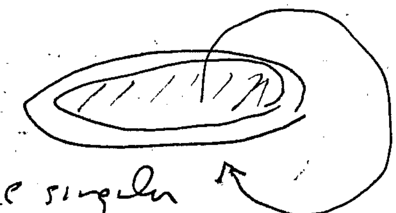
Theory of Magnetic Cheshire Charge



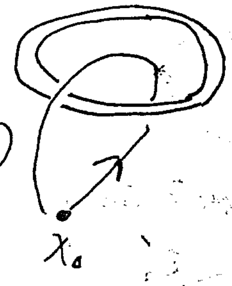
$$\Omega_0 Q \Omega_0^{-1} = -Q$$

⇒ Magnetic charge must flip

⇒ charge is transferred to string

In the singular gauge w/ cut

Magnetic field lines "originating" at the surface ⇒ Magnetic Cheshire charge

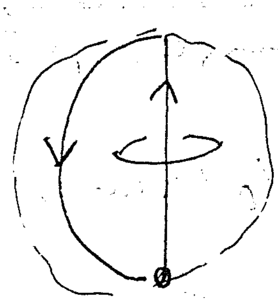


Magnetic charge of a string

$$H_{con} \ni h(C, x_0)\, h^{-1}(C_0, x_0)$$

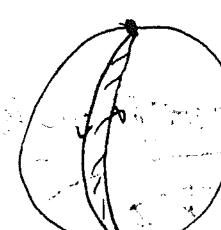
As loop sweeps around string, a closed loop in H_{con}

- on the other hand - describes how the flux trapped in the core of the string "twists" as we follow the string

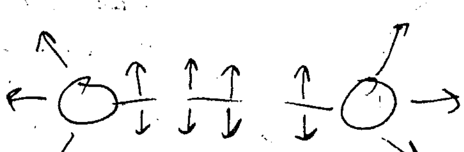


Define the family of paths to ones that share the axis of the string loop

then $C C_0^{-1}$ is



Goes from south to north pole along line of longitude, returns along a different line of longitude

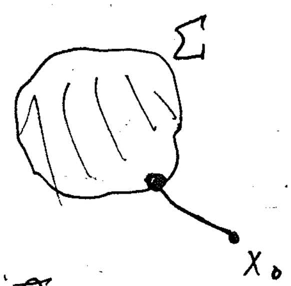


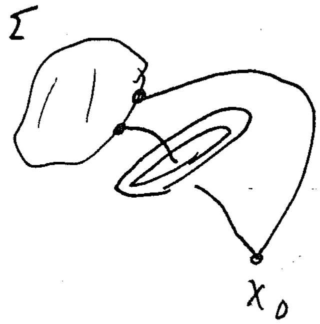
the Higgs field for a slice through the string, if the string has a twist: it is a Hedgehog:

this monopole has an "elongated" core

The basepoint is important!

Assign magnetic charge to Σ. Specify how it is tied to basepoint

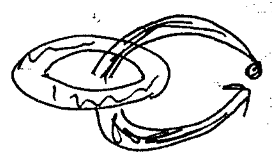


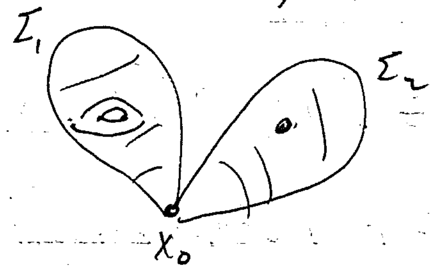


There may be inequivalent ways to do this!

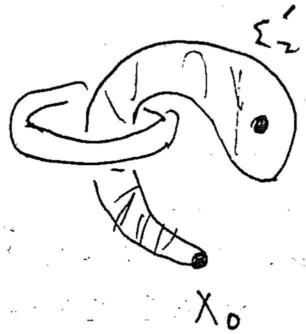
Two surfaces with same basepoint differ by an element of $\pi_1(M, x_0)$

M = space w/ strings and monopoles removed

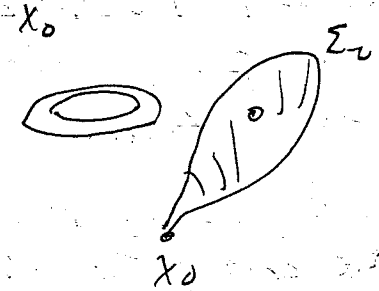




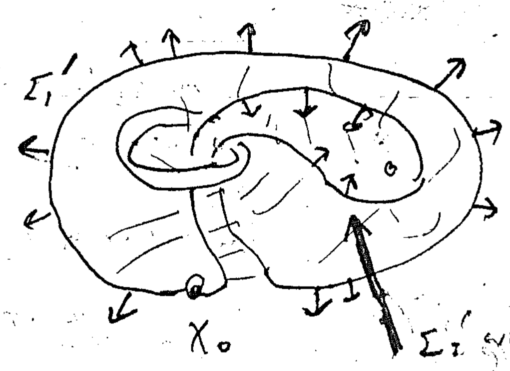
~~Pull~~ Pull monopole through



this surface $\Rightarrow$

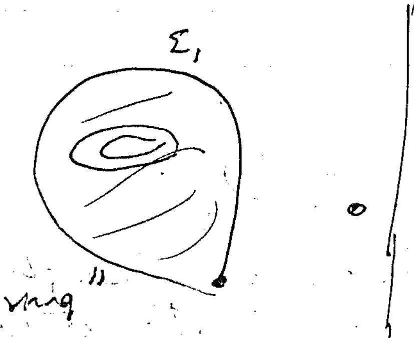


What happens? element $a \in \pi_1(H_{un})$, The magnetic charge, gets conjugated by element $h \in \pi_0(H: \ )$, the string flux



Σ_2' with "wrong" orientation

$\Rightarrow$



elements of $\pi_2(M,$ (which is commutative)

$\Sigma_1' = \Sigma_1 \circ \Sigma_2 \circ (\Sigma_2')^{-1}$

i.e. magnetic charge conservation:

$$(a_1\, a_2) \to (a_1 a_2 (a_2')^{-1},\ a_2')$$

where $a_{1,2}$ are elements of $\pi_1(H_{conn})$

$$a_2' = h a_2 h^{-1}$$

$$h \in \pi_0(H)$$

$g = g_{Dirac} \cdot m \Rightarrow$

$$|m_1, m_2\rangle \to |m_1 + 2m_2, -m_2\rangle$$

It's remarkable — Winding a magnetic monopole through a loop can "twist" the loop!!

The same applies even if symmetries are global — e.g. in a nematic LC.

(You can't say when the "charge transfer" happens, but it definitely does!)

Strings ending on Monopoles

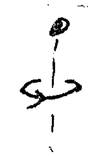
Consider the hierarchy of symmetry breakdown:

$$G \xrightarrow{v_1} H_1 \xrightarrow{v_2} H_2$$

Monopoles in the first stage associated with

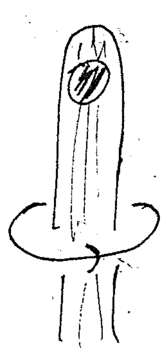
$\pi_1(H_1)/\pi_1(G)$] Its magnetic charge is associated with a noncontractible loop

in H_1 — the effect of transport around the Dirac string of the monopole



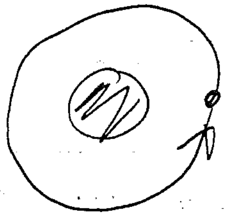
the loop must be contractible in G, so that the Dirac string can terminate at a nonsingular monopole core in which the heavy G gauge fields are excited.

What happens at the next stage of symmetry breaking? Can the monopole survive? Only if the loop in H_1 can be deformed to lie in H_2. What if that is not the case?



If a loop in H_1 cannot be deformed to a loop in H_2 — that is precisely the criterion for a topologically stable string to arise from the breakdown

$$H_1 \xrightarrow{v_2} H_2$$



i.e. outside the core of the string, Higgs order parameter winds in H_1/H_2, represented by $g(\theta)\Phi_0$, $g(\theta) \in H_1$

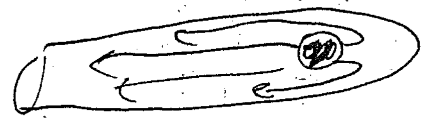
this path is contractible in H_1/H_2 iff $g(\theta)$ can be deformed to lie in H_2

So what happens is — the G/H_1 monopole becomes the endpoint of the H_1/H_2 string!

<u>Examples</u>

(1) $SU(2) \xrightarrow{v_1} U(1) \xrightarrow{v_2} I$

At the first stage there are 't Hooft Polyakov monopoles, at the second there are Nielsen-Olesen (abelian Higgs) strings. The vacuum becomes a superconductor at the second stage. The Meissner effect kicks in, and

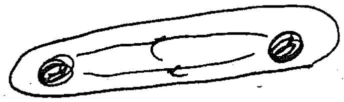


The flux emanating from monopole collapses to a tube.

