|0\rangle = \frac{1}{\sqrt{3}}\left((12) + (23) + (31)\right)$$

$$|1\rangle = \frac{1}{\sqrt{3}}\left((12) + \omega(23) + \bar{\omega}(31)\right) \qquad \omega = e^{2\pi i/3}$$

$$|2\rangle = \frac{1}{\sqrt{3}}\left((12) + \bar{\omega}(23) + \omega(31)\right)$$

Acting on $|1\rangle, |2\rangle$

(123) : $|1\rangle \to \omega|1\rangle$ $\qquad (123) = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$

(123) : $|2\rangle \to \bar{\omega}|2\rangle$ $\qquad (132) = \begin{pmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{pmatrix}$

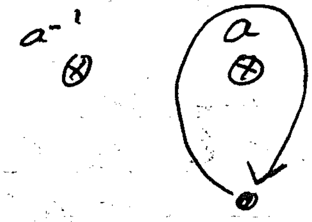
$$(12):\ |1\rangle \leftrightarrow |2\rangle \qquad (12) \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$(13):\ |1\rangle \to \omega|2\rangle,\quad |2\rangle \to \bar\omega|1\rangle \qquad (13) \doteq \begin{pmatrix} 0 & \bar\omega \\ \omega & 0 \end{pmatrix}$$

$$(23):\ |1\rangle \to \bar\omega|2\rangle,\quad |2\rangle \to \omega|1\rangle \qquad (23) = \begin{pmatrix} 0 & \omega \\ \bar\omega & 0 \end{pmatrix}$$

This is discrete Cheshire charge.

A particle with charge and no flux can transfer Cheshire charge to a pair of fluxes that were initially uncharged.



Create a pair of charges in irreps $R, \bar R$ — combined in trivial rep

$$\frac{1}{\sqrt{n_R}}\, |e_i^{\bar R}\rangle \otimes |e_i^R\rangle$$

Now wind particle around flux a

$$\longrightarrow \frac{1}{\sqrt{n_R}}\, |e_i^{\bar R}\rangle \otimes |e_j^R\rangle\, D^{(R)}_{ji}(a)$$

Will the pair be able to annihilate? Find overlap with the singlet pair

$$\frac{1}{\sqrt{n_R}}\, \langle \bar e_k^R| \otimes \langle e_k^R|$$

$$\hookrightarrow \frac{1}{n_R}\sum_i D^R_{ii}(a) = \frac{1}{n_R}\chi^R(a)$$

The probability that no charge transfer takes place is $p^{(0)} = \left|\frac{1}{n_R}\chi^R(a)\right|^2$

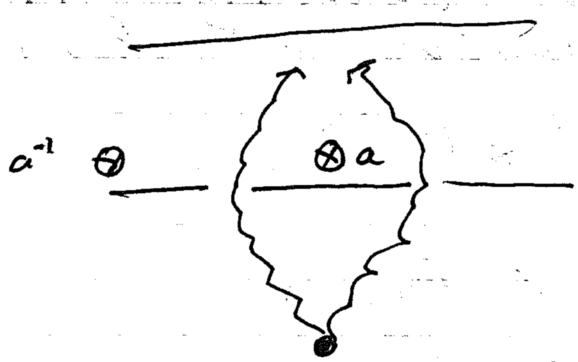
— this is the same for each $a' \in [a]$ so we obtain the same probability if we sum over $a' \in [a]$. This is the probability of "no transfer" between zero-charge flux and charge pairs.

After charge passes through

$|0;[a]\rangle \otimes |0; \text{pair}\rangle$

$\hookrightarrow \sum_R \sqrt{p_R}\ |\text{flux}, \bar{R}\rangle \otimes |\text{pair}, R\rangle$

You can compute p_R (exercise).



