Instantons in Yang-Mills Theory

We have already seen that the Euclidean field equations of Yang-Mills theory have instanton solutions with finite action:

$$S_E = \frac{8\pi^2}{g^2}|\nu| \; ; \quad \nu = -\frac{1}{8\pi^2}\int \mathrm{Tr}\, F^2 = \frac{1}{32\pi^2}\int d^4x \; F_{\mu\nu}^a \tilde{F}^{\mu\nu a}$$

And we have discussed several examples of how solutions to Euclidean equations of motion are useful for a semiclassical analysis of quantum tunneling phenomena. Now we want to understand the interpretation of the YM instanton in terms of quantum tunneling.

First let's recall the field eqns. The curvature two-form is

$$F = dA + A^2 = \tfrac{1}{2}(-ig F_{\mu\nu})\, dx^\mu dx^\nu$$

where $\quad A = (-ig A_\mu)\, dx^\mu$

Under gauge transformation

$$A \to \Omega^{-1} A \Omega + \Omega^{-1} d\Omega$$
$$F \to \Omega^{-1} A \Omega$$

If gauge transformation is infinitesimal

$$\Omega = I + \omega, \qquad A \to (I-\omega) A (I+\omega) + (I-\omega)\, d\omega$$
$$\Omega^{-1} = I - \omega \qquad\qquad = A + [A, \omega] + d\omega$$

or $\omega: A \to A + \delta_\omega A$, where $\delta_\omega A = [D, \omega] \equiv [d + A, \omega]$

One field equation, the Bianchi identity, is nondynamical

$$F = dA + A^2 \Rightarrow dF = dA^2 = dA\,A - A\,dA = -[A, dA]$$

$$\text{or } 0 = dF + [A, dA] = dF + [A, F]$$

$$\Rightarrow \quad \boxed{0 = [D, F]}$$

Let's write the YM action in the form notation:

$$S_E = \frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{\mu\nu a}$$

The Hodge dual of two-form F is

$$*F = \frac{1}{2}(-ig\,\tilde{F}_{\mu\nu})dx^\mu dx^\nu$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$$

$$F*F = -g^2/4\; F_{\mu\nu}dx^\mu dx^\nu \frac{1}{2}\epsilon_{\lambda\sigma\alpha\beta}F^{\alpha\beta}dx^\lambda dx^\sigma$$

and

$$dx^\mu dx^\nu dx^\lambda dx^\sigma = d^4x\, \epsilon^{\mu\nu\lambda\sigma}$$

$$\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\lambda\sigma\alpha\beta} = \delta^\lambda_\alpha\delta^\sigma_\beta - \delta^\lambda_\beta\delta^\sigma_\alpha$$

$$\Rightarrow \quad F*F = -\frac{g^2}{2}F_{\mu\nu}F^{\mu\nu}d^4x$$

$$\text{tr}\, F*F = -g^2/4\; F^a_{\mu\nu}F^{\mu\nu a}d^4x \qquad (\text{tr}\, T^aT^b = \tfrac{1}{2}\delta^{ab})$$

$$\Rightarrow \quad S_E = -\frac{1}{g^2}\int \text{tr}(F*F)$$

The Euler-Lagrange equation is

$$0 = \delta S_E = -\tfrac{2}{g^2} \int \mathrm{tr}\,(\delta F * F)$$

(Trace is cyclic, two-forms commute)

$$\delta F = d(\delta A) + A(\delta A) + (\delta A) A$$

$$\Rightarrow 0 = \int \mathrm{tr}\left[(d\,\delta A + A(\delta A) + (\delta A) A) * F \right]$$

use $d(\delta A * F) = d(\delta A) * F - \delta A\, d(*F)$

$$\Rightarrow d(\delta A) * F = d(\delta A * F) + \delta A (d * F)$$

(Integrates to zero for δA fixed on boundary)

Also — Trace is cyclic, and (-1) for interchanging odd-forms

$$\Rightarrow 0 = -\int \mathrm{tr}(\delta A)\left[d(*F) + A(*F) - (*F) A \right]$$

Vanishes for any δA obeying B.C. $\Rightarrow$

$$\boxed{0 = d(*F) + [A, *F] = [D, *F]}$$

— The YM field equation

In components $[D_\mu, \tilde{F}_{\nu\lambda}]\, dx^\mu dx^\nu dx^\lambda = 0$

$G = 0 \Leftrightarrow *G = 0$, so dual one-form

$$0 = \tfrac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} [D^\mu, \tilde{F}^{\nu\lambda}]\, dx^\sigma$$

i.e. $[D^\mu, F_{\mu\nu}] = 0$

Now consider quantizing the theory in the gauge $A_0 = 0$, as we discussed for the abelian case on p 57 ff. We must impose the A_0 field equation as a constraint on the physical Hilbert space. This is Gauss's law

$$[D^i, E_i] = 0$$ (in pure YM theory w/o matter)

Thus — for any ω, a position dependent element of the Lie algebra, we must have

$$0 = \int_\Sigma \mathrm{tr}\, \omega [D, *F]$$

Σ → integrated over a "time slice" i.e. 3-surface with x^0 = constant

Now perform these manipulations

$$0 = \int_\Sigma \mathrm{tr}\, \omega \left(d{*F} + A{*F} - {*F}A \right)$$

$$= \int_\Sigma d\left[\mathrm{tr}(\omega * F)\right]$$

$$+ \int_\Sigma \mathrm{tr} \left(-d\omega * F + \omega A * F - A\omega * F \right)$$

And $d\omega + A\omega - \omega A = [D, \omega] = \delta_\omega A$

$$\Rightarrow \quad 0 = \int_{\partial\Sigma} \mathrm{tr}(\omega * F) - \int_{\Sigma} \mathrm{tr}(\delta_\omega A) * F$$

Let's rewrite this in standard notation,
with $\Omega = I - ig\omega^a T^a$

$$0 = \int_{S^2_\infty} d^2\vec{S}\cdot(-ig\omega^a)\vec{E}^a - \int d^3x(-ig)\,\delta_\omega \vec{A}^a\cdot\vec{E}^a$$

Furthermore, $E_i^a = \frac{\partial \mathcal{L}}{\partial A_i^a}$ is momentum conjugate to A_i^a

In the Schrödinger representation of the canonical commutation relations

$$E_i^a = -i\frac{\delta}{\delta A_i^a}$$

acting on wave-functional $\Psi[A]$
So infinitesimal gauge transformation acts in the Hilbert space as

$$\delta_\omega = \int d^3x\, \delta_\omega A_i^a \frac{\delta}{\delta A_i^a} = i\int d^3x\, \delta_\omega A_i^a E_i^a$$
$$= i\int_{S^2_\infty} d^2\vec{S}\cdot(\omega^a\vec{E}^a)$$

Thus if $\omega^a \to 0$ as $r\to\infty$, then $\delta_\omega = 0$.
we have

States in $\mathcal{H}_{physical}$ are invariant under gauge transformations with $\omega\big|_{S^2_\infty} = 0$

This also applies to finite gauge transformations that can be built by composing infinitesimals with $\omega\big|_{S^2_\infty} = 0$. That is, gauge trans

$$\exp(-ig\,\omega^a(x)\,T^a)$$

acts trivially on $\mathcal{H}_{physical}$ if $\omega^a(x)\big|_{S^2_\infty} = 0$

Gauge transformations with this property are called "small" gauge transformations.