Overall, the pattern of symmetry breaking is $SU(2) \rightarrow I$, so that

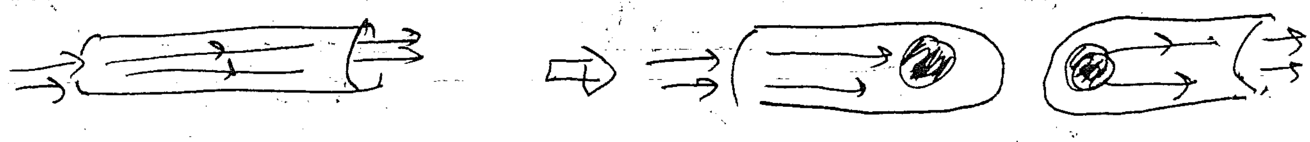
$$\pi_2(G/H_2) = \pi_1(G/H_2) = 0$$

and there are neither topologically stable strings or monopoles. What we have instead are "mesons"



Monopoles and antimonopoles are bound together by flux tubes – i.e. monopoles are confined

For $v_1 \gg v_2$, strings are metastable



A string can break via nucleation of monopole antimonopole pair. But the decay rate per unit time and length is

$$\Gamma/L \sim \exp(-\pi m^2/K)$$

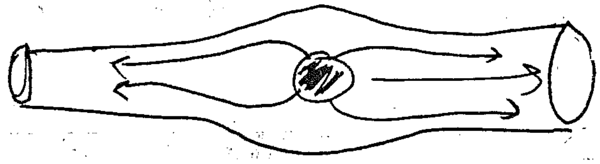
The exponential suppression arises because there is a barrier to penetrate of height $2m$ (m = monopole mass) and width $2m/K$ (where K is the string tension) that is, monopoles must be distance $l = 2m/K$ apart for the savings in string energy Kl to compensate for the energy cost $2m$ of creating the pair.

(2) $\quad SU(2) \xrightarrow{v_1} U(1) \xrightarrow{v_2} Z_2$

Again, in the first stage, an 't Hooft-Polyakov monopole arises with $g = 4\pi/e$.

In the second stage, an $SU(2)$ triplet

condenses, with charge e. Hence the flux quantum in the second stage is $2\pi/e$, half the charge of a monopole



Hence the flux of the monopole collapses to two strings

The monopole is a "bead" that can slide along the string.

The overall pattern of symmetry breaking is

$$G = SU(2) \rightarrow H_2 = Z_2,$$

so there are stable strings $\pi_1(G/H_2) = Z_2$

As noted some time ago, the Z_2 classification means that the vortex and antivortex are topologically equivalent, but not necessarily dynamically equivalent. That is, vortex can be deformed to antivortex as energy remains finite, but there is an energy barrier to surpass. The monopole core is the cost of crossing the barrier, which is large for $v_1/v_2 >> 1$.

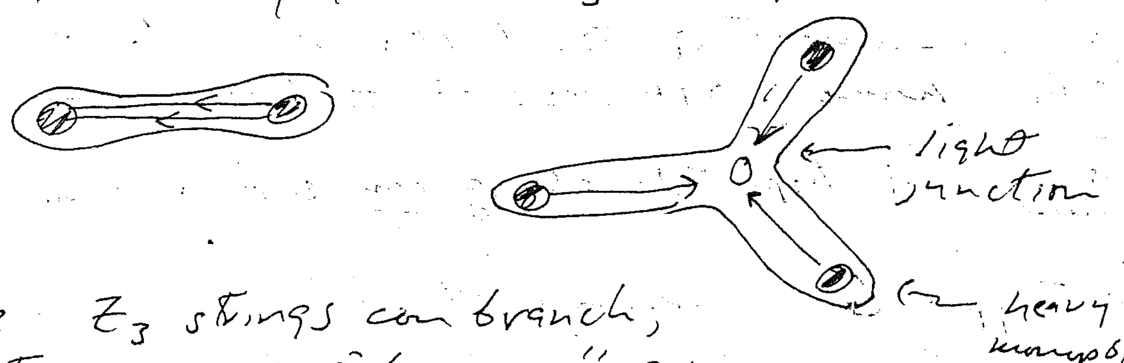
Note that if the superconductor is "type I," it is energetically favored for

the two strings to collapse to one with twice the flux.

(3) $\quad SO(8) \to SU(3)/Z_3 \to I$

The first stage is that studied earlier, giving rise to Z_3 monopoles. In the second stage, $SU(3)/Z_3$ is broken, so there are Z_3 flux tubes.

This example is related to QCD by a sort of electric magnetic duality (except there is no conserved baryon number). Monopoles have Z_3 quantum numbers like the triality of quarks, and are confined by the Z_3 flux tubes



The Z_3 strings can branch, so there are "baryons" as well as meson. But baryons, like mesons, can annihilate into vacuum.

Back to the general theory. Suppose symmetry breaking pattern

$$H_1 \xrightarrow{v_2} H_2$$

occurs, with $\pi_1(H_1/H_2) \neq 0$, and there are topologically stable strings

Can any such string be "destabilized" by introducing some higher scale breakdown $G \xrightarrow{v_1} H_1$?

NO! There are two kinds of strings, one type can be destabilized, the other cannot be

Consider exact sequence

$$\pi_1(H_1) \longrightarrow \pi_1(H_1/H_2) \rightarrow \pi_0(H_2) \rightarrow \pi_0(H_1)$$

string is associated with element of $\pi_1(H_1/H_2)$

- Type-U (can be unstable):

For some choice of H_1 such that matter is in single valued reps.

element of $\pi_1(H_1/H_2)$ is in kernel of homomorphism: $\pi_1(H_1/H_2) \Rightarrow \pi_0(H_2)$

A type-U string can end on a monopole

- Type-S (always stable)

For any choice of H_1 such that matter is in single-valued reps

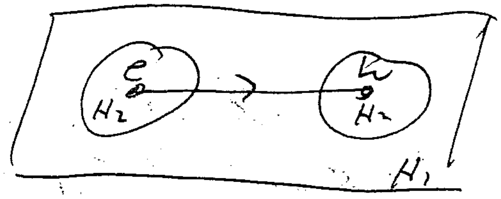
element of $\pi_1(H_1/H_2)$ is not in the kernel (mapped to nontrivial element of $\pi_0(H_2)$).

A type-S string cannot possibly end on a monopole.

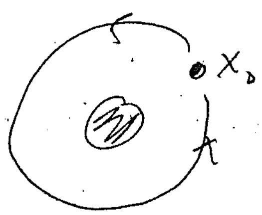
For a type-U string, element of $\pi_1(H_1/H_2)$ is image of element of $\pi_1(H_1)$. The Higgs field outside the core is

$g(\theta)\Phi_0$, where $g(\theta)$ is a closed loop in H_1. We can embed H_1 in simply connected G. Then $\pi_2(G/H_1)$ monopole is endpoint of $\pi_1(H_1/H_2)$ string

But a type-S string is not associated with a closed loop in H_1 — rather an open path in H_1 whose endpoints are in distinct connected components of H_2

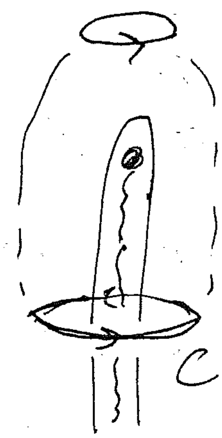


For this type of string, the effect of gauge transport around the string is nontrivial that is,



$$\underline{P}\exp\left(i\int_{C,x_0} A\right) = h \neq e$$

So the string has nontrivial Aharonov-Bohm interactions with particles in H_2 representations. This Aharonov-Bohm effect is associated with an element of H_2 not connected to the identity, so it cannot smoothly turn off if we slip the loop C off the end of the string. Therefore, the string cannot end!



Incidentally, if the string is type-U, a possible way for the string to end is on a black hole that carries the $\pi_1(H_i)$ magnetic flux (in that case, we don't need to introduce a higher scale of Higgs symmetry breakdown).

Another way to understand the type-U / type-S classification is to think about...

Walls ending on strings

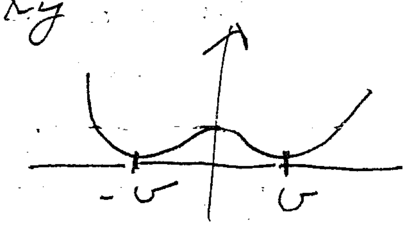
First consider walls. These are associated with spontaneously broken discrete (global) symmetries.

E.g. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)$

ϕ is a real scalar field, and $V(\phi) = V(-\phi)$ so $\phi \to -\phi$ is a Z_2 symmetry

Spontaneously broken if

$\langle \phi \rangle = v \neq 0$



In one spatial dimension, there is a topological conservation law...