If we have a mystery charged particle (with zero flux) — we determine R as follows:

Make a charge zero pair of fluxes, but one behind screen between slits

From visibility and fringe shifts determine

$\langle u|D(a)|u\rangle$ where $|u\rangle$ is charge state

Except — incoherently combine all a in a class

$\Rightarrow \frac{1}{n_H}\sum_{b\in H}\langle u|D(b)D(a)D(b)^{-1}|u\rangle$

$= \frac{1}{n_H}\sum_b u_\ell^* D(b)_{\ell k} D(a)_{km} D(b)^{-1}_{mr} u_r$

Orthogonality

$$\frac{1}{n_H}\sum_b D(b)_{\ell k}\, D(b)^{-1}_{nr} = \frac{1}{n_R}\delta_{\ell r}\delta_{kn}$$

$$\Rightarrow \quad \langle u|D(a)|u\rangle \Rightarrow \frac{\chi(a)}{n_R}$$

— this is the amplitude for no charge transfer to take place — otherwise there is no interference because we can look at the pair of fluxes to extract the which-way information!

Magnetic Monopoles

Lecture 13

We'll now study magnetic monopoles. There is ample motivation. We'll reinforce the impression that "topology is fun," now connecting with physics in three spatial dimensions. Monopoles almost certainly are in the particle spectrum of the real world. The theory of monopoles connects with quark confinement, strongly-coupled gauge dynamics, and string theory. Even cosmology

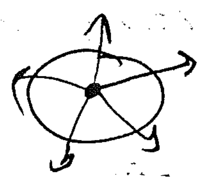
The (classical) Maxwell eqns have an asymmetry between electric and magnetic sources, e.g.

$$\mathrm{div}\, E = \rho$$
$$\mathrm{div}\, B = 0.$$

This can be remedied if we introduce objects with magnetic charge. A magnetic monopole at the origin with magnetic charge g has B field

$$B = \frac{g}{4\pi}\frac{\hat{r}}{r^2}$$

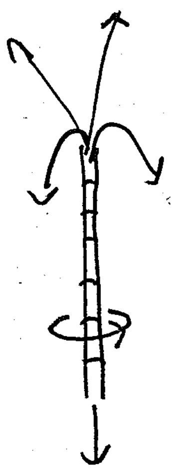
-normalized so flux Φ through a two-sphere is



$$\Phi = \int_{S^2} d\vec{S}\cdot\vec{B} = g$$

(differs by 2π from normalization in Les Houches notes.

Dirac considered the quantum theory of monopoles in 1931, and noticed something interesting: he connected the existence of monopoles with quantization of electric charge



Dirac pictured a monopole as a semi-infinite flux tube (solenoid) with magnetic flux spilling out of one end. As we shrink the flux tube to infinitesimal thickness, only the monopole is left--

But is the tube really invisible?

The flux g through the tube might be detected through an Aharonov-Bohm interaction with a charge e.

$$\text{AB phase} = \exp(ie\Phi) = e^{ieg}$$

- the tube disappears only for $eg = 2\pi n$, n = integer

$$\Longrightarrow \boxed{g = \frac{2\pi}{e} \cdot n}$$

the Dirac quantization condition

- if e is the smallest nonzero charge in the world, all monopoles have quantized charge $g = n g_D$, $g_D = 2\pi/e$ is Dirac's magnetic charge.

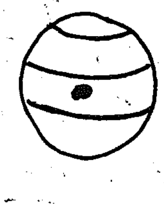
And we can run the argument in reverse: If a monopole exists with $g = g_D = 2\pi/e$, then all electric charges are

$$Q = e \cdot \text{integer}$$

- electric charge is quantized.

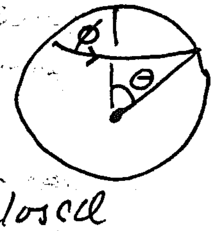
It's a remarkable argument. We will reexamine it from a variety of viewpoints.

Since div $\vec{B} \neq 0$, we cannot write $\vec{B}$ in terms of a vector potential $\vec{A}$ as $\vec{B}$ = curl $\vec{A}$ if $\vec{A}$ is "globally defined." Yet $\vec{A}$, not just $\vec{B}$, is needed in the quantum theory.