Since $\mathbb{R}^3$ is contractible, a general gauge transformation on $\mathbb{R}^3$ can also be built out of infinitesimals. We find the

$$\exp(-ig\,\omega^a(x)\,T^a)$$

is represented acting on the physical Hilbert space by the unitary transformation

$$U[\omega] = \exp\left[i\int d^2\vec{S}\cdot(\omega^a\vec{E}^a)\right]$$

In particular, this formula tells us how <u>global</u> gauge transformations act on $\mathcal{H}_{phys}$. A global gauge transformation has the property

$$\omega^a\Big|_{S^2_\infty} = \text{constant}$$

Define an operator $Q = Q^a T^a$ by

$$Q = \int d^2\vec{S}\cdot\vec{E}.$$

$(\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab} \Rightarrow Q^a = 2\,\mathrm{tr}(T^a Q).)$

Then we have $\exp(-i\omega)$ represented by

$$U[\omega] = \exp\left(i\omega^a \frac{1}{g} Q^a\right) = \exp\left[2i\,\mathrm{tr}\left(\omega\frac{Q}{g}\right)\right]$$

This gauge transformation is trivial if ω is any matrix with eigenvalues $2\pi\cdot$ integer. Then $U[\omega]$ should be trivial acting on $\mathcal{H}_{phys}$; i.e. $2\,\mathrm{tr}\,\omega\left(\frac{Q}{g}\right)$ should have integer eigenvalues for any ω with integer eigenvalues. It is therefore <u>necessary</u> that:

Eigenvalues of Q are $\frac{1}{2}g\cdot$ integer

– this is charge quantization

Vacua of Yang-Mills Theory

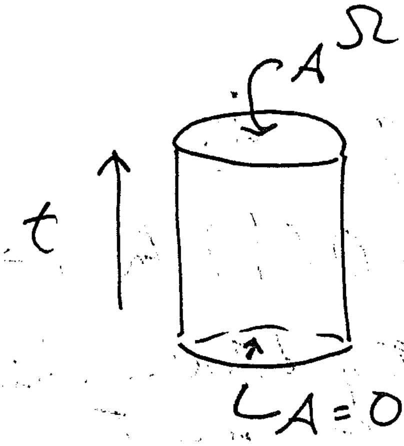
Now we want to understand the semiclassical vacuum structure of Yang-Mills theory. Classically, the configuration with $A=0$ has zero energy. In the quantum theory all states

$$|A = \Omega^{-1} d\Omega\rangle,$$

where Ω is a small gauge transformation, should be identified. They represent the same state in $\mathcal{H}_{phys}$.

But are there other states, distinct from $|A=0\rangle$ in $\mathcal{H}_{phys}$, that *also* have energy $= 0$?

Consider, first, a gauge transformation $\Omega = \exp(-ig\omega_a T^a)$ acting on $A=0$, where $\Omega\big|_{S^2_\infty} \neq$ constant. We will argue that these have no relevance to the vacuum structure, at least as far as semiclassical physics is concerned.



Consider the path-integral evaluation of the amplitude for a process in which one YM vacuum state evolves to another; this is

$$\langle \text{final} | e^{-HT} | \text{initial} \rangle = \int_{B.C.} [dA]\, e^{-S[A]}$$

where path integral is performed with $A = A_{\text{initial}}$ on initial slice and $A = A_{\text{final}}$ on final slice. (Gauge fixing is implicit.)

In $A^0 = 0$ gauge, we still have the freedom to do time-independent gauge transformations, so we can choose $A_{\text{initial}} = 0$ w/o loss of generality. Then the final vacuum state is

$$(-igA) = \Omega^{-1} \partial \Omega = (A=0)^{\Omega}$$

Under the assumption that $\Omega|_{S^2_\infty} \neq$ constant,

Ω has angular dependence as $r \to \infty$,

i.e. $\quad A \simeq A_\phi d\phi \sim \frac{i}{g} \frac{1}{r} \Omega^{-1} \partial_\phi \Omega \, d\phi$

where ϕ is an angular variable on S^2. The important point is that A is $O(\frac{1}{r})$

In $A_0 = 0$ gauge $E_i^a = \dot{A}_i^a$

If $A_{initial} = 0$ and $A_{final} = O(\frac{1}{r})$ for $r \to \infty$,

then any field history interpolating between A_{init} and A_{final} in time $T < \infty$ must have an electric field decaying like $\frac{1}{r}$, and so $S_E = \infty$. E.g., consider one component

$$S_E = \int_0^{T} dt \int d^3x \; \frac{1}{2}\dot{A}^2 + \cdots$$

To minimize just this term, solve Euler-Lagrange equation $\ddot{A} = 0$

$$\Rightarrow \quad A = \frac{t}{T}(A_{final}) \quad (\text{since } A_{init} = 0)$$

$$\dot{A} = \frac{1}{T}(A_{final}),$$

$$S_E \geq T \frac{1}{T^2} \int^{R} d^3x \, (A_{final})^2 \sim R/T \to \infty$$

if $R \to \infty$ with T fixed

As previously emphasized, configurations that dominate path integral actually have finite action density rather than finite action... But if we want to do a semiclassical analysis of vacuum tunneling, configurations with $S_E \to \infty$ and $e^{-S_E} = 0$ are of no interest. The classical vacua that are "semiclassically accessible" are those with

$$A = \Omega^{-1} d\Omega \quad \text{where} \quad \Omega|_{S^2_\infty} = \text{constant}$$

A special case of $\Omega|_{S^2_\infty}$ = constant is

$$\Omega = \exp(-ig\omega^a T^a), \quad \text{where } \omega^a|_{S^2_\infty} = \text{constant};$$

— that is, a global gauge transformation. We have already seen that such an Ω is represented acting on $\mathcal{H}_{phys}$ by

$$U[\omega] = \exp(i\omega^a Q^a)$$

But this $U[\omega] = I$ acting on the charge-zero sector of the theory, $Q = 0$. This sector includes all vacuum states (which have $*F = 0$). Therefore a global gauge transformation acting on $|A = 0\rangle$ leaves the state invariant, rather than taking it to another semiclassically accessible vacuum state.

In fact, any $\Omega_0 \in G$ can be expressed as $\exp(-ig\omega_0^a T^a)$ for some ω_0, so there is a global gauge transformation that has the form

$$\Omega = \exp(-ig\omega^a T^a), \quad \omega^a|_{S^2_\infty} = \omega_0^a = \text{constant}$$

This means that we can fully account for all semiclassically accessible vacua by confining our attention to gauge transformations Ω such that

$$\Omega|_{S^2_\infty} = I.$$

(we just multiply Ω with $\Omega|_{S^2_\infty} = \Omega_0$ by a global gauge transformation, with value Ω_0^{-1} on S^2_∞). Indeed, the gauge transformations of interest are those such that

$$\Omega = \exp(-ig\omega_a T^a) \text{, where}$$

$$\Omega\Big|_{S^2_\infty} = I \text{ ; } \quad \omega\Big|_{S^2_\infty} \neq \text{constant,}$$

and furthermore it is not possible to smoothly deform $\omega(\vec{x})$ to a constant on S^2_∞, while maintaining $\Omega|_{S^2_\infty} = I$ and ω globally defined (i.e. smooth) on $\mathbb{R}^3$. (If the gauge transformation ω is infinitesimal, then the only way to have $\Omega = I$ is $\omega = 0$ ...

If $\Omega = I$ on S^2_∞, we can regard Ω as a function on compactified $\mathbb{R}^3$, i.e. S^3. The semiclassical accessible vacua are in one-to-one correspondence with the homotopy classes of maps

$$S^3 \to G \text{ ,}$$

i.e., the elements of $\pi_3(G)$. And we know that

$$\pi_3(G) = \mathbb{Z} \quad \text{for any simple compact Lie group.}$$

This (infinite and discrete) group $\mathbb{Z}$

can be regarded as a <u>symmetry</u> of the Yang Mills theory: it is the vestige of the gauge symmetry that survives, on S^3, when the small gauge transformations are modded out (small meaning homotopically <u>trivial</u> on S^3), as the small g.t. act trivially on $\mathcal{H}_{phys}$

$$\frac{\{\text{All } \Omega \text{ on } S^3\}}{\{\text{small } \Omega \text{ on } S^3\}} = \pi_3(G) = \mathbb{Z}$$

This group, associated with Ω such that $\Omega|_{S^2_\infty} = 1$, is quite distinct from the <u>continuous</u> group of global gauge transformations (which act trivially on the $A=0$ vacuum).

