Spin Structures

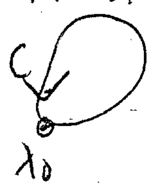
Lecture #21

A Riemannian manifold M has a tangent bundle TM — we associate an n-dim vector space of tangent vectors with each point on the manifold. We can specify a "vielbein" —

$\{\hat{e}_a(x),\ a=1,\dots,n\}$ an orthonormal frame of tangent vectors at point $x \in M$

$x \xrightarrow{P} y$ Parallel transport of a tangent vector defines an $SO(n)$-valued connection on the bundle

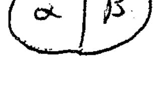
$U(P;y,x)$ is the $SO(n)$ transform that describes the effect of carrying the standard frame from x to y and composing with standard frame at y. What has an



invariant meaning is $U(C,x_0)$ — the effect of parallel transport around a closed path.

(In the nonorientable case, $G = O(n)$.)

Now we want to introduce spinors on M — i.e. lift the $SO(n)$ connection to a $Spin(n)$ connection. Can we?



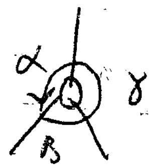
Cover the manifold and consider a double intersection, with matching condition $R_{\alpha\beta} \in SO(n)$

There are two smooth ways to lift to $Spin(n)$;

differing by overall $-I$

$$\pm L_{\alpha\beta} \in Spin(n) \downarrow R_{\alpha\beta} \in SO(n))$$

At each double intersection, we choose either + or − sign — i.e. a Z_2-valued one-cochain



At triple intersections, transport for tangent vectors is smooth, so

$$R_{\alpha\beta} R_{\beta\gamma} R_{\gamma\alpha} = I$$

Lifting, we obtain

$$L_{\alpha\beta} L_{\beta\gamma} L_{\gamma\alpha} = \pm 1 = (-1)^{n_{\alpha\beta\gamma}}$$

at each triple intersection. This is a Z_2-valued 2-cochain. And in fact it is a cocycle. Otherwise, the tangent bundle would have Z_2 monopole singularities.

Transport for spinors will be regular iff we can choose a lifting with $n_{\alpha\beta\gamma} = 0$ at each triple intersection. The $n_{\alpha\beta\gamma}$'s have equivalence classes corresponding to different liftings of the same bundle — these differ by coboundary of a one-cochain

i.e. $$L_{\alpha\beta} \to (-1)^{l_{\alpha\beta}} L_{\alpha\beta}$$

$$\Rightarrow (-1)^{n_{\alpha\beta\gamma}} \to (-1)^{(\delta l)_{\alpha\beta\gamma}} (-1)^{n_{\alpha\beta\gamma}}$$

— these equivalence classes are just the cohomology classes of $H^2(M, \mathbb{Z}_2)$

Thus, associated with the manifold of M is a class in $H^2(M, \mathbb{Z}_2)$ — it is called the "second Stiefel-Whitney class" and denoted

$$w_2(M)$$

An orientable manifold M admits spinors iff $w_2(M)$ is a trivial class

$$w_2(M) = 0$$

— if so, we say that M is a "spin manifold"

Note that w_2 does not depend on the Riemannian connection that we choose for TM — it is an intrinsic topological property of M itself (no smooth deformation of the connection can alter $w_2(M)$).

In effect $w_2(M)$ assigns an $SO(n)$ monopole number to each non-contractible 2-fold in M. All these monopole numbers must be trivial for M to be a spin manifold.

Example: Coset manifold G/H

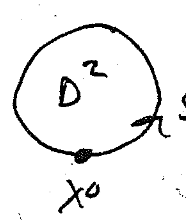
G is a compact Lie group, $H \subseteq G$ is a subgroup. Is G/H a spin manifold

Recall that we can parametrize

$G/H = \{ g\Phi_0 ,\ g \in G \}$ where $h\Phi_0 = \Phi_0$ for $h \in H$

If G is simply connected, then

$\pi_2(G/H) = \pi_1(H)$ .. A noncontractible S^2 in G/H can be represented as a mapping of a disk $D^2 \to G$, where the boundary $\partial D^2 = S^1 \to H$, and the loop in H associated with the boundary is noncontractible:



$g(r,\theta)$, where $g(1,\theta) = h(\theta) \in H$ is noncontractible loop

We want to examine whether the tangent bundle on this $S^2 \subset G/H$ has a nontrivial $SO(n)$ monopole number ($n = \dim G/H$).

If we think of the boundary of the disk as the location of the "Dirac string" on the S^2, the issue is whether the path in $SO(n)$ associated with a loop enclosing the string is contractible: i.e. is it a rotation by an odd or even mult. of 2π.

To study the tangent bundle, consider decomposition of $\mathcal{G}$ = Lie algebra of G

$$\mathcal{G} = \underset{\substack{\Uparrow \\ H\text{-generators} \\ \{T^\alpha\}}}{\mathcal{H}} \oplus \underset{\substack{\nwarrow \text{ broken generators} \\ \{X^a\}}}{\mathcal{K}}$$

We can choose to normalize as

$$\mathrm{tr}(T^\alpha T^\beta) = \delta^{\alpha\beta}, \quad \mathrm{tr}(X^a X^b) = \delta^{ab}, \quad \mathrm{tr}(T^\alpha X^a) = 0$$

At the point Φ_0, tangent space spanned by

$$X^a \cdot \Phi_0, \quad a = 1, 2, \ldots$$

Inner product $(X^a\Phi_0, X^b\Phi_0) = \Delta^{ab}$

We can choose a basis that diagonalizes this.
In fact, if $\Omega \in H$,

$$\Omega X^a \Omega^{-1} = R^a_{\ a'}(\Omega)\, X^{a'}$$ where $R^a_{\ a'}(\Omega)$ is a rep of H

Since $\Omega\Phi_0 = \Phi_0$ and Ω is unitary (H is compact)

$$\Delta^{ab} = (\Omega X^a\Phi_0, \Omega X^b\Phi_0) = R^a_{\ a'} R^b_{\ b'} \Delta^{a'b'}$$

Thus Δ^{ab} commutes with the H-representation it is a multiple of I in each irrep of H. That is

$$(X^a \underline{\Phi}_0, X^b \underline{\Phi}_0) = \Delta^a \delta^{ab}$$

in the basis in which X^a's are in definite H irreps, with Δ^a = constant in each irrep

At another point $\underline{\Phi} = g \underline{\Phi}_0$ in G/H, tangent vectors are

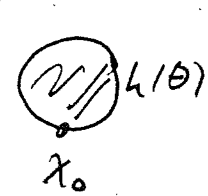
$$g X^a g^{-1} \underline{\Phi}$$

our choice of basis at $\underline{\Phi}$ depends on how we choose the coset rep g (gh would do as well)

Make any smooth choice, and study how the tangent vectors behave on the boundary of the disk

[Remark: this choice of basis implicitly defines an H-valued connection i.e. a notion of parallel transport for the tangent vectors in each irrep of H

Our task is to study how the X^a's rotate under the action of $h(\theta)$.



Suppose $\mathcal{M} = CP^2 = SU(3)/U(2)$

i.e. the space (z_1, z_2, z_3) where $z_{1,2,3} \in \mathbb{C}$, modulo multiplication by a complex scalar

i.e.

Consider adjoint Higgs field in $SU(3)$
$$\Phi_0 = \mathrm{diag}\left(1 \;\; -\tfrac{1}{2} \;\; -\tfrac{1}{2}\right)$$

As we've discussed, the minimal two sphere in G/H ~ minimal loop in
$$H = U(2) = [SU(2) \times U(1)]/\mathbb{Z}_2$$

$Q = \mathrm{diag}(1, -\tfrac{1}{2}, -\tfrac{1}{2})$ is $U(1)$ generator
$T = \mathrm{diag}(0, \tfrac{1}{2}, -\tfrac{1}{2})$ is an $SU(2)$ generator

$\tilde{Q} = Q - T = \mathrm{diag}(1, -1, 0)$ generates minimal loop:
$$h(\theta) = \mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right)$$

How does $h(\theta)$ act on the 4 <u>broken</u> generators?