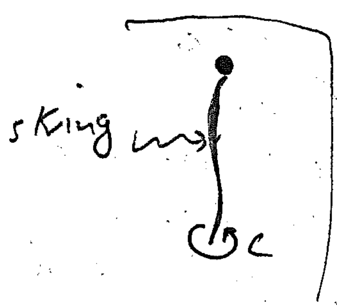
If we construct $\vec{A}$ on a two-sphere enclosing the monopole, then $\vec{A}$ must be singular somewhere (or else we must cut the sphere into pieces and define $\vec{A}$ on each of the patches, as described below). E.g.

$$\vec{A}\cdot d\vec{r} = \frac{g}{4\pi}(1-\cos\theta)\,d\phi$$



then $\oint A = \frac{g}{2}(1-\cos\theta)$ — the flux enclosed by a circle at latitude θ

This $\vec{A}$ has a singularity at the south pole, where $\oint \vec{A}\cdot d\vec{r} = g$ for an infinitesimal loop.

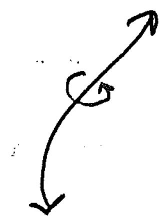


The Dirac quantization condition arises from the requirement that the Wilson loop operator $\exp(ie\oint_C A) = 1$, for the infinitesimal loop C.

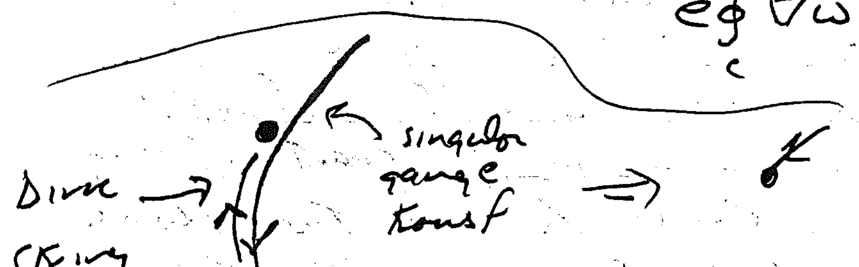
The singularity in $\vec{A}$ is called the "Dirac string" and the point monopole is at its end. We can move the string with a "singular gauge transformation"

$$\vec{A} \to \vec{A} + \vec{\nabla}\omega$$

where the gauge function ω has an infinite (not semi-infinite) string singularity such that



$$e\oint_C \nabla\omega = 2\pi$$ for infinitesimal C enclosing the string

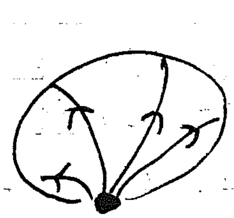


Though it moves the Dirac string (a gauge artifact) the singular gauge transformation can't move the end of the string (the monopole).

Singular gauge transformations are allowed (and so are monopoles) because the wavefunction of charge e particle changes by

$$\exp ie[\omega(2\pi) - \omega(0)] = e^{2\pi i}$$

around the string — What Dirac discovered is that monopoles are allowed if and only if the gauge group is compact — $U(1)$ rather than $\mathbb{R}$ — so that it contains noncontractible loops



Yet another way to say this: we can consider a closed path in the space of closed loops on the 2-sphere. Each loop begins and ends at the south pole. And the initial and final loops are infinitesimal. But the sequence of loops sweeps out S^2 — For each C in the sequence:

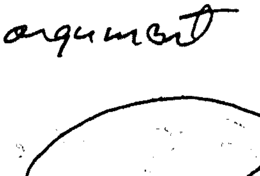
$$\exp\left(ie\oint_{C,x_0} A\right) = h(C,x_0) \in U(1)$$

as C varies we obtain a closed loop in $U(1)$

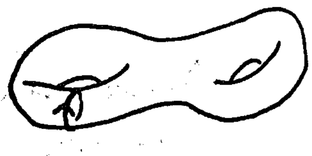
— the magnetic charge in Dirac units is the winding number of this loop.