To be more explicit, consider

$$\Omega : \mathbb{R}^3 \to SU(2)$$

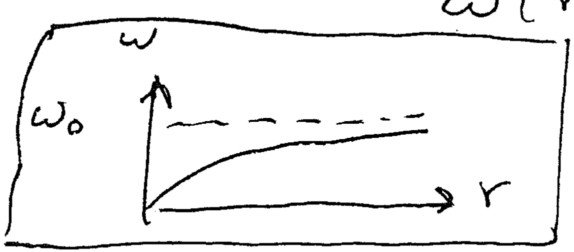
where

$$\Omega(\vec{x}) = \exp\left[i \frac{\omega(r)}{2} \vec{\sigma}\cdot\hat{r}\right]$$

$$= \cos\left(\frac{\omega(r)}{2}\right) + i\,\vec{\sigma}\cdot\hat{r}\,\sin\left(\frac{\omega(r)}{2}\right)$$

where

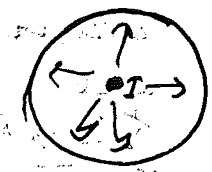
$$\omega(r) = \begin{cases} 0 & r = 0 \\ \omega_0 & r = \infty \end{cases}$$



i.e. $\Omega(\vec{x})$ is a gauge rotation by angle $\omega(r)$ about axis $\hat{r}$

Hence $\Omega|_{S^2_\infty} \neq$ constant unless $\omega_0 = 2\pi p$, p = integer. In fact, for $\omega_0 = 2\pi n$, we have

$$\Omega|_{S^2_\infty} = (-1)^n I$$



and p is the number of times the map covers $SU(2)$ (the <u>winding</u> <u>number</u>).

For $\omega_0 \neq 2\pi p$, the state $|\Omega^{-1} d\Omega = A\rangle$ is not "semiclassically accessible" — only for the isolated values $\omega_0 = 2\pi p$ does the gauge transformation take $|A=0\rangle$ to a state that is relevant to semiclassical physics.

Let $\Omega_1(\vec{x}) = \exp\left(i\frac{\omega(r)}{2}\,\vec{\sigma}\cdot\hat{r}\right)$, $\omega(0) = 0$, $\omega(\infty) = 2\pi$

be generator of $\Pi_3(SU(2))$ (or of $\Pi_3(G)$, if $\vec{\sigma}$ denotes generator of $SU(2)$ subgroup of G), — actually with $\Omega|_{S^2_\infty} = -I$ (rather than I)

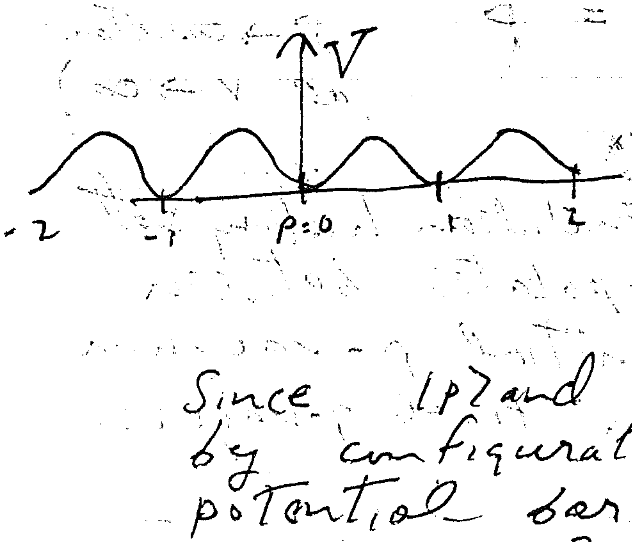
$\Omega_p = \Omega_1^{\,p}$ is winding number p gauge transformation — a general element of the symmetry group $\mathbb{Z}$ (of course, for $p<0$ $\Omega_p = (\Omega_1^{-1})^{|p|}$)

Since Z is abelian, we can decompose $\mathcal{H}_{phys}$ into irreps, that are one dimensional and labeled by an angular variable

$$\theta \in [0, 2\pi)$$

$$\Omega_p |\theta\rangle = e^{ip\theta} |\theta\rangle$$

We have discovered a continuously varying angular parameter in YM theory. Since the θ-parameter actually labels distinct vacua of the theory, different values of θ correspond to distinct superselection sectors (local physics can't change θ), as we'll see more explicitly below.



The Ω_p's acting on $|A=0\rangle$ give distinct classical vacua

$$|p\rangle = \Omega_p |0\rangle$$

Since $|p\rangle$ and $|p\pm 1\rangle$ cannot be connected by configurations of $S_E \to 0$, there is a potential barrier separating the neighboring vacua. But there is a finite amplitude for quantum <u>tunneling</u> from p to $p \pm 1$

Recall $(G = SU(2))$ that

$$p = \frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}(\Omega^{-1} d\Omega)^3 \qquad \text{and that}$$

the Chern-Simons three-form

$$\omega_3 = \mathrm{tr}\left(AF - \tfrac{1}{3}A^3\right)$$

satisfies $\quad d\omega_3 = \mathrm{tr}\, F^2$

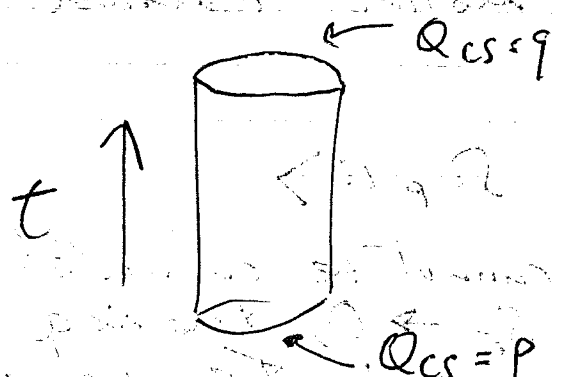
Integrating ω_3 over a 3-dim timeslice Σ gives the "Chern-Simons charge" on that slice

$$Q_{CS} = -\frac{1}{8\pi^2}\int_\Sigma \omega_3$$

And if $A = \Omega^{-1}d\Omega$ is "pure gauge" on the slice with $F = 0$

$$Q_{CS} = \frac{1}{24\pi^2}\int_\Sigma \mathrm{tr}(\Omega^{-1}d\Omega) = p$$

(where $\Omega \to$ constant at $r \to \infty$)



A Euclidean history that interpolates between an initial p-vacuum and a final q-vacuum has

$$\Delta Q_{CS} = q - p = -\frac{1}{8\pi^2}\left[\int_{\Sigma_{final}} \omega_3 - \int_{\Sigma_{init}} \omega_3\right]$$

$$= -\frac{1}{8\pi^2}\int d\omega_3 = -\frac{1}{8\pi^2}\int \mathrm{tr}\, F^2$$

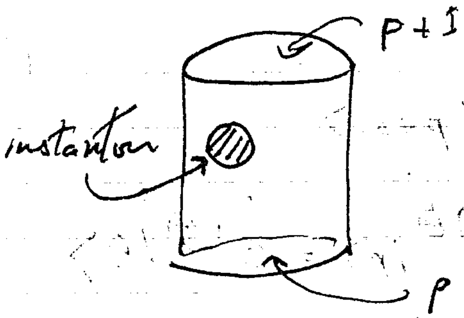
i.e. $\Delta Q_{CS} = \frac{g^2}{32\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv \nu$

S_E in the sector with $\nu \neq 0$ is minimized by an instanton configuration with

$$S_E = \frac{8\pi^2}{g^2}|\nu|$$

(solving $F = \pm\tilde{F}$, where $\pm = \text{sign}(\nu)$.)

