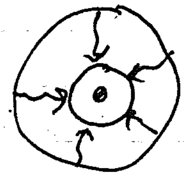
Lecture #14

Naively — both Z_N electric charge (the N-ality of the matter) and Z_N magnetic charge label the superselection sectors of the SU(N) Yang-Mills theory — each characterizes a long range field. This is too naive, because it could be altered by nonperturbative phenomena, e.g. confinement. But we'll eventually see that Z_N magnetic charge sectors survive in the confining theory without quarks (but not in the Higgs theory).

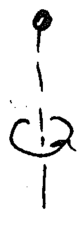


Consider what happens when we imagine contracting the two-sphere to a point. If the magnetic charge $\in \pi_1(H)$ (which is a discrete quantity) is nontrivial, then the charge cannot smoothly go to zero as the sphere contracts — either it must suddenly jump, or it remains nontrivial as the sphere collapses to infinitesimal size. Either way, the magnetic charge resides on a point singularity — the "end of the Dirac string" where $\vec{B}$ diverges.

There is a logically acceptable alternative and we'll soon see how it can be realized. The monopole could have a nonsingular core in which the gauge fields of $G \supset H$ are excited, so that gauge group enlarges from H to G in the core. Then we can avoid the singularity if our loop in H is able to unwind in G.

Before we get to that, though, consider what happens when the gauge group is (at least locally) a product $H_1 \times H_2$

For example, suppose $H = U(1) \times U(1)$. In Dirac's argument, there are two contributions to the Aharonov-Bohm phase as an object. If the magnetic charges of the monopole under the two $U(1)$'s are (g_1, g_2) and the object has electric charges q_1, q_2, then we require



$$e^{iq_1 g_1} e^{iq_2 g_2} = 1$$

or
$$q_1 g_1 + q_2 g_2 = 2\pi \cdot \text{integer}$$

– but we need not have $q_1 g_1 = 2\pi \cdot \text{integer}$ or $q_2 g_2 = 2\pi \cdot \text{integer}$. One nontrivial phase might be compensated by another.

Suppose there are two charged fields ψ_A, ψ_B with
$$(q^1/e_1,\ q^2/e_2) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) \\ (-\frac{1}{2}, \frac{1}{2}) \end{cases}$$

Since there are electric charges that are half integer, we might think that magnetic charges must be even multiples of
$$g_{D1} = \frac{2\pi}{e_1}, \quad g_{D2} = \frac{2\pi}{e_2}$$

That is correct for the monopoles that carry only one of the two $U(1)$ charges, but not for a monopole that carries both.

The allowed magnetic charges are

$$(M_1/g_{D1},\ M_2/g_{D2}) = (m_1, m_2)$$

such that $\frac{1}{2}(m_1 + m_2) = \text{integer}$

$\frac{1}{2}(m_1 - m_2) = \text{integer}$

This allows (odd, odd) charges, as well as the (even, even) charges allowed by Dirac quantization for each $U(1)$ separately.

How do we reconcile this observation with the property that the magnetic charge is an element of $\pi_1(H)$

Here it seems that $H = U(1)\times U(1)$ and $\pi_1(U(1)\times U(1)) = \pi_1(U(1)) \otimes \pi_1(U(1$

— we seem to be missing the (odd, odd) states.

The resolution is that the gauge group is <u>not</u> $U(1)\times U(1)$; rather it is

$$H = [U(1)\times U(1)]/Z_2$$

— because the two $U(1)$'s "intersect" somewhere other than the identity element. Acting on $\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$, a $U(1)_1$ transformation is $\begin{pmatrix} e^{i\omega_1/2} & 0 \\ 0 & e^{-i\omega_1/2}\end{pmatrix}$

and a $U(1)_2$ transformation is $\begin{pmatrix} e^{i\omega_2/2} & 0 \\ 0 & e^{i\omega_2/2}\end{pmatrix}$

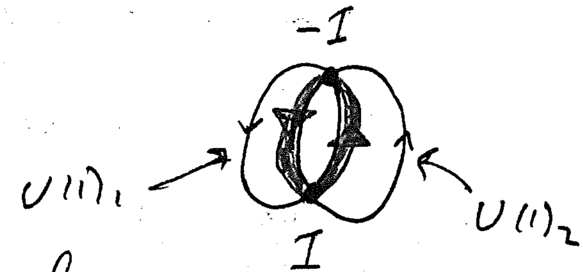
these coincide at both $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $(\omega_1 = \omega_2 = 0)$

and also $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $(\omega_1 = \omega_2 = 2\pi)$

It is wrong to say that the group is $H = U(1)_1 \times U(1)_2$ because this counts $(-I)$ twice. Hence

$$H = [U(1) \times U(1)]/Z_2$$

Now, it is not true that $\pi_1[(U(1)\times U(1))/Z_2] = \pi_1(U(1)) \times \pi_1(U(1))$



There are paths that go half way around $U(1)_1$ (from I to $-I$) and then return to I through $U(1)_2$. These loops in H correspond to the (odd, odd) magnetic charge sectors.

A more interesting example is the standard model. Below the electroweak symmetry breaking scale, its gauge group (locally) is $SU(3)_{color} \times U(1)_{em}$. For one generation, the charges are (in units of e)

$$u = 3^{2/3}$$
$$d = 3^{-1/3}$$
$$e = 1^{-1}$$

Since the quarks have electric charge $2/3$, $-1/3$, our naive expectation is that the magnetic charge quantum is

$$3 g_D = 3\,\frac{2\pi}{e}$$

But that's wrong. Acting on $\begin{pmatrix} u \\ d \end{pmatrix}$ the $U(1)_{em}$ gauge transformation acts as

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \begin{pmatrix} e^{i\frac{2}{3}\omega} & 0 \\ 0 & e^{-i\frac{1}{3}\omega} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

For $\omega = 2\pi$, this is $\mathrm{diag}(e^{-2\pi i/3}, e^{-2\pi i/3})$

But $e^{-2\pi i/3} I$, acting on a color triplet object, is an element of $Z_3 \subset SU(3)$

Hence $U(1)_{em}$ and $SU(3)_{color}$ intersect at Z_3, and the gauge group is really

$$H = [SU(3)_{color} \times U(1)_{em}]/Z_3$$

There are noncontractible loops in H that wind $\frac{1}{3}$ of the way around $U(1)_{em}$, and return to the identity via $SU(3)$. These correspond to magnetic monopoles with charge g_D.

Physically, g_D is allowed because, at long distance quarks are confined and charges are integer (physical states have trivial Z_3), and we can't detect Dirac's string.

If we try to detect it in a quark interference experiment at $\ll 10^{-13}$ cm from the string, the color magnetic flux of the string compensates for the string's $U(1)_{em}$ flux.

Monopoles as solitons

So far we have considered the properties of monopoles observed from afar (their long-range fields). Nothing we have said ensures that these monopoles really exist. The Dirac magnetic charges are singular – formally they have infinite Coulomb energy and infinite mass. But now we will see that there are gauge theory models in which nonsingular monopoles of finite mass arise as classical solutions to field equations.

These nonsingular monopoles are associated with a topological conservation law, and so are closely analogous to the vortex solutions we found in two spatial dimensions. Indeed, we saw that the topological sectors of a model with vortices can be associated with either the winding number of the Higgs field at spatial infinity, or the topological magnetic flux. Now we will find topological sectors in three spatial dimensions – associated with either the winding of the Higgs field, or equivalently, the topological magnetic charge.

As we did with the vortices, we'll start with an example, then proceed to a general analysis.

Our model (three spatial dimensions) has gauge group $G = SU(2)$, and a Higgs field in the triplet irrep of $SU(2)$. The Lagrange density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - U(\phi)$$

where $\qquad U(\phi) = \frac{1}{8}\lambda(\phi^a\phi^a - v^2)^2$

Here

$D_\mu = \partial_\mu - ieA_\mu$ — acting on a triplet

$$D_\mu \phi^a T^a = [\partial_\mu - ieA_\mu^a T^a, \phi^b T^b]$$

$$= \partial_\mu \phi^b T^b - ieA_\mu^a \phi^b \underbrace{[T^a, T^b]}_{= i\varepsilon^{abc}T^c}$$

$$= T^a(\partial_\mu \phi^a + e\varepsilon^{abc} A_\mu^b \phi^c)$$

$$F_{\mu\nu}^a T^a = \frac{-1}{ie}[\partial_\mu - ieA_\mu^b T^b, \partial_\nu - ieA_\nu^c T^c]$$

$$= T^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\varepsilon^{abc} A_\mu^b A_\nu^c)$$

(different sign convention than Les Houches)

For $v^2 > 0$, minimum of Higgs potential is
$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

which breaks $SU(2) \to U(1)$ (as in our discussion of the σ-model) the Higgs mass is found as

$$\frac{1}{8}\lambda\left((v+\phi')^2 - v^2\right)^2 = \frac{1}{8}\lambda(2v\phi')^2 + \cdots$$

$$\Rightarrow \mu_H^2 = \lambda v^2 \qquad = \frac{1}{2}\lambda v^2 \phi'^2 + \cdots$$

the gauge boson masses come from

$$\frac{1}{2} D_\mu \phi^a D^\mu \phi^a = \frac{1}{2} e^2 \left(\varepsilon^{abc} A_\mu^b \phi_0^c\right)\left(\varepsilon^{ade} A^{\mu d} \phi_0^e\right)$$

$$= \frac{1}{2} e^2 v^2 \left((A_\mu^1)^2 + (A_\mu^2)^2\right)$$

So the charged gauge bosons have

$$\mu_\pm^2 = e^2 v^2$$

The manifold of minima of the potential is

$$G/H = SU(2)/U(1) = S^2$$

– a two-sphere parametrized by the unit vector

$$\hat{\phi}^a = \phi^a / v = \frac{\phi^a}{|\phi|}\,; \quad |\phi| = \sqrt{\phi^b \phi^b}$$

Now consider the configurations of finite energy. In these, the Higgs field must approach a minimum of the potential as $r \to \infty$

$$\phi^a(r,\theta,\phi) \xrightarrow[r\to\infty]{} v\,\hat{\phi}^a(\theta,\phi)$$

From the behavior of the Higgs field at spatial infinity, we obtain a mapping

$$S^2 \longrightarrow S^2$$

– from two-sphere (at spatial infinity) to two-sphere (of Higgs field minima) this map has a topological invariant – its winding number – that we have already discussed in connection with the nonlinear sigma model (in two spatial dimensions).