Thinking of magnetic charge in terms of the winding of a Wilson loop has the advantage that we work with gauge invariant quantities, and don't have to discuss the singular $\vec{A}$ (although the singularity is still implicitly there — otherwise $h(C, x_0)$ couldn't wind!) It also emphasizes that the magnetic charge is an intrinsic property of gauge transport on S^2 — we don't have to commit ourselves to any hypothesis about how A behaves inside the sphere. Indeed, we would have a Dirac quantization condition for charged particles that live on a compact two-dimensional surface — i.e. the flux $\Phi = \int B$ integrated over the surface is $n\Phi_0$ — an integral number of flux quanta.

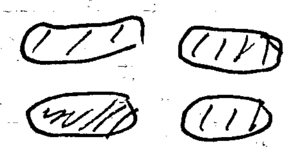
We can use a similar argument for charged particles on a torus — A cycle of the torus can sweep around the donut, and the Wilson loop returns to its original value.



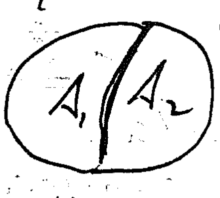
Again, magnetic charge is winding number, but in this case, the loop in $U(1)$ does not necessarily begin and end at the identity



But this definition of the magnetic flux (i.e. this argument for its quantization) does not work for higher genus surfaces — we can't "sweep out" the surface with a loop. Topologists have a general strategy for describing gauge transport (i.e. fiber bundles) in this more general situation.

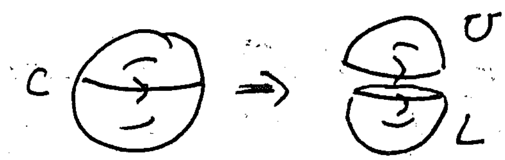


The procedure is to cut the manifold into contractible patches (two-dim disks in the case of the Riemann surface). Given B on each patch, we can find $\vec{A}$ with $B = \vec{\nabla} \times \vec{A}$ on the patch (Poincare lemma). The boundaries between patches are an artifice, so a charged particle crossing the boundary does not notice anything — i.e. the gauge potentials A_1 and A_2 in two patches are related by a gauge transformation at the boundary.



So to characterize transport on the whole surface, we stitch the patches together with gauge transformations (transition functions of the bundle).

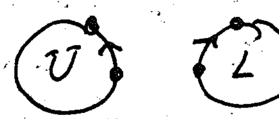
We'll return to higher genus later — first let's see how this procedure is implemented for S^2.



cut the two-sphere at the equator, into upper (U) and lower (L) hemispheres

$\vec{A}_U$ is (nonsingular) vector potential on U

$\vec{A}_L$ is (nonsingular) vector potential on L



At the equator, they are related by a gauge transformation

$$e(A_U - A_L) = -i(\nabla\Omega)\Omega^{-1}$$

where $\Omega = e^{i\omega}$ is single-valued on the circle

In other words, both patches must agree on the value of the equatorial Wilson loop

$$\exp\left(ie\oint_C A_U\right) = \exp\left(ie\oint_C A_L\right)$$

But $\exp\left(ie\oint A_U\right) = e^{ie\Phi_U}$ where Φ_U is flux through U

$\exp\left(ie\oint A_L\right) = e^{-ie\Phi_L}$ → minus sign if flux is defined outward on both patches

So we have $e^{ie\Phi_U} = e^{-ie\Phi_L}$

or $e^{ieg}e^{ie\Phi} = e^{ie(\Phi_U+\Phi_L)} = 1$ ⟹ Dirac quantization

Equivalently

$$\exp ie\Phi = \exp ie\int(A_U - A_L) = \Omega(2\pi)\Omega(0)^{-1} = I$$

or $U(\phi) = \exp\left[ie\int d\phi\,(A_{\phi,U} - A_{\phi,L})\right]$

has a winding number about equator: the mag. charge

The observation that magnetic charge (in Dirac units) is the winding no. of a gauge transformation — a topological property — allows us to extend the discussion to arbitrary gauge groups.

If the gauge group is the compact Lie group H: there is a topological magnetic flux on the two sphere (and, again, it can be extended to higher genus surfaces) if gauge trans. winds.