The p-vacua are semiclassically unstable (like the classical ground states of the double well potential)



In the leading semiclassical approximation, tunneling from $|p\rangle$ to $|p+1\rangle$ is dominated by a single instanton — a configuration of finite size, whose center can be at any 4-dim position. We find

$$\langle p+1|e^{-Ht}|p\rangle = VT\, K e^{-8\pi^2/g^2}$$

VT: from integration over zero modes of instanton

K: integration over small fluctuations about instanton

$e^{-8\pi^2/g^2}$: e^{-S_0}

Lecture #25

We expect that actual energy eigenstates will be the irreps of the $\mathbb{Z}$ symmetry acting as

$$\Omega_K |p\rangle \rightarrow |p+K\rangle$$

These are the $|\theta\rangle$ states, which can be constructed from $|p\rangle$ states by Fourier Transforming

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} e^{-ip\theta} |p\rangle$$

Then $\langle\theta'|\theta\rangle = \frac{1}{2\pi} \sum_{p\in\mathbb{Z}} e^{-ip(\theta-\theta')} = \delta(\theta-\theta')$

and $\Omega_K |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_p e^{-ip\theta} |p+K\rangle$

$$= \frac{1}{\sqrt{2\pi}} \sum_p e^{i(p-K)\theta} |p\rangle = e^{i\theta} |\theta\rangle$$

The "θ-superselection rule" follows from the symmetry property, for any gauge-inv. local operator

$$\langle p|\mathcal{O}|q\rangle = \langle p|\Omega_K^{-1} \mathcal{O} \Omega_K|q\rangle$$

$$= \langle p+K|\mathcal{O}|q+K\rangle$$

so $\langle p|\mathcal{O}|q\rangle = A_{\mathcal{O}}(p-q)$: Just a function of the difference

Then $\langle \theta' | \mathcal{O} | \theta \rangle = \frac{1}{2\pi} \sum_{p,q} e^{ip\theta'} e^{-iq\theta} A_{\mathcal{O}}(p-q)$

shift $p = q + \nu$: $= \frac{1}{2\pi} \sum_{pq} e^{i\nu\theta'} A_{\mathcal{O}}(\nu)\, e^{iq(\theta'-\theta)}$

$= \delta(\theta - \theta')\, A_{\mathcal{O}}(\theta)$

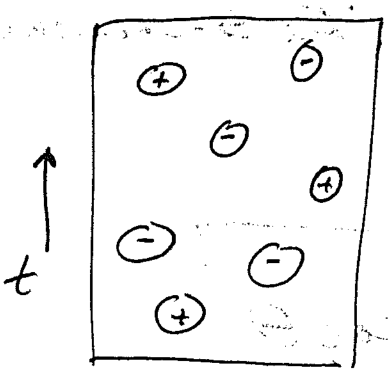
$$A_{\mathcal{O}}(\theta) = \sum_{\nu} e^{i\nu\theta} A_{\mathcal{O}}(\nu)$$

In particular, inserting $\mathcal{O} = I$ we find

$$\langle \theta' | e^{-Ht} | \theta \rangle \propto \delta(\theta' - \theta)$$

— the θ vacua are stable w.r.t. tunneling. Further, all local operators (commuting with Ω_K) respect the θ-sectors $\Rightarrow$ θ superselection.

In a sector labeled by θ, histories are weighted in the path integral by $e^{i\nu\theta}$, where ν is the instanton number. As in our discussion of the double-well potential, in the semiclassical evaluation of the path integral, we should sum over a "dilute gas" of instantons and anti instantons:



Actually, the field strength of an instanton falls off with distance as

$$F \sim O\left(\frac{1}{r^4}\right)$$

— not exponentially as for the double well — but still it is not a bad approximation to say

$$S_E \sim (n+m)\, 8\pi^2/g^2$$

for n widely separated instantons and m anti-instantons, and that small fluctuations nearly factorize, so we get a factor

$$Z_{0,0}\,(K)^{m+n}$$

from the Gaussian integration. If we sum over n and m, including VT for integrating over zero-modes of each instanton and anti-instanton, we obtain

$$\langle e^{-HT}\rangle_\theta = Z_{0,0} \sum_{n,m} \frac{1}{n!}\frac{1}{m!}\left(KVTe^{-S_0}\right)^{n+m} \cdot e^{i\theta(n-m)}$$

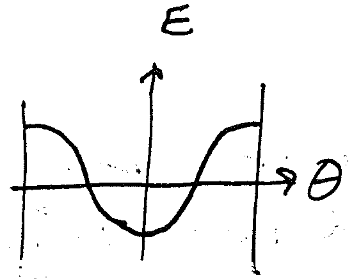
$$= Z_{00}\left(\exp\left(KVTe^{-S_0}e^{i\theta}\right)\cdot \exp\left(KVTe^{-S_0}e^{-i\theta}\right)\right)$$

$$= Z_{0,0}\, \exp\left(VT\, 2Ke^{-S_0}\cos\theta\right)$$

As $T \to \infty$, this is dominated by the energy of the ground state – the θ-vacuum

$\sim e^{-E(\theta)T}$, where

$$\boxed{E(\theta)/V = -2Ke^{-S_0}\cos\theta}$$



(cf. Band structure in a periodic potential in one-dim.)

The sum over n, m is dominated by

$$n, m \sim KVTe^{-S_0}$$

or $$\frac{n}{VT}, \frac{m}{VT} \sim Ke^{-S_0}$$ = "density of instantons"

Our semiclassical approximations are reasonable for a dilute instanton gas; i.e.

$$\text{density} << \rho^{-4}$$

where ρ is the <u>size</u> of the core of the instanton.

However — there is some bad news: the classical YM theory is scale invariant, and the parameter ρ — the instanton scale — is an arbitrary parameter; aside from the 4 zero modes for translation of the center, there is also a zero mode associated with <u>scale transformations</u>.

The action $\frac{1}{4g^2}\int F_{\mu\nu}F^{\mu\nu}$ is

invariant under $A(x) \to \lambda A(\lambda x)$

$$F(x) \to \lambda^2 F(\lambda x)$$

In a particular gauge, $\nu = 1$ solution to $F = *F$ (with center at $x=0$) is

$$-igA_\mu = \frac{x^2}{x^2+\rho^2}\,\Omega^{-1}\partial_\mu\Omega$$

$$\Omega(x) = \frac{x_0 + i\vec{\sigma}\cdot\vec{x}}{(x_0^2+\vec{x}^2)^{1/2}}$$

$$S = \frac{1}{g^2}\int d^4x \frac{48\rho^4}{(x^2+\rho^2)^4} = \frac{48\pi^2}{g^2}\int_0^\infty \frac{y\,dy}{(1+y)^4} \quad (x^2 = \rho^2 y)$$

$$= 8\pi^2/g^2$$

under $A_\mu(x) \to \lambda A_\mu(\lambda x)$

$$-igA_\mu \to \frac{\lambda^2x^2}{\lambda^2x^2+\rho^2}\,\Omega^{-1}(\lambda x)\,\partial_\mu\Omega(\lambda x)$$

or $\rho \to 1/\rho$

Actually, there are 8ν zero modes for the ν instanton solution, $(4+1)\nu$ for position and scale of each instanton, plus another 3ν for the "gauge orientation" of each instanton. The one-instanton solution is rotationally invariant, in the sense that a spatial rotation can be compensated by a gauge rotation. With ν instantons we can rotate the relative orientations (so really $8\nu - 3$ parameters) In $SU(N)$

This becomes $4N\nu$ parameters -- including $4N-5 = N^2-1 - [(N-2)^2-1] - 1$ for the orientation of the instanton's $SU(2)$ inside $SU(N)$.