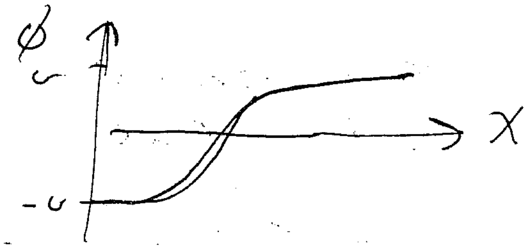
Finite energy requires

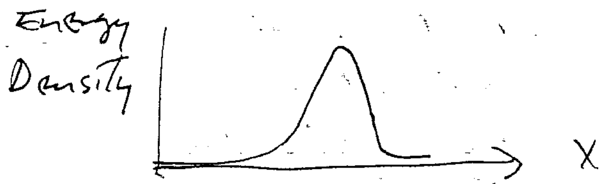
$$\phi(x) \longrightarrow \pm v \quad \text{as } x \to \pm\infty$$

A configuration with, say $\phi(-\infty) = -v$
$\phi(\infty) = v$
cannot be deformed, while energy remains finite, to vacuum, which has $\phi(\infty) = \phi(-\infty)$

Minimizing the energy in this sector, we find a localized "kink" solution

Inside the kink, ϕ must cross from negative to positive value, which costs both potential energy and gradient energy



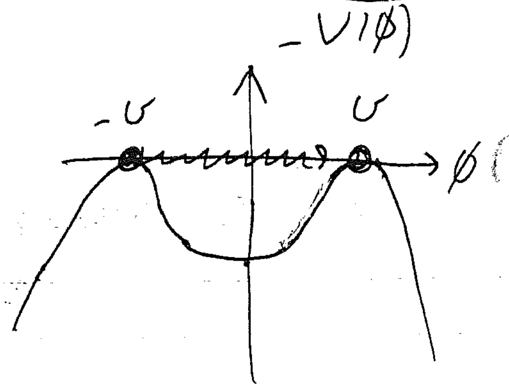


Find the static kink solution by minimizing the energy

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi) \right]$$

This is just like the action for a particle in potential $-V(\phi)$, parametrized by "time" x. A first integral of Euler-Lagrange equation is "energy conservation" equation

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - V(\phi) = \text{constant}$$



If $V(\phi)=0$ at its minimum, then the constant is zero, and

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = V(\phi)$$

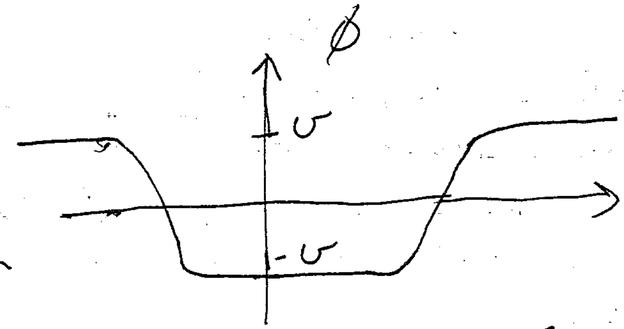
the minimum energy is

$$E_{min} = \int_{-\infty}^{\infty} dx\,\big(2V(\phi)\big) = \int_{-v}^{v} d\phi \,\frac{2V(\phi)}{d\phi/dx}$$

$$= \int_{-v}^{v} d\phi \sqrt{2V(\phi)}$$

This expression is the mass of the kink.

Note that a kink and "anti-kink" can be patched together – these can annihilate to vacuum



In d spatial dimensions, the kink becomes a "domain wall" of codimension one (dimension $d-1$) that separates regions of $\phi=v$ vacuum from $\phi=-v$ vacuum.

Now consider a hierarchy of symmetry breakdown:

$$H_1 \xrightarrow{v_2} H_2 \xrightarrow{v_3} H_3$$

where strings arise at the first stage: $\pi_1(H_1/H_2) \neq 0$

and walls at the second

$$\pi_0(H_2/H_3) \neq 0$$

In the second stage, there are degenerate vacua $h\Phi_0 = \Phi_0$ for $h \in H_3$

cosets in $H_2/H_3 = \{g\Phi_0,\ g \in H_2\}$

label these states, and a jump from one connected component of H_2/H_3 to another requires a wall

A string of S-type has the property that transport around the string induces

$H_2 \ni h = P\exp(i\int A) \neq e$

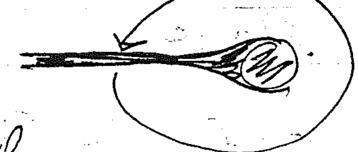
If $h \notin H_3$, then the order parameter must jump as the string is circumnavigated

$\Leftarrow$ Walls end on strings

Example:

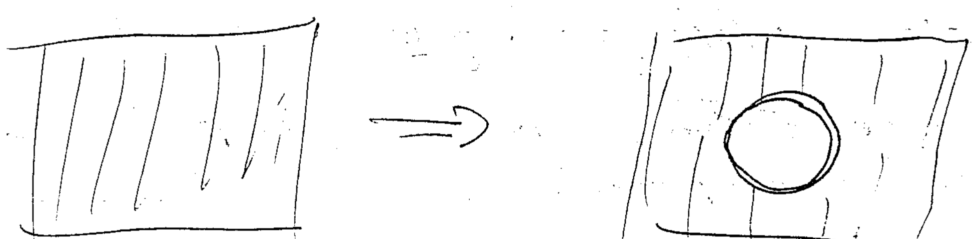
(1) $U(1) \xrightarrow{v_2} Z_2 \xrightarrow{v_3} I$

In the first stage, charge-(2e) field condenses, string carries flux $2\pi/2e$, and charge e particle acquires phase -1 when it circles the string. In the second stage, charge-(e) field condenses.



If $D\phi = 0$ for the charge-0 field, then $\phi \to -\phi$ as it circles the string. Hence ϕ condensate must jump at a wall.

For $v_3 \ll v_2$, domain wall is metastable

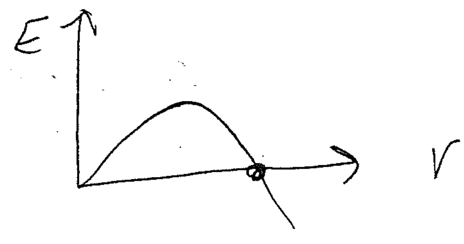


A loop of string can nucleate – a hole in the wall – and subsequently grow. Occurs at rate

$$\Gamma/\text{Area} \simeq \exp\left[-\frac{16\pi}{3} K^3/\sigma^2\right]$$

Crudely, energy of string loop of radius r is

$$E(r) = 2\pi r K - \pi r^2 \sigma$$



Width $r = 2K/\sigma$

Height $= \pi K^2/\sigma$

width·ht $\sim K^3/\sigma^2$

Three ways to characterize Type-S string:

(1) There are charges that have nontrivial Aharonov-Bohm interactions with the string — i.e.

$$P\exp\left(i\int_{C,x_0} A\right) = h \neq e$$

where h acts nontrivially on reps of unbroken gauge group

(2) The string cannot end, irrespective of short distance physics.

(3) For an appropriate symmetry breaking at lower scale, the string becomes boundary of wall.

(3) Follows from (1) — particle with nontrivial Aharonov-Bohm interaction could condense.

And (2) follows from (3). If the string could be boundary of a wall, then the string cannot end — "the boundary of a boundary is zero"

Topological Classification of Gauge Fields

Lecture #19

We have seen that $\pi_1(H)$ provides a complete topological classification of H gauge fields on S^2 — if two H connections have the same topological magnetic flux (element of $\pi_1(H)$), then one can be smoothly deformed to the other.

It is interesting, and sometimes important, to extend this classification to other manifolds, including higher dimensional manifolds.

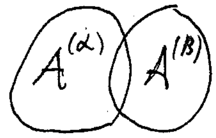
So lets start out with an easier case, $H = U(1)$. A gauge field — i.e. a notion of parallel transport on the manifold M that assigns an element of H to each closed path on M is called an "H vector bundle" on M. In the special case $H = U(1)$, it is also called a complex line bundle. In fact there is a pretty simple topological classification of $U(1)$ bundles on M.

In particular, consider the case where M is an orientable two-dimensional closed manifold (Riemann surface). Then we'll be able to show that the bundle is <u>completely</u> classified by the topological invariant

$$n = \frac{1}{2\pi}\int_M F$$

— just the total magnetic flux through the surface in Dirac units (I've set the gauge coupling $e=1$ — i.e. have absorbed e into the normalization of F.) The two form $F/2\pi$, whose integral over the manifold has an integer value that completely classifies the bundle, is called the "first Chern class" of the bundle.

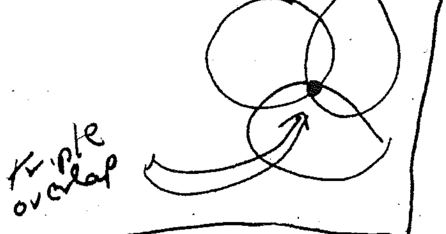
To see how to arrive at a classification of bundles, where M is two dimensional, we start by covering the manifold with overlapping contractible patches. We assign a gauge field to each patch (which can be smoothly deformed to $A=0$, since the patch is contractible).



Note ω defined modulo $2\pi \cdot n$, a point to which we will return

Where two neighboring patches overlap, they are related by a gauge transformation — e.g. $e^{i\omega_{\alpha\beta}} \in U(1)$

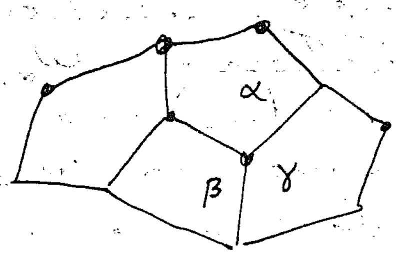
in the overlap of patch α with patch β. We will choose our cover so all double overlaps are contractible.