Adjoint of $SU(3)$ decomposes under $SU(2) \times U(1)$ as
$$8 = 3 \otimes \bar{3} - 1 \rightarrow (2^{-\frac{1}{2}} + 1^{1}) \otimes (2^{\frac{1}{2}} + 1^{-1}) - 1^{0}$$
$$= \underbrace{3^{0} + 1^{0}}_{\text{unbroken}} + \underbrace{2^{3/2} + 2^{-3/2}}_{\text{broken}}$$

$$2^{3/2} \Rightarrow \tilde{Q} = \mathrm{diag}\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) + \mathrm{diag}\left(\tfrac{3}{2}, \tfrac{3}{2}\right)$$
$$= \mathrm{diag}(2, 1)$$

$$2^{-3/2} \Rightarrow \tilde{Q} = \mathrm{diag}\left(\tfrac{1}{2}, -\tfrac{1}{2}\right) + \mathrm{diag}\left(-\tfrac{3}{2}, -\tfrac{3}{2}\right)$$
$$= \mathrm{diag}(-1, -2)$$

Acting on the 4 broken generators

$$L(\theta) = \underbrace{(e^{2i\theta}, e^{-2i\theta}, 1, 1)}_{\text{Rotation by } 4\pi} \cdot (1, 1, \underbrace{e^{i\theta}, e^{-i\theta}}_{\text{Rotation by } 2\pi})$$

In all, a rotation by odd multiple of 2π

$\Rightarrow$ CP^2 is not a spin manifold!

Exercise:
$CP^n = SU(n+1)/U(n)$

Another example
$M = SU(3)/SO(3)$

Here $8 = 3 \otimes 3 - 1 \Rightarrow (3 \otimes 3 - 1) = 3 \oplus 5$

undroken broken

(II) Noncontractible S^2 in M
$\Rightarrow$ Noncontractible loop in $SO(3)$

under $SO(2)$ $\quad 5 \to 2 \oplus 1 \oplus 0 \oplus (-1) \oplus (-2)$

i.e. $h(\theta) = \mathrm{diag}\left(e^{2i\theta}, e^{i\theta}, 1, e^{-i\theta}, e^{-i2\theta}\right)$

Rotation by odd multiple of $2\pi \Rightarrow$

$SU(3)/SO(5)$ is not a spin manifold

$$\boxed{\text{Exercise:} \quad SU(n)/SO(n)}$$

Berry Phase

$H(\vec{\lambda})$ Hamiltonian depends on parameters $\vec{\lambda} = (\lambda_1, \lambda_2, --) \in \underline{M}$

Spectrum: $H(\vec{\lambda})\Psi(\vec{\lambda}) = \epsilon(\vec{\lambda})\Psi(\vec{\lambda})$

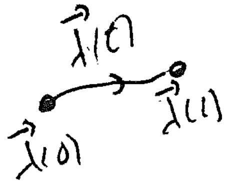
discrete and nondegenerate for all values of $\vec{\lambda}$

$\Psi(\vec{\lambda})$ has an arbitrary phase $\Rightarrow$ a $U(1)$ bundle on M. What are the global topological properties of this bundle?

~~Locally (in patches) choose a standard reference wave function $\phi(\vec{\lambda})$ at each point in M. Normalized~~

$\cancel{(\phi(\vec{\lambda}), \phi(\vec{\lambda})) = 1}$

We'd like a notion of parallel transport on this bundle. A natural one is adiabatic transport. Consider any one of the eigenstates, e.g. the ground state.



Adiabatic theorem: if we change λ slowly, under $H(\lambda(t))$, $\psi(\lambda(0))$ evolves to $\psi(\lambda(1))$.

$$\psi(\vec\lambda(1)) = \text{phase}\cdot \lim_{T\to\infty} \exp\left[-i\int_0^t ds\, H(\lambda(s/T))\right]\psi(\vec\lambda(0))$$

The phase of $\psi(\vec\lambda(1))$ has uninteresting component $\exp\left[-iT\int_0^1 ds\, E(\vec\lambda(s))\right]$. Eliminate it by adding a constant to H

$$H' = H(\vec\lambda) - E(\vec\lambda)$$

so $$H'(\vec\lambda)\,\psi(\vec\lambda) = 0$$

But there is a remaining "geometric phase" or "Berry phase".

The connection that defines transport of the geometric phase is the "Berry connection". Its defining property is that under parallel transport

$$(\psi, d\psi) = 0$$

We have eliminated the "trivial" phase $\propto t$ — what remains is the effect of the <u>rotation</u> of the eigenvector $\psi(\vec{\lambda})$ as $\vec{\lambda}$ varies.

Locally, in patches, choose on M a standard phase at each point $\vec{\lambda}$

i.e. $\phi(\vec{\lambda})$, normalized so that

$$(\phi(\vec{\lambda}), \phi(\vec{\lambda})) = 0$$

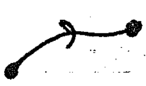
and hence $0 = d(\phi,\phi) = (d\phi,\phi) + (\phi, d\phi)$

$$= (\phi, d\phi) + (\phi, d\phi)^*$$

i.e. $(\phi, d\phi)$ is imaginary, and

$A = i(\phi, d\phi)$ is a real-valued one form on M

In fact A is Berry's connection. The effect of parallel transport along $\lambda(t)$ is written



$$\psi(\vec{\lambda}(t)) = U(\vec{\lambda}(t))\, \phi(\vec{\lambda}(t))$$

(where e.g. $\psi(\vec{\lambda}(0)) = \phi(\vec{\lambda}(0))$). Here

$$U(\vec{\lambda}(t)) = \exp\left(i \int_{\vec{\lambda}(0)}^{\vec{\lambda}(t)} A\right) \in U(1)$$

or $iA = U^{-1} dU$

The defining property $(\psi, d\psi) = 0$
then implies

$$0 = (U\phi,\ dU\phi + U\,d\phi)$$

$$= (\phi,\phi)\,U^{-1}dU + (\phi, d\phi) \Rightarrow U^{-1}dU = iA$$

$$= -(\phi, d\phi)$$

As usual, A is modified by a gauge transformation,
i.e. a change

$$\phi(\vec{\lambda}) \to \Omega(\vec{\lambda})^{-1}\phi(\vec{\lambda})$$

in our standard wavefunctions

$$\Rightarrow (\phi, d\phi) \to (\Omega^{-1}\phi,\ d\Omega^{-1}\phi + \Omega^{-1}d\phi)$$

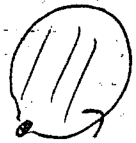
$$= (\phi, d\phi) + \qquad \Omega\, d\Omega^{-1}$$

$$\text{or} \quad A = i(\phi, d\phi) \to A + i\Omega\, d\Omega^{-1}$$

$$= A - i\,\Omega^{-1}d\Omega$$

What has an invariant geometrical meaning
is the effect of transport around a closed
path:

$$\exp\left(i\oint_C A\right) = \exp\left(i\int_\Sigma F\right)$$



$$\text{where} \quad F = dA = i(d\phi, d\phi)$$

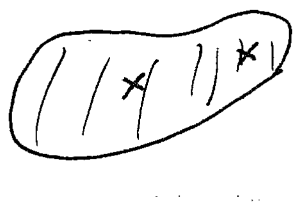
is Berry's curvature.

As for any $U(1)$ bundle, the Berry field strength F has the property

$$\frac{1}{2\pi}\int_{\Sigma} F = \text{integer}$$

for any closed orientable noncontractible manifold $\Sigma \subseteq M$

why might Σ be noncontractible? —



Because it might enclose level crossings where the geometric phase becomes ill-defined (and the adiabatic theorem does not apply) Generically, level crossings have codimension 3.

Where 2 levels approach one another, the effective 2-level Hamiltonian becomes

$$H = E_0 + \vec{a}\cdot\vec{\sigma} \qquad \text{(General } 2\times 2 \text{ Hermitian matrix).}$$

In a 3-parameter manifold, let splitting $\to 0$ i.e. $\vec{a} \to 0$ at $\vec{x}=0$. Expand to linear order in $\vec{x}$, and eliminate E_0

$$\Rightarrow \quad H = \vec{\sigma}\cdot \underset{\sim}{C}\, \vec{x}$$

$\odot$ Now let's integrate Berry curvature over a small sphere.