The classical scale invariance is broken by quantum effects. The determinant has ultraviolet divergences, and we need to introduce a subtraction point μ to define it. Then

$$e^{-S_0} K \Rightarrow \int \frac{d\rho}{\rho^5} e^{-8\pi^2/g_\mu^2} f(\rho\mu)$$

– the $1/\rho^5$ from dimensional analysis, and the μ-dependence of g_μ cancels that of $f(\rho\mu)$

We may choose $\rho\mu = O(1)$ to avoid large logarithms in $f(\rho\mu)$, but then, (if one-loop β function ~~is a good approximation~~)

$$\mu \frac{dg^2}{d\mu} = -\frac{1}{8\pi^2} b g^4$$

[C_A = adjoint Casimir = N for $SU(N)$]

($b = \frac{11}{3} C_A$ in pure YM theory)

$$-8\pi^2 \frac{dg^2}{(g^2)^2} = b\, d(\ln \mu)$$

$$\Rightarrow \frac{8\pi^2}{g_\mu^2} = b \ln \mu/\Lambda \Rightarrow e^{-8\pi^2/g_\mu^2} = \left(\frac{\Lambda}{\mu}\right)^b$$

$$e^{-S_0} K \Rightarrow \Lambda^4 \int \frac{d\rho}{\rho} (\Lambda\rho)^{b-4}$$

– the dilute gas approximation is reasonable

for small instantons $\rho\Lambda \ll 1$;
but for $b > 4$ (true in QCD w/ 3 light quarks: $b = 11 - \frac{2}{3}N_F = 9$) it goes grievously wrong for $\rho\Lambda = O(1)$, where the ρ integral is dominated.

Therefore, semiclassical evaluation of the Θ-dependence of the vacuum energy cannot really be justified.

But our qualitative insight — the existence of Θ superselection sectors — is robust.

Θ as a coupling constant

If $\mathbb{R}^4$ is compactified to S^4, then gauge field configurations are classified by

$$\nu = -\frac{1}{8\pi^2}\int \mathrm{tr}\, F^2 = \text{integer}.$$

Then same is true on $\mathbb{R}^4$ for configurations of finite Euclidean action.

On S^4, the expectation value of a gauge-invariant local operator in a Θ sector is

$$\langle \mathcal{O} \rangle_\Theta = \frac{\sum_\nu e^{i\Theta\nu}\int(dA)_\nu\, \mathcal{O}\, e^{-S_E[A]}}{\sum_\nu e^{i\Theta\nu}\int(dA)_\nu\, e^{-S_E[A]}}$$

where $\int(dA)_\nu$ means sum over histories with specified ν.

But since ν is the integral of a local density we can move the $e^{i\theta\nu}$ inside the integral, replacing

$$S_E \to S_E - i\theta\nu$$

$$= \int \mathrm{tr}\left[-\frac{1}{g^2} F * F + \frac{i\theta}{8\pi^2} F^2\right]$$

$$= \int d^4x_E \left[\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{i\theta g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a}\right]$$

In Lorentzian spacetime e^{-S_E} continues to e^{iS}, where

$$S = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{\theta g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a}\right)$$

$$\left[\; d^4x \to -i\, d^4x_E \qquad \varepsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} \to \left(-i\, \varepsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma}\right)_E \;\right]$$

Thus, we can interpret θ as a coupling constant that appears in the Lagrange density of the Yang-Mills theory. Since

$$\mathrm{tr}\, F^2 = d\omega_3$$

the "θ-term" is a total derivative (locally) and has no effect at all on the classical equations of motion. But as we've seen, it does have nonperturbative quantum effects that are $O(\exp(-C/\hbar))$.

For the theory on $\mathbb{R}^4$ rather than S^4, the dominant contributions to the path integral come from configurations of ∞ action, and ν need not be an integer, but the "θ as coupling constant" formulation provides a way to define the θ parameter that agrees with our semiclassical interpretation.

Notice that the θ-term is imaginary when we continue to S_E. This is characteristic of local operators in the action that violate T (or CP).

	C	P	T	CP	CPT
E	−	−	+	+	+
B	−	+	−	−	+
E·B	+	−	−	−	+

Thus under CP: $\theta \to -\theta$.
The θ term is CP violating unless θ, an angular variable, is $\theta = \pm\pi$

This is a serious problem for QCD phenomenology. The electric dipole moment of the neutron is a CP-odd observable (spin $\sigma \to -\sigma$ under CP, so $\sigma \cdot E \to -\sigma \cdot E$), and the experimental limit

$$EDM \lesssim 10^{-25}\, e\,cm$$

has been used to estimate $|\theta| \lesssim 10^{-9}$.
($\theta \sim \pi$ doesn't seem to work)

Why so small? This is the "θ-puzzle" or "strong CP problem"

Is θ really a periodic variable ∈ [0, 2π)? Our argument for periodicity applies for the theory on S^4, and it is reasonable to expect the identification

$$\theta \sim \theta + 2\pi$$

to apply independent of boundary conditions. When we learn about anomalies, we'll be able to give an even more convincing argument that θ is an angle.

<u>The θ-dependent dyon charge</u>

In the charge-0 (vacuum) sector of YM theory, θ dependence is nonperturbative, and highly suppressed in the semiclassical limit. But curiously, in the presence of a magnetic monopole, θ dependence is more manifest, and <u>not</u> suppressed by $\exp(-8\pi^2/g^2)$.

Consider the 't Hooft-Polyakov model with a Higgs triplet, and

$$SU(2) \to U(1).$$

There is a monopole solution, and the asymptotic behavior of the Higgs field in the "radial gauge" is:

$$\phi^a\Big|_{S^2_\infty} = v\hat{r}^a$$

Now consider in this gauge the SU(2) gauge transformation

$$\Omega(\vec{x}) = \exp\left[i\frac{\omega(r)}{2}\vec{\sigma}\cdot\hat{r}\right]$$

where $\omega(r) = \begin{cases} 0 & r=0 \\ \omega_0 & r=\infty \end{cases}$

In our discussion of the vacuum structure, we dismissed this gauge transformation for $\omega_0 \neq 2\pi\cdot(\text{integer})$ on the grounds that $A = \Omega^{-1}d\Omega$ is nonzero on S^2_∞, and not "semiclassically accessible" from $A=0$. But now the situation is different. $\Omega(\vec{x})$ preserves the asymptotic Higgs field and gauge field, as

$$[D,\omega]\Big|_{S^2_\infty} = 0$$

This is most easily seen if we transform (locally) to the unitary gauge with

$$\phi = v\phi_0 = \text{constant},$$

for in that gauge $\omega = \text{constant}$, and

$$A \to A + d\omega = A.$$

On the monopole vacuum, then, unlike in the vacuum sector,

$$|A^{\Omega}, \phi^{\Omega}\rangle$$

should be regarded as "close to" $|A, \phi\rangle$ for $\Omega = I + \omega$ and $\omega|_{S^2_\infty} \neq 0$, infinitesimal.

What is going on is that the monopole has a "charge zero mode" or collective coordinate. The gauge transformation $\Omega(\vec{x})$ should really be regarded as a <u>global</u> gauge rotation acting on the monopole solution, as it is a transformation in the unbroken $U(1)$ at each point on S^2_∞. Unlike in the vacuum, the (global $U(1)$) does not leave the 't Hooft-Polyakov solution invariant, it takes it to another nearby solution, with a core that has a slightly different orientation in the internal symmetry space.

Formally, we can think of ω_0 as a dynamical variable that must be quantized. We then allow it to be time-dependent

$$\omega_0(t)$$

but with A_0 still fixed at zero, so that $\omega_0(t)$ is no longer merely a gauge transformation. Furthermore, we need to demand that the Gauss law constraint be still satisfied, which determines $\omega(\vec{x},t)$ given its asymptotic value $\omega_0(t)$, and $\omega(\vec{x},t)$, though it preserves ϕ, A asymptotically, deforms the monopole core.

(More details are in §2.7 of the Les Houches lectures.)

Furthermore, in the case $\omega_0 = 2\pi$, $\Omega(\vec{x})$ *is* an $SU(2)$ gauge transformation with winding one, which we know should be represented by phase $e^{i\theta}$ in the θ-sector of the theory.