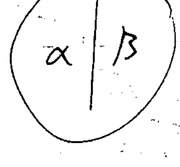
There are also regions where three patches overlap, and we will choose each triple overlap to be contractible as well.

Finally, we can choose the cover so that there are no quadruple overlaps. (We'll take it for granted that such a cover exists.)

We can deform our cover so that double overlaps have negligible thickness — we then obtain a tesselation of the manifold by polygons, where each vertex has coordination number 3. The gauge transformation $e^{i\omega_{\alpha\beta}}$ becomes the "transition function"



that relates

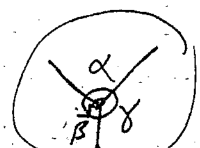


$A^{(\alpha)}$ on one side of the boundary $\alpha\beta$ with $A^{(\beta)}$ on the other

Two bundles with the same transition functions are obviously topologically equivalent. The difference of the two has trivial transition functions so that A is "globally defined" — it can be deformed to zero in each patch. [e.g. $A \to \lambda A$.] Hence the topology of the bundle is completely characterized by the transition functions.

Note that the transition functions have an orientation — i.e. $\omega_{\beta\alpha} = -\omega_{\alpha\beta}$

If $e^{i\omega_{\alpha\beta}}$ is gauge transformation when we step from α to β, $e^{i\omega_{\beta\alpha}} = e^{-i\omega_{\alpha\beta}}$ is transform when we step from β to α (α to β and back again must be the identity). There is



a similar condition at triple intersections

$$e^{i\omega_{\alpha\beta}} e^{i\omega_{\beta\gamma}} e^{i\omega_{\gamma\alpha}} = I$$

— where α, β, γ traversed at the triple intersection in counterclockwise sense

or $\omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma} \in \mathbb{Z}$
— a "Dirac string" at the triple intersection. (Note the compressed notation — $\omega_{\alpha\beta}$ is a function (real valued) defined on a line segment. In the above expression $\omega_{\alpha\beta}, \omega_{\beta\gamma}, \omega_{\gamma\alpha}$ are values of the function at the endpoints of the segment, the triple intersection)

Now an assignment of a real-valued function with $\omega_{\beta\alpha} = -\omega_{\alpha\beta}$ to each double intersection is called a (real-valued) 1-cochain, and the assignment of a value to each triple intersection is called a two-cochain. The operation

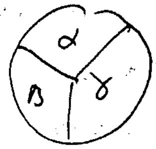
$$\delta : \omega \to \delta\omega = \omega_{\alpha\beta} + \omega_{\beta\gamma} + \omega_{\gamma\alpha} \quad \text{at } \alpha\beta\gamma \text{ triple intersection}$$

is called the coboundary operator: it takes a one-cochain to a two-cochain. An obvious generalization of δ takes a k-cochain to a $(k+1)$-cochain.

In the case we are considering, $\omega_{\alpha\beta}$ is actually a *function* on the double intersection. Still we can use the property $\delta^2 = 0$ to conclude $\delta(\delta\omega) = 0$.

That is, for a general manifold of dimension n, we tesselate it by n-cells which have $(n-1)$-dim double intersections, the $(n-1)$-cells have $(n-2)$ dimensional triple intersections etc. Since k-cochains have an orientation, we find a $(k+1)$-chain by summing $(k+2)$ k-chain values, weighted by signs.

The coboundary has the property $\delta^2=0$



E.g. applied to a 0-cochain l_α

$(\delta l)_{\alpha\beta} = \omega_{\alpha\beta} = l_\alpha - l_\beta$

$(\delta l)_{\beta\gamma} = \omega_{\beta\gamma} = l_\beta - l_\gamma \Rightarrow \delta\omega_{\alpha\beta\gamma} = 0$

$(\delta l)_{\gamma\alpha} = \omega_{\gamma\alpha} = l_\gamma - l_\alpha$

(In two dimensions $(\delta n)_{\alpha\beta\gamma\delta} = 0$ is trivially true, since there are no quadruple intersections)

In any dimension we can extract from the transition functions of a $U(1)$ bundle an integer-valued 2-cocycle i.e. a 2-cochain with vanishing coboundary

Now, though $n = \delta\omega$, and so is the coboundary of a real-valued 1-cochain, it is not necessarily the co-boundary of an integer-valued co-chain.

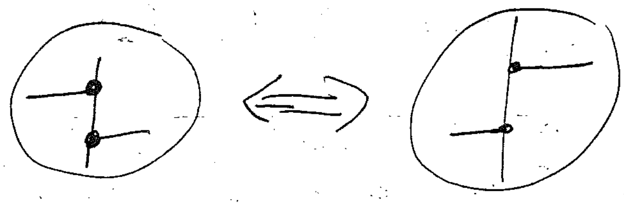
The pth cohomology group of M with integer coefficients is

$$H^p(M,\mathbb{Z}) = \frac{\text{(integer-valued } p\text{-cocycles)}}{\text{(co-boundaries of int-valued } (p-1)\text{-chains)}}$$

- i.e. elements of $H^p(M,\mathbb{Z})$ are equivalence classes of p-cocycles, where two belong to the same class if they differ by a coboundary. The classes have an additive group structure.

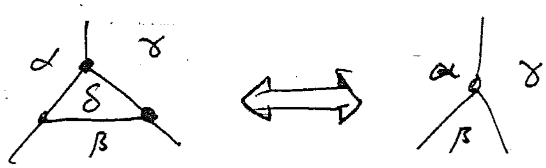
An important fact is that $H^2(M,\mathbb{Z})$ is a property of the manifold, and does not depend on the tessellation. We can convert one tessellation to another with two basic moves (and their inverses)

slide one triple intersection past another:



In a 1-cocycle, integer assigned to "internal line" is determined by external line — sliding can't change the equivalence classes $\Rightarrow$ $H^1(M,\mathbb{Z})$ unchanged

shrink a 2-cell:



Assignments to the edges of δ are determined by external lines up to a constant, i.e. coboundary of a 0-cycle that vanishes outside δ — and we can't shrink δ unless its edges are determined up to coboundary -- so $H^1(M,\mathbb{Z})$ unaffected

Similar argument for $H^p(M,\mathbb{Z})$

We can think of the triple intersections with $n_{\alpha\beta\gamma} \neq 0$ as the locations where a Dirac string pierces the manifold. The freedom to alter $n_{\alpha\beta\gamma}$ by the coboundary of a 1-cochain is freedom to perform a "large" gauge transformation that moves the

string
$$\omega_{\alpha\beta} \to \omega'_{\alpha\beta} = \omega_{\alpha\beta} + 2\pi K_{\alpha\beta}$$
($K_{\alpha\beta}$ = integer), does not alter $e^{i\omega_{\alpha\beta}}$ but changes $n_{\alpha\beta\gamma}$ by

$$n_{\alpha\beta\gamma} \to n'_{\alpha\beta\gamma} = n_{\alpha\beta\gamma} + (\delta K)_{\alpha\beta\gamma}$$

Altering $n_{\alpha\beta\gamma}$ by a coboundary corresponds to moving a string ●—○ ⇒ ○—● from one position to a neighboring position, creating ○—○ ⇒ ⊖—⊕ a string antistring pair or annihilating a pair ⊖—⊕ ⇒ ○—○

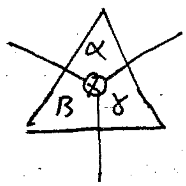
So, given transition functions, there is an unambiguous way of assigning an element of $H^2(M,\mathbb{Z})$ to the bundle. Since this is a discrete quantity, it is a topological invariant of the bundle. Therefore, two $U(1)$ bundles that are associated with distinct elements of $H^2(M,\mathbb{Z})$ surely are not topologically equivalent

To show that $H^2(M,\mathbb{Z})$ classifies the bundles, it remains to show that

- There is a bundle corresponding to each element of $H^2(M,\mathbb{Z})$.

- Bundles corresponding to the same class in $H^2(M,\mathbb{Z})$ are topologically equivalent.

First, let's show that, given a 2-cocycle (on two-manifold), there is a U(1) bundle with Dirac strings given by the $n_{\alpha\beta\gamma}$'s

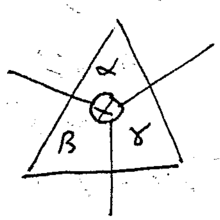


We consider the triangulation dual to the tesselation (each site with coordination number 3 $\Rightarrow$ a triangle)

If the triple intersection has flux $n_{\alpha\beta\gamma}$ (in Dirac units) introduce equal and opposite flux on the dual 2-simplex.

then we can choose $A = 0$ at the boundary of each 2 simplex, and there is not problem gluing the 2-simplices together, defining A on the whole 2-manifold

We can rephrase this in the language of matching conditions (transition functions):



choose $\omega_{\beta\gamma} = \omega_{\gamma\alpha} = 0$ inside the triangle

$\omega_{\alpha\beta} = 0$ at boundary of triangle

$\omega_{\alpha\beta} = 2\pi n_{\alpha\beta\gamma}$ at triple intersection

Then since $\omega = 0$ on all double intersections at the boundaries of all Δ's, we can glue Δ's together.