Since $$n = \frac{1}{2\pi}\int F$$
is a topological property, it is unchanged if we deform $\underset{\sim}{C}$ as long as $\det \underset{\sim}{C} \neq 0$ (no level crossing on sphere)

So deform to $\underset{\sim}{C} = \pm I$ for $\det \underset{\sim}{C} \gtrless 0$

$$H = \pm \vec{\sigma}\cdot\vec{x}$$

Now we can use rotational invariance to evaluate n. Near $x = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$x = (\varepsilon_1, \varepsilon_2, 1) \Rightarrow$
$$H(\varepsilon) = \begin{pmatrix}1 & \varepsilon^* \\ \varepsilon & -1\end{pmatrix} \qquad \varepsilon = \varepsilon_1 + i\varepsilon_2$$

and ground state is $\phi(\varepsilon) = \begin{pmatrix}-\varepsilon^*/2 \\ 1\end{pmatrix}$

$$F = i(d\phi, d\phi) = \frac{i}{4}(d\varepsilon_1 + i d\varepsilon_2)\wedge(d\varepsilon_1 - i d\varepsilon_2)$$
$$= \frac{1}{2}(d\varepsilon_1 \wedge d\varepsilon_2) = \frac{1}{2}d\Omega \quad \text{(area element on sphere)}$$

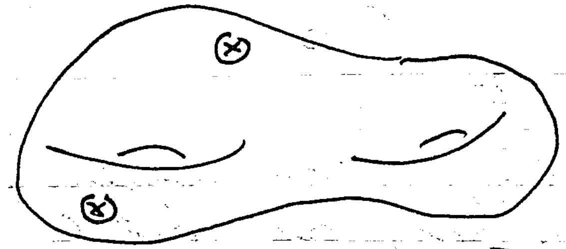
$$n = \frac{1}{2\pi}\int F = \frac{1}{2\pi}\frac{1}{2}4\pi = 1 \quad (\det C > 0)$$

For $\det C < 0$ $\quad \phi(\varepsilon) = \begin{pmatrix}1 \\ \varepsilon/2\end{pmatrix}$ is ground state

$$F = \frac{i}{4}\left(d\varepsilon_1 - i d\varepsilon_2, d\varepsilon_1 + i d\varepsilon_2\right) = -\frac{1}{2}d\varepsilon_1 \wedge d\varepsilon_2$$
$$= -\frac{1}{2}d\Omega$$

$$\Rightarrow n = \frac{1}{2\pi}\int F = -1$$

Σ



For a general noncontractible surface, $dF = 0 \Rightarrow$

$$\Rightarrow \int_{\Sigma} F = \sum_{\substack{\text{level} \\ \text{crossings}}} \text{sign}(\det \underline{C})$$

The way Berry's bundle "twists" is determined by the level crossings. When the net number of crossings $\neq 0$, the phase of $\psi(\vec{\lambda})$ cannot be globally defined on Σ.

~~Berry Phase and QHE~~

~~If H is constrained to be real (time reversal invariant) then level crossings~~

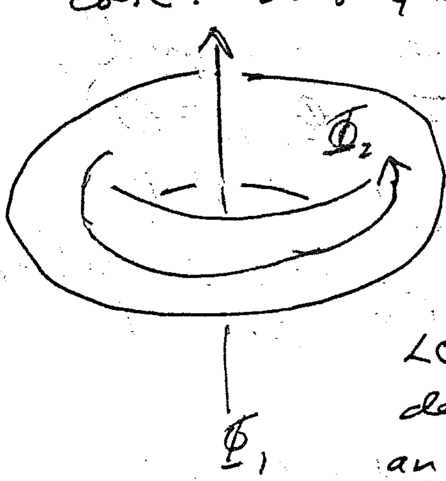
Berry Phase and QHE

Recall our discussion of quantum Hall effect (cf p. 51) - the Hall conductivity is a *topological* property, and therefore remains unchanged when we weakly perturb the Hamiltonian. Why is it topological?

Because of the incompressibility of the system (existence of a mass gap). The ground state is split from the rest of the spectrum and adiabatic transport is well defined.

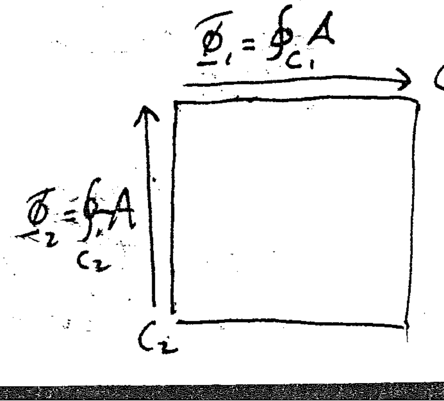
For IQHE, there is no degeneracy. For FQHE, there is a robust q-fold degeneracy at $\nu = p/q$.

In either case, σ_H can be related to Berry phase (actually a $U(q)$ rather than $U(1)$ connection in the FQHE case. It's quantization is *Dirac* quantization.



Consider QHE system on a torus. Two fluxes Φ_1, Φ_2 link torus.

Let Φ_1 be time dependent, providing an emf V around C_1



$\Phi_1 = -Vt$

This emf causes a transverse Hall current to flow along C_2.

Since Hall conductivity is a bulk property, it should not be sensitive to boundary conditions. Hall conductance should not depend on the instantaneous value of Φ_1 or Φ_2.

So imagine ramping up Φ_1 by one flux quantum for fixed Φ_2. And then consider averaging Φ_2 over

$$0 \leq \Phi_2 \leq \phi_0 = 2\pi/e$$

Now there is another torus, the Berry torus parametrized by the two periodic variables

$$0 \leq \Phi_1 \leq 2\pi/e$$

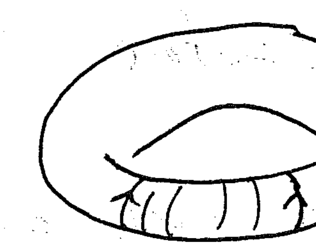
$$0 \leq \Phi_2 \leq 2\pi/e$$

(as far as the electrons are concerned, $\Phi_{1,2} = 2\pi/e$ is equivalent to $\Phi_{1,2} = 0$)

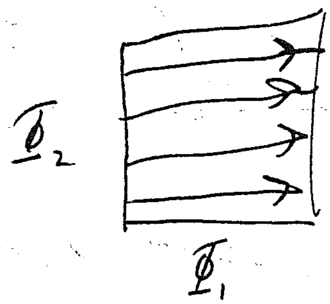
The torus is equipped with its Berry connection and Berry curvatures describing transport of the ground state.

These are not to be confused with the physical torus and its electromagnetic connection...

Recall that the magnetic flux enclosed by a torus is the same thing as the winding number of the flux



$$\left(\oint_C A\right)$$



as the loop C winds around the torus --

For fixed Φ_2, what is the Berry holonomy associated with Φ_1 increasing from 0 to $2\pi/e$?

It is $\exp(-i/\hbar\, \delta H T)$ where T is the time for Φ_1 to turn on, and

$$\delta H = \frac{1}{c}\int d^2x\, J_2 A^2 = \frac{1}{c}\Phi_2 \int dx_1 J_2$$

$$= \frac{1}{c}\Phi_2 \cdot I_2$$

the Hall current flowing in $\hat{x}_2$ direction

This is the Berry phase because it arises only from the time dependence of H (which drives the current). It is trivial when H is static and no current flows.

Now $T = \phi_0/cV$ and $I_2 = \sigma_H V$

$$\Rightarrow \quad \oint A_{\text{Berry}} = \frac{1}{\hbar}\,\frac{1}{c}\Phi_2\,\sigma_H V \frac{\phi_0}{cV} = \frac{1}{\hbar c^2}\sigma_H \phi_0 \Phi_2$$

But -- σ_H does not depend on Φ_2. As Φ_2 varies from 0 to ϕ_0, this must advance by $2\pi \cdot$ integer

$$= 2\pi \frac{h\sigma_H}{e^2}\left(\Phi_2/\phi_0\right)$$

using $\phi_0 = \dfrac{2\pi\hbar c}{e}$

Therefore $\frac{h}{e^2}\sigma_H$ = integer

or Hall conductivity $\sigma_H = e^2/h \cdot$ integer follows from the Dirac quantization condition on the Berry bundle!

What about the fractional effect?