The winding of this map arose in the NLσ model because we were interested in configurations (skyrmions) with $\hat{n}^a(\vec{x})$ taking values on S^2 everywhere in $\mathbb{R}^2$ (not just at ∞). We saw there that

$$q = \frac{1}{8\pi}\int d^2x\ \varepsilon_{abc}\,\varepsilon^{ij}\,\hat{n}^a\partial_i\hat{n}^b\partial_j\hat{n}^c$$

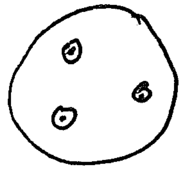
$$= \text{integer}$$

is the no. of times the map covers the target S^2. And that there was an associated topological current (a 3-vector)

$$J^\mu = \frac{1}{8\pi}\varepsilon^{\mu\nu\lambda}\varepsilon_{abc}\,n^a\partial_\nu n^b\partial_\lambda n^c$$

such that $\partial_\mu J^\mu = 0$ is an identity and $q = \int d^2x\ J^0$

It arises in a different way here (from the asymptotic $r \to \infty$ behavior of $\phi^a(\vec{x})$), but the topological charge can again



q on the big sphere is a sum of q's of the little spheres, where each little sphere encloses a zero of ϕ

be related to a conserved current. Consider the 4-vector

$$K^\mu = \frac{1}{8\pi} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abc}\, \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c$$

($\hat{\phi}^a = \phi^a/|\phi|$, so K^μ defined away from zeros of ϕ.)

Because of the antisymmetry of $\varepsilon^{\mu\nu\rho\sigma}$, K^μ is evidently a total derivative

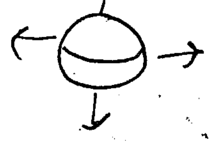
$$K^\mu = \partial_\nu \left[\frac{1}{8\pi} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abc}\, \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c \right]$$

Therefore the integral of K^0 over all space is a surface term

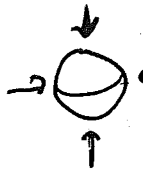
$$\int d^3x\, K^0 = \underbrace{\frac{1}{8\pi} \int_{S^2} d^2 S_i\, \varepsilon^{ijk} \varepsilon_{abc}\, \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c}_{= q \text{ on the } r=\infty \text{ 2-sphere.}}$$

Since q is actually a topological inv., $\int d^3x\, K^0$ is actually a sum over the zeros of ϕ, weighted by ± 1 (the integrand has δ-function singularities at the zeros).

We can find a nontrivial static solution to the field equations by minimizing the energy in the sector $q = 1$. (Actually, we still need to check that there really are configurations of finite energy, i.e. that we can choose a gauge field that makes $D_\mu \phi$ fall off fast enough. - more on that in a minute.)

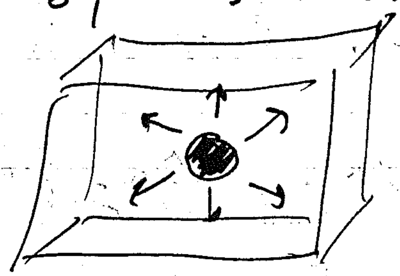


In a suitable gauge the spherically symmetric configuration with $q = 1$ is the identity map $S^2 \rightarrow S^2$. It is a "hedgehog" configuration



"Antihedgehog" has $q=-1$

such that the Higgs field points "radially outward." As with the vortex, we can argue that, somewhere in $\mathbb{R}^3$, the hedgehog must have a zero ($\phi^a=0$) — there is a "core" where the Higgs field departs from the minimum of the potential. Were this not so, we would be able to deform the $q=1$ map to the ($q=0$) constant map by shrinking the sphere to a point.



Thus there is a "lump" of Higgs field potential energy — the <u>core</u> of the hedgehog!

But how is <u>topological</u> charge related to <u>magnetic</u> charge? It is convenient to have a gauge invariant notion of the (abelian) magnetic field associated with the massless photon — i.e. the unbroken $U(1)$ gauge symmetry. Consider

$$F_{\mu\nu} = F^a_{\mu\nu}\hat{\phi}^a + \frac{1}{e}\varepsilon^{abc}\hat{\phi}^a D_\mu\hat{\phi}^b D_\nu\hat{\phi}^c$$

(where $\hat{\phi}^a = \phi^a/|\phi|$.)

— this is evidently invariant under local $SU(2)$ transformations, as $F^a_{\mu\nu}$, $D_\mu\hat{\phi}^b$, $\hat{\phi}^c$ all transform covariantly, and all indices are contracted. In a local region where $|\phi| = \text{const}$ (e.g. a patch of the sphere at ∞, where $|\phi| = v$) we may choose a gauge with

$$\hat{\phi}^a = \hat{e}_3 \quad \text{, i.e.} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{then } \partial_\mu \hat{\phi} = 0.$$

We have $F^a_{\mu\nu} \hat{\phi}^a = F_{\mu\nu} = \partial_\mu A^3 - \partial_\nu A^3 + e\left(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu\right)$

and $\hat{\phi}^a \varepsilon^{abc} D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c = D_\mu \hat{\phi}^1 D_\nu \hat{\phi}^2 - D_\mu \hat{\phi}^2 D_\nu \hat{\phi}^1$

$$= e^2\left(A^2_\mu A^1_\nu - A^1_\mu A^2_\nu\right)$$

Therefore $F_{\mu\nu} = \partial_\mu A^3 - \partial_\nu A^3$
– the commutator term is cancelled by the covariant derivative, and $F_{\mu\nu}$ is the field strength of the unbroken $U(1)$ gauge field. Magnetic charge should be defined as the total flux of this abelian magnetic field

$$g = \int d^2\vec{S} \cdot \vec{B} \qquad \text{(integrated over } S^2 \text{ at } r = \infty\text{)}$$

where $B^i = \frac{1}{2}\varepsilon^{ijk} F_{jk}$

In a finite energy configuration, $D_\mu \phi \sim \frac{1}{r^{3/2+\varepsilon}}$ so that the covariant derivative term does not contribute to the asymptotic flux, and we may also write

$$g = \int d^2 S \cdot \vec{B}^a \hat{\phi}^a$$

where $B^{ai} = \varepsilon^{ijk} F^a_{jk}$

To be explicit, consider the case in which the asymptotic $\hat{\phi}(\theta,\phi)$ is the identity map from S^2 to S^2 — the spherically symmetric hedgehog

$$\phi^a = v\,\frac{r^a}{r} \Rightarrow \partial_i \phi^a = v\,\frac{1}{r^3}\left(r^2\delta_i^{\ a} - r_i r^a\right)$$

without the gauge field $\partial_i \phi^a \sim \frac{1}{r}$ means that the gradient energy

$$\frac{1}{2}\int d^2x\,(\partial_i\phi^a)^2$$

diverges <u>linearly</u>.

To construct a finite energy configuration, we want

$$D_i\phi^a = \partial_i\phi^a + e\varepsilon^{abc}A_i^{\ b}\phi^c \xrightarrow[r\to\infty]{} 0$$

or $\quad e\,\hat{r}^c\varepsilon^{abc}A_i^{\ b} = \frac{-1}{r}\left(\delta_i^{\ a} - \hat{r}_i\hat{r}^a\right) + \cdots,$

which is solved by

$$A_i^{\ b} = \frac{-1}{er}\,\varepsilon_{ibk}\,\hat{r}_k + \cdots$$

check: $\varepsilon_{ibk}\varepsilon^{abc}\hat{r}_k\hat{r}^c = \left(\delta^c_{\ k}\delta^a_{\ i} - \delta^c_{\ i}\delta^a_{\ k}\right)\hat{r}_k\hat{r}^c$

$$= \left(\delta^a_{\ i} - \hat{r}_i\hat{r}^a\right)$$

This implies

$\hat{\phi}^a\vec{B} = -\dfrac{\hat{r}}{er^2}$ — the configuration is an antimonopole with $\boxed{g = -4\pi/e}$

Since the quantum of charge in this model is e, finding a minimal monopole charge of $g = 4\pi/e$ is a mild surprise — the Dirac quantization condition would have allowed $g = 2\pi/e$. Here is a simple explanation. In principle, we could have introduced matter in the spin-$\frac{1}{2}$ irrep of $SU(2)$ into the model, and so obtain charge-$(e/2)$ particles after the breaking of $SU(2)$ to $U(1)$. The magnetic charge must be $g = (4\pi/e)\cdot$ integer to ensure the compatibility of the monopole with the existence of such matter.

Thus, in configurations of finite energy, magnetic charge and topological charge are locked together.

We can construct nonsingular configurations with nonvanishing topological charge and finite energy. By minimizing the energy in the $g=1$ sector, we construct a static monopole solution to the classical field equations.

It is important to appreciate that the vector potential

$$A_i^a = -\varepsilon_{iak}\frac{\hat{r}_k}{er}$$

(in contrast to Dirac's singular vector potential) is globally defined on S^2. This is possible because we have three vector potentials instead of just one. Each component, e.g. $A_i^{a=3}$ has a curl whose divergence (of course) vanishes... But the physical magnetic field observable from afar is $\phi^a B_i^a$ — which can have a nonvanishing integral over S^2.