Next, we want to see that two bundles both associated with the same class in $H^2(M,\mathbb{Z})$ are equivalent topologically. The difference of the bundles has $n_{\alpha\beta\gamma}$ = coboundary, so we can choose $n_{\alpha\beta\gamma}=0$ by choosing the appropriate "singular gauge". Thus

$$\omega_{\alpha\beta}+\omega_{\beta\gamma}+\omega_{\gamma\alpha}=0$$

at each triple intersection. Therefore, there is no obstruction to deforming each $\omega_{\alpha\beta}$ so that it vanishes at both its endpoints, and everywhere in between. Thus the difference of the bundles has vanishing transition functions — they are topologically equivalent.

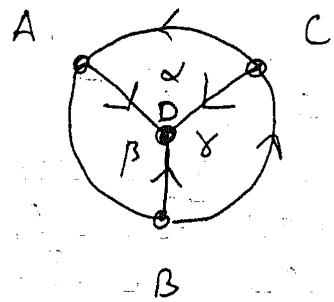
This classification of a $U(1)$-bundle by $H^2(M,\mathbb{Z})$ applies for both orientable and nonorientable manifolds. But for an orientable two-manifold, it can be related to the magnetic flux through the surface.

A class can be characterized by

$$n=-\sum n_{\alpha\beta\gamma}$$

[i.e. $H^2(M,\mathbb{Z})=\mathbb{Z}$ for orientable Riemann surface]

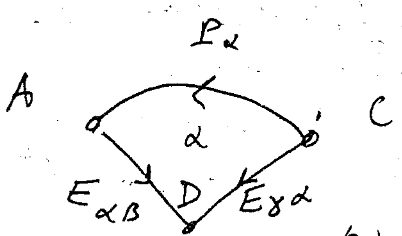
(summed over all triple intersections). E.g. we have the freedom to move all of the strings to a single site, where n will be the total flux. Furthermore



Consider the total flux in regions $\alpha + \beta + \gamma$

$$= \int F^{(\alpha)} + \int F^{(\beta)} + \int F^{(\gamma)}$$

$$= \oint_{C_\alpha} A^{(\alpha)} + \oint_{C_\beta} A^{(\beta)} + \oint_{C_\gamma} A^{(\gamma)}$$



$$A^{(\alpha)} - A^{(\beta)} = -d\omega_{\alpha\beta}$$
$$A^{(\beta)} - A^{(\gamma)} = -d\omega_{\beta\gamma}$$
$$A^{(\gamma)} - A^{(\alpha)} = -d\omega_{\gamma\alpha}$$

That is $\psi^{(\beta)} = e^{i\omega_{\alpha\beta}}\psi^{(\alpha)}$

$$\Rightarrow A^{(\beta)} = A^{(\alpha)} + d\omega^{\alpha\beta}$$

$$= \int_{P_\alpha} A^{(\alpha)} + \int_{P_\beta} A^{(\beta)} + \int_{P_\gamma} A^{(\gamma)}$$

$$+ \int_{E_{\alpha\beta}} \left(A^{(\alpha)} - A^{(\beta)}\right) + \int_{E_{\beta\gamma}} \left(A^{(\beta)} - A^{(\gamma)}\right) + \int_{E_{\gamma\alpha}} \left(A^{(\gamma)} - A^{(\alpha)}\right)$$

$$= \int_{P_\alpha} A^{(\alpha)} + \int_{P_\beta} A^{(\beta)} + \int_{P_\gamma} A^{(\gamma)}$$

$$- \omega_{\alpha\beta}\Big|_A^D - \omega_{\beta\gamma}\Big|_B^D - \omega_{\gamma\alpha}\Big|_C^D$$

$$= -2\pi n_{\alpha\beta\gamma} + ---$$

After we integrate over all patches, only the contributions from the triple intersections remain — and so

$$\frac{1}{2\pi}\int F = -\sum n_{\alpha\beta\gamma} = n$$

The magnetic flux through strings "returns" thru surface

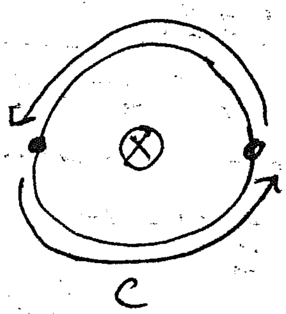
— the 1st Chern class provides a complete classification.

The nonorientable case is different. The sign of F is ambiguous, because a positive magnetic field becomes a negative magnetic field upon traversing an orientation reversing loop. A Dirac string can likewise become an antistring. Hence the only invariant notion is whether the number of strings is even or odd:

$$H^2(M,\mathbb{Z}) = \mathbb{Z}_2 \text{ for } \underline{\text{nonorientable}} \text{ closed Riemann surface}$$

In the nonorientable case, the class cannot be expressed as an integral of the field strength (in fact, the sign of F is ill defined, since an "outward pointing" magnetic field becomes inward pointing when transported around an orientation reversing cycle). Actually there is a representative of the nontrivial class in $H^2(M,\mathbb{Z})$ with $F=0$ everywhere (but nontrivial holonomy around a noncontractible loop)

E.g., consider RP^2, the disk with antipodal points on the boundary identified



Imagine a single dirac string at the center of the disk

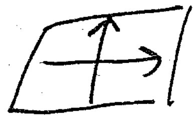
Then

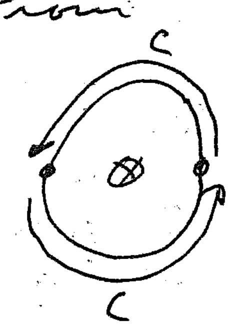
$$g(\theta) = \exp\left(i\int_0^\theta A\right)$$

has winding number 1 for any loop that wraps once around the string, if the connection is flat. But this is not a trivial gauge field. The boundary of the disk is $C + C$, where C is noncontractible

Hence the effect of transport around C is the nontrivial Aharonov-Bohm phase (-1)

Of course, on an orientable surface (e.g. the torus) we can have a flat connection with nontrivial holonomy around cycles, but this connection can be smoothly deformed to one with trivial holonomy, and so is not topologically distinct from $A = 0$.





This flat connection on RP^2 with one Dirac string is different... The holonomy associated with C is locked at (-1) and can't be smoothly trivialized.

i.e., since $2C$ is contractible, for a flat connection, $2C$ must have trivial holonomy. So C itself must have holonomy ± 1.

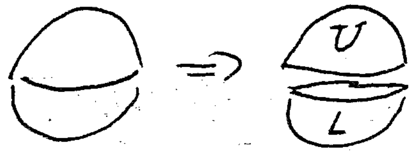
Generalizations

Now we have a complete classification of $U(1)$ gauge fields on 2-manifolds we'd like to discuss how the classification generalizes to

(1) $U(1)$ gauge fields on n-manifolds
(2) G gauge fields on 2-manifolds
(3) G gauge fields on n-manifolds

Lecture #20

We can begin by considering higher dimensional spheres. A G-bundle on S^2 is classified by its monopole number, an element of $\pi_1(G)$. It is a straightforward generalization to see that G-bundles on the k-sphere S^k are classified by $\pi_{k-1}(S^k)$



First of all, given gauge field A on S^k, we can cut S^k into two k-dimensional balls that are glued together at the equator S^{k-1}. The balls are contractible, so $A^{U,L}$ on U and L balls are topologically trivial -- they are patched together on S^{k-1} by a G gauge transformation

$$A_U = \Omega(A_L)\Omega^{-1} - i(d\Omega)\Omega^{-1} \equiv A_L^{\Omega}$$

The map $\Omega: S^{k-1} \to G$ defines an element of $\pi_{k-1}(G)$ associated with the bundle. And since the elements are discrete, we can't smoothly deform a bundle associated with one element to a bundle associated with another distinct element.

To show that $\pi_{k-1}(G)$ classifies G bundles, then, we need to show

1. There is a G-bundle for each element of $\pi_{k-1}(G)$
2. Bundles associated with the same element are topologically equivalent.

Includes $\pi_0(G)$ ... classifies G-bundle on circle

To show ①: Given $\Omega: S^{k-1} \to G$

choose $A_L = 0$

$A_U = \lambda(-i\, d\Omega\, \Omega^{-1})$

i.e. the same function Ω on each of a set of concentric S^{k-1}'s

where $\lambda = 1$ on the boundary of the lower ball and shrinks smoothly to $\lambda = 0$ at the center of the ball.

To show ②:

If $A_U = \Omega A_L \Omega^{-1} - i(d\Omega)\Omega^{-1} \equiv A_L^{\Omega}$

where Ω can be deformed to $\Omega = I$ the contraction of Ω defines a function

$$(\text{Ball})_U \to G$$

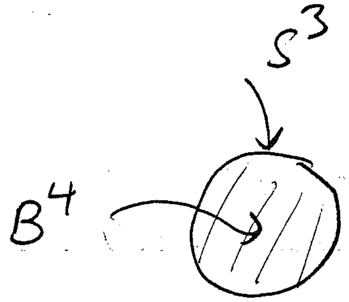
that matches Ω on the boundary of the ball. That means we can perform a gauge transformation on $(\text{Ball})_U$ such that $\Omega = I$; i.e. A is globally defined on S^K. Then the homotopy

$$\lambda A, \quad \lambda \in [0, 1]$$

contracts the gauge field to $A = 0$

In particular, $\pi_{K-1}(U(1)) = 0$

for $K > 2$, and therefore a $U(1)$-bundle on S^K is always topologically trivial for $K \geq 3$. This means, for example, that if we are given a $U(1)$ gauge field on the boundary of a 4-ball



(i.e. S^3) there is no topological obstruction to extending the gauge field into the interior of the ball. Same for B^K if $K \geq 4$. But of course, we can smoothly extend a $U(1)$ gauge field on S^2 into B^3 iff the monopole number on S^2 vanishes.