And this $\Omega(\vec{x})$ should be regarded as a $U(1)$ gauge transformation: Our general formula told us that $\Omega = \exp(i\omega^a T^a)$ is represented on H_{phys} by

$$\exp\left(-\frac{i}{e}\int d^2\vec{S}\cdot(\omega^a \vec{E}^a)\right)$$

In this case $T^a = \frac{1}{2}\sigma^a$ and $\omega^a/_2 = \omega_0 \hat{r}^a = \hat{\phi}^a \omega_0$

and as on p.175, the electric charge associated with the unbroken $U(1)$ is

$$Q = \int d^2 S_i\, \hat{\phi}^a E_i^a$$

We conclude, then, that

$$\exp\left(-i\omega_0 \frac{Q}{e}\right)$$

represents $\Omega(\vec{x})$ in the monopole sector, and for $\omega_0 = 2\pi$ we have

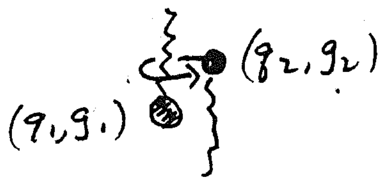
$$\exp\left(-i2\pi \frac{Q}{e}\right) = e^{i\theta} \Rightarrow$$

$$\boxed{Q = ne - \frac{e\theta}{2\pi}} \quad , \; n = \text{integer}$$

!!?

The eigenvalues are shifted away from integer values... the shift respects periodicity of θ, as charge spectrum is invariant under $\theta \to \theta + 2\pi$

— But what about the Dirac quantization condition?



If charge-e particles with no magnetic charge exist, then Dirac $\Rightarrow$

$$g = g_D \cdot \text{integer}, \quad g_D = \frac{2\pi}{e}$$

But for two dyons with charges (electric, mag) = (q, g) we require that each is unable to detect the string of the other, which means

$$q_1 g_2 - q_2 g_1 = 2\pi \cdot \text{integer}$$

or, if $q = \tilde{q} e$ and $g = \frac{2\pi}{e} m$,

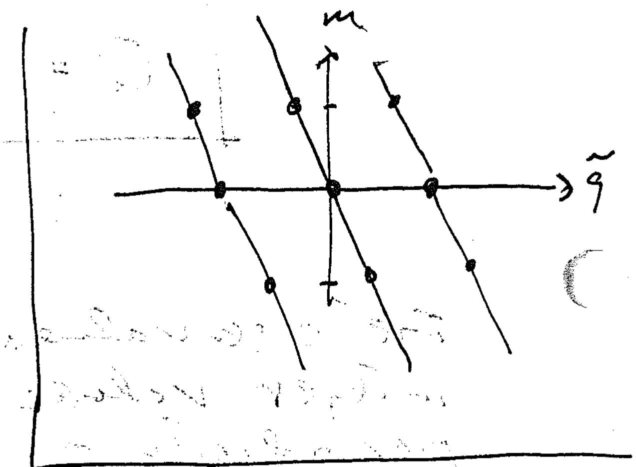
$$\tilde{q}_1 m_2 - \tilde{q}_2 m_1 = \text{integer}$$

And this is solved if

$$\tilde{q} = \text{integer} - \frac{\theta}{2\pi} m .$$

Charge offsets are allowed, if the noninteger electric charge is proportional to the magnetic charge.

And that is just what we have found!



If the asymptotic Higgs field has winding number m on S^2 then

$$\Omega(\vec{x}) = \exp\left(i \frac{\omega(r)}{2} \vec{\sigma} \cdot \hat{\phi}\right)$$

has winding m on $SU(2)$, for $\omega|_{S^2_\infty} = 2\pi$

and we have

$$\exp(-2\pi i \frac{Q}{e}) = e^{im\theta} \Rightarrow \frac{Q}{e} = n - \frac{\theta}{2\pi} m$$

Note that under CP: $\theta \to -\theta$, as $\tilde{q}$ and m have opposite CP properties. Thus the charge spectrum is

CP invariant only if $\Theta = 0, \pi$. We have already seen that the YM theory is CP-noninvariant for $\Theta \neq 0, \pi$

Now

$$\exp\left(2\pi i \frac{Q}{e}\right) = e^{-i\Theta}$$

is not so shocking if the unbroken symmetry is $U(1)$ — which is not simply connected and can have projective representations (Still, it is a surprise since in the 't Hooft - Polyakov model the $U(1)$ is embedded in $SU(2)$ — which is simply connected...)

But the situation is more puzzling if

$$G \longrightarrow H$$

and the unbroken group H is nonabelian. E.g. grand unified monopoles carry

$$[SU(3) \times U(1)]/Z_3$$

magnetic charge. How can an $SU(3)$ electric charge be subjected to a Θ-dependent shift? Then $SU(3)$ generators are quantized to eigenvalue T^a = half integer due to the <u>commutation</u> relations of the generators.

We need to understand this, but first let's derive how the charge spectrum is modified in the non-abelian case. We'll do this by a different method that is instructive anyway, working in the unitary gauge ϕ = constant, and treating θ as a coupling in the Lagrangian.

Including the θ term, the action is

$$\mathcal{L} = \tfrac{1}{2} E_i^a E_i^a - \tfrac{1}{2} B_i^a B_i^a + \frac{e^2\theta}{8\pi^2} E_i^a B_i^a,$$

and the momentum conjugate to A_i^a is

$$\frac{\partial \mathcal{L}}{\partial \dot{A}_i^a} = E_i^a + \frac{e^2\theta}{8\pi^2} B_i^a,$$

so we need to revisit our discussion of what the Gauss law implies about charge quantization

From the Gauss law we still have

$$\Omega = (I - i e \omega^a T^a) \Rightarrow$$

$$0 = \int_{S^2_\infty} d^2\vec{S} \cdot \omega^a \vec{E}^a - \int d^3x \, \delta_\omega \vec{A}^a \cdot \vec{E}^a$$

In addition we have $[D_i, B_i^a] = 0$, so that

$$0 = \int_{S^2_\infty} d^2\vec{S} \cdot \omega^a \vec{B}^a - \int d^3x \, \delta_\omega \vec{A}^a \cdot \vec{B}^a,$$

provided that $\omega(\vec{x}) = 0$ in the vicinity of any point monopole singularity where $[D_i, B_i^a]$ might break down.

From $\frac{\delta}{\delta A_i^a} = i\Pi_i^a = i\left(E_i^a + \frac{e^2\theta}{8\pi^2} B_i^a\right)$,

we take linear combinations of these equations to find

$$\delta_\omega = \int d^3x\, \delta_\omega A_i^a \frac{\delta}{\delta A_i^a} = i\int d^3x\, \delta_\omega A_i^a \left(E_i^a + \frac{e^2\theta}{8\pi^2} B_i^a\right)$$

$$= i\int_{S^2_\infty} d^2\vec{S}\cdot\left[\omega^a\left(\vec{E}^a + \frac{e^2\theta}{8\pi^2}\vec{B}^a\right)\right]$$

which exponentiates to $\Omega = \exp(-ie\omega^a T^a) \Rightarrow$

$$U[\omega] = \exp\left[i\int_{S^2_\infty} d^2\vec{S}\cdot\omega^a\left(\vec{E}^a + \frac{e^2\theta}{8\pi^2}\vec{B}^a\right)\right]$$

For a global gauge transformation with

$$\omega^a\big|_{S^2_\infty} = \omega_0^a = \text{constant},$$

this becomes

$$U[\omega] = \exp\left[i\omega_0^a \int_{S^2_\infty} d^2\vec{S}\cdot\left(\vec{E}^a + \frac{e^2\theta}{8\pi^2}\vec{B}^a\right)\right]$$

$$= \exp\left[i\omega_0^a\left(Q^a + \frac{e\theta}{2\pi} Q_m^a\right)\right],$$

where $\vec{E}\big|_{S^2_\infty} \sim \hat{r}\,\frac{Q}{4\pi r^2}$

$\vec{B}\big|_{S^2_\infty} \sim \hat{r}\,\frac{Q_m}{4\pi r^2}\cdot\left(\frac{4\pi}{e}\right)$ defines operators Q and Q_m

Dirac $\Rightarrow$ Q_m has integer eigenvalues, since $4\pi/e$ is the Dirac quantum if eigenvalues of Q/e are half integer

This is $U[\omega] = \exp\left[2i\,\mathrm{tr}\,\omega\left(Q + \frac{e\theta}{2\pi}Q_m\right)\right]$

representing $\Omega : \exp(-ie\omega)$

Hence we should have $U[\omega] = I$, for

Eigenvalues $\frac{e\omega}{2\pi}$ = integer $\Rightarrow$

Eigenvalues of $\frac{Q}{e} + \frac{\theta}{2\pi}Q_m = \frac{1}{2}(\text{integer})$

Here Q_m has integer eigenvalues, so that $\theta \to \theta + 2\pi$ shifts changes by integers rather than half integers. That makes sense, since in irreducible representations of H, the eigenvalues of the generators have <u>spacing</u> = integer

We have found that the θ-dependent shift in electric charge is proportional to the magnetic charge Q_M. This conclusion sounds peculiar.