Since A_i^a is nonsingular, there is not need for a singularity in the field strength inside the two-sphere (in contrast to Dirac's monopole). E.g., a spherically symmetric Ansatz for ϕ and A is

$$\phi^a(\vec{r}) = v\hat{r}_a h(\mu r)$$

$$A_i^a(\vec{r}) = -\frac{\varepsilon_{iak}\hat{r}_k}{er}\left[1 - K(\mu r)\right]$$

This configuration has no singularities provided

$$h(0) = 0 \qquad h(\infty) = 1$$
$$K(0) = 1 \qquad K(\infty) = 0$$

— the center of the core of the hedgehog, and the gauge field at the center, are well behaved.

As in our discussion of the vortex, we can rescale fields:

$$A = \frac{1}{e}\tilde{A} \qquad \phi = \frac{1}{e}\tilde{\phi}$$

and write

$$\mathcal{L} = \frac{1}{e^2}\left[-\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}(\tilde{D}_\mu\tilde{\phi})^2 - \frac{1}{8}\left(\frac{\lambda}{e^2}\right)(\tilde{\phi}^2 - \mu^2)^2\right]$$

the only length scale is $\mu^{-1} = (ev)^{-1}$, and the dimensionless parameter λ/e^2 appears in the field equations. The semiclassical limit is $e^2 \to 0$, with μ and λ/e^2 fixed

In this limit, the size $\sim \mu^{-1}$ of the monopole stays fixed, while its energy diverges like $1/e^2$

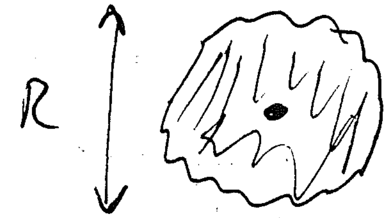
i.e. $\quad$ size $= S\left(\frac{\lambda}{e^2}\right) \mu^{-1}$

$$\text{mass} = M\left(\frac{\lambda}{e^2}\right)\frac{\mu}{e^2}$$

Thus, at weak coupling, a monopole is large and "squishy" compared to its Compton wavelength.

This feature makes the quantum theory of the 't Hooft-Polyakov (soliton) monopole much more tractable than the quantum theory of the singular Dirac monopole. Though the monopole charge $\sim 1/e$ is large when e is small, the quantum corrections to the classical structure of the monopole are small; because the monopole is a coherent excitation involving many elementary quanta that does not easily arise as a quantum fluctuation.

In contrast, the dynamics of point monopoles with $g \sim 1/e$ is a difficult strong-coupling problem.



A monopole, whether pointlike or nonsingular, has intricate structure in a cloud with size $\sim g^2 m^{-1} \sim \frac{1}{e^2 m} = R$ (where m is monopole mass) — since the Coulomb self energy of charge g smeared in this region is $g^2/R \sim m$)

In the 't Hooft-Polyakov case, this cloud can be described by classical field equations, with small quantum corrections. In the Dirac case, the cloud is associated with large quantum fluctuations that are highly intractible.

Let's look more closely at the classical structure of the 't Hooft-Polyakov monopole, following our discussion of the vortex.

We anticipate that the monopole core will have two size scales: a "magnetic" radius r_m that characterizes the size of the region where A_i^a departs significantly from its asymptotic value (K far from 0) and a scalar radius r_s where the Higgs field is far from its asymptotic value (h far from 1). These sizes are determined by the competition among several effects: in

$$E = \int d^3x \left[\frac{1}{2} \vec{B}^a \cdot \vec{B}^a + \frac{1}{2} \vec{D} \phi^a \cdot \vec{D} \phi^a + V(\phi) \right]$$

- The coulomb energy of a magnetically charged sphere:

$$E_{mag} \sim \int_{r_M}^{\infty} 4\pi r^2 dr \, \frac{1}{2}\left(\frac{g}{4\pi r^2}\right)^2 \sim \frac{g^2}{4\pi r_M} \sim \frac{4\pi}{e^2 r_M} = \frac{4\pi v}{e}\left(\frac{1}{\mu r_M}\right)$$

(which makes the magnetic core want to expand)

- The potential energy of the Higgs field core

$$E_{core} \sim \int_0^{r_S} 4\pi r^2 dr \, \frac{\lambda}{8} v^4 \sim 4\pi \lambda r_S^3 v^4 = \frac{4\pi v}{\sqrt{\lambda}} m_H^3 v r_S^3$$

(makes the scalar core want to collapse)

- The gradient energy: For $r_S < r_M$, the circumferential gradient is not screened by the gauge field for $r_S < r < r_M$:

$$E_{grad} \sim \int_{r_S}^{r_M} 4\pi r^2 dr \, \frac{v^2}{r^2} \sim 4\pi v^2 (r_M - r_S)$$
$$= \frac{4\pi v}{e} \mu (r_M - r_S)$$

Altogether:

$$E \sim \frac{4\pi}{e} v \left[\frac{1}{\mu r_M} + \frac{e}{\sqrt{\lambda}} m_H^3 r_S^3 + \underbrace{\mu (r_M - r_S)}_{= \mu r_M - \frac{e}{\sqrt{\lambda}}(r_S m_H)} \right]$$

Minimizing with respect to r_S and r_M, we find

$$r_M \sim \mu^{-1} \qquad \left(\begin{array}{l} \mu = e v, \\ m_H = \sqrt{\lambda} v \end{array} \right)$$
$$r_S \sim m_H^{-1}$$

$$E \sim \frac{4\pi\mu}{e^2} \left[O(1) + O\left(\frac{e}{\sqrt{\lambda}}\right) \right]$$

This estimate is self-consistent for $\mu < m_H$ or $\frac{e}{\sqrt{\lambda}} < 1$, so

$$E = \frac{4\pi}{e^2}\mu\left[O(1)\right] = gv\cdot\left[O(1)\right]$$

In the case $\mu > m_H$ — Again r_s wants to shrink and r_μ wants to expand — which they are free to do until $r_\mu > r_s$, and the gradient energy kicks in. So minimum occurs for $r_s \sim r_\mu \sim \mu$, and again $E \sim 4\pi/e^2\,\mu$.

So the monopole mass is insensitive to the value of the dimensionless parameter λ/e^2. Numerically one finds it is monotonically increasing with λ/e^2

$$M_{monopole} = \frac{4\pi}{e^2}\, m\left(\lambda/e^2\right)$$

$$\text{where} \quad m\left(\lambda/e^2\right) = \begin{cases} 1 & , \ \lambda/e^2 = 0 \\ 1.787\ldots & , \ \lambda/e^2 = \infty \end{cases}$$

In order of magnitude, this is the Coulomb energy

$$g^2/4\pi R$$

of a sphere with radius $R = \mu^{-1}$.
At weak coupling, a monopole is heavy compared to its inverse size because the magnetic charge g is <u>large</u>.

Addendum to page 165:

Asymptotically, $\phi^a = v\hat{r}^a \Rightarrow \partial_i \phi^a = v\frac{1}{r}(\delta_i^{\ a} - \hat{r}_i \hat{r}^a)$

$$A_i^b = \frac{1}{er}\varepsilon_{bij}\hat{r}_j \Rightarrow$$

$$\varepsilon^{abc}A_i^b = \frac{1}{er}\varepsilon^{bca}\varepsilon_{bij}\hat{r}_j = \frac{1}{er}\left[\delta^c_i\delta^a_j - \delta^c_j\delta^a_i\right]\hat{r}_j$$

$$= \frac{1}{er}\left[\delta^c_i\hat{r}^a - \delta^a_i\hat{r}^c\right]$$

Hence

$$e\varepsilon^{abc}A_i^b\phi^c = v\frac{1}{r}\left[\delta^c_i\hat{r}^a - \delta^a_i\hat{r}^c\right]\hat{r}^c$$

$$= v\frac{1}{r}\left[\hat{r}_i\hat{r}^a - \delta^a_i\right]$$

and therefore,

$$D_i\phi^a = \partial_i\phi^a + e\varepsilon^{abc}A_i^b = 0 \quad \text{at } r\to\infty.$$

Now we want to evaluate the magnetic field strength

$$B^a_\ell = \frac{1}{2}\varepsilon_{\ell ij}F^a_{ij},$$

$$F_{ij} = \partial_i A^a_j - \partial_j A^a_i + e\varepsilon^{abc}A^b_i A^c_j$$

$$A^a_j = \frac{1}{e}\varepsilon_{ajk}\frac{r_k}{r^2} \Rightarrow$$

$$\partial_i A^a_j = \frac{1}{e}\varepsilon_{ajk}\partial_i\left(\frac{r_k}{r^2}\right) = \frac{1}{er^2}\varepsilon_{ajk}(\delta_{ik} - 2\hat{r}_i\hat{r}_k)$$

$$\varepsilon_{\ell ij}\partial_i A^a_j = \frac{1}{r^2 e}\varepsilon_{j\ell i}\varepsilon_{jka}(\delta_{ik} - 2\hat{r}_i\hat{r}_k)$$

$$= \frac{1}{r^2 e}(\delta_{\ell k}\delta_{ia} - \delta_{\ell a}\delta_{ik})(\delta_{ik} - 2\hat{r}_i\hat{r}_k)$$

$$= \frac{1}{r^2 e}(\delta_{\ell a} - 3\delta_{\ell a} - 2\hat{r}_a\hat{r}_\ell + 2\delta_{\ell a})$$

So $\varepsilon_{\ell ij}\left(\partial_i A_j^a\right) = \frac{-2}{er^2}\hat{r}_\ell \hat{r}^a$

There is also a commutator term that contributes to B_ℓ^a

$$e\varepsilon^{abc}A_i^b A_j^c$$

$$= \frac{1}{er^2}\left(\varepsilon^{bca}\varepsilon^{bik}\right)\varepsilon^{cj\ell}\hat{r}_K \hat{r}_\ell$$

$$= \frac{1}{er^2}\left(\delta_{ci}\delta_{aK} - \delta_{cK}\delta_{ai}\right)\varepsilon^{cj\ell}\hat{r}_K \hat{r}_\ell$$

$$= \frac{1}{er^2}\left(\varepsilon_{ij\ell}\hat{r}^a \hat{r}_\ell - \varepsilon_{Kj\ell}\hat{r}_K \hat{r}_\ell \delta_{ai}\right)$$

$$= \frac{1}{er^2}\left(\varepsilon_{ij\ell}\hat{r}^a \hat{r}_\ell\right)$$ This contributes $\frac{1}{er^2}\hat{r}^a \hat{r}_\ell$ to B_ℓ^a.