Okay, now consider a $U(1)$ gauge field on an <u>arbitrary</u> n-manifold.

As already discussed, we may tessellate the manifold such that it is covered by contractible n-cells, double intersections are contractible $(n-1)$-cells, triple intersections are contractible $(n-2)$-cells, etc., down to $(n+1)$-fold intersections at 0-cells (i.e. points). We have already seen that, given a $U(1)$-bundle, we can associate with it a 2-cocycle

$$\{n_{\alpha\beta\gamma}\}$$

that assigns an integer (no. of Dirac strings) to each triple-intersection. Since $n_{\alpha\beta\gamma}$ is coboundary of a real-valued 1-cocycle (transition functions), its coboundary vanishes

$$(\delta n)_{\alpha\beta\gamma\delta} = 0.$$

The interpretation of this cocycle condition is that Dirac strings cannot <u>terminate</u> (no point monopoles in a nonsingular bundle).

Furthermore, the "singular gauge transformations" $\omega_{\alpha\beta} \to \omega_{\alpha\beta} + 2\pi K_{\alpha\beta}$ change the cocycle $n_{\alpha\beta\gamma}$ by coboundary of an integer-valued 1-cocycle. And so, with each $U(1)$-bundle on M, we may associate a class in $H^2(M, \mathbb{Z})$. The classes are discrete, so bundles assigned to distinct classes are topologically inequivalent.

To show that $H^2(M, \mathbb{Z})$ provides a complete classification of $U(1)$ bundles, we need to show:

① There is a $U(1)$ bundle for each class
② Bundles in the same class are topologically equivalent.

As in our discussion of 2-manifolds, we consider a triangulation dual to the cover: 0-simplices dual to n-cells, 1-simplices dual to $(n-1)$-cells, etc. down to n-simplices dual to 0-cells. The union of K-simplices is called the K-skeleton.

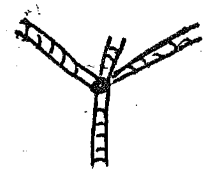
For ①, our earlier construction already tells us, given a 2-cocycle, how to construct a corresponding $U(1)$-bundle on the 2-skeleton. But we must show that the $U(1)$ gauge field can be extended from the 2-skeleton to the rest of the manifold M.

We carry out the extension step by step. First extend from 2-skeleton to 3-skeleton, then from 3-skeleton to 4-skeleton, etc -- up to the n-skeleton $= M$

Consider any 3-simplex. It is a tetrahedron, with a triple intersection piercing the center of each of its four faces.

3 simplex: 

Since $\delta n = 0$ (Dirac strings don't end) the flux entering the tetrahedron matches the flux going out. Thus there is no obstruction to extending the transition functions on double intersections into the interior of the 3-simplex.

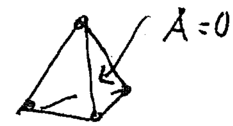
strings: 

In other words, the $U(1)$-bundle on the boundary of the 3-simplex has zero magnetic flux; hence it is topologically trivial, and can be extended into the interior.

Furthermore, once we have a $U(1)$-bundle on the 3-skeleton, $\pi_2(U(1)) = \pi_3(U(1)) = \cdots = 0$ implies that there are no further obstructions that prevent us from extending from 3-skeleton to 4-skeleton, etc, up to n-skeleton.

To show 2 ... Consider two bundles in same $H^2(M,\mathbb{Z})$ class. Our earlier argument for two-manifolds already shows there is a deformation of one to the other on the 2-skeleton this means that the difference of the two bundles has transition functions that vanish where the double intersections cross the 2-skeleton. Can we deform the transition functions so they vanish where double intersections cross 3-skeleton?

Yes... After we have trivialized the difference of the two bundles on the two skeleton (deformed to $A=0$ on the two-skeleton), the gauge field on a 3-simplex has $A=0$ on its boundary — it can be regarded as a $U(1)$ gauge field on S^3, if we identify the boundary as a point. Like any $U(1)$ gauge field on S^3, this is topologically trivial, and can be deformed to $A=0$



By a similar argument $\pi_k(U(1))=0$ means we can deform transition functions to $\omega=0$ on $(k+1)$-simplices, and hence all of M. Therefore A is globally defined on M, and therefore topologically trivial

$\lambda A \quad 0 \le \lambda \le 1$, contracts it to zero.

So now we have shown that $H^2(M,\mathbb{Z})$ provides a complete classification of $U(1)$-bundles. (To check whether one bundle can be deformed to another, it suffices to compare the corresponding classes in $H^2(M,\mathbb{Z})$.)

There are two types of nontrivial classes in $H^2(M,\mathbb{Z})$ (and indeed in $H^p(M,\mathbb{Z})$). Torsion classes are nontrivial cocycles η with the property $k\eta$ = trivial for some integer k, while free classes have the property that $k\eta$ is nontrivial for any k if η is nontrivial.

The free classes in $H^p(M,\mathbb{Z})$ are those that survive (i.e. remain nontrivial i.e. still are not coboundaries) if we allow real coefficients (real-valued cochains). Furthermore, the cohomology of real valued cochains is isomorphic to the (de Rham) cohomology of differential forms.

$$H^p(M,\mathbb{R}) = (\mathbb{R})^{b_p}$$

is a real vector space, spanned by the linearly independent closed p-forms that are not exact:

$$H^p(M,\mathbb{R}) = \frac{(\text{closed } p\text{-forms})}{(\text{exact } p\text{-forms})}$$

— the dimension of this vector space

(number of linearly independent nontrivial closed forms) is called the "pth Betti number" b_p of the manifold M.

Cohomology classes of p-forms can be related to a (partial) classification of closed p-dimensional submanifolds of M. The integral of a closed form ω over a closed manifold Σ depends only on the cohomology class of the form.

If $\omega - \omega' = d\eta$, then

$$\int_\Sigma \omega - \int_\Sigma \omega' = \int_\Sigma d\eta = 0$$ by Stokes, since $\partial\Sigma = 0$

Furthermore, if Σ is a boundary, $\Sigma = \partial\Gamma$, then Stokes $\Rightarrow$

$$\int_\Sigma \omega = \int_{\partial\Gamma} \omega = \int_\Gamma d\omega = 0$$ for any closed form ω

So cohomology classes of p-forms allow us to assign numbers to closed submanifolds that are not boundaries. Roughly speaking, there is a one-to-one correspondence between closed-but-not-exact p-forms and submanifolds that are closed-but-not-a-boundary. So b_p can be regarded as the number of independent non-contractible p-dimensional closed submanifolds of M

What makes this statement rough is that there are two kinds of submanifolds that are noncontractible but cannot be detected by differential forms: Σ could be nonorientable, or it might have the property that $K\Sigma$ is a boundary although Σ is not. In the latter case we must have

$$K \int_\Sigma \omega = \int_{K\Sigma} \omega = \int_{\partial \Gamma} \omega = \int_\Gamma d\omega = 0$$

for any closed form ω, even though Σ is not a boundary. We already encountered the former case in the case of RP^2. The nonorientable two-dimensional submanifolds give rise to the torsion in $H^2(M, \mathbb{Z})$. Submanifolds with $K\Sigma = \partial\Gamma$ are not detected in cohomology with integer coefficients, but are seen if the coefficients are in $\mathbb{Z}_K$, as we'll discuss below.

Anyway, all the free classes in $H^2(M, \mathbb{Z})$ can be represented by differential forms. The two-form $F/2\pi$ (called an "integral cohomology class" because integrating it over any closed manifold yields an integer value) is sufficient to completely identify the free part of the $H^2(M, \mathbb{Z})$ class associated with a $U(1)$ bundle

This "first Chern class" of the bundle assigns an integer

$$\frac{1}{2\pi}\int_\Sigma F = \text{integer}$$

to each noncontractible submanifold $\Sigma \subseteq M$

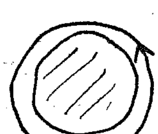
The torsion classes in $H^2(M,\mathbb{Z})$ can always be realized by flat connections on nonorientable submanifolds that have nontrivial holonomy around noncontractible 1-cycles, as we saw for RP^2.

<u>G-bundles</u>

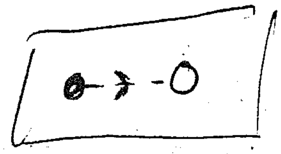
The same ideas can be applied to creating a topological classification of bundles with gauge group G.

It helps to generalize the notion of a "Dirac string" and a "singular gauge transformation" to higher dimensions.