Now Berry's connection lives in
$U(q) = [SU(q) \times U(1)]/Z_q$

— because of the generic q-fold degeneracy. Only the winding of the $U(1)$ phase is relevant to σ_H. But in $U(q)$ the Dirac quantum is smaller by a factor of q — a loop can wind from I to $e^{2\pi i/q} I$ — which is in the center of $SU(q)$, and can return to I through $SU(q)$. Hence

$$\sigma_H = e^2/hq \cdot \text{integer}$$

is the Dirac quantization condition for a system with generic q-fold degeneracy!

Chern Classes

Recall, gauge fields on k-sphere S^k classified by $\pi_{k-1}(G)$.

In the case of S^2 and $G = U(1)$, $\pi_1(G) = \mathbb{Z}$, and classification is magnetic charge in Dirac units:

$$n = \frac{1}{2\pi}\int F.$$

More generally, the $U(1)$ gauge fields on manifold M can be classified by $H^2(M,\mathbb{Z})$, and the free part of $H^2(M,\mathbb{Z})$ is detected by integrating the 2-form F over two-dimensional submanifolds. This 2-form is the "first chern class"

E.g. think of the 2-sphere as a two-dimensional disk, with boundary identified as a point,

and suppose $F_{\mu\nu} = 0$ at the boundary (e.g. consider configurations on $\mathbb{R}^2$ with finite energy - field strength $\to 0$ as $r \to \infty$). The gauge potential A_μ is globally defined on the disk (which is contractible) but is a "pure gauge" (gauge transformation of $A = 0$) at $r = \infty$

$$iA = g^{-1}dg \quad \text{where } g(\theta) \in U(1)$$
$$= i\,d\sigma \qquad g(\theta) = \exp(i\sigma(\theta))$$

This $g(\theta)$ has a winding number

$$n = \frac{1}{2\pi}[\sigma(2\pi) - \sigma(0)] = \frac{1}{2\pi}\oint A = n = \frac{1}{2\pi}\int F,$$

So we can relate Chern class, integrated over S^2, to winding number of a gauge transformation on boundary of a disk

How does this generalize to higher spheres and other gauge groups?

Homotopy groups of Lie groups are quite intricate — but the "free" part is simple. E.g. for fixed K and large enough n, the

$\pi_K(SU(n))$'s become <u>stable</u>:

$$\pi_K(SU(n)) = \begin{cases} \mathbb{Z} & K \text{ odd} \\ 0 & K \text{ even} \end{cases}$$

for $n \geq (K+1)/2$

E.g. $\pi_3(SU(n)) = \mathbb{Z}$

$$\mathbb{Z} = \pi_5(SU(3)) = \pi_5(SU(4)) = \cdots$$
$$\mathbb{Z} = \pi_7(SU(4)) = \pi_7(SU(5)) = \cdots$$
$$\mathbb{Z} = \pi_9(SU(5)) = \pi_9(SU(6)) = \cdots$$

How can we detect this topology by integrating differential forms?

Notation: to avoid annoying (i)'s and (-1)'s, let A denote one-form taking values in antihermitian matrix-valued one-forms

$$A = (-igA_\mu)dx^\mu, \quad A_\mu = A_\mu^a T^a, \quad \mathrm{tr}\, T^a T^b = \tfrac{1}{2}\delta^{ab}$$

Then $D_\mu = (\partial_\mu - igA_\mu) \rightarrow D = d + A$

Gauge transformation property is

$$g \rightarrow \Omega^{-1} g$$

$$D_\mu \rightarrow \Omega^{-1} D_\mu \Omega \Rightarrow (-iA_\mu) \rightarrow \Omega^{-1}(-iA_\mu)\Omega + \Omega^{-1}\partial_\mu \Omega,$$

becomes $\quad A \rightarrow \Omega^{-1} A \Omega + \Omega^{-1} d\Omega$

A "pure" gauge - gauge transform of $A=0$ - is

$$A = g^{-1} dg$$

under gauge transformation: $\quad g \rightarrow g\Omega$

$$A = g^{-1}dg \rightarrow (g\Omega)^{-1} d(g\Omega) = \Omega^{-1} g^{-1}(dg\,\Omega + g\,d\Omega)$$
$$= \Omega^{-1}(g^{-1}dg)\Omega + \Omega^{-1} d\Omega$$

Similarly - there is curvature 2-form

$$F = \tfrac{1}{2}(-igF_{\mu\nu})dx^\mu dx^\nu = [\partial_\mu(-igA_\nu) + (-ig)^2 A_\mu A_\nu]\, dx^\mu dx^\nu$$

$$\left(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right)$$

$$= dA + A^2 \equiv F$$

e.g.
$$A_\mu^a A_\nu^b A_\lambda^c A_\sigma^d \cdot \underbrace{tr(T^a T^b T^c T^d)}_{\text{cyclic}} \cdot \underbrace{dx^\mu dx^\nu dx^\lambda dx^\sigma}_{(-1) \text{ under cycle}}$$

Now, for map $g: S^K \to G$, there is a topological invariant, an integral of a K-form over the sphere

$$\int_{S^K} tr\left[g^{-1}dg \wedge \cdots \wedge g^{-1}dg \right]$$

Note that the integrand vanishes trivially if K is even, since the trace is cyclic and the wedge product is antisymmetric. For K even, we pick up an odd number of (-1)'s anticommuting $g^{-1}dg$ through the $(K-1)$ other $g^{-1}dg$'s. So this invariant is interesting only for odd K.

Why is it invariant? $g^{-1}dg$ transforms like a connection one-form under right multiplication

$$g \to g\Omega,$$

but it is simpler to study the invariance of the integral under left multiplication

$$g \to \Omega g,$$

under which

$$g^{-1}dg \to g^{-1}\Omega^{-1}(d\Omega\, g + \Omega dg) = g^{-1}dg + g^{-1}(\Omega^{-1}d\Omega)g$$

We can deform the map $g: S^k \to G$ by an infinitesimal amount by left multiplying by infinitesimal map

$$g \to \Omega g \Rightarrow g^{-1}dg \to g^{-1}\Omega^{-1}(d\Omega\, g + \Omega\, dg)$$

or $\quad g^{-1}dg \to g^{-1}dg + g^{-1}(\Omega^{-1}d\Omega)g$

Consider $\quad \Omega = I + \varepsilon + \cdots \quad$ (ε infinitesimal)

$$\Omega^{-1}d\Omega = d\varepsilon + \cdots$$

Consider the case $K=3$

$$g^{-1}dg \to g^{-1}dg + g^{-1}(d\varepsilon)g$$

$\mathrm{tr}\,(g^{-1}dg \quad g^{-1}dg \wedge g^{-1}dg)$

$$\to \text{same} + 3\,\mathrm{tr}\,(g^{-1}d\varepsilon g \quad g^{-1}dg \quad g^{-1}dg)$$

$$= \text{same} + 3\,\mathrm{tr}\,(d\varepsilon \quad dg\,(g^{-1}dg\,g^{-1}))$$

But note that

$$0 = d\,(g^{-1}g) = d(I) = g^{-1}dg + d(g^{-1})\,g \Rightarrow$$

$$d(g^{-1}) = -g^{-1}(dg)\,g^{-1}$$

So $\quad \delta\,\mathrm{tr}[(g^{-1}dg)^3] = -3\,\mathrm{Tr}[d\varepsilon \; dg \; d(g^{-1})]$

$$= -3\,d\,[\mathrm{tr}\,(\varepsilon \; dg \; d(g^{-1}))]$$

— this is a total derivative, whose integral vanishes over S^3 — Hence

$$\int_{S^3} \mathrm{tr}\,(g^{-1}dg)^3$$

is a homotopy invariant.

As is $\quad \int_M \mathrm{tr}(g^{-1}dg)^3 \quad$ where $\partial M = 0$, M any 3-manifold

Exercise: $\int_{S^k} \mathrm{tr}\,(g^{-1}dg)^k$

is homotopy invariant of map $S^k \to G$ for other odd values of K

Now recall for $G=U(1)$ on S^2,

$F = dA$ where A may not be globally defined on S^2. For $G=U(1)$ on four manifold, write the four-form

$$F^2 = dA\,dA = d(A\,dA) = d(AF)$$

– the three form $\omega_3 = AF$ is the Chern-Simons form for $G=U(1)$

Want to generalize to other gauge groups

$$F = dA + A^2$$

$$\Rightarrow \quad dF = +(dA)A - A\,dA = [dA, A] = [F, A]$$

$$\Rightarrow dF + [A, F] = [d+A, F] = [D, F] = 0$$

This is Bianchi identity $[D_\sigma, F_{\mu\nu}]\,dx^\sigma dx^\mu dx^\nu = 0$

This means that $\mathrm{tr}\,F^m$ is a closed form

$$d\,\mathrm{tr}(F^m) = m\,\mathrm{tr}(dF\,F^{m-1}) = -m\,\mathrm{tr}[A,F]\,F^{m-1} = -\,\mathrm{tr}[A, F^m] = 0$$

While $\mathrm{tr}\, F^m$ is closed, it might not be exact (= exterior derivative of a globally defined form) so it can represent a nontrivial cohomology class.