Combining the terms we find

$$\boxed{B_\ell^a = -\frac{1}{er^2}\hat{r}^a \hat{r}_\ell}$$

Thus: $B_\ell^a \hat{\phi}^a = B_\ell^a \hat{r}^a = -\frac{1}{er^2}\hat{r}_\ell \Rightarrow g = -\frac{4\pi}{e}$

Note that $B_\ell^a \hat{r}_\ell = -\frac{1}{er^2}\hat{r}^a$, so that

$\int d^2 s^\ell B_\ell^a = 0$ — each of the components $\vec{B}^a$ has trivial flux,

which enables A_i^a to be nonsingular on two-sphere.

Bogomol'nyi Bound

Actually, in the limit $\lambda \to 0$, we can find the monopole solution (and compute its mass) analytically, because the field equations can be reduced to first order equations

If we can ignore the potential energy ($\lambda = 0$), the energy of a static configuration with $D_0 \phi = 0 = \vec{E}^a$ is

$$\mathcal{E} = \int d^3x \left[\tfrac{1}{2} B_i^a B_i^a + \tfrac{1}{2} D_i \phi^a D_i \phi^a \right]$$

Now note that

$$0 \leq \int d^3x \, \tfrac{1}{2} \left(B_i^a \mp D_i \phi^a \right)^2$$

$$= \int d^3x \left(\tfrac{1}{2} B_i^a B_i^a + \tfrac{1}{2} D_i \phi^a D_i \phi^a \right) \mp \int d^3x \, B_i^a D_i \phi^a$$

But the cross term can be expressed as a surface integral, because

$$D_i B_i^a = 0$$

– this is the Bianchi identity, or source free Yang-Mills equation. It arises from the Jacobi identity applied to commutators of covariant derivatives:

$$0 = [D_i, [D_j, D_k]] + [D_j, [D_k, D_i]] + [D_k, [D_i, D_j]]$$

and since $F_{jk} = \frac{-i}{e} [D_j, D_k] = \varepsilon_{jk} B_k$

and $D_j B_k^a = [D_j, B_k]^a$, this is $D_i B_i^a = 0$

Since $B_i^a \phi^a$ is gauge-invariant,

$$D_i (B_i^a \phi^a)$$
$$= B_i^a D_i \phi^a = \partial_i (B_i^a \phi^a).$$

Then our cross term becomes

$$\int_{S^2_\infty} d^2 S^i \, B_i^a \phi^a$$

Asymptotically as $r \to \infty$, $\phi^a \to v \hat{\phi}^a$, and we have seen that the magnetic charge is

$$g = \int_{S^2_\infty} dS^i \, B_i^a \hat{\phi}^a$$

So we have found

$$0 \le \mathcal{E} \mp g v \Rightarrow \mathcal{E} \geq \pm g v$$

Choosing the sign so $\pm g = |g|$, we conclude that

$$\mathcal{E} \geq |g| v$$

— this is the BPS (Bogomolnyi-Prasad-Sommerfield) bound on the mass of the monopole. The condition for equality is

$$B_i^a = D_i \phi^a \qquad g > 0,$$
$$B_i^a = -D_i \phi^a \qquad g < 0.$$

(as well as $D_0 \phi^a = E_i^a = 0$).

A simple generalization applies to dyon configurations with both magnetic and electric charge. Consider

$$0 \le \int d^3x \, \frac{1}{2}\left[\left(B_i^a - (\cos\alpha) D_i\phi^a\right)^2 + \left(E_i^a - \sin\alpha \, D_i\phi^a\right)^2 \right]$$

$$= \int d^3x \, \frac{1}{2}\left[B_i^a B_i^a + E_i^a E_i^a + D_i\phi^a D_i\phi^a \right]$$

$$- \int d^3x \left[(\cos\alpha) B_i^a + (\sin\alpha) E_i^a \right] D_i$$

we can use the Bianchi identity to write the first cross term in terms of the magnetic charge, as before. The electric field does not obey the Bianchi identity, but we can use Gauss's law (the A_0^a field equation) — since we know that the minimal energy config will satisfy the field equations.

Field equation:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu^a} = \frac{\partial \mathcal{L}}{\partial A_\nu^a}$$

$$\Rightarrow \partial_\mu\left(-F^{\mu\nu a}\right) = -\frac{1}{2} F^{\mu\lambda b} \underbrace{\frac{\partial}{\partial A_\nu^a} F_{\mu\lambda}^b}_{2e\varepsilon^{bca}\eta^\nu{}_\lambda A_\mu^c} + D^\lambda \phi^b \underbrace{\frac{\partial}{\partial A_\nu^a} D_\lambda \phi^b}_{e\varepsilon^{bac}\eta_\lambda^\nu \phi}$$

$$\Rightarrow -\left(\partial_\mu F^{\mu\nu a} + e\,\varepsilon^{abc} A_\mu^b F^{\mu\nu c}\right) = -\varepsilon^{abc} D^\nu \phi^b \phi^c$$

The electric field is $E_i^a = F_{0i}^a$

Our metric convention is $\partial_i E^i = -\partial_i E_i$
so the A_0 eqn (Gauss law) is

$$D_i E_i^a = \varepsilon^{abc} D^0 \phi^b \phi^c$$

Therefore

$$D_i (E_i^a \phi^a) = E_i^a D_i \phi^a + \underbrace{\phi^a D_i E_i^a}$$

$$= \partial_i (E_i^a \phi^a) \qquad \hookrightarrow \varepsilon^{abc} D^0 \phi^b \phi^c \phi^a = 0$$

(since $E_i^a \phi^a$ is a gauge singlet)

We have

$$\int d^3x\, E_i^a D_i \phi^a = \int_{S^2_\infty} d^2 S^i (E_i^a \phi^a) \quad \text{– a surface integral}$$

As with the magnetic field, the gauge-invariant electric field is

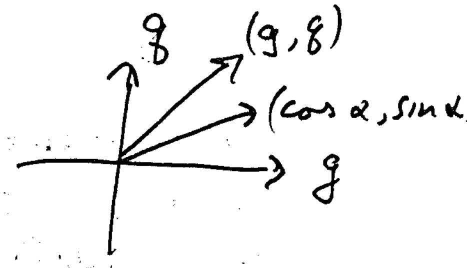
$$E_i = F_{0i} = F_{0i}^a \hat{\phi}^a + \frac{1}{e} \varepsilon^{abc} \hat{\phi}^a D_0 \hat{\phi}^b D_i \hat{\phi}^c$$

$$= F_{0i}^a \hat{\phi}^a \qquad \text{if } D_0 \hat{\phi}^b = 0.$$

So the electric charge is

$$q = \int d^2 S^i E_i = \int d^2 S^i E_i^a \hat{\phi}^a, \text{ and}$$

we have $\quad \mathcal{E} \geq (g \cos\alpha \quad + q \sin\alpha \quad) v$



We can choose the angle α as we please, and we get the strongest bound if (g, q) and $(\cos\alpha, \sin\alpha)$ align, or

$$\tan\alpha = q/g \ , \quad \cos\alpha = g/\sqrt{g^2+q^2} \ , \quad \sin\alpha = q/\sqrt{g^2+q^2}$$

Then

$$\mathcal{E} \geq \sqrt{g^2+q^2} \cdot v \ ,$$

with equality for

$$B_i^a = (\cos\alpha) D_i \phi^a$$
$$E_i^a = (\sin\alpha) D_i \phi^a$$
$$D_0 \phi^a = 0$$

This is the BPS (Bogomol'nyi - Prasad - Sommerfield) bound ~~in~~ its more general form. The monopoles or dyons that saturate the bound are called BPS states, or BPS - saturated states. These solve the first order Bogomol'nyi equations.

Solutions with winding number $\pm$ one (the minimal magnetic charge $g = \pm 4\pi/e$) can be written down explicitly in terms of special functions. These are monopoles (or dyons) with mass:

$$M_{monopole} = |g| v \ , \quad M_{dyon} = (g^2+q^2)^{1/2} v$$

We have derived these relations using the classical field equations, but we will see later that, in a supersymmetric extension of the model, there are exact relations in the quantum theory.

Static solutions can also be constructed with $n > 1$ monopoles, i.e.

$$g = \frac{4\pi}{e} \cdot n$$

There is a repulsive Coulomb interaction between monopoles, but this is precisely cancelled by an attractive interaction due to the massless Higgs scalar, so that monopoles are non interacting. (The deviation of the Higgs field from the vacuum value also falls off like $1/r$.)

The n-monopole solution has $4n$ continuously varying real parameters, called the moduli of the solution. $3n$ of these parameters specify the positions of the centers of the n monopoles. The remaining n parameters specify an "orientation" of each monopole in the unbroken $U(1)_{em}$ gauge group. We'll return a little later to the interpretation of this parameter.