A magnetic monopole field on S^2 can be described in terms of hemispheres with a transition function, or in terms of a disk ("punctured sphere") where the gauge field at the boundary of the disk is pure gauge —

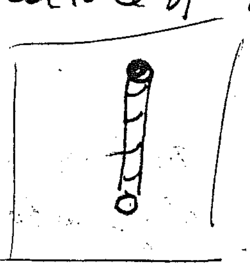


— the connection at one point on the boundary is related to the connection elsewhere on the boundary by a gauge transformation $g(\theta)$, $0 \le \theta \le 2\pi$. This gauge transformation has a "winding number" (element of $\pi_1(G)$) that we identify as the magnetic flux on S^2

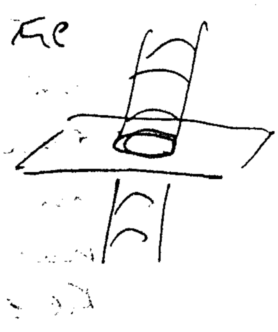


We can move the Dirac string singularity from one point on a manifold to another with a singular gauge transformation g "winds" by a nontrivial element of $\pi_1(G)$ upon crossing the dotted line that connects the 2 points

Similarly, we can describe a gauge field on S^n either in terms of "hemispheres" B^n and transition functions at the equator, or in terms of an n-sphere that is "punctured" by removing a tiny n-ball. The gauge field is nonsingular, so it is a "pure gauge" on the surface of this n-ball — and so the n-ball is associated with an element of $\pi_{n-1}(G)$ — if the element is nontrivial, there is a "string singularity" located at the puncture.



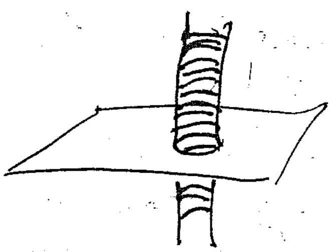
We can "move" the singularity by performing a singular gauge transformation that acts in a tube of infinitesimal thickness that connects the two S^{n-1}'s — On the cross section of the tube, the gauge transformation "winds" as described by an element of $\pi_{n-1}(G)$
(it is trivial on the edges of the tube, nontrivial on it's $(n-1)$-dim cross section.)



Consider a "good cover" of a manifold M, as described earlier. With double intersections we associate elements of $\pi_0(G)$, with triple intersections elements of $\pi_1(G)$, etc -- with

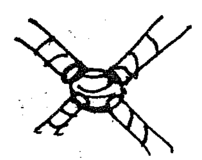
$K+2$ fold intersections, which have codimension $K+1$, we may associate elements of $\pi_K(G)$. These are the "Dirac strings" of various dimensionalities —

Now we have the freedom to perform a singular gauge transformation on $(K+1)$-fold intersections, which have codimension K — they are "tubes"...



On a K dimensional slice through the tube, the gauge transformation on the tube is classified by $\pi_K(G)$.

— The strings are a $(K+1)$-cochain taking values in $\pi_K(G)$. Furthermore, its coboundary is trivial — this is the statement that a Dirac string can never terminate in a nonsingular bundle. If the "flux in" does not match the "flux out" at a codimension $(K+2)$, $(K+3)$ fold intersection, then there is an infinitesimal $(K+1)$-ball on whose boundary the gauge field is a topologically nontrivial "pure gauge." It can't be pure gauge inside the ball, or else the g.t. inside the ball would define



a homotopy that contracts the nontrivial map $S^K \to G$ on the boundary of the ball. So the gauge field has a codim-$(K+2)$ singularity

— A singular gauge transformation on $(K+1)$-fold intersections changes the strings by the coboundary of a K-cochain.

So -- given a bundle on M, we may associate with it classes in

$$\left.\begin{array}{c} H^1(M, \pi_0(G)) \\ H^2(M, \pi_1(G)) \\ \vdots \\ H^K(M, \pi_{K-1}(G)) \\ \vdots \end{array}\right]$$

that describe the configurations of strings modulo gauge transformations

Claim: Two bundles are topologically equivalent iff all of the $H^{K+1}(\pi_K(G))$'s match.

The "only if" is the usual statement -- since the classes are discrete, we can't smoothly deform a bundle in one class to a bundle in another.

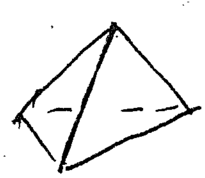
To show the iff, consider the difference of the two bundles -- all classes are trivial, so with an appropriate choice of singular gauge transformations on intersections, all Dirac strings can be removed. Now we wish to show that the bundle can be trivialized — i.e. deformed smoothly to one with $A=0$ everywhere.

We carry out the trivialization of the bundle in stages, setting

- $A=0$ on the 0-skeleton
- $A=0$ on the 1-skeleton
- $A=0$ on the 2-skeleton

$\vdots$

- $A=0$ on the K-skeleton.



To proceed from a trivialization on the K-skeleton to one on the K+1-skeleton, we consider a particular (K+1)-simplex.

If we already have $A=0$ on the K-skeleton, that means that $A=0$ on the boundary of the (K+1)-simplex — we can identify the boundary as a point, and obtain a G-bundle on S^{K+1} classified by $\pi_K(G)$.

The element of $\pi_K(G)$ is determined by the "Dirac string" that intersects the (K+1)-simplex in its interior. — This string is trivial, so the G bundle on the K+1 simplex is trivial, and can be deformed to $A=0$.

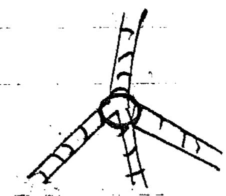
So... the classes $H^K(M, \pi_{K-1}(G))$ describe the possible obstructions that might arise as we attempt to trivialize a bundle. If no such obstruction is encountered, then we can deform to $A=0$.

What kinds of classes can arise in $H^K(M, \pi_{K-1}(G))$? We have already discussed e.g. $H^2(M, \mathbb{Z})$ which has free classes (detectable in the cohomology of forms) and torsion classes

Torsion classes occur in $H^2(M, \mathbb{Z})$ if e.g. M has $\pi_1(M) = \mathbb{Z}_n$ — then there is a loop C that is not a boundary, where nC is a boundary. Flat connections with holonomy $\exp(2\pi i/n)$ on C represent the nontrivial class.

In e.g. $H^2(M, \mathbb{Z}_m)$ there are some nontrivial classes that are just mod m reductions of the classes in $H^2(M, \mathbb{Z})$.

These are said to be "integral" classes. But there can also be additional classes in $H^2(M, \mathbb{Z}_m)$ that cannot be obtained as a reduction of a $H^2(M, \mathbb{Z})$ class — "nonintegral classes." Nonintegral classes can occur because a $\mathbb{Z}_m$-valued 2-cocycle might not lift to a $\mathbb{Z}$-valued 2-cocycle; this might happen because the total magnetic flux at a quadruple intersection is required to sum to zero only modulo n



Nonintegral classes in $H^2(M, \mathbb{Z}_m)$ arise if

$$\pi_2(M) = \mathbb{Z}_m$$

(Recall, we have discussed examples of manifolds with this property, the coset space

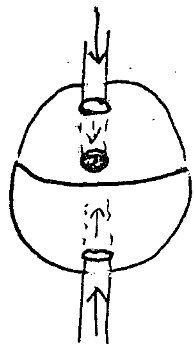
$$M = G/H = SO(M^2-1)\Big/[SU(m)/\mathbb{Z}_m] \;;$$

with $\pi_2(M) = \mathbb{Z}_m$

or $$\pi_2\left(SU(3)/SO(3)\right) = \mathbb{Z}_2 \;. \Big)$$

E.g., if $\pi_2(M) = \mathbb{Z}_2$, there is a 2-sphere in M that is not a boundary, but if we cover the 2-sphere <u>twice</u>, then it is a boundary

Consider a bundle with a single (or odd number) of $\mathbb{Z}_2$ Dirac strings on the noncontractible Σ -- this is a nontrivial class, as there is no way to smoothly remove the $\mathbb{Z}_2$ string on this noncontractible 2-manifold



Now consider the ball in M whose boundary is 2Σ. The 2 Z_2 strings on 2Σ can be extended into the ball, where they meet (at a quadruple intersection)

But there is no corresponding class in $H^2(M,\mathbb{Z})$. If there is one $\mathbb{Z}$-valued string on Σ, there are two on 2Σ - we can't extend these strings into the ball, because their orientations won't match up.

In other words, if Σ is noncontractible but $n\Sigma$ is contractible, it is possible to have Z_n magnetic flux on Σ, since the flux on $n\Sigma$ is trivial. But it is not possible to have $\mathbb{Z}$-valued magnetic flux on Σ, because the flux on the contractible surface $n\Sigma$ would be nontrivial.

FYI - From "Encyclopedic Dictionary of Math.," Appendix A. Notation: ∞ means $\mathbb{Z}$, 2 means $\mathbb{Z}_2$, etc

(VI) The Homotopy Groups $\pi_k(G)$ of Compact Connected Lie Groups G

Here the group G is one of the following:

$SO(n)$ $(n \geqslant 2)$, $Spin(n)$ $(n \geqslant 3)$, $U(n)$ $(n \geqslant 1)$, $SU(n)$ $(n \geqslant 2)$,
$Sp(n)$ $(n \geqslant 1)$, G_2, F_4, E_6, E_7, E_8.