For example, a GUT monopole with $[SU(3)\times U(1)]/Z_3$ (standard model) magnetic charge, if minimal, has (in a particular gauge)

$$Q_M = \mathrm{diag}\left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right) + \mathrm{diag}\left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\right)$$

$$= \mathrm{diag}(0, 0, -1),$$

acting on a quark triplet. Since this is a linear combination of an $SU(3)_C$ generator and a $U(1)_{em}$ generator, both electric charge and color electric charge are shifted by θ. But how can this be? Eigenvalues of the (normalized) $SU(3)$ generators are required to be quantized as a consequence of the commutation relations that they satisfy...?!

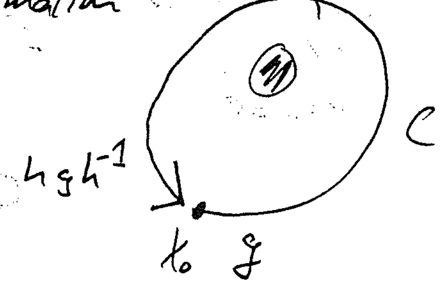
To resolve this puzzle, we must think carefully about what is meant by a global gauge transformation in the background of a magnetic monopole — or more physically, how we would measure the electric charge of a monopole...

Indeed, the situation is reminiscent of the problem of measuring the charge of a nonabelian vortex, which we considered earlier.

If H is an unbroken nonabelian gauge symmetry, and a vortex has flux

$$P\exp\left(i\int_{C,x_0} A\right) = h \in H$$

then there is an obstruction to constructing a global g gauge transformation if g does not commute with H. Mathematically,

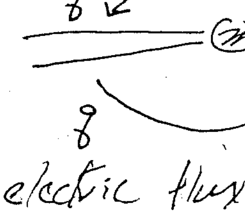


a g gauge transformation, transported about a path C that encloses the vortex, becomes a $hgh^{-1} \neq g$ gauge transformation.

Physically, we could measure the charge by observing the AB interaction of the charge with a g-flux, but there is no AB interference if the g flux changes its identity upon circumnavigating the charge. Only a charge in $N(h) \subset H$, the normalizer of the flux, makes mathematical or physical sense.

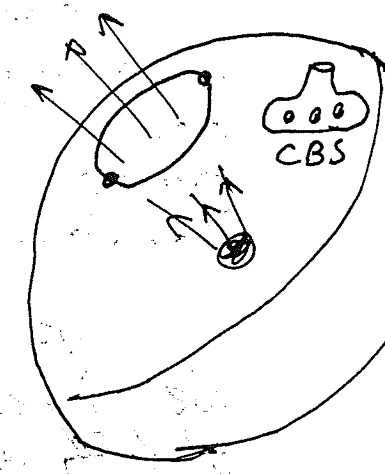
Or — in the case of the Alice vortex, we measure the local electric field by observing the response of a test charge q to the local electric field.

But a charge q becomes charge $-q$ when carried around the vortex — and the sign of E has no global meaning



— we can't define or measure the electric flux through a circle

On physical grounds, we may anticipate difficulty measuring (and hence defining) the electric charge of a nonabelian magnetic monopole. I measure an electric charge by observing the response of colored test charges to the color electric field. But I need to establish global conventions for the color of the test charges on a large two-sphere.



I can do that by transporting test charges to a "charge Bureau of standards" where standard R, Y, B quarks are carefully preserved. Furthermore, if there is no magnetic charge, all B fields fall off with distance faster than $1/r^2$, so on a large sphere, there is no obstruction to establishing global conventions. (When we carry a red quark around a large closed path on the big two-sphere, it is still red when it returns home.

This isn't so if there is magnetic charge and $B \sim \frac{1}{r^2}$ — then the flux enclosed by a path that spans nonzero solid angle is nonzero, and a red quark might return home as a yellow quark. This difficulty prevents us from measuring a charge Q that does not commute with the magnetic charge Q_m — only charges that commute with Q_m are realizable globally on the large two-sphere.

Again, there is a corresponding mathematical statement: a topological obstruction to implementing a global gauge transformation on the monopole background.

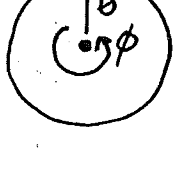
To implement a global H-transformation, we must have H generators that are "globally defined" on the two-sphere. That is — generators

$$T^a(\theta,\phi)$$

that satisfy the proper commutation relations at each point (θ,ϕ) of S^2, and that vary "smoothly" on S^2. On a monopole background "smoothly" means that $T^a(\theta,\phi)$ continuous on upper/lower patches, and respect the matching conditions at the equator:

$$T^a_U = \Omega(\phi)\, T^a_L\, \Omega(\phi)^{-1} \quad \text{at } \theta = \tfrac{\pi}{2} \ \text{(equator)}$$

Alternatively (shrinking the lower hemisphere to infinitesimal size), we may represent the sphere as a disk, with



$T^a(\theta,\phi)$ continuous on the disk,

while at the boundary:

$T^a(\theta=\pi,\phi) = \Omega(\phi)\, \tilde{T}_0^a\, \Omega(\phi)^{-1}$, where

$\tilde{T}_0^a$ is a (ϕ-independent) constant, and $\Omega(\phi)$ is a noncontractible loop in H (whose homotopy class $\in \pi_1(H)$ is the magnetic charge). One way to interpret this condition is that, if we do a "singular gauge transformation" to remove the Dirac string at the south pole ($\theta=\pi$), then T^a must be continuous at $\theta=\pi$.

Recall (cf page 201) that for H semisimple, an automorphism of the H Lie algebra (map $T'^a = f(T^a)$ that preserves the structure constants) that is connected to the identity automorphism in the group of automorphisms is an <u>inner</u> automorphism

$$T^a \to \Sigma T^a \Sigma^{-1}, \quad \Sigma \in H.$$

We may write, on the disk

$$T^a(\theta,\phi) = \Sigma(\theta,\phi)\, T_0^a\, \Sigma(\theta,\phi)^{-1}$$

where T_0^a constant (independent of θ,ϕ). The matching requirement

$$T^a(\theta=\pi,\phi) = \Omega(\phi)\, \tilde{T}_0^a\, \Omega(\phi)^{-1}$$
$$= \Sigma(\theta=\pi,\phi)\, \tilde{T}_0^a\, \Sigma(\theta=\pi,\phi)^{-1}$$

then becomes
$$\tilde{T}_0^a = \Sigma(\theta=\pi,\phi)^{-1}\Omega(\phi)\, T_0^a\, \Omega(\phi)^{-1}\Sigma(\theta=\pi,\phi);$$

i.e., $\Sigma^{-1}(\theta=\pi,\phi)\Omega(\phi) = \Omega_0$ is a <u>constant</u> inner automorphism, independent of ϕ

and hence $\Sigma^{-1}(\theta=\pi,\phi)\,\Omega(\phi)\,\Omega_0^{-1}$
is a trivial inner automorphism, i.e.

commutes with all generators, and so lies in center of H. But the center of semisimple H is discrete, so we conclude that

$$\Sigma^{-1}(\theta=\pi,\phi)\,\Omega(\phi)\,\Omega_0^{-1} = h \in \text{center}(H)$$

is a constant group element, or

$$\Omega(\phi) = \Sigma(\theta=\pi,\phi)\,\tilde{\Omega}_0 \qquad (\tilde{\Omega}_0 = h\Omega_0)$$

Since $\Sigma(\theta,\phi)$ is continuous on the disk, varying θ from π to 0 (contracting the boundary to the north pole) defines a homotopy that deforms $\Omega(\phi)$ to a constant. Thus, $\Omega(\phi)$ is homotopically trivial, and the H-magnetic charge vanishes. We have

<u>Theorem</u>: For H semisimple, H gauge transformations are globally realizable only if H magnetic charge is <u>zero</u>.