For now, let us just note that if the monopoles are moving very slowly, then it is a good approximation to say that the configuration looks like one of the static solutions at each instant, but with moduli that slowly vary. This is called the "moduli-space approximation." We may study the dynamics of slowly moving monopoles with an effective action that governs how the moduli change with time.

E.g., Let X^m, $m = 1, \dots 4n$

denote coordinates on the moduli space. The static solutions are

$$A_\mu = A_\mu(\vec{x}, X)$$
$$\phi = \phi(\vec{x}, X)$$

In the moduli-space approximation, the time dependent configurations are

$$A_\mu = A_\mu(\vec{x}, X(t))$$
$$\phi = \phi(\vec{x}, X(t)).$$

Plugging into the action

$$S = \int dt \int d^3x \, \mathcal{L}(A, \phi),$$

we obtain the effective action for the moduli

$$S = \int dt \left(\tfrac{1}{2} g_{mn}(X) \dot{X}^m \dot{X}^n \right)$$

(coming from $\int d^4x \, \tfrac{1}{2}(E^2 + (D_0\phi)^2)$.)

(there can be no velocity independent terms, as S is a constant independent of moduli when the configuration is static.)

The quantity $g_{mn}(\underset{\sim}{X})$ can be interpreted as a Riemannian metric on the moduli space. Indeed, the equations of motion in the moduli space approximation are geodesic equations for this metric. That is, the dynamical evolution of the multimonopole configuration is a geodesic on the moduli space, parametrized by the time, which is the affine parameter of the geodesic

Euler-Lagrange equation for

$$L = \tfrac{1}{2} g_{mn}(\underset{\sim}{X}) \dot{X}^m \dot{X}^n$$

is

$$\frac{d}{dt} g_{mn} \dot{X}^n = \tfrac{1}{2} g_{kl,m} \dot{X}^k \dot{X}^l$$

$$= g_{mn,l} \dot{X}^l \dot{X}^n + g_{mn} \ddot{X}^n$$

$$\Rightarrow \quad g_{mn} \ddot{X}^n = \left(\tfrac{1}{2} g_{kl,m} - g_{mk,l}\right) \dot{X}^k \dot{X}^l$$

We get the same eqn from

$$L = \int ds = \int \left(g_{mn}(X)\, dX^m dX^n\right)^{\frac{1}{2}}$$

$$= \int d\lambda \left(g_{mn}(X) \frac{dX^m}{d\lambda} \frac{dX^n}{d\lambda}\right)^{\frac{1}{2}}$$

<u>if</u> after varying the action we choose λ so that $\left(g_{mn}(X) \frac{dX^m}{d\lambda}\frac{dX^n}{d\lambda}\right)^{\frac{1}{2}} = ds/d\lambda$ = constant

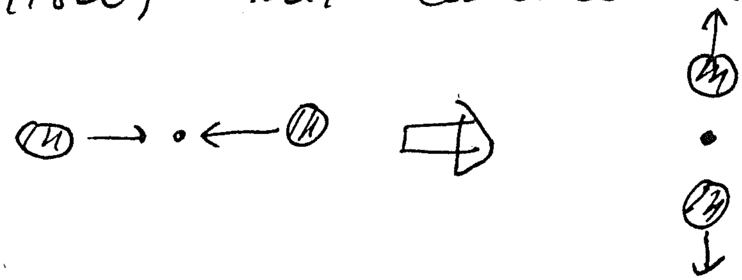
(i.e. affine parametrization).

The moduli space and its metric have a beautiful geometrical structure that has been much studied. See for much more detailed discussion:

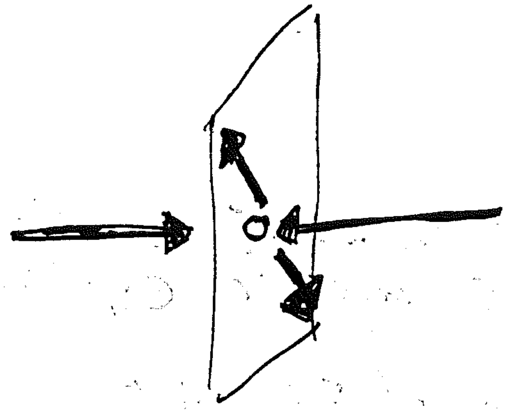
M. Atiyah + N. Hitchin, "The geometry and dynamics of magnetic monopoles," (Princeton, 1988).

For the 2-monopole configuration, the geodesics can be explicitly calculated, and hence the problem of monopole-monopole scattering at low velocity can be solved.

An interesting special case is that of two identical monopoles (zero electric charge and matching $U(1)_{em}$ orientation) that collide head on:



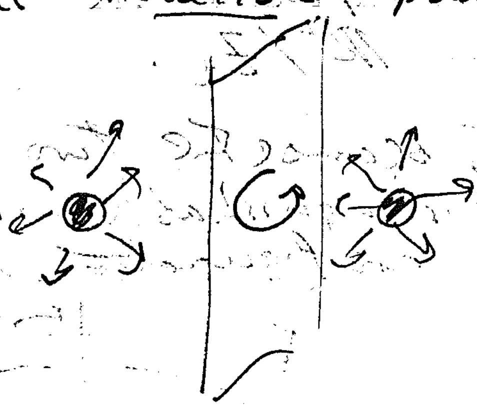
Surprisingly, there is scattering by a 90° angle.



Axial symmetry is evidently broken: the outgoing monopoles pick a particular axis in the plane $\perp$ to axis of incoming monopoles.

What determines the plane of scattering?

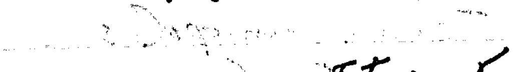
After the CM motion (3 parameters) and an overall global $U(1)_{em}$ gauge transformation are separated out, the 2-monopole solution has 4 parameters. 3 are for the *relative* position. What is the 4th?



It is a *relative* $U(1)_{em}$ rotation of the two monopoles. This is *not* just a gauge transformation. We can "cut" the solution in two along a plane $\perp$ to the axis connecting the monopoles. Rotate one half by $U(1)_{em}$ (preserving the Higgs field) and then glue together again. We obtain a *new* soln, not a mere gauge transformation of the original.

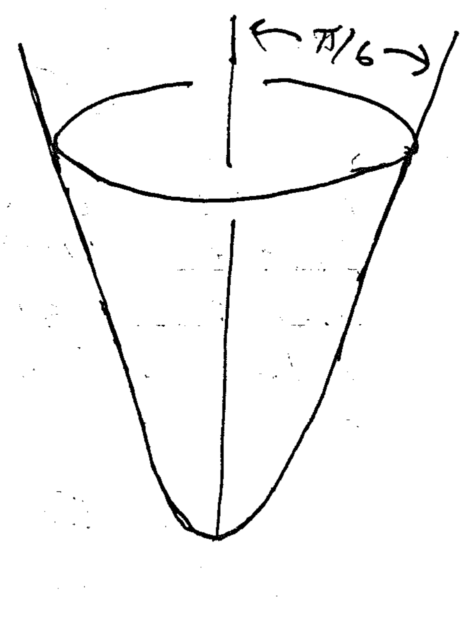
So the metric is nontrivial on a *4-manifold* (relative position + relative charge). What Atiyah and Hitchin noticed is that this metric is "self-dual" – i.e. a "gravitational "instanton" — that is what enabled them to find the metric explicitly.

It is the orientation of this internal $U(1)$ coordinate that picks out the scattering plane of two monopoles that collide head on.



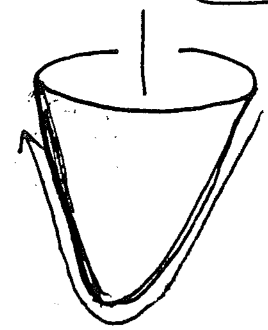
For low velocity scattering of monopoles with zero impact parameter, and a particular choice of the initial $U(1)_{em}$ relative coordinate, the geodesic motion is confined to a 2-dimensional submanifold. When the monopoles are far apart, the metric is flat on $\mathbb{R}^2/\mathbb{Z}_2$.

The $\mathbb{Z}_2$ is modded out, because the two monopoles are indistinguishable; if they change places the configuration is unchanged. So asymptotically it is a cone with half-opening angle $\pi/6$



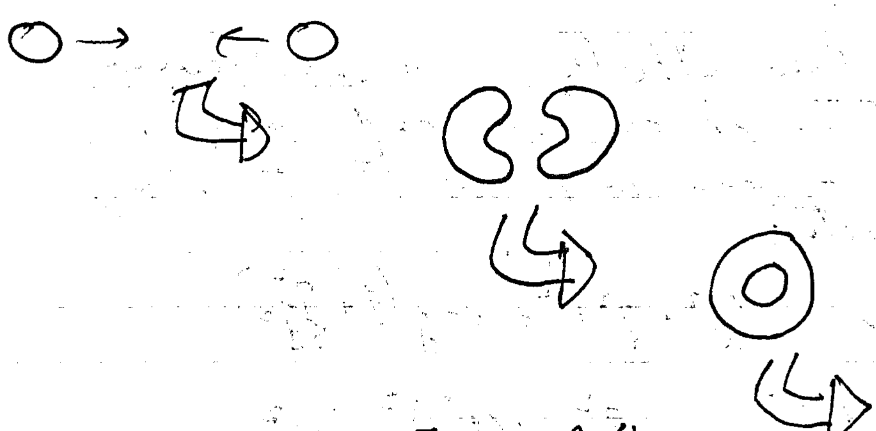
($\sin \pi/6 = \frac{1}{2}$, so circumference $\sim \pi \times$ (separation)).