Consider

$$\mathrm{tr}\, F^2 = \mathrm{tr}\,(dA + A^2)(dA + A^2)$$

Note that $\mathrm{tr}(A^4) = 0$, since trace is cyclic and wedge product is antisymmetric

$$\mathrm{tr}\, F^2 = \mathrm{tr}\left(dA\,dA + A^2 dA + dA A^2\right)$$
$$= d\, \mathrm{tr}\left(A dA + \tfrac{2}{3} A^3\right) = d\omega_3$$
$$\omega_3 = \mathrm{tr}\left(A dA + \tfrac{2}{3} A^3\right)$$
$$= \mathrm{tr}\left(AF - \tfrac{1}{3} A^3\right)$$

Note: $\mathrm{tr}(A^2 dA) \propto c^{abc} A^a_\mu A^b_\nu \partial_\lambda A^c_\sigma \, dx^\mu dx^\nu dx^\lambda dx^\sigma$

i.e. $\mathrm{tr}\, A dA A = -\mathrm{tr}\, A^2 dA$

is Chern-Simons 3-form.

Now consider a G-gauge field on S^4. Represent S^4 as B^4 with boundary S^3 identified as a point, and suppose $F = 0$ on boundary. E.g. The Euclidean action on $\mathbb{R}^4$

$$\frac{1}{4g^2} \int F_{\mu\nu} F^{\mu\nu} = \frac{1}{2g^2} \int F * F$$

is finite if F falls off faster than $\frac{1}{r^2}$ as $r \to \infty$

Pure gauge $A = g^{-1} dg$ falls off like $1/r$

Then $\int \operatorname{tr} F^2 = -\frac{1}{3} \int_{S^3} \operatorname{tr} A^3$

$$= -\frac{1}{3} \int_{S^3} \operatorname{tr}(g^{-1} dg)^3$$

Let's evaluate for $G = SU(2)$.
We can write

$$g = (a_0 + i\vec{a}\cdot\vec{\sigma}) \qquad a_0, a_1, a_2, a_3 \in \mathbb{R}$$
$$a_0^2 + \vec{a}^2 = 1$$

Then
$$g^\dagger g = (a_0 - i\vec{a}\cdot\vec{\sigma})(a_0 + i\vec{a}\cdot\vec{\sigma})$$
$$= a_0^2 + a_i a_j \sigma^i \sigma^j = (a_0^2 + \vec{a}^2) I$$

So $SU(2)$ is topologically S^3. For the identity map $S^3 \to SU(2)$, we can use spherical symmetry to simplify evaluation of integral

i.e. $g(x) = (x_0 + i\vec{x}\cdot\vec{\sigma})$ where $(\vec{x}_0, \vec{x})$, $x_0^2 + \vec{x}^2 = 1$, parametrizes S^3

Then, in vicinity of $x_0 = 1$
$$g^{-1} dg = i\, d\vec{x}\cdot\vec{\sigma}$$

$$\operatorname{tr}(g^{-1}dg)^3 = (i)^3 dx^a dx^b dx^c \underbrace{\operatorname{tr}(\sigma^a \sigma^b \sigma^c)}_{2i\varepsilon^{abc}}$$

$$= 2\, dx^1 dx^2 dx^3\, \varepsilon^{abc} \varepsilon^{abc} = 12\, dV$$

where dV is volume element on the three-sphere. So -- for identity map

$$\int_{S^3} \mathrm{tr}(g^{-1}dg)^3 = 12\ (\text{Volume of } S^3)$$
$$= 12(2\pi^2) = 24\pi^2$$

More generally

$$\frac{1}{24\pi^2}\ \mathrm{tr}\left(g^{-1}dg\right)^3$$

is just the (signed) invariant volume element on the group $SU(2)$ and

$$\frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}(g^{-1}dg)^3 = n$$

is winding number: i.e. no. of times the map $S^3 \to SU(2)$ covers the group

For our gauge field on the 4-sphere, the form

$$\frac{-1}{8\pi^2}\ \mathrm{tr}\, F^2$$

is an integral cohomology class

$$\frac{-1}{8\pi^2}\int \mathrm{tr}\, F^2 = \text{integer}$$

Not just for S^4, but any 4 manifold. This is the "second chern class" of the bundle.

[Exercise: higher chern classes]

In our old notation

$$\frac{-1}{8\pi^2}\int tr F^2 = \frac{1}{8\pi^2}(-ig)^2 \int \tfrac{1}{2} F^a_{\mu\nu} dx^\mu dx^\nu \tfrac{1}{2} F^b_{\lambda\sigma} dx^\lambda dx^\sigma \; tr(T^a T^b)$$

$$= \frac{-g^2}{32\pi^2}\int dx^0 dx^1 dx^2 dx^3 \tfrac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma}$$

Denoting $\tilde{F}^{\mu\nu a} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} F^a_{\lambda\sigma}$ (dual tensor)

$$\nu = \frac{g^2}{32\pi^2}\int d^4x \; F^a_{\mu\nu}\tilde{F}^{\mu\nu a} = \text{integer}$$

For a map $S^3 \to G$, there is a continuous deformation so that image is in an $SU(2)$ subgroup of G, so ν can be interpreted as winding about the $SU(2)$ subgroup

Instantons

Why do we care about nontrivial classes in 4d Euclidean spacetime? Because they correspond to physical processes that have interesting qualitative effects, and that can be analyzed semiclassically.

Green functions "Wick rotated" to Euclidean spacetime $dx^0 = -idt$ are generated by Euclidean path integral. E.g.

$$S = \int d^D x \left[\tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)\right]$$

$$\to \int -i d^D x_E \left[\tfrac{-1}{2}(\nabla\phi)^2 - V(\phi)\right]$$

Hence $e^{iS} \to e^{-S_E}$, where S_E is positive definite

$$S_E = \int \left[\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)\right] d^D x_E$$

In pure Yang-Mills theory (no matter)

$$S_E = \frac{1}{4}\int d^4x \, F_{\mu\nu} F^{\mu\nu} + \text{sources} + \text{gauge-fixing}$$

Semiclassical approximation: $\hbar \to 0$

$$\int e^{-S_E/\hbar}$$ evaluated by steepest descent

i.e. integrate over small fluctuations about stationary points of the action. Expanding around $A = 0$ (minimum of S_E) generates ordinary Feynman diagram P.T.

But there are also other local minima of S_E, and associated semiclassical corrections to naive perturbation theory. These are small

$$\sim e^{-S_{E,0}/\hbar}$$

where $S_{E,0}$ is Euclidean action of nontrivial solution to field eqns.

But if they produce effects that differ qualitatively from effects seen in p.t., they can still be interesting.

In YM, there is a "topological conservation law" in 4D. It is not really a conserved topological charge. But we can find finite action configurations that can't be smoothly deformed to $A=0$.

Minimizing action in these sectors, we find nonvacuum solns to the Euclidean field eqns — i.e. new semiclassical corrections.