(1) The Fundamental Group $\pi_1(G)$.

$$\pi_1(G) \cong \begin{cases} \infty & (G = U(n)\ (n \geqslant 1),\ \ SO(2)), \\ 2 & (G = SO(n)\ (n \geqslant 3)), \\ 0 & (\text{for all other groups } G). \end{cases}$$

(2) Isomorphic Relations $(k \geqslant 2)$.

$\pi_k(U(n)) \cong \pi_k(SU(n))$ $(n \geqslant 2)$, Continued):

2
210
9240+6+2
$90+2^2$
$180+2^2$
$72+2^3$
$1440+24+2^4$
$144+2^3$
144+6+2
$48+2^2$
$2016+12+2^2$
$16+2^2$
$16+2^2$
$16+2^3$
$240+16+2^3$
$16+2^3$
$16+2^2$
$16+2^2$
$16+2^2$
$8+2^2$
$4+2^2$
2^3
2^2

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$\pi_k(U(1)) \cong \pi_k(SO(2)) \cong 0.$

$\pi_k(Spin(n)) \cong \pi_k(SO(n))\ (n \geqslant 3),$

$\pi_k(Spin(3)) \cong \pi_k(Sp(1)) \cong \pi_k(SU(2)) \cong \pi_k(S^3),$

$\pi_k(Spin(4)) \cong \pi_k(Spin(3)) + \pi_k(S^3),$

$\pi_k(Spin(5)) \cong \pi_k(Sp(2)),$

$\pi_k(Spin(6)) \cong \pi_k(SU(4)).$

(3) The Homotopy Group $\pi_k(G)$ $(k \geqslant 2)$.

$\pi_2(G) \cong 0.$

$\pi_3(G) \cong \infty \quad (G \neq SO(4)), \quad \pi_3(SO(4)) \cong \infty + \infty.$

$$\pi_4(G) \cong \begin{cases} 2+2 & (G = SO(4),\ \ Spin(4)), \\ 2 & (G = Sp(n),\ \ SU(2),\ \ SO(3),\ \ SO(5),\ \ Spin(3),\ \ Spin(5)), \\ 0 & (G = SU(n)\ (n \geqslant 3),\ \ SO(n)\ (n \geqslant 6),\ \ G_2,\ \ F_4,\ \ E_6,\ \ E_7,\ \ E_8). \end{cases}$$

$$\pi_5(G) \cong \begin{cases} 2+2 & (G = SO(4),\ \ Spin(4)), \\ 2 & (G = Sp(n),\ \ SU(2),\ \ SO(3),\ \ SO(5)\ \ Spin(3),\ \ Spin(5)), \\ \infty & (G = SU(n)\ (n \geqslant 3),\ \ SO(6),\ \ Spin(6)), \\ 0 & (G = SO(n),\ \ Spin(n)\ (n \geqslant 7),\ \ G_2,\ \ F_4,\ \ E_6,\ \ E_7,\ \ E_8). \end{cases}$$

$\pi_k(G)$, $k \geqslant 6$.

G \ k	6	7	8	9	10	11	12	13	14	15
$Sp(1)$	12	2	2	3	15	2	2^2	$12+3$	$84+2^2$	2^2
$Sp(2)$	0	∞	0	0	120	2	2^2	$4+2$	1680	2
$Sp(3)$	0	∞	0	0	0	∞	2	2	10080	2
$Sp(4)$	0	∞	0	0	0	∞	2	2	0	∞
$SU(2)$	12	2	2	3	15	2	2^2	$12+3$	$84+2^2$	2^2
$SU(3)$	6	0	12	3	30	4	60	6	$84+2$	36
$SU(4)$	0	∞	24	2	$120+2$	4	60	4	$1680+2$	$72+2$
$SU(5)$	0	∞	0	∞	120	0	360	4	1680	6
$SU(6)$	0	∞	0	∞	0	∞	720	2	$5040+2$	6
$SU(7)$	0	∞	0	∞	0	∞	0	∞	5040	0
$SU(8)$	0	∞	0	∞	0	∞	0	∞	0	∞
$SO(5)$	0	∞	0	0	120	2	2^2	$4+2$	1680	2
$SO(6)$	0	∞	24	2	$120+2$	4	60	4	$1680+2$	$72+2$
$SO(7)$	0	∞	2^2	2^2	8	$\infty+2$	0	2	$2520+8+2$	2^4
$SO(8)$	0	$\infty+\infty$	2^3	2^3	$24+8$	$\infty+2$	0	2^2	$2520+120+8+2$	2^7
$SO(9)$	0	∞	2^2	2^2	8	$\infty+2$	0	2	$8+2$	$\infty+2^3$
$SO(10)$	0	∞	2	$\infty+2$	4	∞	12	2	8	$\infty+2^2$
$SO(11)$	0	∞	2	2	2	∞	2	2^2	8	$\infty+2$
$SO(12)$	0	∞	2	2	0	$\infty+\infty$	2^2	2^2	$24+4$	$\infty+2$
$SO(13)$	0	∞	2	2	0	∞	2	2	8	$\infty+2$
$SO(14)$	0	∞	2	2	0	∞	0	∞	4	∞
$SO(15)$	0	∞	2	2	0	∞	0	0	2	∞
$SO(16)$	0	∞	2	2	0	∞	0	0	0	$\infty+\infty$
$SO(17)$	0	∞	2	2	0	∞	0	0	0	∞
G_2	3	0	2	6	0	$\infty+2$	0	0	$168+2$	2
F_4	0	0	2	2	0	$\infty+2$	0	0	2	∞
E_6	0	0	0	∞	0	∞	12	0	0	∞
E_7	0	0	0	0	0	∞	2	2	0	∞
E_8	0	0	0	0	0	0	0	0	0	∞

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(4) Stable Homotopy Groups. For sufficiently large n for fixed k, the homotopy groups for classical compact simple Lie groups $G=Sp(n)$, $SU(n)$, $SO(n)$ become stable. We denote them by the following notations. Here we assume $k \geqslant 2$.

$$\pi_k(Sp)=\pi_k(Sp(n)) \qquad (n \geqslant (k-1)/4),$$
$$\pi_k(U)=\pi_k(U(n)) \cong \pi_k(SU(n)) \qquad (n \geqslant (k+1)/2),$$
$$\pi_k(O)=\pi_k(SO(n)) \qquad (n \geqslant k+2).$$

Bott periodicity theorem

$$\pi_k(Sp) \cong \begin{cases} \infty & (k \equiv 3,7 \pmod 8), \\ 2 & (k \equiv 4,5 \pmod 8), \\ 0 & (k \equiv 0,1,2,6 \pmod 8). \end{cases}$$

$$\pi_k(O) \cong \begin{cases} \infty & (k \equiv 3,7 \pmod 8), \\ 2 & (k \equiv 0,1 \pmod 8), \\ 0 & (k \equiv 2,4,5,6 \pmod 8). \end{cases}$$

$$\pi_k(U) \cong \begin{cases} \infty & (k \equiv 1 \pmod 2), \\ 0 & (k \equiv 0 \pmod 2). \end{cases}$$

(5) Some Special Cases.

(a,b) means the greatest common divisor of two integers a and b.

$$\pi_{2n}(SU(n)) \cong n!.$$

$$\pi_{2n+1}(SU(n)) \cong \begin{cases} 2 & (n \text{ even}), \\ 0 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+2}(SU(n)) \cong \begin{cases} (n+1)!+2 & (n \text{ even}, \geqslant 4), \\ (n+1)!/2 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+3}(SU(n)) \cong \begin{cases} (24,n) & (n \text{ even}), \\ (24,\pi+3)/2 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+4}(SU(n)) \cong \begin{cases} (n+2)!(24,n)/48 & (n \text{ even}, \geqslant 4), \\ (n+2)!(24,n+3)/24 & (n \text{ odd}). \end{cases}$$

$$\pi_{2n+5}(SU(n)) \cong \pi_{2n+5}(U(n+1)).$$

$$\pi_{2n+6}(SU(n)) \cong \begin{cases} \pi_{2n+6}(U(n+1)) & (n \equiv 2,3 \pmod 4, n \geqslant 3), \\ \pi_{2n+6}(U(n+1))+2 & (n \equiv 0,1 \pmod 4). \end{cases}$$

$$\pi_{4n+2}(Sp(n)) \cong \begin{cases} (2n+1)! & (n \text{ even}), \\ 2(2n+1)! & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+3}(Sp(n)) \cong 2.$$

$$\pi_{4n+4}(Sp(n)) \cong \begin{cases} 2+2 & (n \text{ even}), \\ 2 & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+5}(Sp(n)) \cong \begin{cases} (24,n+2)+2 & (n \text{ even}), \\ (24,n+2) & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+6}(Sp(n)) \cong \begin{cases} (2n+3)!(24,n+2)/12 & (n \text{ even}), \\ (2n+3)!(24,n+2)/24 & (n \text{ odd}). \end{cases}$$

$$\pi_{4n+7}(Sp(n)) \cong 2.$$

$$\pi_{4n+8}(Sp(n)) \cong 2+2.$$