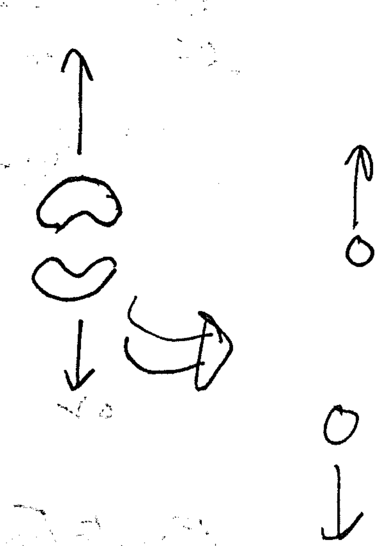
But the vertex of the cone is rounded and nonsingular.



obviously, geodesic motion, for a trajectory heading toward the vertex, is as shown. But this is 90° scattering, as points on opposite sides of the cone differ in orientation by $\frac{1}{2}\cdot\pi$.

If we follow the energy contour of monopole cores, as they approach one another, they are adiabatically distorted:





when the cores "coincide," the configuration is not spherically symmetric; its core is toroidal, and oriented along the scattering plane...

It is also interesting to consider scattering with nonzero impact parameter (angular momentum). Classically, the orbital angular momentum can be transferred to internal U(1)em angular momentum — i.e. electric charge. So a monopole pair can scatter and become a dyon pair!

What of conservation of $\vec{J}$? There is a kinetic and field contribution to $\vec{J}$ (more on that later). For 2 dyons, the field contribution is nonzero. So angular momentum is conserved, but transferred from kinetic to field --

Quantum effects - in particular quantization of charge and angular momentum - are important at very low velocity:

$$m_{dyon} \sim v(g^2+q^2)^{\frac{1}{2}} = vg\left(1+\frac{q^2}{2g^2}\right)$$

$$q \sim e \qquad \frac{q^2}{2g^2} \approx \frac{1}{2}\left(\frac{e^2}{4\pi}\right)^2 \sim \frac{1}{2}\alpha^2$$
$$g \sim \frac{4\pi}{e}$$

or $\quad m_{dyon} = M_{monopole} + \frac{1}{2}\alpha^2 M_{monopole}$

Excitation of the charge of the dyon is not energetically allowed for $v < \alpha$

For more detailed discussion: see Les Houches lectures

Lecture #16

Monopoles and solitons:

General connection between winding: $\pi_2(G/H)$

mag. charge: $\pi_1(H)$

$$G/H = \{\underline{\Phi},\ \underline{\Phi} = \Omega \underline{\Phi}_0,\ \Omega \in G\},$$

where $H\underline{\Phi}_0 = \underline{\Phi}_0$

What is π_2? choose a basepoint $x_0 \in M$

$\pi_2(M, x_0)$: equiv classes

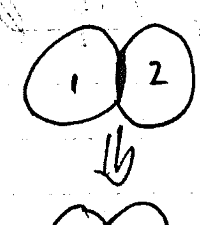
Maps Knot Take North pole to x_0

Has (abelian) group structure

E.g. "puncture" $S^2 \Rightarrow$ a disk

But boundary $\to x_0$

[π_n is abelian for $n \geq 2$]



we can glue two disks together — order does not matter because boundary is connected

cf $\pi_1(M, x_0)$

$\bullet \xrightarrow{1} \circ\, \bullet \xrightarrow{2} \circ$

we can cut circle open, too but boundary not connected

to $\bullet \xrightarrow{2} \circ\, \bullet \xrightarrow{1} \circ$ order might matter!

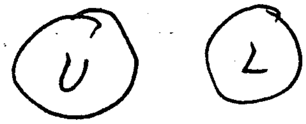
element of π_2 = topological charge of soliton...

Discrete $\Rightarrow$ conserved

How to compute π_2?

$\underline{\Phi}(\theta, \phi)$: function $S^2 \to G/H$

cut at the equator



Two contractible patches

"unitary gauge" imposed locally

$$\Omega_U(\theta,\phi)\, \underline{\Phi}(\theta,\phi) = \underline{\Phi}_0 \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

$$\Omega_L(\theta,\phi)\, \underline{\Phi}(\theta,\phi) = \underline{\Phi}_0 \qquad \tfrac{\pi}{2} \le \theta \le \pi$$

At the equator

$$\underline{\Phi}(\tfrac{\pi}{2},\phi) = \Omega_L^{-1}(\tfrac{\pi}{2},\phi)\, \underline{\Phi}_0$$

$$\Rightarrow \underline{\Phi}_0 = \Omega_U(\tfrac{\pi}{2},\phi)\, \underline{\Phi}(\tfrac{\pi}{2},\phi) = \Omega_U(\tfrac{\pi}{2},\phi)\, \Omega_L^{-1}(\tfrac{\pi}{2},\phi),$$

$$\Omega_U(\tfrac{\pi}{2},\phi)\, \Omega_L^{-1}(\tfrac{\pi}{2},\phi) = \Omega(\phi) \in H$$

- A loop in H
- Contractible in G

Inclusion homomorphism $\pi_1(H) \to \pi_1(G)$

Because $\Omega_U(\theta,\phi)$, $\Omega_L(\theta,\phi)$ are "contractions" of the loops on the equator

Homomorphism

$$\pi_2(G/H) \to \pi_1(H)/\pi_1(G)$$

well defined – representatives of equiv classes smoothly deformable to one another

Claim: it is an isomorphism: $\Big\{$ onto, trivial kernel

- Kandel u Kiviol
  that is, contractible loops in H means contractible surface in G/H

why? $\tilde{\Omega}^{-1}(\theta,\phi)\,\Omega_U(\theta,\phi) = \Omega'_U(\theta,\phi)$

for $\tilde{\Omega}^{-1} \in H$ also takes $\Phi \to \Phi_0$ (unitary gauge)

$\rightarrow \hat{\Omega}(\theta,\phi)$ contracts loop in H i.e.

$\tilde{\Omega}(\pi/2,\phi) = \Omega(\phi)$

that means $\Omega'_U = \Omega_L$ on equator $\theta = \pi/2$

i.e. $\quad \underbrace{\Omega(\theta,\phi)}\,\Phi(\theta,\phi) = \Phi_0$

$\therefore$ an S^2 in G

$\Phi(\theta,\phi) = \underbrace{\Omega(\theta,\phi)^{-1}}\,\Phi_0$

$\pi_2(G) = 0 \Rightarrow$ deform to point

✓

- Homo is onto
  for any loop in H contractible in G can be obtained on the equator for some $\Phi(\theta,\phi)$

$\Omega(\phi)$ = loop in H, contractible in G

$$\Omega_U(\pi/2,\phi) = \Omega(\phi) \qquad \Phi(\theta,\phi) = \begin{cases} \Omega_U^{-1}\Phi_0 & (U) \\ \Phi_0 & (L) \end{cases}$$

$$\Omega_L = I$$

Okay so now we know what top charge is
Relate to magnetic charge --

Finite energy $\Rightarrow$ $D_i \Phi = (\partial_i - ie A_i)\Phi = 0$
on $S^2(\infty)$

Go to unitary gauge in
U, L hemispheres
$\Phi = \Phi_0 \Rightarrow \partial_i \Phi = 0$

$[A_i, \Phi] = 0$

only gauge
potential in H is excited

A nonsingular in original gauge $\Rightarrow$
smooth at equator $\Rightarrow$
Afterward $A_{U,L}$ related by $\Omega(\phi)$ at
equator $\Rightarrow$ $\pi_1(H)/\pi_1(G)$

~ Magnetic Charge

We have a nonsingular
monopole — fields are
smooth in its core — how?

As we shrink S^2, A lives in G
$\Rightarrow$ g.t. on equator takes values
in G and loop becomes contractible

G gauge fields are excited in core!!

more generally:
exact sequence —

$$\to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \pi_{n-1}(G) -$$

Image of each map is kernel of the next.

E.g., for $\pi_n(G/H)$, cut n-sphere into hemispheres, and trivialize Φ on each. Obtain an H-valued gauge transformation on equatorial S^{n-1}.

Other generators of $\pi_n(G/H)$ due to nontrivial G-valued gauge transformation on S^n: these are trivial on equatorial S^{n-1}.

special case —

$$\pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G)$$

Hence, our earlier conclusion that $\pi_1(G) = 0$ implies

$$\pi_1(G/H) = \pi_0(H) / \pi_0(G)$$

Examples

$G = SU(3)$ Φ = octet (adjoint) irrep

i.e. Φ = hermitian traceless 3×3

what invariants in the Higgs potential?

$\mathrm{tr}\,\Phi^2$ $(\mathrm{tr}\,\Phi^2)^2$ $(\mathrm{tr}\,\Phi^4)$ Is it clear that's all?

How do we minimize it?