The nontrivial sectors are those with

$$\nu = -\frac{1}{8\pi^2}\int \star F^2 \neq 0$$

We can find the minimum action in a sector with $\nu \neq 0$, because there is a Bogomolnyi bound:

$$0 \leq \left(F^a_{\mu\nu} \mp \tilde{F}^a_{\mu\nu}\right)^2 = 2\left(F^a_{\mu\nu} F^{\mu\nu a} \mp F^a_{\mu\nu} \tilde{F}^{\mu\nu a}\right)$$

because

$$\tilde{F}^{\mu\nu a} \tilde{F}^a_{\mu\nu} = \frac{1}{4} \underbrace{\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\alpha\beta}}_{\frac{1}{2}\left(\delta^\lambda_\alpha \delta^\sigma_\beta - \delta^\lambda_\beta \delta^\sigma_\beta\right)} F^a_{\lambda\sigma} F^{a\alpha\beta}$$

$$= F^a_{\lambda\sigma} F^{a\lambda\sigma}$$

Therefore $F^a_{\mu\nu} F^{\mu\nu a} \geq \pm F^a_{\mu\nu} \tilde{F}^{\mu\nu a}$

$$S_E = \frac{1}{4} \int d^4x \, F^a_{\mu\nu} F^{\mu\nu a} \geq \pm \frac{1}{4} \frac{32\pi^2}{g^2} \int d^4x \, \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a}$$

$$= \pm \frac{8\pi^2}{g^2} \nu$$

Choosing appropriate sign

$$S_E \geq \frac{8\pi^2}{g^2} |\nu|$$

Inequality saturated for

$\nu > 0$: $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ "self-dual"

$\nu < 0$: $F_{\mu\nu} = -\tilde{F}_{\mu\nu}$ "anti-self-dual"

Since these configurations minimize S_E in sector with $\nu \neq 0$, they solve YM field equations (S_E is stationary)

Second-order field eqns reduce to first-order self-duality condition.

For $\nu = \pm 1$, we have

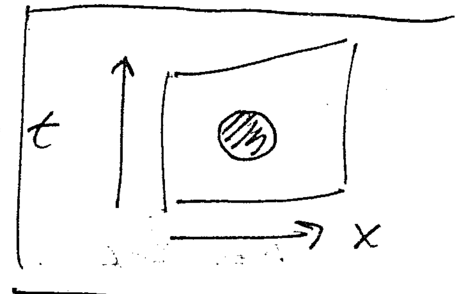
$$e^{-S_{E,0}/\hbar} = e^{-8\pi^2/\hbar g^2}$$

- exponentially small semiclassical corrections. (Smaller than any finite order of perturbation theory)

Instantons and Quantum Tunneling

A recurring theme of this course is that field configurations may fall into distinct sectors. E.g. finite energy configurations — then we can construct static solutions to field eqns by minimizing energy in a nontrivial sector, obtaining solitons (monopoles, vortices, kinks) or extended objects (strings, domain walls).

Now we have encountered another such classification — configurations of finite Euclidean action fall into topological sectors. By minimizing in a nontrivial sector, we find solutions to the Euclidean field eqns, solutions localized in space and time (hence the name "instanton"). These are of interest for evaluating path integral in semiclassical (small-$\hbar$) approximation:



$$\text{contribution} \sim O(e^{-S_0/\hbar})$$

where $S_0 > 0$ is action of instanton. These are effects smaller than those seen in Feynman diagram perturbation theory ($\sim \hbar^{\text{power}}$), but still of interest if qualitatively distinct from perturbative effects.

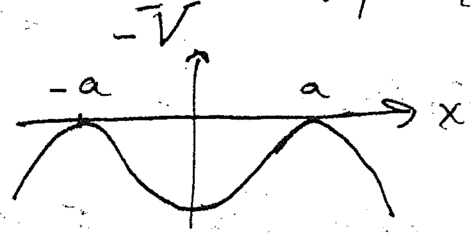
Note that what is a soliton in one dimensionality may be an instanton in another:

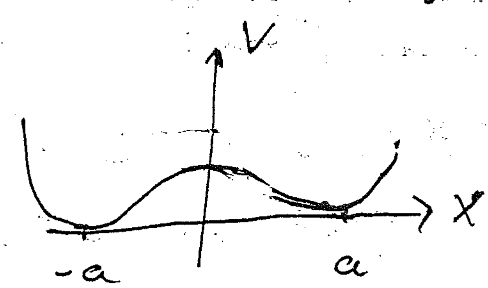
	0+1	1+1	2+1	3+1	4+1
Kink (codim 1)	instanton	particle	string	wall	—
vortex (codim 2)		instanton	particle	string	wall
monopole (codim 3)			instanton	particle	string
Yang-Mills instanton (codim 4)				instanton	particle

Let's consider the kink - cf discussion on p. 218. Think of ϕ as the position x of a particle, parametrized by Euclidean time t. Euclidean action

$$S_E = \int_{-\infty}^{\infty} dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \right]$$

has the same form as considered earlier for energy of 1+1 dimensional field $\phi(x)$.





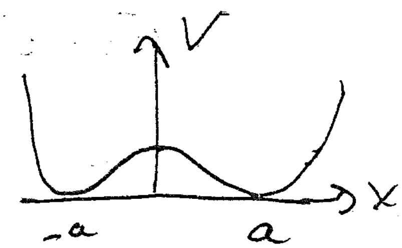
stationary action $\delta S_E = 0 \Longrightarrow$ trajectory of a particle in inverted potential

E.g. solution with $x \to -a$ as $t \to -\infty$:

$$\frac{1}{2}\left(\frac{dx}{dt}\right) = V(x) \qquad S_0 = \int_{-\infty}^{\infty} dt \, 2V(x) = \int_{-a}^{a} dx \sqrt{2V(x)}$$

This solution gives the leading contribution (as $\hbar \to 0$) to

$$\langle a | e^{-HT} | -a \rangle \sim K e^{-S_0/\hbar}$$



Note that

$$\exp\left[-\frac{1}{\hbar}\int_{-a}^{a} dx \sqrt{2V}\right]$$

is WKB tunneling factor associated with tunneling through barrier, i.e.

$$\frac{1}{2}\hbar^2 K^2 = E - V = -V$$

$$\Rightarrow K = i\sqrt{2V}/\hbar$$

Let's make this connection more firmly. First K arises from integrating over small fluctuations about the instanton solution $X_0(t)$. If

$$x(t) = X_0(t) + \delta x(t)$$

then $S_E = \int dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right]$

$$= S_0 + \int dt\, \delta x \left[-\frac{1}{2}\frac{d^2}{dt^2} + \frac{1}{2}V''(x_0)\right]\delta x + O(\delta x^3)$$

(no linear term, because $\delta S_E = 0$ at x_0.) Integrating over small fluctuations about local minimum of S_E:

$$\sim e^{-S_0/\hbar}\int d(\delta x)\exp\left[-\frac{1}{\hbar}(\delta x, A\,\delta x) + O(\delta x)^3\right]$$

We rescale $\delta x = \hbar \delta \tilde{x}$, expand $\exp(-\hbar^{\frac{1}{2}} O[(\delta x)^3])$ and do Gaussian integrals $\Longrightarrow$

$$\sim N e^{-S_0/\hbar} (\det A)^{-\frac{1}{2}} [1 + O(\hbar)]$$

where A is the differential operator

$$A = -\frac{d^2}{dt^2} + V''(x_0)$$

However, there is a subtlety — the operator A has a zero mode (eigenstate with eigenvalue zero). Since

$$\ddot{x}_0 = V'(x_0)$$

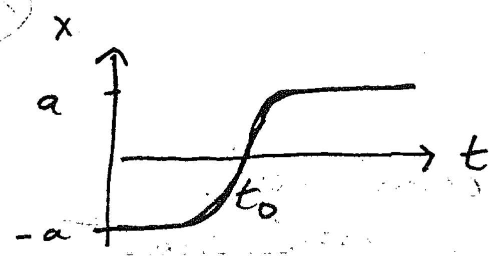
$$\Rightarrow \frac{d}{dt}\ddot{x}_0 = \frac{d^2}{dt^2}\dot{x}_0 = V''(x_0)\dot{x}_0,$$

we have $$\left[-\frac{d^2}{dt^2} + V''(x_0)\right]\dot{x}_0 = 0$$

The zero mode arises because of time translation symmetry of S_E — If $x_0(t)$ is a solution to $\delta S_E = 0$, then so is

$$x_0(t+\Delta) = x_0(t) + \Delta \dot{x}_0(t)$$

Therefore $(\det A)^{-\frac{1}{2}}$ is ill defined; we need to treat the integration along the mode $\delta x \propto \dot{x}_0$ separately



the "center" t_0 of the instanton can be placed anywhere in time interval T.