To be explicit:

$$\vec{A}_U \cdot d\vec{r} = g/4\pi\,(1-\cos\theta)\,d\phi \qquad 0 \le \theta \le \pi/2$$

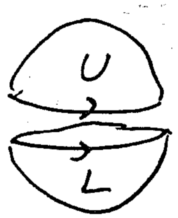
$$\vec{A}_L \cdot d\vec{r} = -g/4\pi\,(1+\cos\theta)\,d\phi \qquad \tfrac{\pi}{2} \le \theta \le \pi$$

are vector potentials, each nonsingular, for the Dirac monopole on U and L patches. At the equator

$$(\vec{A}_U - \vec{A}_L)\cdot d\vec{r} = \frac{g}{2\pi}d\phi = -i(\partial_\phi \Omega)\Omega^{-1}\,d\phi$$

where $\Omega(\phi) = \exp(\ ieg\,\phi/2\pi$

is single valued g.t. on equator, with winding no. 1



Now, the gauge transformation at the equator defines a loop in H. The magnetic charge is the homotopy class of this path — an element of $\pi_1(H)$

Magnetic Charge ~ $\pi_1(H)$

Alternatively, in the singular string gauge, the flux running through the string is classified by $\pi_1(H)$ – winding of a gauge transformation on an infinitesimal path that encloses the string.

What is $\pi_1(H)$ for a compact Lie group?

Any Lie group has a simply-connected universal covering group (eg. $\mathbb{R}$ for $U(1)$, $SU(2)$ for $SO(3)$, etc.) A theorem of Cartan says that given a Lie algebra (Lie group in the vicinity of identity), there is unique simply connected Lie group with that Lie algebra.

Therefore if the simply-connected covering group is $\bar{H}$, any Lie group H with the same Lie algebra as $\bar{H}$ is

$$H = \bar{H}/K,$$

where K is an invariant discrete subgroup of $\bar{H}$

What are the discrete invariant subgroups of a connected Lie group?

For $h \in \bar{H}$, $k \in K$ we must have

$$h k h^{-1} = k' \in K$$

But if K is discrete and $\bar{H}$ is connected K' must be independent of h. We can deform h to e and then

$$K' = eKe = K.$$

We conclude that $hKh^{-1} = K$ for for each $h \in \bar{H}$, $k \in K$. That is, K is contained in the <u>center</u> of $\bar{H}$

Furthermore, each $\bar{H}$ has a faithful representation that is irreducible – the "defining" representation. K commutes with this irrep, so by Schur's lemma, for each $K \in K$

$$K = cI$$

– a multiple of identity. So K is the subgroup of $\bar{H}$ represented by a multiple of I in the defining irrep.

Example: $\bar{H} = SU(N)$ is simply connected. Its center is

$$Z_N = \{e^{2\pi i k/N} I \quad k = 0, \ldots, N-1\}.$$

Any group with the $SU(N)$ Lie algebra is $H = SU(N)/K$ where $K \subseteq Z_N$

E.g. For $SU(6)$, Z_6 has the subgroups

$$K = \{I\}, Z_2, Z_3, Z_6$$

What does it mean to say that the gauge group is $H = \bar{H}/K$ rather than $\bar{H}$? It means that all matter fields in the theory transform as single-valued reps of H, so in other words transform trivially under K. E.g. each irrep of $SU(N)$ can be assigned an "N-ality" n that characterizes how it transforms under Z_N

$$D^R(e^{2\pi i k/N} I) = (e^{2\pi i k/N})^n I$$

— Thus N-ality is the number (modulo N) of copies of the defining N-dimensional irrep we need to fuse to obtain R. If all fields have trivial N-ality, then $H = SU(N)/Z_N$, and magnetic monopoles are classified by

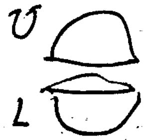
$$\pi_1\left[SU(N)/Z_N\right] = Z_N$$

For example: For $\bar{H} = SU(2)$, we say that $H = SO(3) = SU(2)/Z_2$ if all matter reps. have integer spin (trivial under Z_2) & if so, topological Z_2 monopoles are allowed.