Diagonalize: eigenvalues $\lambda_1, \lambda_2 \quad -(\lambda_1+\lambda_2)$

$$\mathrm{tr}\,\Phi^2 = \lambda_1^2 + \lambda_2^2 + (\lambda_1+\lambda_2)^2 \qquad \text{Blah Blah}$$

Let's suppose

$$\langle \Phi \rangle = \Phi_0 = v\left(\tfrac{1}{2}, \tfrac{1}{2}, -1\right)$$

$G \to H$ ← degen eigenvalues

Locally: $H \sim SU(2)\times U(1)$
(same Lie algebra)

$\left(\begin{array}{c|c} SU(2) & \\ \hline & \end{array}\right)$ $\underbrace{\begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{2} & \\ & & -1 \end{pmatrix} = Q}_{\text{unbroken } U(1)}$

But... $e^{2\pi i Q} = \mathrm{diag}(-1,-1,1)$
— nontrivial element of center

$H = [SU(2)\times U(1)]/Z_2$ Also known as $U(2)$

What does this mean?
Fundamental (3) of $SU(3)$,
when decomposed $3 \to 2^{\frac{1}{2}} + 1^{-1}$

SU(2) $(\supset U(1))$

Charge can be $\frac{1}{2}e$, but only if
SU(2) 2-ality is nontrivial

As already discussed: minimal

$$e A \cdot dr = \underset{\sim}{P}\ \tfrac{1}{2}(1-\cos\theta)\,d\phi$$

Loop goes around U(1) halfway and SU(2) halfway $\Rightarrow$

$$g = \frac{2\pi}{e} \quad \text{where } e = \text{charge of a U(1) \underline{singlet}}$$

the minimal monopole has SU(2) (Z_2) and U(1) mag. charge

Dirac: $\exp(2\pi i \underset{\sim}{P}) = I$ where $\underset{\sim}{P}$ is SU(3) generator $\underset{\sim}{P} = (\tfrac{1}{2}, -\tfrac{1}{2}, 0) + (\tfrac{1}{2}, \tfrac{1}{2}, -1) = (1, 0, -$

<u>SU(5) Model</u>: A GUT

$$\Phi = \text{adjoint} \quad \langle \Phi \rangle = v\,\mathrm{diag}\left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$$G = SU(5) \longrightarrow H \sim SU(3) \times SU(2) \times U(1)$$

But $Q = \mathrm{diag}\left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)$

$$\exp(2\pi i Q) = \mathrm{diag}\left(e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}, -1, -1\right)$$

$$\in Z_3 \times Z_2 = Z_6$$

$$= \text{center of } SU(3) \times SU(2)$$

i.e. $H = [SU(3) \times SU(2) \times U(1)]/Z_6$

$$5_L \rightarrow \bar{3}_L^{\frac{1}{3}} + 2_L^{-\frac{1}{2}} = \bar{d} + \binom{e}{\nu}$$

$$\overline{10}_L \rightarrow (\bar{5} \times \bar{5})_{\text{anti sym}} = \left[(3^{-\frac{1}{3}} + 2^{\frac{1}{2}}) \times (3^{-\frac{1}{3}} + 2^{\frac{1}{2}})\right]_{\text{ant sym}}$$

$$= \bar{3}_L^{2/3} + (3,2)_L^{\frac{1}{6}} + \cancel{(1)}\ (1)_L^{1}$$

Thus: Q/e = integer for <u>trivial</u> $Z_3 \times Z_2$

The minimal monopole will be:

$g = 2\pi/e$: with Z_2 and Z_3 magnetic charge

$$e^{2\pi i \frac{P}{m}} = 1$$

$$\left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{1}{2}\right) + \left(-\tfrac{1}{3}, -\tfrac{1}{3}, +\tfrac{2}{3}, 0 0\right) + \left(0 0 0, \tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$$= (0\ 0\ 1\ 0\ -1)$$

integer eigenvalues

d_L does not see the string, because its charge $U(1)_{em} \in [SU(2) \times U(1)]/Z_2$ compensates for color —

Not much affected by the EW symmetry breaking ("Nor W/Z one")

[Tiny core $\sim 10^{-28}$ cm

Mass $\sim \frac{4\pi}{e^2}\mu \sim \frac{\mu}{\alpha_{GUT}} \gtrsim 10^{16}$ GeV

where the massive Higgs and SU(5) gauge bosons are excited

(e.g. bound on proton lifetime)!

$\Rightarrow$ Mass of bacterium

K.E. of charging Rhino!

What happens to the Z_3?

In a confining theory -- it is "screened"

But a monopole is a hadron

[Where are they??]

Lecture #17

We've seen that nonsingular topological solitons with magnetic charge always arise in

<u>semisimple</u> $G \to H \times U(1)$.

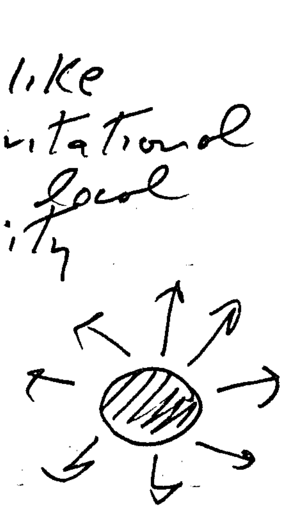
So (heavy!) where does L18 start? monopoles are a generic feature of grand unification.

Probably, the exactly stable particle spectrum of the world is:
- Graviton (massless)
- Photon (massless)
- Lightest neutrino (lightest fermion)
- Electron (lightest electrical charge)
- Magnetic Monopole (lightest magnetic charge)

— everything else (e.g. proton) ought to be able to decay!

Actually, even if one does not accept GUTs — there is another persuasive argument why magnetic monopoles should exist. In the classical theory of gravitation coupled to $U(1)_{em}$ (Einstein/Maxwell theory), there are magnetically charged black holes...

These Reissner-Nordström black holes have a singularity, where both curvature and the electromagnetic field strength diverge. But in GR (unlike Yang-Mills) the inertial/gravitational mass is not the integral of a local energy density. The singularity is shielded by a black hole horizon, and the mass is finite.



As finite-mass magnetic charges exist, and magnetic charge is exactly conserved, there is some lightest magnetically charged object, which must be stable.

The stable monopole could be an 't Hooft-Polyakov type object, slightly perturbed by gravitational corrections. But if none such exist, it is an extremal Reissner-Nordström black hole, which has vanishing surface gravity and therefore zero Hawking temperature (it does not radiate)

this has $$g^2/4\pi = G m^2 = m^2/m_{Pl}^2$$

or $$m^2 = \frac{1}{4\pi}\left(\frac{2\pi}{e}\right)^2 m_{Pl}^2 = \frac{1}{4}\left(\frac{4\pi}{e^2}\right) m_{Pl}^2$$

$$\Rightarrow m_{\text{monopole}} \sim \frac{1}{2\sqrt{\alpha_{em}}}\, m_{Pl} \sim 10^{20}\ \text{GeV}$$

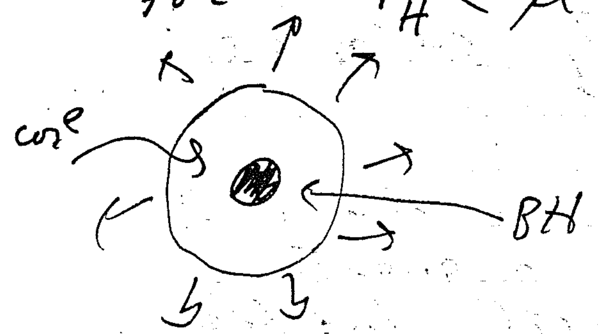
— for a monopole w/ Dirac charge

(At least for α_{em} small, the quantum gravity corrections to this relation need not be large — as the extremal black hole has horizon "radius"

$$r_H = GM = 2\frac{1}{\sqrt{\alpha_{em}}} L_{Pl},$$

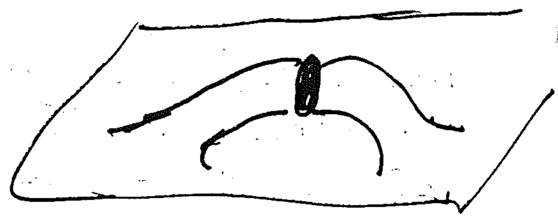
which is large compared to Planck size.

If there are 't H-P monopoles, then the RN solution actually becomes unstable for $r_H < \mu^{-1}$ (μ = vector mass)



then the classically stable solution is actually a black hole embedded inside the monopole core — Quantum mechanically, the BH radiates and disappears, leaving an 't H-P object behind.

The argument that any theory containing gravity and electromagnetism (GUT or not) must contain stable monopoles is not air-tight. It might be possible to consistently "truncate" them from the theory. The issue is — in a sufficiently energetic collision, what is the amplitude to pair create monopoles. If it is nonzero, no matter how small, monopoles exist. They need not exist if it is <u>exactly</u> zero.



And -- this question is related to the question: does quantum gravity allow changes in the topology of spacetime. In the spontaneous nucleation of a pair of extremal black holes, the horizons of the black holes are initially identified — so there is a wormhole in space. This "throat" subsequently collapses, producing a singularity.

If the spontaneous nucleation of a wormhole due to a quantum fluctuation is strictly forbidden, then perhaps monopoles need not exist (if there is no grand unification).

More examples: [Cf: 'tH-P objects — we can't truncate, since $M\bar{M}$ pair smoothly connected to vacuum

A monopole with Z_2 magnetic charge. Consider Higgs symmetry breaking

$$G = SU(3) \rightarrow H = SO(3)$$

How? 6-dimensional irrep — i.e. symmetric tensor Φ_{ab}

$\Omega \in SU(3)$: $\Phi \rightarrow \Omega \Phi \Omega^T$

then $\Phi_0 = v I \Rightarrow$

unbroken group

$$H = \{\Omega \in SU(3),\ \Omega\Omega^T = I\} = SO(3)$$

Since $SO(3) = SU(2)/Z_2 \Rightarrow \pi_1(H) = Z_2$,
we have

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G) = \pi_1(H) = Z_2$$

– there are topologically stable Z_2 monopoles.
Monopole and antimonopole are topologically equivalent – i.e. a pair of monopoles can annihilate

It's interesting to consider hierarchy of symmetry breakdown:

$$G \xrightarrow{v_1} H_1 \xrightarrow{v_2} H_2$$

where $v_2 \ll v_1$. E.g.,

$$G = SU(3) \longrightarrow H_1 = SO(3) \longrightarrow H_2 = U(1)$$

E.g. Higgs fields $(6) + (3)$

$$\langle \Phi_{ab} \rangle = v_1 \delta_{ab}\ ;\ \langle \Phi_a \rangle = v_2 \delta_{a3}$$

The second stage is just like the 't Hooft-Polyakov monopole, with Higgs triplet under $SO(3)$

There is associated magnetic monopole

$$\pi_2(H_1/H_2) = \frac{\pi_1(U(1))}{\pi_1(SO(3))}$$

These are light monopoles, $m \sim \frac{4\pi v_2}{e}$, corresponding to noncontractible loops in $U(1)$ that are contractible in $SO(3)$

But these are not the only monopoles. There are also noncontractible loops in $U(1)$ that remain noncontractible in $SO(3)$, but are of course contractible in $SU(3)$. These are heavy monopoles associated with 1st stage of breakdown

$$\pi_2(G/H_1) = \pi_1(SO(3)), \quad m \sim \frac{4\pi v_1}{e}$$

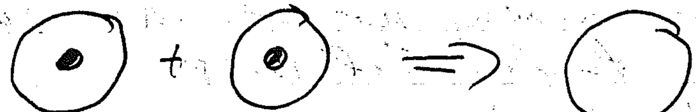
As the universe "cools" a Z_2 monopole chooses to become either a $g = 2\pi/e$ monopole or a $g = -\frac{2\pi}{e}$ monopole (it flips a coin).