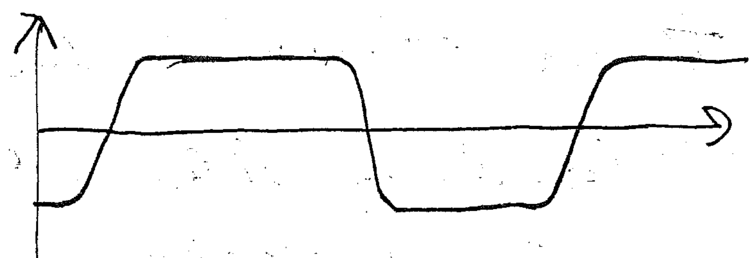
We obtain for the one-instanton contribution to the path integral

$$T K e^{-S_0/\hbar}$$

where $K = N(\det{}' A)^{-\frac{1}{2}}$

$\Longleftarrow$ det′ is product of nonzero eigenvalues

The divergence as $T \to \infty$ is a signal that there are other important contributions that we have not yet included. These are due to widely separated instantons



etc..

Each instanton contributes a separate factor $K e^{-S_0}$ to the path integral (if instantons are far apart and hence "noninteracting"), apart from the zero mode integration. Integrating over the center of each instanton generates a factor of

$$\frac{1}{n!} T^n$$

if there are n instantons arranged in a specified

order (to avoid overcounting).
For $\langle a | e^{-HT} | -a \rangle$, there must be an odd number of instantons, and so we find

$$\langle a | e^{-HT} | -a \rangle = C \sum_{n\,odd} \frac{1}{n!} \left(K T e^{-S_0} \right)^n$$

$$C \tfrac{1}{2} \left[\exp\left(T K e^{-S_0} \right) - \exp\left(- T K e^{-S_0} \right) \right]$$

To interpret, consider summing over energy eigenstates

$$\langle a | e^{-HT} | -a \rangle = \sum_n \langle a | e^{-HT} | n \rangle \langle n | -a \rangle$$

$$= \sum_n \langle a | n \rangle \langle n | -a \rangle \; e^{-E_n T}$$

- dominated for $T \to \infty$ by low lying states

We find two low lying states, with energies

$$E = \pm K e^{-S_0},$$

the usual splitting of ground state degeneracy due to quantum tunneling.

Note that the dominant contribution to the path integral comes not from configurations that have finite action as $T \to \infty$, but rather finite action <u>density</u> per unit of Euclidean time.

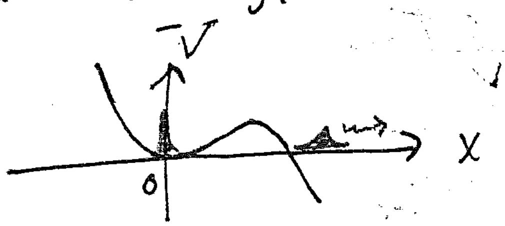
the sum over n is peaked at

$$n \sim K T e^{-S_0} \quad \text{or} \quad \frac{n}{T} \sim K e^{-S_0}$$

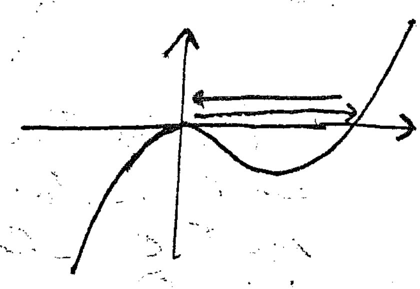
our approximations (neglecting instanton interactions) are reasonable for

$$\frac{n}{T} \sim K e^{-S_0/\hbar} \ll (\text{instanton size})^{-1}$$

It is also interesting to consider, not the lifting of classical degeneracy due to tunneling, but the decay of an unstable state.



How quickly does a state initially localized at $x=0$ tunnel through the potential barrier?

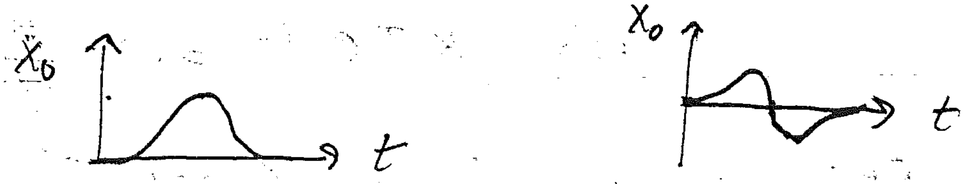


Solution in inverted potential is a "bounce" that approaches $x=0$ for $t \to \pm\infty$, and bounces off the wall at $t=t_0$. The bounce has action S_B.
There is no restriction that number of bounces be even or odd. So summing "dilute" bounces gives, naively

$$\langle 0|e^{-HT}|0\rangle \sim \exp\left(T K e^{-S_B/\hbar}\right)$$

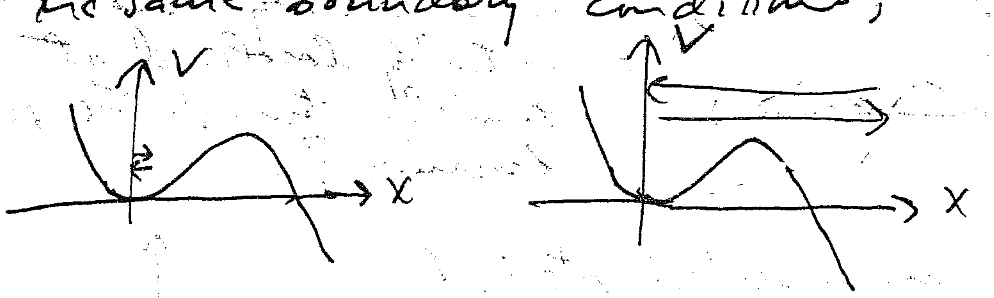
But again there is a subtlety. This time $\left[-\frac{d^2}{dt^2} + V''(x_0)\right]$ has not, just the zero mode

$\dot{X}_0(t)$, but also a single negative mode



Since $\dot{X}_0$ has one node, it is not the lowest eigenstate of $-\frac{d^2}{dt^2}+V''(X_0)$ — there is one lower (hence negative) mode.

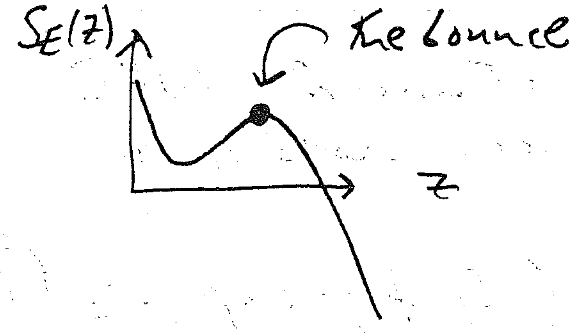
Unlike an isolated instanton, the bounce "starts" and "finishes" at the same point $X=0$. There are histories of lower action obeying the same boundary conditions,



e.g. histories that stay near $X=0$ at all times, and histories that spend a lot of time in the classically allowed region where $V<0$ (for these, we may have $S_E \to 0$)

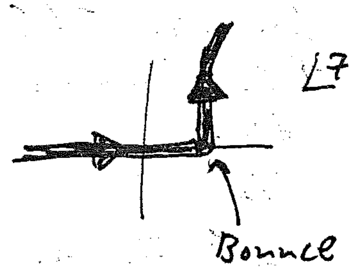
So we need to handle with care, not just the integration over the zero mode, but also the integral over the negative mode.

There is a direction (parametrized by z) in field configuration space along which the bounce is a local maximum

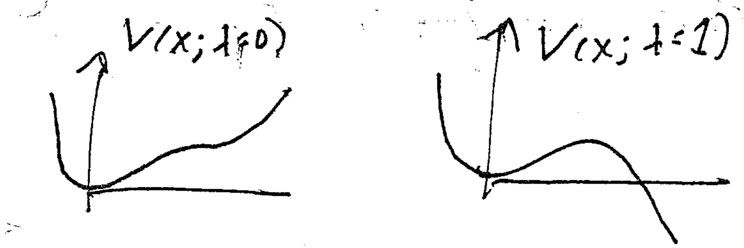


the integral $\int dz\, e^{-S_E(z)/\hbar}$ actually diverges

But we can hope to define it as an analytic continuation of a convergent integral; this is a standard procedure in the theory of asymptotic expansions — an integral representation of a function $f(\lambda) = \int dz\, e^{g(\lambda,z)}$ can be continued away from the region in λ such that the integral converges, if we distort the contour of the z integration as λ varies.