For QCD with quarks, $H = SU(3)$ and there are no monopoles (quarks would be able to see the string.) But for the $SU(3)$ gauge theory with gluons only, or with color octet matter, $H = SU(3)/Z_3$, and Z_3 monopoles are allowed.

SU(4) is Spin(6), the covering group of SO(6) (which has two 4-dimensional spinor irreps (4) and $(\bar{4})$) the vector (6) of SO(6) has 4-ality 2. So the pure glue SU(4) theory admits Z_4 monopoles, the SO(6) theory with vector matter has Z_2 monopoles.

To be explicit (in the patching picture) vector potentials on U and L in the case of the $SU(N)/Z_N$ monopole can be expressed as



$$e\vec{A}_U \cdot d\vec{r} = \underset{\sim}{P}\ \tfrac{1}{2}(1-\cos\theta)\,d\phi \qquad 0 \le \theta \le \pi/2$$

$$e\vec{A}_L \cdot d\vec{r} = -\underset{\sim}{P}\ \tfrac{1}{2}(1+\cos\theta)\,d\phi \qquad \tfrac{\pi}{2} \le \theta \le \pi$$

where P is an SU(N) generator

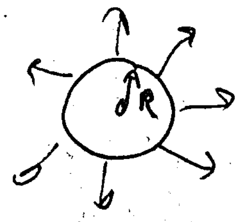
The SU(N) gauge transformation associated with winding about the equator is

$$\underset{\sim}{P} \exp\left(ie\int_\phi (\vec{A}_U - \vec{A}_L)\cdot d\vec{r}\right) = \exp(i\underset{\sim}{P}\phi)$$

we require $\quad \exp(i 2\pi \underset{\sim}{P}) = e^{2\pi i m/N}\, \underline{I}$

where m (mod N) is the Z_N magnetic charge. (In fact, any long range monopole field that solves the YM field eqns has this form, for a proper gauge choice.) $\underset{\sim}{P}$ is traceless, but could have the form

$$\underset{\sim}{P}_0 = \operatorname{diag}\Big(\overbrace{\tfrac{m}{N}, \dots, \tfrac{m}{N}}^{N-m \text{ times}}, \overbrace{\tfrac{m-N}{N}, \dots \tfrac{m-N}{N}}^{m \text{ times}}\Big)$$



The magnetostatic energy of the field outside a shell is

$$\int d^3x \; \tfrac{1}{2}\, \vec{B}^a \cdot \vec{B}^a = \int_{r>R} d^3x \; \mathrm{tr}\, B_r^2$$

and $B_r = \frac{1}{2c}\, \underset{\sim}{P}\, \frac{\hat{r}}{r^2} \Rightarrow \mathrm{tr}\, B_r^2 = \frac{1}{4c^2}\, \frac{1}{r^4}\, \mathrm{tr}\, \underset{\sim}{P}^2$

$$\frac{1}{4c^2}(\mathrm{tr}\, P^2) \int_R^\infty \frac{4\pi r^2 dr}{r^4} = \frac{\pi}{c^2 R}\, \mathrm{tr}(\underset{\sim}{P}^2)$$

thus the Z_N monopole of lowest energy has minimal $\mathrm{tr}\, \underset{\sim}{P}^2$.

we have the freedom to shift eigenvalues of $\underset{\sim}{P}$ by integer values, while preserving the constraint that the eigenvalues sum to one.
But the only changes to $\underset{\sim}{P}_0$ that don't increase the energy are just permutations of the eigenvalues. In fact a perturbative stability analysis shows that a monopole configuration is unstable if any difference of eigenvalues of $\underset{\sim}{P}$ has $|\lambda_1 - \lambda_2| \geq 2$

E.g. for $a > 0$ $\underset{\sim}{P} = (a, a-2, \text{---})$

will want to decay to $\underset{\sim}{P} = (a-1, a-1, \text{---})$.

which has lower Coulomb energy.
The decay proceeds via the emission of classical nonabelian gluon radiation. But this radiation cannot carry away the Z_N magnetic charge which is topological (discrete) and cannot be altered by evolution under the $SU(N)$ Yang-Mills equations