More generally, what if we wish to globally realize only the subgroup $H'\subset H$? Our argument shows that $\Sigma(\theta,\phi)$ defines a deformation of the loop $\Omega(\phi)$ in H to a loop that commutes with the H' generators. We have:

$H' \subset H$ is globally realizable only if $\Omega(\phi)$ can be deformed to commute with H'

This conclusion is in accord with expectations. Consider a long range magnetic field that can be described (in a particular gauge) by

$$e\vec{A}\cdot d\vec{r} = Q_m \tfrac{1}{2}(1-\cos\theta)\,d\phi$$

then the H' that commutes with Q_m is globally realizable. The H' generators commute with the magnetic field

$$B = Q_m \frac{\hat{r}}{er^2}$$

so that H' test charges can be globally calibrated on S^2_∞.

So, for example, a Z_3 monopole in $H = SU(3)/Z_3$ has

$$Q_m = (1, 1, -2)$$

Charges on this monopole background do not

comprise complete $SU(3)/Z_3$ multiplets
Rather, the globally realizable symmetry
commutes with Q_m — it is

$$H' = [SU(2) \times U(1)]/Z_2$$

Or -- consider the pattern of Higgs symmetry breakdown

$$G = SU(3) \to H = [SU(2) \times U(1)]/Z_2,$$

discussed e.g. on p.190, driven by Adjoint Higgs field

$$\langle \Phi \rangle = \Phi_0 = v\left(\tfrac{1}{2}, \tfrac{1}{2}, -1\right)$$

The minimal monopole has

$$Q_m = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right) + \left(\tfrac{1}{2}, \tfrac{1}{2}, -1\right) = (1\ 0\ -1)$$

$\uparrow$ T_3 of $SU(2)$ $\quad$ $\uparrow$ $U(1)$ gen. $= Q$

Naively, charge on the monopole background would be an irrep of H $[SU(2) \times U(1)]/Z_2$, but now we know that is wrong, only an

$$H' = [U(1) \times U(1)]/Z_2$$

generated T_3, Q is globally realizable

What about the charge spectrum that arises from the semiclassical quantization of the charge collective coordinate of the monopole?

The monopole solution is just the 't Hooft Polyakov solution embedded in the SU(2) subgroup of SU(3) with

$$Q' = Q_m = T_3 + Q = (1, 0, -1)$$

as diagonal generator, while

$$Q'' = 3T^3 - Q = (1, -2, 1)$$

acts trivially on the monopole — the collective coordinate lives in Q' and only Q' charges occur when we quantize (i.e. $3T^3 = Q$)

The lowest lying charge excitations have $T^3 = \pm\frac{1}{2}$, $Q = \pm\frac{3}{2}$] these are just the quantum numbers of the heavy gauge bosons, as generators of SU(3) decompose under H as

$$8 \to 3^0 + 1^0 + 2^{3/2} + 2^{-3/2}$$

Crudely speaking, the charge arises from heavy gauge bosons excited in and bound to the monopole core.

Low lying dyons can't decay by emitting heavy vectors, as

$$M_{dyon} \sim \sqrt{g^2+q^2}\, v$$ (cf Bogomolnyi bound) — page 176

or $M_{dyon} \sim g v \sqrt{1 + q^2/g^2}$

$$M_{dyon}(q) - M_{monopole} \sim g v \frac{q^2}{2g^2}$$

$q \sim e$, $g \sim \frac{4\pi}{e}$ $\Rightarrow$ $\Delta M \sim \frac{e^3 v}{4\pi}$] the right order of magnitude for $\frac{Q^2}{4\pi R}$, $Q \sim e$, $R \sim (ev)^{-1}$

$\Delta m \sim \frac{e^2}{4\pi} e v \sim \alpha \mu$

$\Rightarrow$ of order $1/\alpha$ stable excitations, if massive vector is lightest charged particle

Now what happens if $\Theta_{SU(3)} \neq 0$?
This becomes Θ for $SU(2)$ and $U(1)$

Eigenvalues of $\frac{Q}{e} + \frac{\Theta}{2\pi} Q_m = \frac{1}{2} \cdot$ integer

(where $Q = Q^a T^a$, $\mathrm{tr}\, T^a T^b = \frac{1}{2}$)

Since $\frac{1}{2} Q'$ obeys this normalization, and $Q_m = Q'$, we have

$$\frac{1}{2} \frac{Q'}{e} + \frac{\Theta}{2\pi} Q =$$

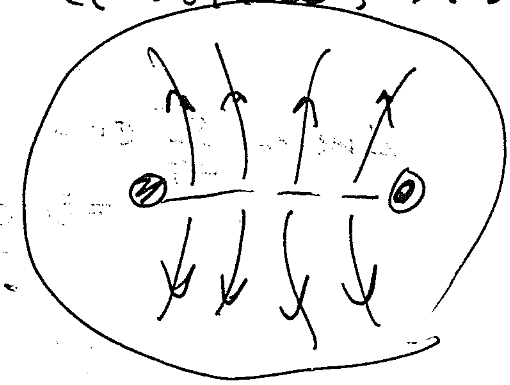
$$\frac{Q' T'}{e} + \frac{\Theta}{2\pi} 2T' = \left(\frac{Q'}{e} + \frac{\Theta}{\pi}\right) T' = \text{half integer}$$

$$\Rightarrow \frac{Q'}{e} = \text{integer} - \frac{\Theta}{\pi}$$

In fact, while vector has $Q' = 2, 1, -1, -2$, the dyon spectrum has $Q' =$ even, so spectrum invariant when $\theta \to \theta + 2\pi$.

(Since $Q = 3T^3$, $Q' = T_3 + Q = 4T^3$ = even integer in the dyon spectrum).

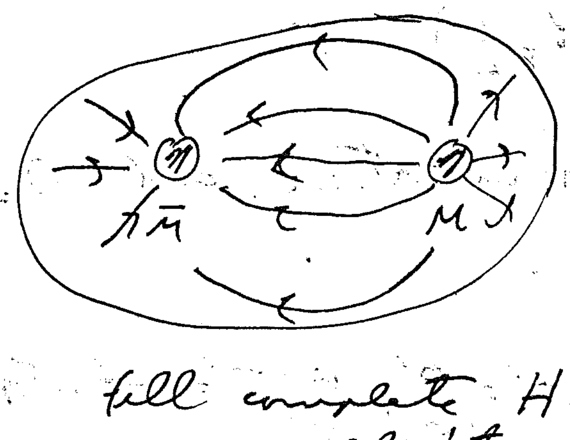
In the core of Alice vortices, recall the phenomenon of Cheshire charge–



For an isolated vortex, charge is ill defined.

But a pair of vortices can have charge with no localized source (in particular, the charge is not localized on the vortex core...) The coulomb energy → 0 as pair separates

Is there an analog of Cheshire charge for a pair of monopoles?



The monopole pair has vanishing magnetic charge, and so admits global H transformations – states fill complete H irreps. But not all states in a multiplet are localized near a monopole core. If we semiclassically quantize a configuration with $M, \bar{M}$ at fixed positions, global gauge trans generator T^a with $[T^a, Q_m] \neq 0$ acts on the long-range field of the monopole, while those with

$[T^a, Q_m] = 0$ rotate the monopole core. The "missing" excitations in the H multiplets are excitations of the long range field, with energy $\to 0$ as MM separation increases.