E.g. suppose the potential is $V(x;\lambda)$, and



nonneg for $\lambda=0$, but goes negative for $\lambda=1$. We need to distort the z-contour into the imaginary direction to obtain a convergent integral — which has an imaginary part

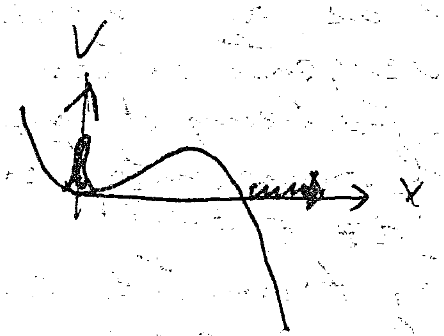
The imaginary part of the energy obtained via analytic continuation of an energy eigenstate can be interpreted as a width or decay rate of the state

$$-\mathrm{Im}\, E = \Gamma/2 = \tfrac{1}{2}|K|e^{-S_B/\hbar}$$

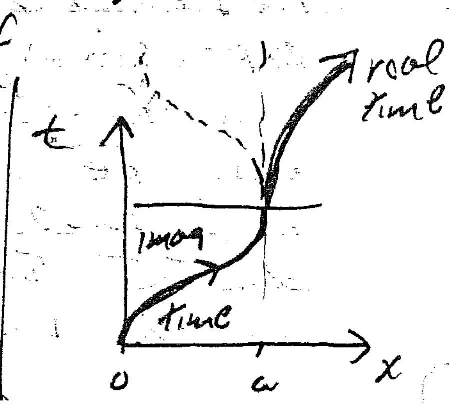
i.e. pole in $G(E)$ located at $E_0 - i\Gamma/2$

So we recover the standard WKB expression for decay rate of an unstable state

$$\Gamma \propto \exp(-S_B/\hbar) = \exp\left(-2\int_0^a dx\,\sqrt{2V(x)}\right)$$



The bounce is symmetric in time. If we cut it in half, it describes the penetration in imaginary time through the barrier. At the turning point where $\frac{dx_0}{dt}=0$, it can be "matched on" to real time evolution in the classically allowed region.

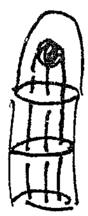


Examples

For the (0+1)-dimensional case, the instanton methods reproduce results that can be obtained as easily with more standard methods. But in higher dimensional systems, the instanton approach to quantum tunneling is far more convenient and efficient.

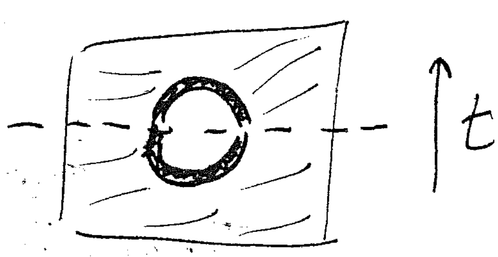
E.g. consider, as discussed previously, a gauge symmetry breakdown with a hierarchy, e.g.

$$SU(2) \xrightarrow{v_1} U(1) \xrightarrow{v_2} I, \quad v_1 \gg v_2.$$



strings arising in the second stage can terminate on monopoles that arise at the first stage.

Hence a string can break via nucleation of a monopole pair. At what rate does this process occur?



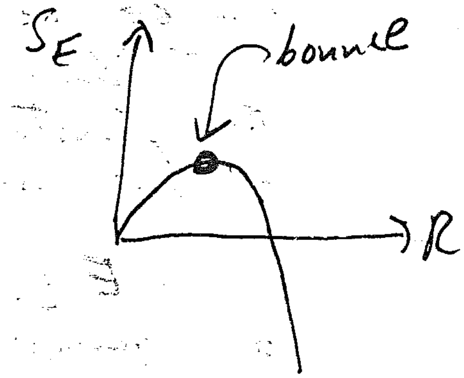
The "bounce" is an unstable stationary point of S_E in which the world line of the monopole is the boundary of a hole in the world sheet of the string. The action is

$$S_E = m \cdot (\text{perimeter}) - K \cdot (\text{area})$$

where m is monopole mass, K is string tension

For a hole of fixed area, the shape that minimizes the action is a circle. For a circle of radius R

$$S_E = 2\pi R m - \pi R^2 K$$



There is a local maximum (the bounce) for

$$\frac{dS_E}{dR} = 0 = 2\pi m - 2\pi R K$$

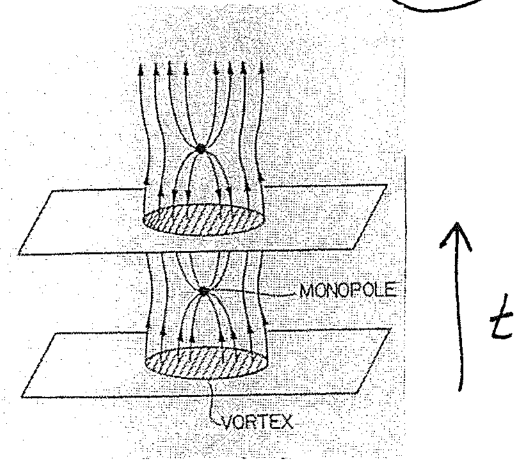
$$\Rightarrow R = m/K$$

and $\quad S_B = \pi m^2 / K$

Now the bounce has 2 zero modes, because of translation invariance in the world sheet (the decay can occur anywhere along the string). We obtain a decay rate per unit length

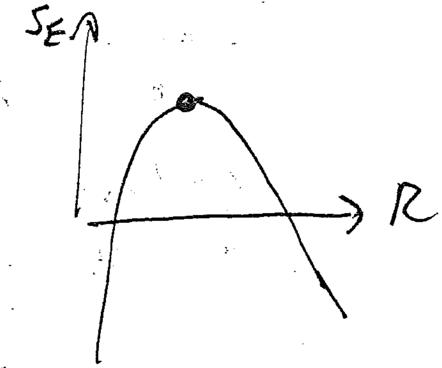
$$\Gamma/L \propto \exp\left(-\pi m^2 / \hbar K\right)$$

The instability has a different interpretation in the same model in 2+1 dimensions. Were it not for the higher scale of symmetry breakdown, the vortex would be a stable particle. The monopole allows the vortex to decay and dissipate — i.e. a finite action config changes the flux by one flux quantum.



The bounce in this case is a monopole-antimonopole pair, with separation comparable to the string thickness. The Coulomb attraction of the pair is balanced by the string tension. Roughly speaking,

$$S_E \sim 2m - g^2/4\pi R - KR.$$



The separation R that maximizes S_E is given by

$$g^2/4\pi R^2 = K$$

or $R^2 \sim \frac{1}{e^2K} \sim \frac{1}{m_V^2}$ (m_V = mass of vector meson ~ inverse size of string)

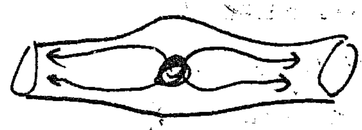
and $S_B \sim 2m \Rightarrow \Gamma \sim e^{-2m/\hbar}$

is the vortex decay rate. The monopole "mass" m is $m \sim v/e$, which has dimensions of $\hbar$ in 2 spatial dimensions.

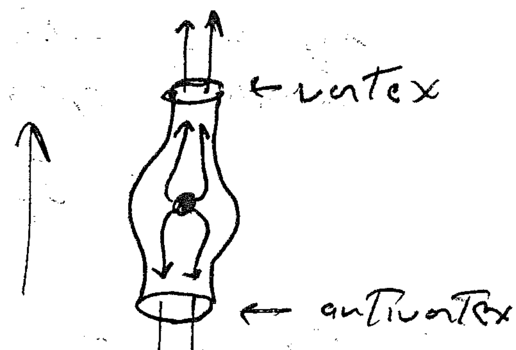
The configuration on the plane that cuts through the center of the bounce is the field configuration that the vortex tunnels to. It has zero total flux and can relax to vector and Higgs particles (e.g. $\frac{v^2}{ev} \sim \frac{v}{e}$ vectors).

Now consider the hierarchy
$$SU(2) \xrightarrow{v_1} U(1) \xrightarrow{v_2} Z_2 ,$$

for which we have observed there can be a "bead" on a string.



In 2+1 dimensions, the bead becomes an instanton, that lifts the degeneracy of vortex and antivortex via quantum tunneling.



We have $S_0 \sim m$, and hence energy splitting $\Delta E \propto e^{-m/\hbar}$