Remember, in 'tH-P model, we had $g = 4\pi/e$, because there might have been matter in $SU(2)$ irreps of half integer spin. Here though we can have $g = 2\pi/e$, because all irreps of $SO(3)$ must be integer spin. That's because they are descended from underlying irreps of $SU(3)$, with $SO(3)$ embedded so that

$$(3) \text{ of } SU(3) \longrightarrow (3) \text{ of } SO(3)$$

(not equivalent to the embedding of $SU(2)$ in $SU(3)$ such that $(3) \to (2) + (1)$)

Note very heavy singly charged monopoles (each with $g = 2\pi/e$)



can be combined — their "inner cores" annihilate, giving the much lighter doubly charged ($g = 4\pi/e$) monopole. We can think of the doubly-charged monopole as a very deeply bound composite of two singly-charged monopoles --

Z_N Monopoles

What symmetry breaking pattern will yield a topologically stable Z_3 monopole?

Here's one way:

$$SO(8) \longrightarrow SU(3)/Z_3$$

$SU(3)/Z_3$ embedded in $SO(8)$ so that vector (8) of $SO(8)$ transforms as adjoint (8) of $SU(3)/Z_3$.

How? 3-index antisymmetric tensor of $SO(8)$, where

$$\langle \Phi_{abc} \rangle = v\, C_{abc}$$

— C_{abc} = $SU(3)$ structure constants.

Think of it this way -- T^a's are conventionally normalized $SU(3)$ generators in fundamental irrep

$$\mathrm{tr}\, T^a T^b = \tfrac{1}{2}\delta^{ab} \qquad [T^a, T^b] = i C^{abc} T^c$$

$$C^{abc} = (-2i)\, \mathrm{tr}[T^a, T^b] T^c$$

(totally antisymmetric)

The T^a's are basis for 8-dim space of 3×3 hermitian matrices

Realize $SO(8)$ as orthogonal transformations acting on this basis

$$T^a \to R^a{}_b T^b$$

R is real and $R^T R = I$

The adjoint irrep of SU(3), acting on the T^a's, is a special type of orthogonal transformation, an <u>inner</u> <u>automorphism</u> of the SU(3) Lie algebra

$$\Omega \in SU(3): \quad T^a \to \Omega T^a \Omega^{-1} = R^a{}_b(\Omega)\, T^b$$

"Automorphism" means that the structure constants are preserved:

$$\mathrm{tr}([T'^a, T'^b]\, T'^c) = \mathrm{tr}([T^a, T^b]\, T^c)$$

"Inner" means that the group itself induces the automorphism. In particular for Ω infinitesimal, $\Omega = I + i\varepsilon^b T^b$

$$T^a \to \Omega T^a \Omega^{-1} = (I + i\varepsilon^b T^b)\, T^a\, (I - i\varepsilon^b T^b)$$
$$= T^a + i\varepsilon^b [T^b, T^a]$$
$$= T^a + i\varepsilon^b\, i C^{bac}\, T^c$$
$$= T^a + C^{abc}\, \varepsilon^b\, T^c$$

– the generators are the structure constants.

Now, it's a (not so difficult) theorem, that the infinitesimal automorphisms of a semisimple Lie algebra are inner.
In fact the group Inn of inner automorphisms is a normal (invariant) subgroup of the group Aut of all automorphisms

[Note: Inn(G) = G/K, where K is center of G.]

— i.e. $a \in Aut$, $b \in Inn$

$$\Rightarrow \quad a b a^{-1} \in Inn$$

(i.e. b is a G group element, taken to another group element under conjugation by a)

the factor group $\frac{Aut}{Inn}$ = finite group

for any semisimple Lie group (and in fact it never has more than 3! elements — It is S_3 for G = Spin(8).)

Okay — so the point is — If

$$\langle \Phi^{abc} \rangle = \sigma C^{abc}$$

then the unbroken group is the subgroup of SO(8) such that

$$R^a_{a'} R^b_{b'} R^c_{c'} C^{a'b'c'} = C^{abc}$$

— i.e. these are the automorphisms of the SU(3) Lie algebra —
Hence, the unbroken group, at least locally, is $SU(3)/Z_3$

And hence it is precisely $SU(3)/Z_3$, as the only Lie groups that are locally isomorphic to SU(3) are SU(3) itself and $SU(3)/Z_3$

Since $SU(3)/Z_3$ has no double cover, the pattern is
$Spin(10) \rightarrow SU(3)/Z_3 \Rightarrow Z_3$ monopoles

Note that there is an obvious generalization

$$G = SO(N^2-1) \longrightarrow H = SU(N)/Z_N,$$

again, with a 3-index antisym tensor Higgs field, and

$$\langle \Phi^{abc} \rangle = v\, C^{abc}$$

where the C^{abc}'s are the $SU(N)$ structure constants. Here $N \geq 3$.
In this model, there are topologically stable solitons that carry Z_N magnetic charge.
(For N even, actually $Spin(N^2-1) \to SU(N)/Z_{N/2}$, so $\pi_2(G/H) = Z_{N/2}$ — i.e. $Z_{N/2}$ monopoles.)

The $SO(10)$ Model

is a realistic G.U.T. in which monopoles can have a heavy inner core as discussed above

$SO(10)$ has (double-valued) 16-dim spinor irrep, which decomposes under $SU(5)$ as

$$\left. 16 \longrightarrow 10 + \bar{5} + 1 \right] \text{ a generation + a right-handed neutrino}$$

(where $SU(5)$ embedded such that

$$10 \to 5 + \bar{5})$$

Consider breaking $SO(10)$ to standard model $[SU(3) \times SU(2) \times U(1)]/Z_6$ in two stages

First stage $\quad \underline{\Phi}_{ab} = (54) =$ traceless symmetric tensor

$$\underline{\Phi} \to \Omega \underline{\Phi} \Omega^T$$

for $\Omega \in SO(10)$

Suppose $\quad \langle \underline{\Phi} \rangle = \underline{\Phi}_0 = v \cdot \mathrm{diag}(\overbrace{2, - 2}^{6 \text{ times}}, \underbrace{-3, - -3}_{4 \text{ times}})$

The unbroken group is locally

$$SO(6) \times SO(4) \sim SU(4) \times SU(2) \times SU(2),$$

but what is its global structure

To see how $SU(4) \times SU(2) \times SU(2)$ embeds in $Spin(10)$, note that

$$16 \to (4, 1, 2) + (\bar{4}, 2, 1)$$

Since the spinor represents simply connected $Spin(10)$ faithfully — and

$$(-I_4, -I_2, -I_2) \quad \text{acts trivial on } (16),$$

we see pattern is

$$G = Spin(10) \xrightarrow{v_1} [SU(4) \times SU(2) \times SU(2)]/Z_2 = H$$

and $\quad \pi_2(G/H) = \pi_1(H)/\pi_1(G) = \pi_1(H) = Z_2$

— there are Z_2 monopoles

there is a noncontractible loop in H, which winds in each of $SU(4)$, $SU(2)$, $SU(2)$ — that is contractible in G

Now the next stage --
suppose Higgs $(16) \supset (4,1,2)$ gets a vev
(the "RH neutrino" component, with no standard model q. nos.)
This is the "Pati-Salam" pattern of symmetry breaking:

$$H_1 \quad SU(4)\times SU(2)_L \times SU(2)_R / \mathbb{Z}_2$$
$$\xrightarrow[v_2]{} [SU(3)\times SU(2)_L \times U(1)_Y]/\mathbb{Z}_6 \text{ st}$$

The $(4,1,2)$ multiplet looks like

$$SU(4)\left\{\begin{bmatrix} u_1 & d_1 \\ u_2 & d_2 \\ u_3 & d_3 \\ \nu & e \end{bmatrix}_R\right. \qquad \underbrace{\qquad}_{SU(2)_R}$$

The unbroken $U(1)_Y$ is linear combination of broken $SU(4)$ and broken $SU(2)_R$

$SU(4)$ generator: $\qquad$ $SU(2)_R$ gen: $\qquad$ $U(1)_Y$ generator

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \\ 0 & -1 \end{bmatrix}$$

We look at how $16 = (4,1,2) + (\bar{4},2,1)$ transforms: since it is standard model $+ (1,1)^0$
Group is $\quad H_2 = [SU(3)\times SU(2)_L \times U(1)_Y]/\mathbb{Z}_6$

So -- the monopole charge spectrum is as in SU(5) model -- But the minimal uncontractible loop in H_2 is also uncontractible in H_1

So the doubly charged monopole, $m \sim \frac{4\pi}{e} v_2$, can be much lighter than singly charged, $m \sim \frac{4\pi}{e} v_1$